# The geodesic flow of a nonpositively curved graph manifold 

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#### Abstract

We consider discrete cocompact isometric actions $G \stackrel{\rho}{\curvearrowright} X$ where $X$ is a locally compact Hadamard space ${ }^{1}$, and $G$ belongs to a class of groups ("admissible groups") which includes fundamental groups of 3 -dimensional graph manifolds. We identify invariants ("geometric data") of the action $\rho$ which determine, and are determined by, the equivariant homeomorphism type of the action $G \stackrel{\partial_{\infty} \rho}{\curvearrowright} \partial_{\infty} X$ of $G$ on the ideal boundary of $X$. Moreover, if $G \stackrel{\rho_{i}}{\curvearrowright} X_{i}$ are two actions with the same geometric data and $\Phi: X_{1} \rightarrow X_{2}$ is a $G$ equivariant quasi-isometry, then for every geodesic ray $\gamma_{1}:[0, \infty) \rightarrow X_{1}$, there is a geodesic ray $\gamma_{2}:[0, \infty) \rightarrow X_{2}$ (unique up to equivalence) so that $\lim _{t \rightarrow \infty} \frac{1}{t} d_{X_{2}}\left(\Phi \circ \gamma_{1}(t), \gamma_{2}([0, \infty))\right)=0$. This work was inspired by (and answers) a question of Gromov in [Gro93, p. 136].


## 1. Introduction

As a consequence of the Morse lemma on quasi-geodesics, geodesic flows are especially simple and well understood in the Gromov hyperbolic case :
a. If $\phi: M_{1} \rightarrow M_{2}$ is a homotopy equivalence between closed negatively curved manifolds, then there is an orbit equivalence $\hat{\phi}: S M_{1} \rightarrow S M_{2}$ between the unit sphere bundles, which covers $\phi$ up to homotopy [Gro76].
b. If $G$ is a hyperbolic group, $G \stackrel{\rho_{i}}{\curvearrowright} X_{i}$ is a discrete, cocompact, isometric action on a Hadamard space $X_{i}$ for $i=1,2$, and $\Phi: X_{1} \rightarrow X_{2}$ is a $G$-equivariant quasiisometry, then $\Phi$ maps each geodesic $\gamma_{1} \subset X_{1}$ to a subset at uniformly bounded Hausdorff distance from a geodesic $\gamma_{2} \subset X_{2}$. Moreover, $\Phi$ induces an equivariant homeomorphism $\partial_{\infty} \Phi: \partial_{\infty} X_{1} \rightarrow \partial_{\infty} X_{2}$ between ideal boundaries, [Gro87].

[^0]c. When $G \stackrel{\rho}{\curvearrowright} X$ is a discrete, cocompact action of a hyperbolic group on a Hadamard space $X$, then the induced action $G \stackrel{\partial_{\infty} \rho}{\curvearrowright} \partial_{\infty} X$ of $G$ on the boundary of $X$ is a finitely presented dynamical system, [Gro87, CDP90].

Naturally one may ask if properties $b$ and c hold without the assumption of Gromov hyperbolicity. It turns out that they do not: one can readily produce examples of pairs of discrete, cocompact, isometric actions $G \curvearrowright X_{1}, G \curvearrowright X_{2}$ where $G$-equivariant quasi-isometries $X_{1} \rightarrow X_{2}$ do not induce boundary homeomorphisms ${ }^{2}$ (this was observed independently by Ruane [Rua96]). In [Gro93] Gromov asked whether two actions $G \curvearrowright X_{i}$ induce $G$-equivariantly homeomorphic boundary actions $G \curvearrowright \partial_{\infty} X_{i}$. The answer to this is also no: S. Buyalo [Buy98] and the authors independently found pairs of actions which induce inequivalent boundary actions ${ }^{3}$. Finally, we remark that the boundary action $G \curvearrowright \partial_{\infty} X$ is finitely presented if and only if $G$ is hyperbolic ${ }^{4}$.

In this paper we examine actions $G \curvearrowright X$ where $G$ belongs to a class of groups which generalize fundamental groups of 3-dimensional graph manifolds. We develop a kind of "coding" for geodesic rays in $X$, which allows us to understand the boundary action $G \curvearrowright \partial_{\infty} X$ and the Tits metric on $\partial_{\infty} X$. Before stating our main result in complete generality, we first formulate it for nonpositively curved 3-dimensional graph manifolds.

By the theorem of [Sch86], when $M$ is a 3 -dimensional graph manifold with a nonpositively curved Riemannian metric, then $M$ has the following structure. There is a collection $M_{1}, \ldots, M_{k}$ of compact nonpositively curved 3-manifolds with nonempty totally geodesic boundary (the geometric Seifert components of $M$ ), and Seifert fibrations $M_{i} \xrightarrow{p_{i}} N_{i}$ where the metric on $M_{i}$ has local product structure compatible with the fibration $p_{i}$, and the $N_{i}$ are nonpositively curved orbifolds; $M$ is obtained from the disjoint union $\amalg_{i} M_{i}$ by gluing boundary components isometrically in pairs via gluing isometries which are incompatible with the boundary fiberings. In what follows we will only consider graph manifolds whose Seifert fibered components have orientable fiber. Note that for each $1 \leq i \leq k$, the universal cover of $M_{i}$ is isometric to a Riemannian product $\tilde{N}_{i} \times \mathbb{R}$; the action of $\pi_{1}\left(M_{i}\right)$ on $\tilde{M}_{i}$ preserves this product structure and so there is an induced action of $\pi_{1}\left(M_{i}\right)$ on the $\mathbb{R}$ factor by translations. Hence we get a homomorphism $\tau_{i}: \pi_{1}\left(M_{i}\right) \rightarrow \mathbb{R}$ for each $i$. We may also define a class function $M L S_{i}: \pi_{1}\left(M_{i}\right) \rightarrow \mathbb{R}_{+}$by taking the minimum of the displacement function for the induced action $\pi_{1}\left(M_{i}\right) \curvearrowright \tilde{N}_{i}$, i.e. $\operatorname{MLS} S_{i}(g)=\inf \left\{d_{\tilde{N}_{i}}(g x, x) \mid x \in \tilde{N}_{i}\right\}$; this corresponds to the marked length spectrum of the nonpositively curved orbifold $N_{i}$.

Now suppose $M$ and $M^{\prime}$ are graph manifolds as above, and $f: M \rightarrow M^{\prime}$ is a homotopy equivalence. Embedded incompressible tori in Haken manifolds are determined up to isotopy by their fundamental groups up to conjugacy [Lau74], so we may assume after isotoping $f$ that it is a homeomorphism which induces homeomorphisms

[^1]$f_{i}: M_{i} \rightarrow M_{i}^{\prime}$ from the Seifert components of $M$ to the Seifert components of $M^{\prime}$ (and hence isomorphisms on the corresponding fundamental groups). We may then use the maps $f_{i}$ to compare the invariants $\tau_{i}, \tau_{i}^{\prime}$ and $M L S_{i}, M L S_{i}^{\prime}$.

Theorem 1.1. The following are equivalent:

1. The functions $M L S_{i}$ and $\tau_{i}$ are preserved up to scale by $f_{i}$ : for $i=1, \ldots, k$ there are constants $a_{i}$ and $b_{i}$ so that $M L S_{i}=a_{i} f^{*}\left(M L S_{i}^{\prime}\right)$ and $\tau_{i}=b_{i} f^{*}\left(\tau_{i}^{\prime}\right)$.
2. Any lift $\tilde{f}: \tilde{M} \rightarrow \tilde{M}^{\prime}$ of $f$ extends continuously to a map $\bar{f}: M \cup \partial_{\infty} M \rightarrow$ $M^{\prime} \cup \partial_{\infty} M^{\prime}$ between the standard compactifications.
3. If $\tilde{f}: \tilde{M} \rightarrow \tilde{M}^{\prime}$ is any lift of $f$, then $\tilde{f}$ maps geodesic rays to geodesic rays, up to uniform sublinear error: there is a function $\theta: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$with $\lim _{r \rightarrow \infty} \theta(r)=0$ so that if $\gamma:[0, \infty) \rightarrow \tilde{M}$ is a unit speed geodesic ray, then there is a ray $\gamma^{\prime}:[0, \infty) \rightarrow \tilde{M}^{\prime}$ where $d\left(\tilde{f} \circ \gamma(t), \gamma^{\prime}([0, \infty))\right)<(1+t) \theta(t)$.
4. If we identify $\pi_{1}(M)$ with $\pi_{1}\left(M^{\prime}\right)$ via $f$, then the induced boundary actions $\pi_{1}(M) \curvearrowright \partial_{\infty} \tilde{M}$ and $\pi_{1}(M) \curvearrowright \partial_{\infty} \tilde{M}^{\prime}$ are equivariantly homeomorphic (by a unique equivariant homeomorphism).

If 1 holds and in addition the constants $a_{i}$ and $b_{i}$ are independent of $i$, then the unique equivariant homeomorphism $\partial_{\infty} \tilde{M} \rightarrow \partial_{\infty} \tilde{M}^{\prime}$ in 4 is an isometry with respect to Tits metrics.

In general (see Lemma 6.2) the structure of $\pi_{1}(M)$ forces the $a_{i}$ 's and $b_{i}$ 's in condition 1 to satisfy $\#\left\{a_{1}, \ldots, a_{k}, b_{1}, \ldots, b_{k}\right\} \leq 2$, and except in special circumstances they all coincide. The condition $M L S_{i}=a_{i} f^{*}\left(M L S_{i}^{\prime}\right)$ means that the homotopy equivalence $N_{i} \rightarrow N_{i}^{\prime}$ induced by $f_{i}$ preserves the marked length spectrum of the nonpositively curved orbifolds up to the scale factor $a_{i}$. Although closed nonpositively curved surfaces with the same marked length spectrum are isometric by [Cro90, Ota90, CFF92], compact nonpositively curved surfaces with geodesic boundary can have the same marked length spectrum without being isometric:

Example 1.2. Let $N$ be a pair of pants with a (constant curvature - 1 ) hyperbolic metric where the boundary components are geodesics with length $L$, and let $\left\{c_{1}, c_{2}\right\} \subset$ $N$ be the fixed point set of the order 3 isometry of $N$. If $L$ is sufficiently large (so that $N$ looks like a bikini) then a closed geodesic in $N$ cannot pass near $\left\{c_{1}, c_{2}\right\}$. This means that one can change the metric near $\left\{c_{1}, c_{2}\right\}$ without disturbing the marked length spectrum of $N$. Note that one can modify this example slightly so that the metric is flat in a neighborhood of the boundary geodesics.

Suppose $M$ is a nonpositively curved graph manifold with a Seifert component $M_{i}$ isometric to $N \times S^{1}$, where $N$ is as in the example. One can change the metric on the $N$ factor as in example 1.2 to get a Riemannian manifold $M^{\prime}$ so that the conditions of Theorem 1.1 hold (with $f=i d$ ), but $M^{\prime}$ is not isometric to $M$.

In section 8 we give an example to show that the uniform sublinear divergence estimate in condition 3 cannot be improved to a bounded distance estimate as in the Gromov hyperbolic case.

We now sketch some of the main points in the proof of Theorem 1.1.
First consider a single nonpositively curved graph manifold $M$ with geometric Seifert components $M_{1}, \ldots, M_{k}$. The universal cover $\tilde{M}_{i}$ is isometric to $\tilde{N}_{i} \times \mathbb{R}-\mathrm{a}$
nonpositively curved 3-manifold with a countable collection of totally geodesic boundary components isometric to $\mathbb{E}^{2}$. The universal cover $\tilde{M}$ of $M$ is tiled by a countable collection of copies of the universal covers $\tilde{M}_{i}$ for $i=1, \ldots k$; we call these subsets vertex spaces. We refer to boundary components of vertex spaces as edge spaces. Two vertex spaces are either disjoint, or intersect along an edge space. Let $T$ be the incidence graph for the collection of vertex spaces: $T$ is the graph which has one vertex for each vertex space, and an edge joining two vertices whenever the corresponding vertex spaces intersect. $T$ is isomorphic to the Bass-Serre tree of the graph of groups associated with the decomposition $\amalg_{i} M_{i} \rightarrow M$ (see section 2.5). If $v \in V:=\operatorname{Vertex}(T)$ (resp. $e \in E:=E d g e(T)$ ) we will use the notation $\tilde{M}_{v}$ (resp. $\tilde{M}_{e}$ ) for the vertex space (resp. edge space) associated with $v$ (resp. e); and we let $\tilde{M}_{v} \simeq \tilde{N}_{v} \times \mathbb{R}$ be the Riemannian product decomposition of $\tilde{M}_{v}$. It is not difficult to check (Lemma 3.23) that if $G_{v}:=\operatorname{Stabilizer}\left(\tilde{M}_{v}\right) \subset G \equiv \pi_{1}(M)$, then the center $Z\left(G_{v}\right)$ of $G_{v}$ is isomorphic to $\mathbb{Z}$, and the fixed point set of $Z\left(G_{v}\right)$ in $\partial_{\infty} \tilde{M}$ is just $\partial_{\infty} \tilde{M}_{v}$; similarly, if $\tilde{M}_{e}$ is an edge space then the fixed point set of $\mathbb{Z}^{2} \simeq G_{e}:=\operatorname{Stabilizer}\left(\tilde{M}_{e}\right) \subset G$ in $\partial_{\infty} \tilde{M}$ is $\partial_{\infty} \tilde{M}_{e}$.

Let $p \in \tilde{M}$ be an interior point of a vertex space, pick $\xi \in \partial_{\infty} M$, and let $\overline{p \xi}$ denote the geodesic ray starting at $p$ which is asymptotic to $\xi$. The ray $\overline{p \xi}$ encounters a (possibly finite) sequence of vertex and edge spaces called the itinerary of $\overline{p \xi}$. The convexity of vertex and edge spaces forces the itinerary $v_{0}, e_{1}, v_{1}, e_{2}, \ldots$ of $\overline{p \xi}$ to be the sequence of successive vertices and edges of a geodesic segment or ray in $T$. In order to understand the rays with itinerary $v_{0}, e_{1}, v_{1}, e_{2}, \ldots$, we construct a piecewise flat complex $\mathcal{T}$ - a template - in $\tilde{M}$ as follows. First let $\gamma_{i} \subset \tilde{M}_{v_{i}}$ be a shortest geodesic from $\tilde{M}_{e_{i}}$ to $\tilde{M}_{e_{i+1}}$ for $i>0$, and let $\gamma_{0}$ be a shortest path from $p \in \tilde{M}_{v_{0}}$ to $\tilde{M}_{e_{1}}$. For $i \geq 0$ define $\mathcal{S}_{i} \subset \tilde{M}_{v_{i}}$ to be the flat strip which is the union of the geodesics in $\tilde{M}_{v_{i}}$ which are parallel to the $\mathbb{R}$-factor of $\tilde{M}_{v_{i}}$ and which pass through $\gamma_{i}$. We define $\mathcal{T}$ to be the union of the edge spaces $\left\{\tilde{M}_{e_{i}}\right\}$ with the strips $\left\{\mathcal{S}_{i}\right\}$; then $\mathcal{T}$ is a Hadamard space with respect to the induced path metric. A key technical step in the proof of Theorem 1.1 is Theorem 5.1, which shows that for any geodesic ray $\overline{p \zeta}$ in the Hadamard space $\mathcal{T}$, there is a unique geodesic ray $\overline{p \zeta^{\prime}}$ in $\tilde{M}$ with the property that for all $x \in \overline{p \zeta}$,

$$
d_{\tilde{M}}\left(x, \overline{p \zeta^{\prime}}\right) \leq \theta\left(d_{\mathcal{T}}(x, p)\right)\left(1+d_{\mathcal{T}}(x, p)\right)
$$

for some function $\theta:[0, \infty) \rightarrow[0, \infty)$ with $\lim _{r \rightarrow \infty} \theta(r)=0$ which is independent of the choice of itinerary. Using Theorem 5.1 one finds that the set of boundary points $\zeta \in \partial_{\infty} \tilde{M}$ for which the ray $\overline{p \zeta}$ has a given infinite itinerary $v_{0}, e_{1}, \ldots$ is homeomorphic to the set boundary points $\zeta^{\prime} \in \mathcal{T}$ so that the $\mathcal{T}$-ray $\overline{p \zeta^{\prime}}$ passes through $\tilde{M}_{e_{i}}$ for every $i$. One sees (Proposition 7.3) that the latter is either a single point or is homeomorphic to a closed interval, depending on the geometry of $\mathcal{T}$ (which depends, in turn, on the choice of itinerary and the geometry of $M$ ).

We now consider a second nonpositively curved 3-manifold $M^{\prime}$, and use primes to denote the vertex spaces, edge spaces, etc for $M^{\prime}$. Let $f: M \rightarrow M^{\prime}$ be a homeomorphism as in Theorem 1.1, and identify the deck groups $G:=\pi_{1}(M) \simeq \pi_{1}\left(M^{\prime}\right)$ via a lift $\tilde{f}$ of $f$. Then $\tilde{f}$ maps vertex (resp. edge) spaces of $M$ homeomorphically to vertex (resp. edge) spaces of $M^{\prime}$, so we may use $\tilde{f}$ to identify the incidence tree $T^{\prime}$ with $T$. Suppose $\phi: \partial_{\infty} \tilde{M} \rightarrow \partial_{\infty} \tilde{M}^{\prime}$ is a $G$-equivariant homeomorphism. Using the remarks about fixed point sets made above, it follows that $\phi\left(\partial_{\infty} \tilde{M}_{v}\right)=\partial_{\infty} \tilde{M}_{v}^{\prime}$
and $\phi\left(\partial_{\infty} \tilde{M}_{e}\right)=\partial_{\infty} \tilde{M}_{e}^{\prime}$ for every $v \in V$ and every $e \in E$. Also, if $p \in \tilde{M}_{v_{0}}$, and $v_{0}, e_{1}, v_{1}, e_{2}, \ldots$ is an infinite itinerary, then $\phi(S)=S^{\prime}$ where $S \subset \partial_{\infty} \tilde{M}$ and $S^{\prime} \subset \partial_{\infty} \tilde{M}^{\prime}$ are the subsets corresponding to the itinerary $v_{0}, e_{1}, v_{1}, e_{2}, \ldots$ (Corollary 5.28); in particular, either $S$ and $S^{\prime}$ are both points or they are both intervals. By considering all possible infinite itineraries and exploiting this correlation, we are able to see (section 7) that the invariants $M L S_{i}, M L S_{i}^{\prime}$ and $\tau_{i}, \tau_{i}^{\prime}$ must agree as in condition 1 of Theorem 1.1. Conversely, if condition 1 holds and $p \in \tilde{M}$, one shows (section 6) that for each itinerary the corresponding templates in $\tilde{M}$ and $\tilde{M}^{\prime}$ have sufficiently similar geometry that their geodesics are "similar"; and this implies that $\tilde{f}$ extends to the compactifications as in 2 of Theorem 1.1.

Our main result generalizes Theorem 1.1 and applies to admissible groups, a class of (fundamental groups of) graphs of groups, see section 3.1 for the precise definition. When an admissible group $G$ acts discretely and cocompactly on a Hadamard space $X$ then we associate geometric data to each vertex group $G_{v} \subset G$ consisting of a class function $M L S_{v}: G_{v} \rightarrow \mathbb{R}^{+}$and a homomorphism $\tau_{v}: G_{v} \rightarrow \mathbb{R}$ (see section 3.2).

Theorem 1.3. Let $G \curvearrowright X$ be a discrete, cocompact, isometric action of an admissible group on a Hadamard space $X$. Then for every vertex $v, M L S_{v}$ and $\tau_{v}$ are determined up to scale factors $a_{v}$ and $b_{v}$ by the topological conjugacy class of the boundary action $G \curvearrowright \partial_{\infty} X$, and vice-versa. If $G \curvearrowright X^{\prime}$ is another such action, then the following are equivalent:

1. $G \curvearrowright X$ and $G \curvearrowright X^{\prime}$ have the same geometric data up to scale.
2. G-equivariant quasi-isometries $X \rightarrow X^{\prime}$ extend canonically to the compactifications $X \cup \partial_{\infty} X \rightarrow X^{\prime} \cup \partial_{\infty} X^{\prime}$.
3. The boundary actions $G \curvearrowright \partial_{\infty} X$ and $G \curvearrowright \partial_{\infty} X^{\prime}$ are $G$-equivariantly homeomorphic (by a unique ${ }^{5} G$-equivariant homeomorphism).
4. If $f: X \rightarrow X^{\prime}$ is a $G$-equivariant quasi-isometry, then there is a function $\theta: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$with $\lim _{r \rightarrow \infty} \theta(r)=0$ so that for every unit speed geodesic ray $\gamma$ : $[0, \infty) \rightarrow X$ there is a ray $\gamma^{\prime}:[0, \infty) \rightarrow X^{\prime}$ with $d\left(f \circ \gamma(t), \gamma^{\prime}(t)\right)<(1+t) \theta(t)$.

Furthermore, if there is a single scale factor s so that $M L S_{v}^{\prime}=s M L S_{v}$ and $\tau_{v}^{\prime}=$ $s \tau_{v}$ for every vertex $v$, then the unique $G$-equivariant homeomorphism $\partial_{\infty} X \rightarrow \partial_{\infty} X^{\prime}$ is an isometry with respect to the Tits metrics.

The authors proved theorem 1.3 while attempting to digest the negative answer to Gromov's question about boundary actions. A key factor in our example was the (unanticipated) presence of intervals in the Tits boundary. After the examples and their properties had been announced, similar structure was found in other manifolds, [HS98]. The paper [BS] also contains some discussion of the Tits boundary of universal covers of nonpositively curved graph manifolds.

Open questions. The results in this paper raise a number of questions. First of all, for each group $G$ one may ask for a generalization of Theorem 1.3, where the

[^2]geometric data $M L S_{v}$ and $\tau_{v}$ are replaced with suitable substitutes. Our methods actually yield more information about the behavior of geodesics than is stated in Theorem 1.3 alone. We are able to give a good description of all the geodesic rays in the Hadamard space $X$ in terms of concrete geometric information; it seems likely that other classes of groups are amenable to a similar treatment. The fundamental groups of the real-analytic manifolds considered in [HS98] are natural candidates for this, as they have structure similar to graph manifold groups. Here are two other questions:

1. What determines the (non-equivariant) homeomorphism type of $\partial_{\infty} X$, when $X$ is a Hadamard space with an action $G \curvearrowright X$ by an admissible group $G$ ?
2. Does part 4 of Theorem 1.3 have an analog where rays are replaced by complete geodesics? This seems within reach.

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## 2. Preliminaries

### 2.1. Coarse geometry

Let $X$ and $X^{\prime}$ be metric spaces, and let $\Phi: X \rightarrow X^{\prime}$ be a map.
Definition 2.1. 1. $\Phi$ is $(L, A)$-Lipschitz if for all $x_{1}, x_{2} \in X$,

$$
d\left(\Phi\left(x_{1}\right), \Phi\left(x_{2}\right)\right) \leq L d\left(x_{1}, x_{2}\right)+A .
$$

$\Phi$ is coarse Lipschitz if it is $(L, A)$-Lipschitz for some $L, A>0$.
2. $\Phi$ is an $(L, A)$-quasi-isometric embedding if it is $(L, A)$-Lipschitz and for all $x_{1}, x_{2} \in X$,

$$
d\left(\Phi\left(x_{1}\right), \Phi\left(x_{2}\right)\right) \geq L^{-1} d\left(x_{1}, x_{2}\right)-A .
$$

The constants $(L, A)$ will often be suppressed. A quasi-geodesic (respectively segment/ray) is a quasi-isometric embedding $\Phi: \mathbb{R} \rightarrow X$ (respectively $\Phi:[a, b] \rightarrow X$, $\Phi:[0, \infty) \rightarrow X)$. We sometimes refer to the image of a quasi-geodesic as a quasigeodesic.
3. $\Phi$ is an $(L, A)$-quasi-isometry if it is an $(L, A)$-quasi-isometric embedding and for all $x^{\prime} \in X^{\prime}, d\left(x^{\prime}, \Phi(X)\right)<A$.
4. $\Phi$ is a $D$-Hausdorff approximation if it is a $(1, D)$-quasi-isometry.

We will use the following well-known lemma:
Lemma 2.2. If $G \curvearrowright X$ is a discrete, cocompact, isometric action of a group $G$ on a length space $X$, then there is a $G$-equivariant quasi-isometry $\Phi: \operatorname{Cayley}(G) \rightarrow X$, where Cayley $(G)$ is any Cayley graph of $G$.

### 2.2. Hadamard spaces

We refer the reader to [Bal95] for the material recalled here.
Geodesics and the boundary. Let $X$ be a locally compact Hadamard space. If $p, q \in X$ then $\overline{p q} \subset X$ denotes the segment from $p$ to $q$. If $p, x, y \in X$ and $p \notin\{x, y\}$, then $\widetilde{Z}_{p}(x, y)$ (respectively $\left.\angle_{p}(x, y)\right)$ denotes the comparison angle (respectively angle) of the triangle $\Delta p x y$ at $p$. We will use $\partial_{\infty} X$ to denote the set of asymptote classes of geodesic rays in $X$, with the cone topology. If $p \in X$ and $\xi \in \partial_{\infty} X$, then $\overline{p \xi}$ denotes the ray leaving $p$ in the asymptote class of $\xi . \bar{X}:=X \cup \partial_{\infty} X$ denotes the usual compactification: a sequence $x_{i} \in \bar{X}$ converges if and only if for any basepoint $p \in X$ the sequence of geodesic segments/rays $\overline{p x_{i}}$ converges in the compact open topology. We denote the Tits angle between $\xi_{1}, \xi_{2} \in \partial_{\infty} X$ by $\angle_{T}\left(\xi_{1}, \xi_{2}\right)$, and $\partial_{T} X$ denotes the underlying set of $\partial_{\infty} X$ equipped with the Tits angle metric (which usually induces a topology different from the one defined above). The metric space $\partial_{T} X$ is a $C A T(1)$ space with respect to this metric. When $\xi_{1}, \xi_{2} \in \partial_{T} X$ and $\angle_{T}\left(\xi_{1}, \xi_{2}\right)<\pi$ then there is a segment between $\xi_{1}$ and $\xi_{2}$ in $\partial_{T} X$, which we denote by $\overline{\xi_{1} \xi_{2}} \subset \partial_{T} X$. This segment
is the limit set in $\bar{X}$ of any sequence of segments $\overline{x_{1}^{k} x_{2}^{k}}$ where $x_{i}^{k}$ tends to infinity along a ray asymptotic to $\xi_{i}$. We will not use the Tits path metric. We recall that $\partial_{\infty}$ and $\partial_{T}$ behave nicely with respect to products: $\partial_{\infty}\left(X_{1} \times X_{2}\right)=\partial_{\infty} X_{1} \circ \partial_{\infty} X_{2}$ and $\partial_{T}\left(X_{1} \times X_{2}\right)=\partial_{T} X_{1} \circ \partial_{T} X_{2}$ where in the first case $\circ$ represents the topological join and in the second the $\frac{\pi}{2}$-metric join. We will use this in the case where $X_{2}=\mathbb{R}$.

We will let $N_{R}(C)$ be the closed metric tubular neighborhood of radius $R$ of a subset $C \subset X$. A closed convex subset $C \subset X$ is also a locally compact Hadamard space as is $N_{R}(C)$, since it is also convex.

Standard comparison arguments show the following.
Lemma 2.3. Let $X$ be a locally compact Hadamard space, and let $C \subset \underline{X}$ be a closed convex subset. Then for any $R>0, \xi \in \partial_{\infty} N_{R} C$, and $z \in C$ we have $\overline{z \xi_{\infty}} \subset C$. In particular, $\partial_{\infty} C=\partial_{\infty} N_{R} C$.

One consequence is:
Lemma 2.4. Let $X$ be a locally compact Hadamard space, and let $C \subset X$ be a closed convex subset. If $p \in X, \xi_{i} \in \partial_{\infty} X$, and $\xi_{i} \rightarrow \xi_{\infty}, \overline{p \xi_{i}} \cap C \neq \emptyset$ for all $i$, then either $\overline{p \xi_{\infty}} \cap C \neq \emptyset$ or $\xi_{\infty} \in \partial_{\infty} C$.

Proof. Pick $x_{i} \in \overline{p \xi_{i}} \cap C$. If lim inf $d\left(x_{i}, p\right)<\infty$ then a subsequence of $x_{i}$ converges to $x_{\infty} \in C \cap \overline{p \xi_{\infty}}$. On the other hand, if $\lim \inf d\left(x_{i}, p\right)=\infty$ then for some subsequence $\overline{p x_{i}} \rightarrow \overline{p \xi_{\infty}}$. By the convexity of $N_{R}(C), \overline{p \xi_{\infty}} \subset N_{R}(C)$ for $R=d(p, C)$ and hence Lemma 2.3 yields the result.
Lemma 2.5. Let $\Phi: X \rightarrow X^{\prime}$ be a quasi-isometric embedding, and assume there is a point $x \in X$ and a function $\theta: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$with $\lim _{r \rightarrow \infty} \theta(r)=0$ so that for every $y \in X, z \in \overline{x y}$, we have

$$
\begin{equation*}
d_{X^{\prime}}(\Phi(z), \overline{\Phi(x) \Phi(y)}) \leq\left(1+d_{X}(z, x)\right) \theta\left(d_{X}(z, x)\right) \tag{2.6}
\end{equation*}
$$

Then there is a unique extension $\bar{\Phi}: \bar{X} \rightarrow \bar{X}^{\prime}$ of $\Phi$ which is continuous at $\partial_{\infty} X$, and $\partial_{\infty} \Phi:=\left.\bar{\Phi}\right|_{\partial_{\infty} X}$ is a topological embedding.

Proof. Let $\Phi$ be an $(L, A)$-quasi isometric embedding, and $\xi \in \partial_{\infty} X$, and $y_{k} \in X$ be such that $\overline{x y_{k}}$ converges to $\overline{x \xi}$. By the convergence we can choose $R_{k} \rightarrow \infty$ so that for all $k \geq n$ we have $d\left(y_{k}, x\right) \geq R_{n}$ and $y_{k n}:=\overline{x y_{k}} \cap S\left(x, R_{n}\right) \subset N_{1}(\overline{x \xi})$. Note that the point on $\xi$ closest to $y_{k n}$ lies in $\xi\left(\left[R_{n}-1, R_{n}+1\right]\right)$, hence by the triangle inequality $d\left(y_{k n}, y_{l n}\right) \leq 4$ for $k, l \geq n$. Using (2.6), for every $k \geq n$ choose $y_{k n}^{\prime} \in \overline{\Phi(x) \Phi\left(y_{k}\right)}$ with $d_{X^{\prime}}\left(y_{k n}^{\prime}, \Phi\left(y_{k n}\right)\right) \leq\left(1+R_{n}\right) \theta\left(R_{n}\right)$. Then for every $k, l \geq n$ we have $d_{X^{\prime}}\left(\Phi\left(y_{k n}\right), \Phi\left(y_{l n}\right)\right) \leq 4 L+A$, and so $d_{X^{\prime}}\left(y_{k n}^{\prime}, y_{l n}^{\prime}\right) \leq 2\left(1+R_{n}\right) \theta\left(R_{n}\right)+4 L+A$. This, along with the fact that $d\left(\Phi(x), y_{k n}^{\prime}\right) \geq L^{-1} R_{n}-A-\left(1+R_{n}\right) \theta\left(R_{n}\right)$, forces $\tilde{Z}_{\Phi(x)}\left(y_{k n}^{\prime}, y_{l n}^{\prime}\right)$ to zero as $n \rightarrow \infty$. This in turn forces $\overline{\Phi(x) \Phi\left(y_{k}\right)}$ to converge to a ray $\overline{\Phi(x) \Phi(\xi)}$ since for each $R>0$ we have for large enough $k$ that the sequence $\left\{\overline{\Phi(x) \Phi\left(y_{k}\right)} \cap S(\Phi(x), R)\right\}$ is Cauchy and hence converges. This proves that $\Phi$ has a unique extension $\bar{\Phi}: \bar{X} \rightarrow \bar{X}^{\prime}$ which is continuous at $\partial_{\infty} X$. The map $\partial_{\infty} \Phi:=\left.\bar{\Phi}\right|_{\partial_{\infty} X}$ clearly has the property that for all $\xi \in \partial_{\infty} X$ and all $y \in x \xi$,

$$
\begin{equation*}
d\left(\Phi(y), \overline{\Phi(x) \partial_{\infty} \Phi(\xi)}\right) \leq(1+d(x, y)) \theta(d(x, y)) \tag{2.7}
\end{equation*}
$$

When $\xi_{1}, \xi_{2} \in \partial_{\infty} X$ are distinct, the rays $\overline{x \xi_{i}}$ diverge linearly, and hence $\Phi\left(\overline{x \xi_{1}}\right)$ and $\Phi\left(\overline{x \xi_{2}}\right)$ diverge linearly since $\Phi$ is a quasi isometric embedding. Now if $\partial_{\infty} \Phi\left(\xi_{1}\right)=$ $\partial_{\infty} \Phi\left(\xi_{2}\right)$ then (2.7) would imply that $\Phi\left(\overline{x \xi_{1}}\right)$ and $\Phi\left(\overline{x \xi_{2}}\right)$ would each diverge sublinearly from $\overline{\Phi(x) \partial_{\infty} \Phi\left(\xi_{1}\right)}$ and hence diverge sublinearly from each other. Thus we conclude that $\partial_{\infty} \Phi\left(\xi_{1}\right) \neq \partial_{\infty} \Phi\left(\xi_{2}\right)$.

### 2.3. Groups acting on Hadamard spaces.

Let $X$ be a Hadamard space. We denote the displacement function of an isometry $g: X \rightarrow X$ by $d_{g}$, and the infimum of $d_{g}$ by $\delta_{g}$. When $g$ is axial, we let $\operatorname{Minset}(g)$ denote the convex subset where $d_{g}$ attains its minimum. We recall that $\operatorname{Minset}(g)$ splits as a metric product $C \times \mathbb{R}$ where $C$ is convex and $g$ acts trivially on the $C$ factor and by translation on the $\mathbb{R}$ factor.

Let $G \curvearrowright X$ be a discrete, cocompact, isometric action of a group $G$ on a Hadamard space $X$. If $H \subset G$ is a subgroup isomorphic to $\mathbb{Z}^{k}$, we let $\operatorname{Minset}(H):=$ $\cap_{h \in H} \operatorname{Minset}(h)$. We recall that $\operatorname{Minset}(H)=\cap_{h \in S} \operatorname{Minset}(h)$ for any generating set $S \subset H$, and that $\operatorname{Minset}(H)$ splits isometrically as a metric product $C \times \mathbb{E}^{k}$ so that $H$ acts trivially on the $C$ factor, and as a translation lattice on the $\mathbb{E}^{k}$ factor. The centralizer $Z(H, G)$ of $H$ in $G$ preserves $\delta_{h}$ for every $h \in H$, and hence also $\operatorname{Minset}(H)$. If $S \subset H$ is a finite generating set, then the function $\sum_{h \in S} \delta_{h}: X \rightarrow \mathbb{R}$ descends to a proper function on $X / Z(H, G)$; in particular, $\operatorname{Minset}(H) / Z(H, G)$ is compact.

### 2.4. Gromov hyperbolic groups and spaces

For background on the material in this section see [Gro87], [GdlH90], and [CDP90]. Some standard facts that we will use: If a Gromov hyperbolic group $G$ acts cocompactly on a Hadamard space $X$ then (since $X$ is then quasi-isometric to Cayley $(G)$ and Gromov hyperbolicity is a quasi isometry invariant) $X$ is Gromov hyperbolic (i.e. $\delta$-hyperbolic for some $\delta$ ). Further $\partial_{\infty} X$ is homeomorphic to $\partial_{\infty} G$, and all infinite order elements $g \in G$ are axial. The Tits metric $\partial_{T} X$ is the discrete metric with any two distinct points having distance $\pi$.

In this section we make use of the Morse lemma for quasi-geodesic segments (see [Gro87, CDP90]):

Lemma 2.8. (Morse Lemma) Given $\delta>0, L>0$ and $A \geq 0$ there is a constant $C=C(\delta, L, A)$ such that if $\gamma_{1}$ and $\gamma_{2}$ are $(L, A)$-quasi-geodesic segments with the same endpoints sitting in a $\delta$-hyperbolic space, then their Hausdorff distance satisfies $d_{\mathcal{H}}\left(\gamma_{1}, \gamma_{2}\right)<C$.

Two geodesics $\gamma_{1}$ and $\gamma_{2}$ in a Hadamard space $X$ are parallel if they stay a bounded distance apart. The parallel set $P(\gamma) \subset X$ of a geodesic $\gamma$ is the union of all geodesics parallel to $\gamma$. By the flat strip theorem, $P(\gamma)$ is a convex subset of $X$, and is isometric to $C_{\gamma} \times \mathbb{R}$ where $C_{\gamma} \subset X$ is convex. A bounded convex set $C$ always contains a unique circumcenter: the center of the smallest metric ball containing $C$.
Lemma 2.9. Let $X$ be a $\delta$-hyperbolic Hadamard space. Then

1. If $\gamma \subset X$ is a geodesic and $P(\gamma) \simeq C_{\gamma} \times \mathbb{R}$ is its parallel set, then $\operatorname{Diam}\left(C_{\gamma}\right)<\delta$. In particular $P(\gamma)$ contains a canonical geodesic $z \times \mathbb{R} \subset C_{\gamma} \times \mathbb{R}$ where $z \in C_{\gamma}$ is the circumcenter of $C_{\gamma}$.
2. If $\gamma_{1}, \gamma_{2} \subset X$ are geodesics, $x_{i} \in \gamma_{i}$, then $\gamma_{1} \cup \overline{x_{1} x_{2}} \cup \gamma_{2}$ is $2 \delta$-quasi-convex ${ }^{6}$.
3. Suppose $\gamma_{1}, \gamma_{2} \subset X$ are geodesics with $\partial_{\infty} \gamma_{1} \cap \partial_{\infty} \gamma_{2}=\emptyset$, and let $\eta$ a minimal geodesic segment between $\gamma_{1}$ and $\gamma_{2}$. Then any geodesic segment running from $\gamma_{1}$ to $\gamma_{2}$ will pass within distance $D=D\left(\gamma_{1}, \gamma_{2}\right)$ of both endpoints of $\eta$; when $d\left(\gamma_{1}, \gamma_{2}\right)>4 \delta$ then we may take $D=2 \delta$.

Proof. 1 follows from the fact that a $\delta$-hyperbolic Euclidean strip has width at most $\delta$. 2 and 3 follow from repeated application of the $\delta$-thinness property of geodesic triangles.

In the following lemma is a slight variation on results from [Gro87]. It shows that discrete isometric actions on Gromov hyperbolic spaces behave like free group actions on trees.

Lemma 2.10. Let $X$ be a $\delta$-hyperbolic Hadamard space, and let $\star \in X$. Suppose $\left(g_{i}\right)_{i \in \mathbb{Z}}$ is a periodic sequence of axial isometries of $X$ with period $k$ (i.e. $g_{i+k}=g_{i}$ for all $i$; and in particular $g_{0}=g_{k}$ and $g_{-1}=g_{k-1}$ ), and let the attracting (respectively repelling) fixed point of $g_{i}$ be $\xi_{i}^{+} \in \partial_{\infty} X$ (respectively $\xi_{i}^{-} \in \partial_{\infty} X$ ). If for every $i$ we have $\xi_{i}^{-} \neq \xi_{i+1}^{+}$, then there are constants $L, A, N$, and $D$ with the following property.

1. If $\left(m_{i}\right)_{i \in \mathbb{Z}}$ is a sequence with $m_{i}>N$, then the broken geodesic with vertices

$$
\begin{equation*}
\ldots, v_{-2}=\left(g_{-1}^{-m_{-1}} g_{-2}^{-m_{-2}}\right)(\star), v_{-1}=g_{-1}^{-m_{-1}}(\star), v_{0}=\star, v_{1}=g_{0}^{m_{0}}(\star), v_{2}=\left(g_{0}^{m_{0}} g_{1}^{m_{1}}\right)(\star), \ldots \tag{2.11}
\end{equation*}
$$

is an $(L, A)$ quasi-geodesic, and

$$
\begin{equation*}
\left|d\left(v_{i}, v_{i+l}\right)-\sum_{j=i}^{i+l-1} m_{j} \delta_{g_{j}}\right|<l D \tag{2.12}
\end{equation*}
$$

2. If $\left(m_{i}\right)_{i \in \mathbb{Z}}$ is a sequence with $m_{i}>N$ and period $k$, then $g:=g_{0}^{m_{0}} \ldots g_{k-1}^{m_{k-1}}$ is an axial isometry with an axis $\gamma$ within Hausdorff distance $D$ of the $g$-invariant broken geodesic with vertices (2.11), and the minimal displacement of $g$ satisfies

$$
\begin{equation*}
\left|\delta_{g}-\left(m_{0} \delta_{g_{0}}+\ldots+m_{k-1} \delta_{g_{k-1}}\right)\right|<D \tag{2.13}
\end{equation*}
$$

Furthermore, as $m_{0} \rightarrow \infty$ (respectively $m_{k-1} \rightarrow \infty$ ), the attracting (respectively repelling) fixed point of $g$ tends to $\xi_{0}^{+}$(respectively $\xi_{k-1}^{-}$).

Proof. Since $\xi_{i}^{-} \neq \xi_{i+1}^{+}$for all $i \in \mathbb{Z}$, there are constants $L_{1}, A_{1}$, and $N_{1}$, so that when $m_{ \pm}>N_{1}$ then for any $i$ the broken geodesic with vertices $g_{i}^{-m_{-}}(\star), \star, g_{i+1}^{m_{+} \star}$ is an $\left(L_{1}, A_{1}\right)$ quasi-geodesic segment. Let $\left(m_{i}\right)_{i \in \mathbb{Z}}$ be a sequence, and let $\eta: \mathbb{R} \rightarrow X$ be the broken geodesic with vertices (2.11). By the local characterization of quasi-geodesics given in [CDP90, Chapitre 3], we get constants $L=L\left(L_{1}, A_{1}, \delta\right)$, $A=A\left(L_{1}, A_{1}, \delta\right)$,

[^3]and $N_{2}=N_{2}\left(L_{1}, A_{1}, \delta,\left\{g_{i}\right\}_{i \in \mathbb{Z}}\right) \geq N_{1}$ so that $\eta$ is an $(L, A)$ quasi-geodesic provided $m_{i} \geq N_{2}$ for all $i$.

We now assume that $m_{i} \geq N_{2}$ for all $i$. By the Morse lemma there is a $D_{1}=$ $D_{1}(L, A, \delta)$ so that there is a geodesic at Hausdorff distance at most $D_{1}$ from $\eta(\mathbb{R})$, and any geodesic $\gamma \subset X$ with $\partial_{\infty} \gamma=\partial_{\infty} \eta$ has Hausdorff distance at most $D_{1}$ from $\eta$. Fix such a geodesic $\gamma \subset X$, and for each $i \in \mathbb{Z}$ let $w_{i} \in \gamma$ be the point in $\gamma$ nearest $v_{i}$. By the triangle inequality we have

$$
\begin{equation*}
\left|d\left(v_{i}, v_{i+1}\right)-d\left(w_{i}, w_{i+1}\right)\right| \leq 2 D_{1} \tag{2.14}
\end{equation*}
$$

Choose $c_{1}=c_{1}\left(\star,\left\{g_{i}\right\}\right)$ so that the distance from $\star$ to the nearest axis of $g_{i}$ is less than $c_{1}$; then for all $i \in \mathbb{Z}$

$$
\begin{equation*}
\left|d\left(v_{i}, v_{i+1}\right)-m_{i} \delta_{g_{i}}\right|=\left|d\left(\star, g_{i}^{m_{i}}(\star)\right)-m_{i} \delta_{g_{i}}\right|<2 c_{1} . \tag{2.15}
\end{equation*}
$$

Since for each $i$, the broken segment with vertices $v_{i-1}, v_{i}, v_{i+1}$ is an $\left(L_{1}, A_{1}\right)$ quasigeodesic, the Morse Lemma gives

$$
\begin{equation*}
d\left(v_{i-1}, v_{i+1}\right) \geq d\left(v_{i-1}, v_{i}\right)+d\left(v_{i}, v_{i+1}\right)-2 D_{1} \tag{2.16}
\end{equation*}
$$

This gives

$$
\begin{equation*}
d\left(w_{i-1}, w_{i+1}\right) \geq d\left(w_{i-1}, w_{i}\right)+d\left(w_{i}, w_{i+1}\right)-8 D_{1} \tag{2.17}
\end{equation*}
$$

Therefore there is an $N=N\left(\star,\left\{g_{i}\right\}\right) \geq N_{2}$ so that if $m_{i} \geq N$ then $w_{i}$ lies between $w_{i-1}$ and $w_{i+1}$ for all $i$. So when $m_{i} \geq N$ we have

$$
\begin{align*}
& \left|d\left(v_{i}, v_{i+l}\right)-\sum_{j=i}^{j+l-1} m_{j} \delta_{g_{j}}\right| \leq 2 D_{1}+\left|d\left(w_{i}, w_{i+l}\right)-\sum_{j=i}^{j+l-1} m_{j} \delta_{g_{j}}\right| \\
& \quad \leq 2 D_{1}+\sum_{j=i}^{j+l-1}\left|d\left(w_{j}, w_{j+1}\right)-m_{j} \delta_{g_{j}}\right| \leq 2 D_{1}(l+1)+2 l c_{1} . \tag{2.18}
\end{align*}
$$

We now set $D:=(2 k+4) D_{1}+2 k c_{1}$, and note that we have proved 1 . When the sequence $\left(m_{i}\right)$ has period $k, m_{i} \geq N$ for all $i$, and $g:=g_{0}^{m_{0}} \ldots g_{k}^{m_{k-1}}$, then we may take $\gamma$ to be an axis for $g$. We have $\delta_{g}=d\left(w_{0}, w_{k}\right)$, and (2.13) follows from (2.18). The last assertion follows immediately from the fact that as $m_{0} \rightarrow \infty$ and $m_{k-1} \rightarrow \infty$, the segments $\overline{\star g_{0}^{m_{0}}(\star)}$ and $\overline{\star g_{k-1}^{-m_{k-1}}(\star)}$ converge to the rays $\overline{\star \xi_{0}^{+}}$and $\overline{\star \xi_{k-1}^{-}}$respectively.

Lemma 2.19. Let $G \stackrel{\rho}{ค} X$ be a discrete, cocompact isometric action of a hyperbolic group $G$ on a Hadamard space $X$. There is a constant $D=D(\delta, \rho)$ so that for every $x_{1}, x_{2} \in X$ there is a $g \in G$ and an axis $\gamma$ for $g$ with $d\left(x_{i}, \gamma\right)<D$ for $i=1,2$, and $d\left(g\left(x_{1}\right), x_{2}\right)<D$.

Proof. If $G$ is elementary, then either $G$ is finite (in which case the result holds trivially) or there is a hyperbolic element $g \in G$ with an axis $\gamma$ so that $X=N_{R}(\gamma)$ for some $R$; this implies 2 in this case. So we may assume that $G$ is nonelementary, and hence $G$ does not fix any $\xi \in \partial_{\infty} X$.

Pick $\star \in X$ and a finite generating set $\Sigma \subset G$. Fix $\sigma_{0} \in \Sigma$, let $\Sigma^{\prime}=\left\{\sigma_{0}\right\} \cup\left\{\sigma \sigma_{0} \mid \sigma \in\right.$ $\Sigma\}$, and let $C\left(\Sigma^{\prime}\right)=\min \left\{d\left(\star, \sigma^{\prime}(\star)\right) \mid \sigma^{\prime} \in \Sigma\right\}$.

We note that by the cocompactness of the action it is sufficient to prove the theorem when $x_{1}$ is $\star$; for then (with a larger $D$ ) if $g_{1}\left(x_{i}\right)$ is near $\star$ (within the diameter of the fundamental domain) and $g$ is the solution for $\star$ and $g_{1}\left(x_{2}\right)$ then $g_{1}^{-1} g g_{1}$ works for $x_{1}$ and $x_{2}$ (since $g_{1}^{-1}\left(\gamma_{g}\right)$ is an axis for $g_{1}^{-1} g g_{1}$ ).
Claim. There are constants $L_{1}, A_{1}$ such that for all $g \in G$, there is a $\sigma^{\prime} \in \Sigma^{\prime}$ so that the broken geodesic with vertices $\left.\left(g \sigma^{\prime}\right)^{-1}\right)(\star), \star,\left(g \sigma^{\prime}\right)(\star)$ is an $\left(L_{1}, A_{1}\right)$-quasi-geodesic.
Proof of claim. If not, there is a sequence $g_{k} \in G$ with $d\left(g_{k}(\star), \star\right) \rightarrow \infty$, so that for every $\sigma^{\prime} \in \Sigma^{\prime}$ the broken geodesic with vertices $\left.\left(g_{k} \sigma^{\prime}\right)^{-1}\right)(\star)$, $\star,\left(g_{k} \sigma^{\prime}\right)(\star)$ is not a $(k, k)$-quasi-geodesic. This clearly implies that after passing to a subsequence (which works for all $\sigma^{\prime}$ ), the segments $\overline{\star\left[\left(g_{k} \sigma^{\prime}\right)(\star)\right]} \overline{\star\left[\left(g_{k} \sigma^{\prime}\right)^{-1}(\star)\right]}$ converge to some ray $\overline{\star \xi\left(\sigma^{\prime}\right)}$. But since $d\left(g_{k}(\star),\left(g_{k} \sigma^{\prime}\right)(\star)\right) \leq C\left(\Sigma^{\prime}\right)$ we see $\left(g_{k}\right)(\star)$ converges to $\xi\left(\sigma^{\prime}\right)$, so $\xi\left(\sigma^{\prime}\right)=\xi$ is independent of $\sigma^{\prime}$. The fact that $\left(g_{k} \sigma^{\prime}\right)^{-1}(\star)$ converges to $\xi$ tells us (by applying $\sigma^{\prime}$ to the sequence) that $g_{k}^{-1}(\star)$ converges to $\sigma^{\prime}(\xi)=\xi^{\prime}$ which is again independent of the choice of $\sigma^{\prime}$. Thus $\sigma^{\prime}(\xi)=\xi^{\prime}$ for all $\sigma^{\prime} \in \Sigma$, and hence for every $\sigma \in \Sigma$, $\sigma\left(\xi^{\prime}\right)=\left(\sigma \sigma_{0}\right) \sigma_{0}^{-1}\left(\xi^{\prime}\right)=\xi^{\prime}$, which is a contradiction.
Proof of Lemma 2.19 continued. By the claim and an application of [CDP90, Chapitre 3] as in the proof of Lemma 2.10, we see that there are constants $L, A$, and $D_{1}$ so that if $g \in G$ and $d(g(\star), \star)>D_{1}$, then there is a $\sigma^{\prime} \in \Sigma^{\prime}$ so that the broken geodesic with $i^{\text {th }}$ vertex $\left(g \sigma^{\prime}\right)^{i}(\star)$ is an $(L, A)$-quasi-geodesic. By the Morse Lemma and Lemma 2.9, $\left(g \sigma^{\prime}\right)$ has an axis at distance $<D_{2}(L, A, \delta)$ from $\star$ and $\left(g \sigma^{\prime}\right)(\star)$. Since $d\left(\left(g \sigma^{\prime}\right)(\star), g(\star)\right)=d\left(\sigma^{\prime}(\star), \star\right) \leq C\left(\Sigma^{\prime}\right)$, the lemma clearly follows.
Lemma 2.20. Consider two discrete, cocompact, isometric actions $G \stackrel{\rho}{\curvearrowright} X, G \stackrel{\rho^{\prime}}{\curvearrowright}$ $X^{\prime}$ where $X$ and $X^{\prime}$ are $\delta$-hyperbolic Hadamard spaces. Assume that the minimum displacement of any $g \in G$ in $X$ is the same as the minimal displacement in $X^{\prime}$. Then any $G$-equivariant $(L, A)$-quasi-isometry $\Phi: X \rightarrow X^{\prime}$ maps unit speed geodesics $\gamma$ to within $D=D\left(L, A, \delta, \rho, \rho^{\prime}\right)$ of a unit speed geodesic $\gamma^{\prime}$, that is $d\left(\Phi(\gamma(t)), \gamma^{\prime}(t)\right) \leq D$.

Proof. By the Morse lemma on quasi-geodesics, it suffices to show that $\Phi$ is a $D_{1}=$ $D_{1}\left(L, A, \delta, \rho, \rho^{\prime}\right)$ Hausdorff approximation. Let $D_{5}=\max \left\{D_{4}(\delta, \rho), D_{4}\left(\delta, \rho^{\prime}\right)\right\}$ where the $D_{4}$ 's come from Lemma 2.19. For $x_{1}, x_{2} \in X\left(\right.$ resp. $\left.\Phi\left(x_{1}\right), \Phi\left(x_{2}\right) \in X^{\prime}\right)$ let $g \in G$ (resp $g^{\prime} \in G$ ) be the elements guaranteed by Lemma 2.19. Since $x_{1}$ is $D_{5}$ close to an axis of $g$ we know that $2 D_{5}+\delta_{g} \geq d\left(x_{1}, g\left(x_{1}\right)\right) \geq \delta_{g}$ and hence $3 D_{5}+\delta_{g} \geq d\left(x_{1}, x_{2}\right) \geq$ $\delta_{g}-D_{5}$. Now

$$
\begin{gathered}
d\left(\Phi\left(x_{1}\right), \Phi\left(x_{2}\right)\right) \geq d\left(\Phi\left(x_{1}\right), \Phi\left(g\left(x_{1}\right)\right)\right)-d\left(\Phi\left(g\left(x_{1}\right)\right), \Phi\left(x_{2}\right)\right) \geq \\
\geq \delta_{g}-L D_{5}-A \geq d\left(x_{1}, x_{2}\right)-(L+3) D_{5}-A
\end{gathered}
$$

and similarly

$$
\begin{gathered}
d\left(x_{1}, x_{2}\right) \geq d\left(x_{1}, g^{\prime}\left(x_{1}\right)\right)-d\left(g^{\prime}\left(x_{1}\right), x_{2}\right) \geq \\
\geq \delta_{g^{\prime}}-L D_{5}-A \geq d\left(\Phi\left(x_{1}\right), \Phi\left(x_{2}\right)\right)-(L+3) D_{5}-A .
\end{gathered}
$$

Hence we can take $D_{1}=(L+3) D_{5}+A$.

Lemma 2.21. Let $G \stackrel{\rho}{\curvearrowright} X$ be a discrete, cocompact isometric action on a $\delta$-Hyperbolic Hadamard space $X$.

1. If $\left\{\gamma_{g} \mid g \in S \subset G\right\}$ is a collection of distinct axis of distinct elements $g \in G$ such that $\left\{\delta_{g}\right\}$ is bounded then $\left\{\gamma_{g}\right\}$ forms a discrete set of geodesics.
2. If two axial elements $g_{1}, g_{2} \in G$ have a common fixed point in $\partial_{\infty} X$ then they have a common axis (i.e. they have both fixed points in common).

Proof. Assume some sequence $\gamma_{g_{i}}$ converges to a geodesic $\gamma$ and let $\star \in \gamma$ then the boundedness of $\left\{\delta_{g_{i}}\right\}$ says that there is a $C$ such that $d\left(\star, g_{i}(\star)\right)<C$ but this cannot be true for infinitely many distinct $g_{i}$.

To see the second statement we can assume that the attracting and repelling fixed points satisfy $\xi_{1}^{+}=\xi_{2}^{+}$and $\xi_{1}^{-} \neq \xi_{2}^{-}$(the other cases are similar). In this case $\left\{g_{1}^{-k} g_{2} g_{1}^{k}\right\}$ are distinct since they have distinct fixed point sets $\left\{\xi_{1}^{+}, g_{1}^{-k}\left(\xi_{2}^{-}\right)\right\}$in $\partial_{\infty} X$ while the axes converge (after taking a subsequence) to an axis of $g_{k}$. But again there is a $C$ such that $d\left(\star, g_{1}^{-k} g_{2} g_{1}^{k}(\star)\right)<C$ giving the desired contradiction.

### 2.5. Graphs of groups and their Bass-Serre trees

For the remainder of the paper, all group actions on simplicial trees will be assumed to be simplicial actions which do not invert edges, and geodesic segments/rays in simplicial trees will be unions of edges.

References for the material in this section are [Ser80, SW79, DD89].
Definition 2.22. A graph of groups is a connected graph $\mathcal{G}$ together with a group $G_{\sigma}$ labeling each $\sigma \in \operatorname{Vertex}(\mathcal{G}) \cup \operatorname{Edge}(\mathcal{G})$, and a monomorphism $G_{e} \rightarrow G_{v}$ for each pair $(e, v)$ consisting of an oriented edge $e$ entering a vertex $v$. An isomorphism of two graphs of groups is an isomorphism of labeled graphs which is compatible with edge monomorphisms.

Let $G \stackrel{\rho}{\curvearrowright} T$ be an action of a group $G$ on a simplicial tree $T$. We can define an associated graph of groups $\mathcal{G}$ as follows. We let the graph underlying $\mathcal{G}$ be $G / T$. For each $\sigma \in \operatorname{Vertex}(\mathcal{G}) \cup \operatorname{Edge}(\mathcal{G})$ we may label $\sigma$ with the stabilizer of a lift $\hat{\sigma} \subset T$ of $\sigma$. An for each pair $(e, v)$, where $e \subset T / G$ is an oriented edge with terminus $v \in T / G$, we can define an edge monomorphism $G_{e} \rightarrow G_{v}$ by composing the inclusion $G_{e}:=G_{\hat{e}} \rightarrow G_{g \hat{v}}(g \hat{v} \in T$ is the terminus of $\hat{e})$ with the isomorphism $G_{g \hat{v}} \rightarrow G_{\hat{v}}=G_{v}$ induced by conjugation by $g^{-1}$. We refer to this as the graph of groups associated with the action $G \curvearrowright T$.

Lemma 2.23. If $\mathcal{G}$ is a graph of groups, then there is a group $G$, a simplicial tree $T$, and an action $G \stackrel{\rho}{\curvearrowright} T$ so that:

1. If $\overline{\mathcal{G}}$ is the graph of groups associated with the action $G \stackrel{\rho}{\curvearrowright} T$, then $\overline{\mathcal{G}} \simeq \mathcal{G}$.
2. If $G^{\prime} \stackrel{\rho^{\prime}}{\curvearrowright} T^{\prime}$ is another action on a simplicial tree satisfying 1, then there is an isomorphism $G^{\prime} \simeq G$ so that the actions $\rho$ and $\rho^{\prime}$ become simplicially isomorphic.

The (isomorphism class of the) group $G$ is the fundamental group of $\mathcal{G}$, and the tree $T$ (or really the action $G \stackrel{\rho}{ค} T$ ) is called the Bass-Serre tree of $\mathcal{G}$. We note that if
$v$ is a vertex of $T$ and $G_{v}$ is its stabilizer, then the $G_{v}$-orbits of $\operatorname{Link}(v)$ correspond bijectively to the elements of $\operatorname{Link}(\bar{v})$ where $\bar{v} \in \mathcal{G}$ is the corresponding vertex of $G / T \simeq \mathcal{G}$, and the stabilizer of $\xi \in \operatorname{Link}(v)$ is just $G_{e}$ where $e$ is the edge associated with $\xi$.

Let $\mathcal{G}$ be a graph of groups and let $\bar{e}$ be an edge of $\mathcal{G}$ with endpoints $\bar{v}_{1}$ and $\bar{v}_{2}$. We let $\mathcal{G}^{\prime}$ be the graph of groups determined by $\bar{e}$. The fundamental group of $\mathcal{G}^{\prime}$ is a free product with amalgamation if $\bar{e}$ is embedded in $\mathcal{G}$ and an $H N N$ extension if $\bar{e}$ is a loop. Choose a lift $e=\overline{v_{1} v_{2}} \subset T$ of $\bar{e}$ to the Bass-Serre tree $T$. We may identify $G_{\bar{v}_{i}}$ with $G_{v_{i}}$ and $G_{\bar{e}}$ with $G_{e}$ in a fashion compatible with the edge inclusions $G_{\bar{e}} \rightarrow G_{\bar{v}_{i}}$, $G_{e} \rightarrow G_{v_{i}}$. When $\bar{e}$ is a loop we may choose $t \in G=\pi_{1}(\mathcal{G})$ so that $t\left(v_{1}\right)=v_{2}$ and the composition $G_{e} \rightarrow G_{v_{1}} \xrightarrow{t(\cdot) t^{-1}} G_{v_{2}}$ agrees with the edge monomorphism $G_{e} \rightarrow G_{v_{2}}$. Set $G^{\prime}:=\left\langle G_{v_{1}}, G_{v_{2}}\right\rangle$ when $\bar{e}$ is embedded and set $G^{\prime}:=\left\langle G_{v_{1}}, t\right\rangle$ when $\bar{e}$ is a loop. Then the orbit $T^{\prime}:=G^{\prime}(e) \subset T$ is a $G^{\prime}$-invariant subtree of $T$, and the action $G^{\prime} \curvearrowright T^{\prime}$ is the Bass-Serre action for $\mathcal{G}^{\prime}$. When $\bar{e}$ is embedded we choose subsets $\Sigma_{i} \subset G_{v_{i}}$ which intersect each right coset of $G_{e}$ exactly once; then any $g \in G^{\prime}$ can be written uniquely in the form

$$
\begin{equation*}
s_{1} \ldots s_{k} r \tag{2.24}
\end{equation*}
$$

where $r \in G_{e}, s_{i} \notin G_{e}$, and the $s_{i}$ 's belong alternately to $\Sigma_{1}$ and $\Sigma_{2}$. The combinatorial distance from the edge $g(e)$ to $e$ is $k$ and $e_{i}=s_{1} \ldots s_{i}(e)$ is the sequence of edges along the path from $e$ to $g(e)$. When $\bar{e}$ is a loop, we choose a cross-section $\Sigma_{1} \subset G_{v_{1}}$ (respectively $\Sigma_{-1}$ ) of the right cosets of $G_{e}$ (respectively $t^{-1} G_{e} t$ ). Then any $g \in G^{\prime}$ can be written uniquely in the form

$$
\begin{equation*}
s_{1} t^{\epsilon_{1}} s_{2} t^{\epsilon_{2}} \ldots s_{k} t^{\epsilon_{k}} r \tag{2.25}
\end{equation*}
$$

where for $i=1, \ldots, k, \epsilon_{i}= \pm 1, s_{i} \in \Sigma_{\epsilon_{i}}, r \in G_{v_{1}}$, and if $s_{i} \in G_{e}$ then $\epsilon_{i-1}=-\epsilon_{i}$.

## 3. Graphs of groups and the structure of Hadamard spaces on which they act

### 3.1. Admissible groups and actions

Definition 3.1. A graph of groups $\mathcal{G}$ is admissible if

1. $\mathcal{G}$ is a finite graph with at least one edge.
2. Each vertex group $\bar{G}_{v}$ has center $Z\left(\bar{G}_{v}\right) \simeq \mathbb{Z}, \bar{H}_{v}:=\bar{G}_{v} / Z\left(\bar{G}_{v}\right)$ is a nonelementary hyperbolic group, and every edge subgroup $\bar{G}_{e}$ is isomorphic to $\mathbb{Z}^{2}$.
3. Let $e_{1}, e_{2}$ be distinct directed edges entering a vertex $v$, and for $i=1,2$ let $K_{i} \subset \bar{G}_{v}$ be the image of the edge homomorphism $\bar{G}_{e_{i}} \rightarrow \bar{G}_{v}$. Then for every $g \in \bar{G}_{v}$, $g K_{1} g^{-1}$ is not commensurable with $K_{2}$, and for every $g \in \bar{G}_{v}-K_{i}, g K_{i} g^{-1}$ is not commensurable with $K_{i}$.
4. For every edge group $\bar{G}_{e}$, if $\alpha_{i}: \bar{G}_{e} \rightarrow \bar{G}_{v_{i}}$ are the edge monomorphisms, then the subgroup generated by $\alpha_{1}^{-1}\left(Z\left(\bar{G}_{v_{1}}\right)\right)$ and $\alpha_{2}^{-1}\left(Z\left(\bar{G}_{v_{2}}\right)\right)$ has finite index in $G_{e} \simeq \mathbb{Z}^{2}$. A group $G$ is admissible if it is the fundamental group of an admissible graph of groups.

Let $G$ be the fundamental group of an admissible graph of groups $\mathcal{G}$, and let $G \curvearrowright T$ be the action of $G$ on the associated Bass-Serre tree. We let $V:=\operatorname{Vertex}(T)$ and $E:=E d g e(T)$ denote the vertex and edge sets of $T$, and when $\sigma \in V \cup E$ we let $G_{\sigma} \subset G$ denote corresponding stabilizer. Properties 1-4 of definition 3.1 imply :
Lemma 3.2. 1. $T$ is an unbounded tree with infinite valence at each vertex, and $G$ acts on $T$ with quotient $\mathcal{G}$.
2. Each vertex group $G_{v}$ has center $Z\left(G_{v}\right) \simeq \mathbb{Z}, H_{v}:=G_{v} / Z\left(G_{v}\right)$ is a nonelementary hyperbolic group, and every edge subgroup $G_{e}$ is isomorphic to $\mathbb{Z}^{2}$.
3. If $e_{1}, e_{2}$ are distinct edges emanating from $v \in V$, then $G_{e_{1}}$ is not commensurable with $G_{e_{2}}$. In particular, $Z\left(G_{v}\right) \subset G_{e_{i}}$ since $g \in Z\left(G_{v}\right)$ implies $G_{e_{i}}=g G_{e_{i}} g^{-1}=$ $G_{\text {ge }_{i}}$ forcing ge $e_{i}=e_{i}$, i.e. $g \in G_{e_{i}}$.
4. If $e \in E$ has endpoints $v_{1}, v_{2} \in V$, then $\left(Z\left(G_{v_{1}}\right) \cup Z\left(G_{v_{2}}\right)\right) \subset G_{e}$ generates a finite index subgroup of $G_{e}$.

Most of the time we will work with the action $G \curvearrowright T$ and ignore the graph of groups that produced it.

Examples of admissible groups:

1. (Graph manifolds) Let $M$ be a 3 -dimensional nonpositively curved graph manifold as in Theorem 1.1, and let $M_{i}, i=1, \ldots k$ be the geometric Seifert components of $M$. Let $\mathcal{G}$ be the graph of groups which has one vertex labeled with $\pi_{1}\left(M_{i}\right)$ for each $i$, and an edge labeled by $\mathbb{Z}^{2}$ for each pair of totally geodesic boundary tori in the disjoint union $\cup_{i} M_{i}$ which are glued to form $M$. The edge monomorphisms come from the two different embeddings of a gluing torus into Seifert components.
2. (Torus complexes) Let $T_{0}, T_{1}, T_{2}$ be flat two-dimensional tori. For $i=1,2$, we choose primitive closed geodesics $a_{i} \subset T_{0}$ and $b_{i} \subset T_{i}$ with length $\left(a_{i}\right)=\operatorname{length}\left(b_{i}\right)$, and we glue $T_{i}$ to $T_{0}$ by identifying $a_{i}$ with $b_{i}$ isometrically. We assume that $a_{1}$ and $a_{2}$ lie in distinct free homotopy classes, and intersect at an angle $\alpha \in\left(0, \frac{\pi}{2}\right]$. Let $\mathcal{G}$ be the graph of groups associated with the decomposition $\left(T_{0} \cup T_{1}\right) \amalg\left(T_{0} \cup T_{1}\right) \rightarrow \cup T_{i}$. Note that $T_{0} \cup T_{1}$ is homeomorphic to $S^{1} \times\left(S^{1} \wedge S^{1}\right)$, so $\pi_{1}\left(T_{0} \cup T_{i}\right)=\mathbb{Z} \times F_{2}$ where $F_{2}$ is the free group on two generators.

Lemma 3.3. 1. If $e_{1}, e_{2} \in E$ are distinct edges incident to $v \in V$, then $Z\left(G_{v}\right) \simeq \mathbb{Z}$ is a finite index subgroup of $G_{e_{1}} \cap G_{e_{2}}$. In particular $G_{e_{1}} \cap G_{e_{2}} \simeq \mathbb{Z}$.
2. If $v_{1}, v_{2} \in V$ are the endpoints of an edge $e \in E$, then $Z\left(G_{v_{1}}\right) \cap Z\left(G_{v_{2}}\right)=\{i d\}$.

Proof. Note that $G_{e_{1}} \cap G_{e_{2}}$ has infinite index in each $G_{e_{i}}$, for otherwise the $G_{e_{i}} \simeq \mathbb{Z}^{2}$ would be commensurable, contradicting 3 of Lemma 3.2. Thus $G_{e_{1}} \cap G_{e_{2}} \simeq \mathbb{Z}$. Also by 3 of Lemma 3.2 we have $Z\left(G_{v}\right) \subset G_{e_{1}} \cap G_{e_{2}}$, so both are rank 1 free abelian groups and 1 follows.

2 follows immediately from 4 of Lemma 3.2 , since the $Z\left(G_{v_{i}}\right)$ are rank 1 subgroups of $G_{e} \simeq \mathbb{Z}^{2}$ which generate a finite index subgroup of $G_{e}$.

Lemma 3.4. If $v_{1}, v_{2} \in V$, then

1. If $d\left(v_{1}, v_{2}\right)>2$ then $G_{v_{1}} \cap G_{v_{2}}=\{i d\}$.
2. If $d\left(v_{1}, v_{2}\right)=2$ and $v \in V$ is the vertex between them, then $Z\left(G_{v}\right)$ is a finite index subgroup of $G_{v_{1}} \cap G_{v_{2}}$.
3. If $d\left(v_{1}, v_{2}\right)=1$, then $G_{v_{1}} \cap G_{v_{2}}=G_{e}$ where $e=\overline{v_{1} v_{2}}$.

Proof. We will prove the assertions in reverse order. Part 3 is immediate since the action $G \curvearrowright T$ does not invert edges, see section 2.5. To prove 2 we let $e_{i}$ be the edge between $v_{i}$ and $v$. Clearly $G_{v_{1}} \cap G_{v_{2}}=G_{e_{1}} \cap G_{e_{2}}$, which by Lemma 3.3 contains $Z\left(G_{v}\right)$ as a subgroup of finite index. To prove 1 , let $e_{1}, e, e_{2} \in E$ be three consecutive edges of the segment $\overline{v_{1} v_{2}}$, and let $w_{1}, w_{2} \in V$ be the endpoints of $e$. Then by 1 of Lemma 3.3, $Z\left(G_{w_{i}}\right)(\simeq \mathbb{Z})$ has finite index in $G_{e_{i}} \cap G_{e}(\simeq \mathbb{Z})$, so we have $\left(G_{e_{1}} \cap G_{e}\right) \cap\left(G_{e} \cap G_{e_{2}}\right)=\emptyset$ since otherwise $Z\left(G_{w_{1}}\right) \cup Z\left(G_{w_{2}}\right)$ would generate a cyclic group in $G_{e}\left(\simeq \mathbb{Z}^{2}\right)$ contradicting 4 of Lemma 3.2. Since

$$
G_{v_{1}} \cap G_{v_{2}} \subset\left(G_{e_{1}} \cap G_{e}\right) \cap\left(G_{e} \cap G_{e_{2}}\right)
$$

1 follows.
Lemma 3.5. For $v \in V$, the fixed point set of $Z\left(G_{v}\right)$ is the closed star $\overline{\operatorname{Star}(v)}$. For $e \in E$, the fixed point set of $G_{e} \frac{\text { is e. In particular, for any } \sigma \in V \cup E, G_{\sigma} \text { leaves no }}{\text { Stal }}$ point in $\partial_{\infty} T$ fixed. Further, if $\overline{\operatorname{Star}(v)}$ is invariant under $Z\left(G_{\sigma}\right)$ then $\sigma \in \overline{\operatorname{Star}(v)}$.

Proof. Fix a vertex $v$. Since $Z\left(G_{v}\right) \subset G_{e}$ for all $e$ in the star it is clear that the closed star is in the fixed point set of $Z\left(G_{v}\right)$. On the other hand 1 of Lemma 3.4 says we need only consider vertices $v_{1}$ such that $d\left(v, v_{1}\right)=2$. Let $w$ be the vertex between $v$ and $v_{1}$. Now 2 of Lemma 3.4 says that $Z\left(G_{w}\right)$ is a finite index subgroup of $G_{v} \cap G_{v_{1}}$, while 2 of Lemma 3.3 says that $Z\left(G_{v}\right) \cap Z\left(G_{w}\right)=\emptyset$. Thus $Z\left(G_{v}\right) \cap G_{v_{1}}=\emptyset$ and the first statement follows.

For an edge $e=\overline{v_{1} v_{2}}$ the first part (since $G_{v_{i}} \subset G_{e}$ ) says that the fixed point set of $G_{e}$ is contained in $\overline{\operatorname{Star}\left(v_{1}\right)} \cap \overline{\operatorname{Star}\left(v_{2}\right)}=e$. So the second statement follows.

The last two statements follow from the first two.
Lemma 3.6. If $v \in V$ then the centralizer of $Z\left(G_{v}\right)$ in $G$ is just $G_{v}$. If $e \in E$ the centralizer of $G_{e}$ in $G$ is $G_{e}$.

Proof. By Lemma 3.5, the fixed point set of $Z\left(G_{v}\right)$ in $T$ is just the closed star of $v$ in $T$. Hence any $g \in G$ which commutes with $Z\left(G_{v}\right)$ must take the star of $v$ to itself and hence fix $v$.

If $e \in E$ and $e=\overline{v_{1} v_{2}}$, then a finite index subgroup $G_{e}$ is generated by $Z\left(G_{v_{1}}\right) \cup$ $Z\left(G_{v_{2}}\right)$. So the centralizer of $G_{e}$ in $G$ is a subgroup of the intersection of the centralizers of $Z\left(G_{v_{1}}\right)$ and $Z\left(G_{v_{2}}\right)$, i.e. $G_{v_{1}} \cap G_{v_{2}}$ which is $G_{e}$ itself.
Lemma 3.7. (Uniqueness of decomposition) Let $G \curvearrowright T$ and $G^{\prime} \curvearrowright T^{\prime}$ be the BassSerre actions associated with two admissible graphs of groups, and suppose $G \xrightarrow{\phi} G^{\prime}$ is an isomorphism. Then after identifying $G$ with $G^{\prime}$ via $\phi$, the trees $T$ and $T^{\prime}$ become $G$-equivariantly isomorphic.

Proof. We will use primes to denote the vertex and edge set of $T^{\prime}$. Pick $v \in V$.
Claim: $G_{v}$ fixes a unique vertex in $T^{\prime}$. Let $g \in Z\left(G_{v}\right)$ be a generator. If Fix $\left(g, T^{\prime}\right)$ is empty, then $g$ translates a unique geodesic $\gamma \subset T^{\prime}$, and since $g \in Z\left(G_{v}\right)$ the whole vertex group $G_{v}$ must preserve $\gamma$, and act on it by translations. The signed translation distance yields a homomorphism $G_{v} \rightarrow \mathbb{Z}$ with nontrivial kernel. But then
$\operatorname{Ker}\left(G_{v} \rightarrow \mathbb{Z}\right)$ fixes $\gamma$ pointwise, which contradicts 1 of Lemma 3.4. Consequently Fix $\left(g, T^{\prime}\right)$ is nonempty, and by 1 of Lemma 3.4 this is a subcomplex of $T^{\prime}$ with diameter at most 2. So $G_{v}$ must fix the center of $\operatorname{Fix}\left(g, T^{\prime}\right)$. It can fix nothing more, since no edge stabilizer can contain the nonabelian $G_{v}$. Thus we have proved the claim.

Now consider the $G$-equivariant map $f: V \rightarrow V^{\prime}$ which assigns to each $v \in V$ the unique vertex in $T^{\prime}$ fixed by $G_{v}$; and define a map $f^{\prime}: V^{\prime} \rightarrow V$ by reversing the roles of $T$ and $T^{\prime}$. For all $v \in V, G_{v}$ fixes $f^{\prime} \circ f(v)$, so we must have $f^{\prime} \circ f(v)=v$; similar reasoning applies to $f \circ f^{\prime}$, and we see that $f$ and $f^{\prime}$ are inverses. The maps $f$ and $f^{\prime}$ are adjacency preserving since two vertices are adjacent iff their stabilizers intersect in a subgroup isomorphic to $\mathbb{Z}^{2}$. It is now straightforward to see that $f$ defines a $G$-equivariant isomorphism $T \rightarrow T^{\prime}$.

Lemma 3.7 justifies use of the phrase " $G \stackrel{\rho}{\curvearrowright} T$ is the Bass-Serre tree of the admissible group $G$."

### 3.2. Vertex spaces, edge spaces, and geometric data for admissible actions

Definition 3.8. We say that $G \stackrel{\rho}{\curvearrowright} X$ is an admissible action if $G$ is an admissible group, $X$ is a Hadamard space, and the action is discrete, cocompact, and isometric.

For the remainder of this section $G \curvearrowright X$ will be a fixed admissible action. In particular, all constants depend on $G \curvearrowright X$ (i.e. the group $G$, the Riemannian manifold $X$, and the action) in addition to other explicitly mentioned quantities. By Lemma 3.7 there is an essentially unique admissible graph of groups associated with $G$, and we will let $G \curvearrowright T$ be the corresponding Bass-Serre tree.

We refer the reader to section 2.3 for properties of Minsets that we use here. For each $v \in V$ we let $Y_{v}:=\operatorname{Minset}\left(Z\left(G_{v}\right)\right):=\cap_{g \in Z\left(G_{v}\right)} \operatorname{Minset}(g)$ (this will be the Minset of a generator), and for every $e \in E$ we let $Y_{e}:=\operatorname{Minset}\left(G_{e}\right):=$ $\cap_{g \in G_{e}} \operatorname{Minset}(g)$.
$\operatorname{Minset}(\alpha)$ is a convex subset of $X$, invariant under the centralizer of $\alpha$, which is a metric product of $\mathbb{R}$ with a Hadamard space. If $\alpha$ belongs to a group of isometries that acts cocompactly on $X$ then the centralizer of $\alpha$ acts cocompactly on $\operatorname{Minset}(\alpha)$ (see section 2.3). Thus $Y_{v}$ is the product of $\mathbb{R}$ with a Hadamard space $\bar{Y}_{v} . Z\left(G_{v}\right)$ acts by translation on the $\mathbb{R}$ factor and the induced action of $H_{v}$ on $\bar{Y}_{v}$ is discrete and cocompact. $Y_{e}$ is the product of $\mathbb{R}^{2}$ with a compact Hadamard space $\bar{Y}_{e}$, and $G_{e}=\mathbb{Z}^{2}$ acts by translations on the $\mathbb{R}^{2}$ factor (section 2.3).

Note that the assignments $v \mapsto Y_{v}$ and $e \mapsto Y_{e}$ are $G$-equivariant with respect to the natural $G$ actions. The minimal displacement of a generator of $Z\left(G_{v}\right)$ is the same as that of a generator of $Z\left(G_{g(v)}\right)=g Z\left(G_{v}\right) g^{-1}$. By the finiteness of $\mathcal{G}$ there is a number $C$ such that for all $v \in V$ the minimal displacement of a generator of $Z\left(G_{v}\right)$ is less than C.

Definition 3.9. Let $G \stackrel{\rho}{\curvearrowright} X$ be an admissible action, and let $T$ be the Bass-Serre tree for $G$. For each $v \in V$ we choose a generator $\zeta_{v} \in Z\left(G_{v}\right)$ in a $G$-equivariant way. We have an isometric splitting $Y_{v} \simeq \bar{Y}_{v} \times \mathbb{R}$, which is preserved by $G_{v}$. The choice of generator $\zeta_{v}$ defines an orientation of the $\mathbb{R}$ factor of $Y_{v}$. We have a map $M L S_{v}: G_{v} \rightarrow \mathbb{R}_{+}$which assigns to each $g \in G_{v}$ the minimum displacement of the
induced isometry $\bar{Y}_{v} \rightarrow \bar{Y}_{v}$. $M L S_{v}$ descends to $G_{v} / Z\left(G_{v}\right) \simeq H_{v}$ since $Z\left(G_{v}\right)$ acts trivially on $\bar{Y}_{v}$. We define a homomorphism $\tau_{v}: G_{v} \rightarrow \mathbb{R}$ by sending $g \in G_{v}$ to the signed distance that $g$ translates the $\mathbb{R}$ factor of $Y_{v} \simeq \bar{Y}_{v} \times \mathbb{R}$. The collections of functions $M L S_{v}$ and $\tau_{v}$ constitute the geometric data of the action. Both $M L S_{v}$ and $\tau_{v}$ descend to functions of the vertex groups of the graph of groups $\mathcal{G}$ defining $G$; we will sometimes find it more convenient to think of the geometric data in this way.

We remark that it follows from the discreteness of the action $H_{v} \curvearrowright \bar{Y}_{v}$ that $g \in G_{v}$ then $M L S_{v}(g)=0$ iff $g$ projects to an element of finite order in $H_{v}$.

Lemma 3.10. The collections $\left\{Y_{v}\right\}_{v \in V}$ and $\left\{Y_{e}\right\}_{e \in E}$ are locally finite. More precisely, for every $R$ there is an $N$ so that if $x \in X$ then there are at most $N$ elements $\sigma \in V \cup E$ so that $Y_{\sigma} \cap B(x, R) \neq \emptyset$.

Proof. Suppose $v \in V$ and $p \in Y_{v}$. Then $p$ has displacement $<C$ under the generators of $Z\left(G_{v}\right)$. Therefore if $p \in B(x, R)$, then $x$ has displacement $<2 R+C$ under the generators of $Z\left(G_{v}\right)$. But there are only finitely many $g \in G$ with $d(g \cdot x, x)<2 R+C$; since $Z\left(G_{v_{1}}\right) \cap Z\left(G_{v_{2}}\right)=\{e\}$ when $v_{1} \neq v_{2}$, the local finiteness of $\left\{Y_{v}\right\}_{v \in V}$ follows. Similar reasoning proves the local finiteness of $\left\{Y_{e}\right\}_{e \in E}$. The fact that $N$ can be chosen independent of $x$ follows from the cocompactness of the $G$ action.

The lemma implies that for any $D$, the collection of $D$-tubular neighborhoods of the Y.'s is locally finite. We also have the following consequences:

Lemma 3.11. For every $D$ there are only finitely many pairs $\left(\sigma_{1}, \sigma_{2}\right) \in(V \cup E) \times$ $(V \cup E)$ - modulo the diagonal action of $G$ - with $N_{D}\left(Y_{\sigma_{1}}\right) \cap N_{D}\left(Y_{\sigma_{2}}\right) \neq \emptyset$.

Proof. By the finiteness of $[V \cup E] / G$ we need only show that for fixed $\sigma$ there are only finitely many $\sigma_{2}$ modulo $G_{\sigma}$ such that $N_{D}\left(Y_{\sigma}\right) \cap N_{D}\left(Y_{\sigma_{2}}\right) \neq \emptyset$. This follows from Lemma 3.10, since $G_{\sigma}$ acts cocompactly on $N_{D}\left(Y_{\sigma}\right)$ and hence for some $g \in G_{\sigma}$ $N_{D}\left(Y_{\sigma}\right) \cap\left(N_{D}\left(Y_{g\left(\sigma_{2}\right)}\right)\right.$ intersects a fixed ball.
Lemma 3.12. For $\sigma_{1}, \sigma_{2} \in V \cup E, G_{\sigma_{1}} \cap G_{\sigma_{2}}$ acts cocompactly on the intersection $N_{D}\left(Y_{\sigma_{1}}\right) \cap N_{D}\left(Y_{\sigma_{2}}\right)$. Thus, in particular, the diameter of $N_{D}\left(Y_{\sigma_{1}}\right) \cap N_{D}\left(Y_{\sigma_{2}}\right) /\left[G_{\sigma_{1}} \cap G_{\sigma_{2}}\right]$ is uniformly bounded by a function of $D$.

Proof. This follows from the local finiteness of the family $\left\{Y_{\sigma}\right\}_{\sigma \in V \cup E}$ and the discreteness of the cocompact action $G \curvearrowright X$. Pick $D>0$ and $\sigma_{1}, \sigma_{2} \in V \cup E$. If $x_{k} \in N_{D}\left(Y_{\sigma_{1}}\right) \cap N_{D}\left(Y_{\sigma_{2}}\right)$, we may choose a sequence $g_{k} \in G_{\sigma_{1}}$ such that $g_{k}\left(x_{k}\right) \rightarrow x_{\infty}$ for some $x_{\infty} \in N_{D}\left(Y_{\sigma_{1}}\right)$. Then $g_{k}\left(\sigma_{2}\right)$ lies in a finite subset of $V \cup E$ (since $g_{k}\left(N_{D}\left(Y_{\sigma_{2}}\right)\right.$ ) intersects some ball $B\left(x_{\infty}, R\right)$ for all $k$ ) so after passing to a subsequence if necessary we may assume that $g_{k} \sigma_{2}$ is constant. Then $g_{1}^{-1} g_{k} \in G_{\sigma_{1}} \cap G_{\sigma_{2}}$ and $\left(g_{1}^{-1} g_{k}\right)\left(x_{k}\right) \rightarrow$ $g_{1}^{-1}\left(x_{\infty}\right)$. Thus $G_{v_{1}} \cap G_{v_{2}}$ acts cocompactly. The second statement now follows from Lemma 3.11.

Lemma 3.13. For every $D$ there is a $D^{\prime}$ (depending only on $D$ ) such that if $\sigma \in V \cup E$ separates $\sigma_{1} \in V \cup E$ from $\sigma_{2} \in V \cup E$, then $N_{D}\left(Y_{\sigma_{1}}\right) \cap N_{D}\left(Y_{\sigma_{2}}\right) \subset N_{D^{\prime}}\left(Y_{\sigma}\right)$. In particular, if $T_{1}$ and $T_{2}$ are the closures of distinct connected components of $T-\sigma$ then

$$
\left[\cup_{\hat{\sigma} \subset T_{1}} N_{D}\left(Y_{\hat{\sigma}}\right)\right] \cap\left[\cup_{\hat{\sigma} \subset T_{2}} N_{D}\left(Y_{\hat{\sigma}}\right)\right] \subset N_{D^{\prime}}\left(Y_{\sigma}\right)
$$

Proof. Pick $D>0$. Suppose $\left(\sigma_{1}, \sigma_{2}, \sigma\right)$ is a triple with $\sigma_{i}, \sigma \in V \cup E, \sigma$ separates $\sigma_{1}$ from $\sigma_{2}$ in $T$, and $N_{D}\left(Y_{\sigma_{1}}\right) \cap N_{D}\left(Y_{\sigma_{2}}\right) \neq \emptyset$. Then $G_{\sigma_{1}} \cap G_{\sigma_{2}} \subset G_{\sigma}$ and $G_{\sigma_{1}} \cap G_{\sigma_{2}}$ acts cocompactly on $N_{D}\left(Y_{\sigma_{1}}\right) \cap N_{D}\left(Y_{\sigma_{2}}\right)$ by Lemma 3.12 ; hence $d\left(Y_{\sigma}, \cdot\right)$ is bounded on $N_{D}\left(Y_{\sigma_{1}}\right) \cap N_{D}\left(Y_{\sigma_{2}}\right)$. By Lemma 3.11 there are only finitely many such triples $\left(\sigma_{1}, \sigma_{2}, \sigma\right)$ modulo $G$, so the lemma follows.

Definition 3.14. Since $G$ acts cocompactly on $X$ we can now fix a $D$ so that $\cup_{v \in V} N_{D}\left(Y_{v}\right)=\cup_{e \in E} N_{D}\left(Y_{e}\right)=X$. We define $X_{v}:=N_{D}\left(Y_{v}\right)$ for all $v \in V$. Let $D^{\prime}$ denote the constant in the previous lemma, $D^{\prime \prime}=\max \left(D, D^{\prime}\right)$, and set $X_{e}:=N_{D^{\prime \prime}}\left(Y_{e}\right)$ for all $e \in E$. We will refer to the $X_{v}$ 's and $X_{e}$ 's as vertex spaces and edge spaces respectively.

We note that Lemma 2.3 implies that for any $\sigma \in V \cup E$ we have $\partial_{\infty} X_{\sigma}=\partial_{\infty} Y_{\sigma}$. We summarize the properties of vertex and edge spaces:

Lemma 3.15. There is a constant $C_{1}$ with the following property.

1. $\cup_{v \in V} X_{v}=\cup_{e \in E} X_{e}=X$.
2. If $\hat{e} \in E$ and $T_{1}$ and $T_{2}$ are the distinct connected components of $T-\operatorname{Int}(\hat{e})$, then $\left[\cup_{v \in T_{1}} X_{v}\right]-X_{\hat{e}}$ and $\left[\cup_{v \in T_{2}} X_{v}\right]-X_{\hat{e}}$ are disjoint closed and open subsets of $X-X_{\hat{e}}$; and $\cup_{e \in T_{1}} X_{e}-N_{C_{1}}\left(X_{\hat{e}}\right)$ and $\cup_{e \in T_{2}} X_{e}-N_{C_{1}}\left(X_{\hat{e}}\right)$ are disjoint closed and open subsets of $X-N_{C_{1}}\left(X_{\hat{e}}\right)$.
3. If $\sigma_{1}, \sigma_{2} \in V \cup E$ and $X_{\sigma_{1}} \cap X_{\sigma_{2}} \neq \emptyset$ then $d_{T}\left(\sigma_{1}, \sigma_{2}\right)<C_{1}$.

Proof. 1 and 2 follow from the definition of vertex/edge spaces and Lemma 3.13. By Lemma 3.11 we can choose $C_{1}$ so that 3 holds.

Corollary 3.16. For any $v \in V, \partial_{T} X_{v}$ is isometric to the metric suspension of an uncountable discrete space, and for every $e \in E, \partial_{T} X_{e}$ is isometric to a standard circle. Pick $v_{1}, v_{2} \in V$.

1. If $d\left(v_{1}, v_{2}\right)>2$, then $\partial_{T} X_{v_{1}} \cap \partial_{T} X_{v_{2}}=\emptyset$.
2. If $d\left(v_{1}, v_{2}\right)=2$ and $v$ is the vertex in between $v_{1}$ and $v_{2}$, then $\partial_{T} X_{v_{1}} \cap \partial_{T} X_{v_{2}}=$ $\partial_{T} \gamma$ where $\gamma \subset Y_{v}$ is a geodesic of the form $\{p\} \times \mathbb{R} \subset \bar{Y}_{v} \times \mathbb{R}=Y_{v}$; i.e. $\partial_{T} X_{v_{1}} \cap \partial_{T} X_{v_{2}}$ is the pair of suspension points of $\partial_{T} X_{v}$.
3. If $d\left(v_{1}, v_{2}\right)=1$, then $\partial_{T} X_{v_{1}} \cap \partial_{T} X_{v_{2}}=\partial_{T} X_{e} \simeq S^{1}$, where $e:=\overline{v_{1} v_{2}}$.

Proof. Since $Y_{v} \simeq \bar{Y}_{v} \times \mathbb{R}$, we have $\partial_{T} X_{v}=\partial_{T} Y_{v}=\Sigma\left(\partial_{T} \bar{Y}_{v}\right)$, and since $\bar{Y}_{v}$ admits a discrete cocompact action by the non-elementary hyperbolic group $H_{v}:=G_{v} / Z\left(G_{v}\right)$, $\partial_{T} Y_{v}$ is a discrete set with the cardinality of $\mathbb{R}$. For all $e \in E, Y_{e} \simeq \bar{Y}_{e} \times \mathbb{R}^{2}$ where $\bar{Y}_{e}$ is compact, so $\partial_{T} X_{e}=\partial_{T} Y_{e} \simeq \partial_{T} \mathbb{R}^{2}$, and the latter is the standard circle.

Pick $v_{1}, v_{2} \in V$, and choose $R$ large enough that $Z:=N_{R}\left(X_{v_{1}}\right) \cap N_{R}\left(X_{v_{2}}\right) \neq \emptyset$. Then $\partial_{T} X_{v_{1}} \cap \partial_{T} X_{v_{2}}=\partial_{T} Z$. The lemma now follows from Lemmas 3.4 and 3.12.
Lemma 3.17. There is a constant $C_{2}$ with the following property. Suppose $v, v^{\prime} \in V$, $e_{1}, \ldots, e_{n} \in E$ are the consecutive edges of the segment $\overline{v v^{\prime}} \subset T, x \in X_{v}$ and $y \in X_{v^{\prime}}$. Then for $1 \leq i \leq n$ we can find points $z_{i} \in \overline{x y}$ such that

1. $d\left(z_{i}, X_{e_{i}}\right)<C_{2}$
2. For all $1 \leq i \leq j \leq n$ we have $d\left(z_{i}, x\right) \leq d\left(z_{j}, x\right)$.
3. For every $p \in X$ we have $\#\left\{z_{i} \in B(p, 1)\right\}<C_{2}$.

Proof. Pick $v, v^{\prime} \in V, x \in X_{v}$, and $y \in X_{v^{\prime}}$. Suppose $\hat{e} \in E$ and let $T_{1}$ and $T_{2}$ be the two connected components of $T-\operatorname{Int}(\hat{e})$. If $\overline{x y} \cap N_{C_{1}}\left(X_{\hat{e}}\right)=\emptyset$ (hence in particular $\overline{x y} \cap X_{\hat{e}}=\emptyset$ ) then by the first part of 2 of Lemma 3.15, $\overline{x y}$ is contained in one of the two disjoint open sets $\left(\cup_{e \subset T_{i}} X_{e}\right)-N_{C_{1}}\left(X_{\hat{e}}\right)$ for $i=1$ or $i=2$. It follows that $\overline{x y} \cap N_{C_{1}}\left(X_{e_{i}}\right)$ is nonempty for every $1 \leq i \leq n$. Let $w_{i} \in \overline{x y}$ be the point in $\overline{x y} \cap N_{C_{1}}\left(X_{e_{i}}\right)$ closest to $x$. Let $z_{1}=w_{1}$, and let $z_{i}$ be the element of $\left\{w_{i}, \ldots, w_{n}\right\}$ closest to $x$. So we have either $z_{i}=w_{i}$, or $z_{i}=w_{i^{\prime}}$ for some $i^{\prime}>i$. In the latter case $\overline{x z_{i}} \subset \cup_{e \subset T^{\prime}} X_{e}$ (hence in particular $z_{i}=w_{i^{\prime}} \in \cup_{e \subset T^{\prime}} X_{e}$ ) where $T^{\prime} \subset T$ is the component of $T-\operatorname{Int}\left(e_{i}\right)$ containing $e_{1}$, so by Lemma 3.13 we have $z_{i} \in N_{D^{\prime}}\left(X_{e_{i}}\right)$ where $D^{\prime}$ depends only on $C_{1}$. If $p \in X$ and $z_{i} \in B(p, 1)$, then $X_{e_{i}} \cap B\left(p, 1+D^{\prime}\right) \neq \emptyset$ and thus $Y_{e_{i}} \cap B\left(p, 1+D^{\prime}+D^{\prime \prime}\right) \neq \emptyset$ so by Lemma 3.10 we have $\#\left\{z_{i} \in B(p, 1)\right\}<N$ where $N$ depends only on $D^{\prime}$. Setting $C_{2}:=\max \left\{D^{\prime}, N\right\}$, the lemma follows.

### 3.3. Itineraries

Our next objective is to associate an itinerary to any ray $\overline{p \xi} \subset X$ which is not contained in a finite tubular neighborhood of a single vertex space; the itinerary of $\overline{p \xi}$ is a ray in $T$ which (roughly speaking) records the sequence of vertex spaces visited by $\overline{p \xi}$.

Let $\rho: X \rightarrow V \subset T$ be a $G$-equivariant coarse Lipschitz map from the Hadamard space $X$ to the vertex set of the tree $T$ with the property that for every $x \in X$ we have $x \in X_{\rho(x)}$. Such a $\rho$ may be constructed as follows. Let $\Sigma \subset X$ be a set theoretic cross-section for the free action $G \curvearrowright X$; define $\rho_{0}: \Sigma \rightarrow T$ so that $\sigma \in X_{\rho_{0}(\sigma)}$ for every $\sigma \in \Sigma$, and then extend $\rho_{0}$ to an equivariant map $X \rightarrow T$. Let $L$ be such that $N_{2 D}\left(Y_{v_{1}}\right) \cap N_{2 D}\left(Y_{v_{2}}\right) \neq \emptyset$ implies $d\left(v_{1}, v_{2}\right)<L$, which exists by Corollary 3.11. In particular if $d(x, y)<2 D$ then $d(\rho(x), \rho(y)) \leq L$. In general, by dividing $\overline{x y}$ into less than $\frac{d(x, y)}{2 D}+1$ segments of length less than $2 D$ and adding the previous estimates we see that $\rho$ will be coarse Lipschitz; i.e. $d(\rho(x), \rho(y)) \leq \frac{L}{2 D} d(x, y)+L$.
Lemma 3.18. If $\gamma:[0, \infty) \rightarrow X$ is a geodesic, then $\rho \circ \gamma:[0, \infty) \rightarrow T$ has the bounded backtracking property ${ }^{7}$.

Proof. Let $e \in E$ be an edge in $T$, and let $T_{1}, T_{2} \subset T$ be the connected components of $T-\operatorname{Int}(e)$. Suppose $\rho \circ \gamma\left(t_{1}\right) \in T_{1}$, and $\rho \circ \gamma\left(t_{2}\right) \in T_{2}-N_{C_{1}}(e)$, where $t_{2}>t_{1}$. By Lemma 3.13 we have

$$
\left[\cup_{v \in T_{1}} X_{v}\right] \cap\left[\cup_{v \in T_{2}} X_{v}\right] \subset X_{e}
$$

Therefore there is a $t_{3} \in\left[t_{1}, t_{2}\right)$ such that $\gamma\left(t_{3}\right) \in X_{e}$. Since $d\left(\rho \circ \gamma\left(t_{2}\right), e\right) \geq C_{1}$, the choice of $C_{1}$ and lemma 3.15 implies that $\gamma\left(t_{2}\right) \notin X_{e}$. Hence the convexity of $X_{e}$ gives $\gamma\left(\left[t_{2}, \infty\right)\right) \subset X-X_{e}$, which forces $\rho \circ \gamma\left(\left[t_{2}, \infty\right)\right) \subset T_{2}$. This property clearly implies uniformly bounded backtracking.
Lemma 3.19. If $\gamma:[0, \infty) \rightarrow X$ is a geodesic ray, then one of the following holds:

1. $\rho \circ \gamma:[0, \infty) \rightarrow T$ is unbounded, and $\rho \circ \gamma([0, \infty))$ lies in a uniform tubular neighborhood of a unique geodesic ray, $\tau$, in $T$ starting at $\rho(\gamma(0))$. The geodesic $\gamma$ intersects $X_{e}$ for all but finitely many edges e of $\tau$.

[^4]2. $\rho \circ \gamma:[0, \infty) \rightarrow T$ is bounded, and $\gamma$ eventually lies in $N_{D^{\prime}}\left(Y_{v}\right)$ (where $D^{\prime}$ comes from Definition 3.14) for some $v \in V$. In this case there is a subcomplex $T_{\gamma} \subset T$ defined by the property that for each simplex $\sigma$ in $T, \sigma \in T_{\gamma}$ if and only if $\gamma$ is asymptotic to $X_{\sigma}$. The possibilities for $T_{\gamma}$ are: a single vertex $v \in V$, a single edge $e \in E$ along with its vertices, or the closed star $\overline{\operatorname{Star}(v)}$ for some $v \in V$.

Proof. Pick $v \in V$. By the convexity of $N_{D^{\prime}}\left(Y_{v}\right)$, either $\gamma$ is eventually contained in $N_{D^{\prime}}\left(Y_{v}\right)$, or $\gamma$ is eventually contained in $X-N_{D^{\prime}}\left(Y_{v}\right)$. In the latter case $\rho \circ \gamma$ eventually remains in a unique component of $T-v$, by Lemma 3.13.

If for every $v \in V$ the ray $\gamma$ eventually lies in $X-N_{D^{\prime}}\left(Y_{v}\right)$, then clearly $\rho \circ \gamma$ is unbounded and hence it must lie within uniform distance of a ray in $T$ by the bounded backtracking property. So we may assume that $\gamma$ is eventually contained in $N_{D^{\prime}}\left(Y_{v}\right)$ for some $v \in V$. We note that if $e \in T_{\gamma}$ then any vertex $v^{\prime}$ of $e$ must also be in $T_{\gamma}$ since $Y_{e} \subset Y_{v^{\prime}}$. Also if $v, v^{\prime} \in V$ and $d\left(v, v^{\prime}\right)>2$ then part 1 of Lemma 3.4 along with Lemma 3.12 says that for any $K, N_{K} X_{v} \cap N_{K} X_{v^{\prime}}$ is compact so $\gamma \notin N_{K} X_{v} \cap N_{K} X_{v^{\prime}}$ and hence at most one of $v$ and $v^{\prime}$ can be in $T_{\gamma}$.

If there are vertices $v_{1}$ and $v_{2}$ in $T_{\gamma}$ with $d\left(v_{1}, v_{2}\right)=2$ and $v$ is the vertex between them then part 1 of Lemma 3.4 along with Lemma 3.12 says that $Z\left(G_{v}\right)$ acts cocompactly on $N_{K} X_{v_{1}} \cap N_{K} X_{v_{2}}$ which contains $\gamma$ for some $K$, and hence there is a $K^{\prime}$ such that for all $t>0$ there is a $g_{t} \in Z\left(G_{v}\right)$ such that $d\left(\gamma(t), g_{t}(\gamma(0))<K^{\prime}\right.$ and hence $\gamma$ stays a distance at most $K^{\prime}+d\left(\gamma(0), Y_{v}\right)$ from a geodesic in the $\mathbb{R}$ direction of $Y_{v}=\bar{Y}_{v} \times \mathbb{R}$ (since $Z\left(G_{v}\right)$ translates the $\mathbb{R}$ direction). Thus for every $e$ with $v$ as a vertex we have $\gamma$ is asymptotic to a geodesic in $Y_{e}$ and hence $e \in T_{\gamma}$. Thus $\overline{\underline{S t a r}(v)} \subset T_{\gamma}$. But since vertices in $T_{\gamma}$ are at most distance 2 apart we see that $\overline{\operatorname{Star}(v)}=T_{\gamma}$.

The only cases left for $T_{\gamma}$ are the two mentioned and the case of two vertices a unit distance apart. But in the final case a similar argument shows that if $e$ is the edge between them then $\gamma$ stays a bounded distance from $Y_{e}$ and hence $e$ must also be in $T_{\gamma}$.

Definition 3.20. Let $\gamma$ be a geodesic ray in $X$. If case 1 of Lemma 3.19 applies then we will say that $\gamma$ has itinerary $\tau$, and otherwise we say that the itinerary of $\gamma$ is the subtree $T_{\gamma} \subset T$ described in case 2 of the lemma. In either case we denote the itinerary of $\gamma$ by $\mathcal{I}(\gamma)$.

One immediate consequence of the proof of Lemma 3.19 is
Corollary 3.21. If $\mathcal{I}(\gamma)=\overline{\operatorname{Star}(v)}$ then $\gamma$ is asymptotic to either the positive or the negative $\mathbb{R}$ direction in the decomposition $Y_{v}=\bar{Y}_{v} \times \mathbb{R}$.
Lemma 3.22. If $\overline{p_{1} \xi}, \overline{p_{2} \xi} \subset X$ are asymptotic geodesic rays, then either both $\mathcal{I}\left(\overline{p_{1} \xi}\right)$ and $\mathcal{I}\left(\overline{p_{2} \xi}\right)$ are finite subtrees, in which case they agree, or both $\mathcal{I}\left(\overline{p_{1} \xi}\right)$ and $\mathcal{I}\left(\overline{p_{2} \xi}\right)$ are rays, in which case $\partial_{\infty} \mathcal{I}\left(p_{1} \xi\right)=\partial_{\infty} \mathcal{I}\left(p_{2} \xi\right)$. In other words, $\mathcal{I}\left(p_{1} \xi\right)$ is a ray in $T$ if and only if $\mathcal{I}\left(\overline{p_{2} \xi}\right)$ is a ray in $T$ asymptotic to $\mathcal{I}\left(\overline{p_{1} \xi}\right)$.

Proof. $\overline{p_{1} \xi}$ lies in a tubular neighborhood of some $Y_{v}$ if and only if $\overline{p_{2} \xi}$ does, thus the case where $\mathcal{I}\left(\overline{p_{1} \xi}\right)$ (or $\mathcal{I}\left(\overline{p_{2} \xi}\right)$ ) is finite follows. Thus $\overline{p_{1} \xi}$ has itinerary a ray $\tau_{1}$ if and only if $\overline{p_{2} \xi}$ has itinerary $\tau_{2}$ for some ray $\tau_{2} \subset T$. But the sets $\rho\left(\overline{p_{1} \xi}\right)$ and $\rho\left(\overline{p_{2} \xi}\right)$ are
at finite Hausdorff distance from one another since $\rho$ is coarse Lipschitz; hence the $\tau_{i}$ are asymptotic.

By the lemma we have a well-defined $G$-equivariant map from $\partial_{\infty} X$ to the union

$$
\partial_{\infty} T \cup(\text { finite subsets of } T)
$$

which assigns to each $\xi \in \partial_{\infty} X$ either $\partial_{\infty} \mathcal{I}(\overline{p \xi}), p \in X$ if $\mathcal{I}(\overline{p \xi})$ is a ray or $\mathcal{I}(\overline{p \xi})$ otherwise; we will also denote this map by $\mathcal{I}$. If $\eta \in \partial_{\infty} T$, we use $\partial_{\infty}^{\eta} X$ to denote the corresponding subset: $\partial_{\infty}^{\eta} X:=\mathcal{I}^{-1}(\eta) \subset \partial_{\infty} X$. We will say that $\partial_{\infty}^{\eta} X$ is trivial if $\partial_{\infty}^{\eta} X$ is a point or nontrivial otherwise; (in the latter case we will see that $\partial_{\infty}^{\eta} X$ is homeomorphic to a closed interval and is in fact an interval in the Tits metric.).

In particular $\partial_{\infty} X=\left(\cup_{v \in V} \partial_{\infty} X_{v}\right) \cup\left(\cup_{\eta \in \partial_{\infty} T} \partial_{\infty}^{\eta} X\right)$, where $\cup_{v \in V} \partial_{\infty} X_{v}$ is disjoint from $\cup_{\eta \in \partial_{\infty} T} \partial_{\infty}^{\eta} X$.

The cone topology and Tits metric on $\partial_{\infty} X_{v}=\partial_{\infty} Y_{v}=\partial_{\infty}\left(\bar{Y}_{v} \times \mathbb{R}\right)$ is described in sections 2.2 and 2.4. We see that in the cone topology $\partial_{\infty} X_{v}$ is just the suspension $\Sigma\left(\partial_{\infty} H_{v}\right)$ and is independent of the metric on $X$. The Tits metric is just the metric suspension of the discrete metric.

We now study the dynamics of the action of $G$ on $\partial_{\infty} X$.
Lemma 3.23. 1. For every $v \in V$, the fixed point set of $Z\left(G_{v}\right)$ in $\partial_{\infty} X$ is $\partial_{\infty} X_{v}$; this set is homeomorphic to the suspension of $\partial_{\infty} H_{v}$ where $H_{v}$ is the nonelementary hyperbolic group $G_{v} / Z\left(G_{v}\right)$.
2. For every $e \in E$, $\operatorname{Fix}\left(G_{e}, \partial_{\infty} X\right)=\partial_{\infty} X_{e}$ which is homeomorphic to a circle.

Proof. Let $\xi \in \partial_{\infty} X$ be fixed by $Z\left(G_{v}\right)$ and $p \in Y_{v}$. If $\mathcal{I}(\overline{p \xi})$ is a ray then by Lemma $3.22 \partial_{\infty} \mathcal{I}(\overline{p \xi})$ is fixed by $Z\left(G_{v}\right)$. But this can not happen since by Lemma $3.5 Z\left(G_{v}\right)$ leaves no point in $\partial_{\infty} T$ fixed. So $\mathcal{I}(\overline{p \xi})$ is a finite subtree which by Lemma 3.22 is invariant under $Z\left(G_{v}\right)$. Thus by Lemma 3.19 and Lemma $3.5 v \in \mathcal{I}(\overline{p \xi})$. Thus $\overline{p \xi} \subset Y_{v}$ and hence $\xi \in \partial_{\infty} X_{v}=\partial_{\infty} Y_{v}$. On the other hand geodesics rays $\overline{p \xi}$ in $Y_{v}$ are translated by a fixed amount by elements $g \in G_{v}$, so $g(\overline{p \xi})$ is asymptotic to $\overline{p \xi}$ and so $\xi$ is fixed by $g$. The rest of part 1 follows from sections 2.2 and 2.4 as above.

Let $\xi$ be fixed by $G_{e}$ and $p \in Y_{e}$. Again since $G_{e}$ leaves no point in $\partial_{\infty} T$ fixed, by Lemma 3.5 $\mathcal{I}(\overline{p \xi})$ is a finite subtree that is invariant under $G_{e}$ and hence by Lemma 3.19 and Lemma 3.5 must contain $e$. Hence we see $\overline{p \xi} \subset Y_{e}$ and $\xi \in \partial_{\infty} Y_{e}=\partial_{\infty} X_{e}$. Again, since $G_{e}$ acts by translations on $Y_{e}$, we see that if $\xi \in \partial_{\infty} X_{e}=\partial_{\infty} Y_{e}$ then it is left fixed by $G_{e}$. Since $Y_{e}=\mathbb{R}^{2} \times \overline{Y_{e}}$ where $\overline{Y_{e}}$ is compact, $\partial_{\infty} Y_{e}=\partial_{\infty} \mathbb{R}^{2}$ is homeomorphic to a circle.

## 4. Templates and the behavior of their geodesics

### 4.1. Templates

In this section we study "Templates". These are piecewise Euclidean Hadamard spaces (which can be embedded in $\mathbb{R}^{3}$ ) which approximate certain subspaces of the spaces we are studying, and carry much of the information about the spaces at infinity.

A template is a Hadamard space $\mathcal{T}$ obtained from a disjoint collection of Euclidean planes $\{W\}_{W \in W_{\text {all }}^{\mathcal{T}}}$ (called walls) and directed Euclidean strips ${ }^{8}\{\mathcal{S}\}_{\mathcal{S} \in \text { Strip }_{\mathcal{T}}}$ by isometric gluing ${ }^{9}$ subject to the following conditions:

1. The boundary geodesics of each strip $\mathcal{S} \in \operatorname{Strip}_{\mathcal{T}}$, which we will refer to as singular geodesics, are glued isometrically to distinct walls in $W a l_{\mathcal{T}}$.
2. Each wall $W \in W a l_{\mathcal{T}}$ is glued to at most two strips, and the gluing lines are not parallel.
3. $\mathcal{T}$ is connected.

One can think of $\mathcal{T}$ as sitting in $\mathbb{R}^{3}$ so that its walls are parallel planes and the strips meet the walls orthogonally. Two walls $W_{1}, W_{2} \in W$ all $\mathcal{T}_{\mathcal{T}}$ are adjacent if there is a strip $\mathcal{S} \in \operatorname{Strip}_{\mathcal{T}}$ with $S \cap W_{i} \neq \emptyset$. The incidence graph $\operatorname{Graph}(\mathcal{T})$ of $\mathcal{T}$ - the graph with vertex set $W a l_{\mathcal{T}}$ and one edge for each pair of incident walls - is a graph isomorphic to a connected subcomplex of $\mathbb{R}$ with the usual triangulation (where the vertices are the integers). A wall is an interior wall if it is incident to two strips, and a strip is an interior strip if it is incident to two interior walls; Wall ${ }_{\mathcal{T}}^{o}$ and Strip $_{\mathcal{T}}^{o}$ denote the interior walls and strips respectively. For every interior wall $W \in W$ allo we have a distinguished point $o_{W}:=W \cap \mathcal{S}_{1} \cap \mathcal{S}_{2}$, where $\mathcal{S}_{i} \in \operatorname{Strip}_{\mathcal{T}}, i=1,2$, are the strips incident to $W$. Let $S t r i p_{\mathcal{T}}^{+}$be the collection of oriented interior strips; an orientation of a strip $\mathcal{S} \in$ Strip $_{\mathcal{T}}$ combines with the direction of $\mathcal{S}$ to give an orientation of the interval factor of $\mathcal{S} \simeq \mathbb{R} \times I$, and also an ordering of the two incident walls. We can define a function $\epsilon:$ Strip $_{\mathcal{T}}^{+} \rightarrow \mathbb{R}$ as follows: if $W_{-}, W_{+}$are incident to $\mathcal{S}^{+} \in$ Strip $_{\mathcal{T}}^{+}$ and $W_{-}<W_{+}$with respect to the ordering defined by $S^{+}$, then $\epsilon\left(\mathcal{S}^{+}\right) \in \mathbb{R}$ is defined to be the signed distance that $o_{W_{+}}$lies "above" $o_{W_{-}}$in the strip $\mathcal{S}^{+}$. We also have a strip width function $l: \operatorname{Strip}_{\mathcal{T}} \rightarrow(0, \infty)$ and an angle function $\alpha:$ Wall $_{\mathcal{T}}^{o} \rightarrow(0, \pi)$ which give the angle between the oriented lines $W \cap \mathcal{S}_{i}$ where $\mathcal{S}_{1}, \mathcal{S}_{2}$ are incident to $W$.

We will sometimes enumerate the consecutive walls and strips of $\mathcal{T}$ so that $W$ all $\mathcal{T}_{\mathcal{T}}=$ $\left\{W_{i}\right\}_{a<i<b}$ and Strip $_{\mathcal{T}}=\left\{\mathcal{S}_{i}\right\}_{a<i<b-1}$ where $a \in\{-\infty, 0\}$ and $b \in \mathbb{N} \cup\{\infty\}$. We then define $L_{i}^{-}:=W_{i} \cap \mathcal{S}_{i-1}$ for $a+1<i<b$ and $L_{i}^{+}:=W_{i} \cap \mathcal{S}_{i}$ for $a<i<b-1$.

An equivalence between two templates $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ is an isometry $\psi: \mathcal{T}_{1} \rightarrow \mathcal{T}_{2}$ which respects strip directions. Two templates are equivalent if and only if there is an incidence preserving bijection Wall $_{\mathcal{T}_{1}} \cup$ Strip $_{\mathcal{T}_{1}} \rightarrow$ Wall $_{\mathcal{T}_{2}} \cup$ Strip $_{\mathcal{T}_{2}}$ which respects the functions $l, \epsilon$, and $\alpha$. We will call a template uniform if there is a $\frac{\pi}{2} \geq \beta>0$ so that the angle function $\alpha:$ Wall ${ }_{\mathcal{T}}^{o} \rightarrow(0, \pi)$ satisfies $\pi-\beta \geq \alpha \geq \beta$, and if the strip widths are bounded away from zero. We are mostly interested in uniform templates. A template $\mathcal{T}$ is full if $\operatorname{Graph}(\mathcal{T}) \simeq \mathbb{R}$, half if $\operatorname{Graph}(\mathcal{T}) \simeq \mathbb{R}_{+}$, and finite if $\mid$ Wall $_{\mathcal{T}} \mid<\infty$.

If $W \in W$ all ${ }_{\mathcal{T}}^{o}$ and $\mathcal{S}_{1}, \mathcal{S}_{2}$ are the incident strips, then the oriented lines $W \cap \mathcal{S}_{1}$ and $W \cap \mathcal{S}_{2}$ divide the plane $W$ into four sectors which we call Quarter Planes and which we label as $Q_{I}, Q_{I I}, Q_{I I I}$, and $Q_{I V}$ as usual. If we are given a choice $Q_{W}$ of quarter planes in $W$ for each $W \in$ Wall $_{\mathcal{T}}$ then there is an isometric immersion $\mathcal{D}:\left[\cup_{W \in \text { Wall }_{\mathcal{T}}} Q_{W}\right] \cup\left[\cup_{\mathcal{S} \in \text { Strip }_{\mathcal{T}}} \mathcal{S}\right] \rightarrow \mathbb{R}^{2}$ (the development) which takes any geodesic ray $\gamma \subset \mathcal{T}$ such that $\gamma \cap W \subset Q_{W}$, to a Euclidean ray (see Figure 3 in section 7.2 for an example of the developement of a special kind of template).

[^5]When $\mathcal{T}$ is a half template we will be primarily interested in geodesic rays $\gamma \subset \mathcal{T}$ that start at a given base point and intersect all but finitely many walls of $\mathcal{T}$. From the separation properties of walls it is clear that such a ray intersects the walls $W \in W$ all $\mathcal{T}_{\mathcal{T}}$ in order. We let $\partial_{\infty}^{\infty} \mathcal{T}$ (resp. $\partial_{T}^{\infty} \mathcal{T}$ ) denote the corresponding subset of $\partial_{\infty} \mathcal{T}$ (resp. $\partial_{T} \mathcal{T}$ ). In section 7.1 we will show that $\partial_{T}^{\infty} \mathcal{T}$ is isometric to either a point, in which case $\mathcal{T}$ is called trivial, or an interval of length $<\pi$.
Remark 4.1. One can show directly that any two half templates such that corresponding angles agree, and both corresponding strip widths and displacements differ by a bounded amount will have $\partial_{\infty}^{\infty}$ 's with the same Tits length. We will only need a weaker version (that will follow from Theorem 5.1) in this paper so we will not digress to prove it here.

### 4.2. Templates associated with itineraries in $T$

We now return to the setting of our paper: $G$ is an admissible group with a discrete cocompact isometric action on a Hadamard space $X$. We now want to associate a template with each geodesic segment/ray in $T$; these templates capture the asymptotic geometry of geodesic segments/rays in $X$ which pass near the corresponding edge spaces.

We first choose, in a $G$-equivariant way, a plane $F_{e} \subset Y_{e}$ for each edge $e \in E$. Then for every pair of adjacent edges $e_{1}, e_{2}$ we choose, again equivariantly, a minimal geodesic from $F_{e_{1}}$ to $F_{e_{2}}$; by the convexity of $Y_{v}=\bar{Y}_{v} \times \mathbb{R}, v:=e_{1} \cap e_{2}$, this geodesic determines a Euclidean strip ${ }^{10} \mathcal{S}_{e_{1}, e_{2}}:=\gamma_{e_{1}, e_{2}} \times \mathbb{R}$ for some geodesic segment $\gamma_{e_{1}, e_{2}} \subset$ $\bar{Y}_{v}$; note that $\mathcal{S}_{e_{1}, e_{2}} \cap F_{e_{i}}$ is an axis of $Z\left(G_{v}\right)$. Hence if $e, e_{1}, e_{2} \in E, e_{i} \cap e=v_{i} \in V$ are distinct vertices, then the angle between the geodesics $\mathcal{S}_{e_{1}, e} \cap F_{e}$ and $\mathcal{S}_{e_{2}, e} \cap F_{e}$ is bounded away from zero (since only finitely many angles show up). We also note that Definition 3.14 tells us that $d_{H}\left(F_{e}, X_{e}\right)$ is bounded by $D^{\prime \prime}+\operatorname{Diam}\left(\bar{Y}_{e}\right)$ which, since there are only finitely many $e$ up to the action of $G$, is uniformly bounded.

The significance of the strips $\mathcal{S}_{e_{1}, e_{2}}$ can be seen in the next two lemmas.
Lemma 4.2. There is a constant $C_{3}$ so that if $e_{1}=\overline{v v_{1}}$ and $e_{2}=\overline{v v_{2}}$ are adjacent edges, then $X_{e_{1}} \cup \mathcal{S}_{e_{1}, e_{2}} \cup X_{e_{2}}$ and $X_{v_{1}} \cup \mathcal{S}_{e_{1}, e_{2}} \cup X_{v_{2}}$ are $C_{3}$-quasi-convex.

Proof. Since the Hausdorff distance $d_{H}\left(X_{\hat{e}}, F_{\hat{e}}\right)$ is uniformly bounded for $\hat{e} \in E$, it suffices to show that there is a constant $C$ so that the unions $F_{e_{1}} \cup \mathcal{S}_{e_{1}, e_{2}} \cup F_{e_{2}}$ are $C$-quasi-convex for all pairs of adjacent edges. But if $e_{1}, e_{2} \in E$ are adjacent then $F_{e_{1}} \cup \mathcal{S}_{e_{1}, e_{2}} \cup F_{e_{2}} \subset Y_{v} \simeq \bar{Y}_{v} \times \mathbb{R}$, and we are reduced to showing that $\bar{F}_{e_{1}} \cup \gamma_{e_{1}, e_{2}} \cup \bar{F}_{e_{2}} \subset$ $\bar{Y}_{v}$ is uniformly quasi-convex, where $\bar{F}_{e_{i}}$ is the image of $F_{e_{i}}$ under the projection $Y_{v} \simeq Y_{v} \times \mathbb{R} \rightarrow Y_{v}$. This follows from Lemma 2.9. This gives the first statement.

The interesting part of the second statement is when we consider $\overline{x y}$ when $x \in X_{v_{1}}$ and $y \in X_{v_{2}}$. In this case Lemma 3.17 gives us $z_{1} \in N_{C_{2}} X_{e_{1}}$ and $z_{2} \in N_{C_{2}} X_{e_{2}}$ on $\overline{x y}$. Now the first statement along with the convexity of $N_{C_{2}+D^{\prime \prime}}\left(X_{v_{i}}\right)$ (note $z_{i} \in N_{C_{2}} X_{e_{i}} \subset$ $N_{C_{2}+D^{\prime \prime}} X_{v_{i}}$ ) yields the second statement.

[^6]Lemma 4.3. There is a constant $C_{4}$ so that if $e_{1}, \ldots, e_{n} \in E$ is a geodesic edge path in $T$ with initial vertex $v_{1}$ and terminal vertex $v_{n}$, then

$$
Z:=X_{e_{1}} \cup \mathcal{S}_{e_{1}, e_{2}} \cup X_{e_{2}} \cup \ldots \cup X_{e_{n-1}} \cup \mathcal{S}_{e_{n-1}, e_{n}} \cup X_{e_{n}}
$$

and

$$
Z^{\prime}:=X_{v_{1}} \cup \mathcal{S}_{e_{1}, e_{2}} \cup X_{e_{2}} \cup \ldots \cup X_{e_{n-1}} \cup \mathcal{S}_{e_{n-1}, e_{n}} \cup X_{v_{n}}
$$

are $C_{4}$-quasi-convex.
Proof. Pick $x, y \in Z$. We may assume without loss of generality that $x \in X_{e_{1}} \cup \mathcal{S}_{e_{1}, e_{2}} \cup$ $X_{e_{2}} \subset X_{v}$ where $v=e_{1} \cap e_{2}$, and $y \in X_{e_{n-1}} \cup \mathcal{S}_{e_{n-1}, e_{n}} \cup X_{e_{n}} \subset X_{v^{\prime}}$ where $v^{\prime}=e_{n-1} \cap e_{n}$. Applying Lemma 3.17 we get points $z_{i} \in \overline{x y} \cap N_{C_{2}}\left(X_{e_{i}}\right)$ for $2 \leq i \leq n-1$, with $d\left(z_{i}, x\right) \leq d\left(z_{j}, x\right)$ when $i \leq j$. If $C_{4}:=C_{2}+C_{3}$, then by Lemma 4.2 we have $\overline{x z_{2}} \subset N_{C_{4}}\left(X_{e_{1}} \cup \mathcal{S}_{e_{1}, e_{2}} \cup X_{e_{2}}\right), \overline{z_{i} z_{i+1}} \subset N_{C_{4}}\left(X_{e_{i}} \cup \mathcal{S}_{e_{i}, e_{i+1}} \cup X_{e_{i+1}}\right)$ for $i=2, \ldots n-1$, and $\overline{z_{n-1} y} \subset N_{C_{4}}\left(X_{e_{n-1}} \cup \mathcal{S}_{e_{n-1}, e_{n}} \cup X_{e_{n}}\right)$.

We omit the proof that $Z^{\prime}$ is quasi-convex, as it is similar.
Lemma 4.3 suggests that we will understand the geodesic geometry of $X$ if the geometry of the sets $Z$ (as in the lemma) can be easily modeled. To this end, we "approximate" $Z$ with a template.

Definition 4.4. Suppose $\gamma \subset T$ is a geodesic segment or ray. Let $\mathcal{T}$ be a template with walls $\{W\}_{W \in \text { Wall }_{\mathcal{T}}}$ and strips $\{\mathcal{S}\}_{\mathcal{S}_{\in S t r i p_{\mathcal{T}}}}$, let $f: W_{\text {all }}^{\mathcal{T}} \boldsymbol{\rightarrow E}$ be an adjacency preserving bijection between the walls of $\mathcal{T}$ and the edges of $\gamma$, and let $\phi: \mathcal{T} \rightarrow X$ be a (not necessarily continuous) map. Then the triple ( $\mathcal{T}, f, \phi$ ) is a $K$-template for $\gamma$ if for all $W \in W$ all $\mathcal{T}_{\mathcal{T}}$ we have $\phi(W) \subset N_{K}\left(X_{f(W)}\right)$ and $X_{f(W)} \subset N_{K}(\phi(W))$ and the following conditions are met for every $\mathcal{S} \in \operatorname{Strip}_{\mathcal{T}}$.

1. $\operatorname{Width}(\mathcal{S}) \geq 1$.
2. If $\mathcal{S}$ is incident to $W_{1}, W_{2} \in W$ all $\mathcal{T}$ and $x, y \in W_{1} \cup \mathcal{S} \cup W_{2}$ then we have $\left|d_{X}(\phi(x), \phi(y))-d_{\mathcal{T}}(x, y)\right|<K$ and if $\gamma_{1}:[0,1] \rightarrow \overline{x y}$ and $\gamma_{2}:[0,1] \rightarrow \overline{\phi(x) \phi(y)}$ are constant speed parameterizations, then $d\left(\phi \circ \gamma_{1}(t), \gamma_{2}(t)\right)<K$ for all $t \in[0,1]$.
3. If $W_{1}, W_{2} \in W$ all $\mathcal{T}_{\mathcal{T}}$ are adjacent to $\mathcal{S}$ then the Hausdorff distance $d_{H}\left(\left(X_{f\left(W_{1}\right)} \cup\right.\right.$ $\left.\left.\mathcal{S}_{f\left(W_{1}\right), f\left(W_{2}\right)} \cup X_{f\left(W_{2}\right)}\right), \phi\left(W_{1} \cup \mathcal{S} \cup W_{2}\right)\right)<K$.

Often when the value of $K$ is not relevant we will refer to the triple $(\mathcal{T}, f, \phi)$ as a template for $\gamma$, by which we mean a $K$-template for some $K$.

Let $\mathcal{T}^{\prime}$ be another template such that the angles on corresponding walls agree with those of $\mathcal{T}$ and such that the other data (strip widths and displacements) differ from $\mathcal{T}$ by a bounded amount. Then there is a natural (discontinuous) map $F: \mathcal{T}^{\prime} \rightarrow \mathcal{T}$ which is an isometry on each wall and simply stretches the width of the strips. It is easy to check that using $\phi^{\prime}=\phi \circ F$ that we get a $K^{\prime}$ template ( $\mathcal{T}^{\prime}, f, \phi^{\prime}$ ) for $\gamma$ (see step 3 of the proof of Lemma 7.14).

For a suitably large $K$ we describe a construction that gives a $K$-template for any geodesic segment or geodesic ray $\gamma \subset \mathcal{T}$. We will refer to these $K$-templates as standard $K$-templates. We begin with a disjoint collection of walls $W_{e}$ and an isometry $\phi_{e}: W_{e} \rightarrow F_{e}$ for each edge $e \subset \gamma$. For every pair $e, e^{\prime}$ of adjacent edges of $\gamma$, we let $\hat{\mathcal{S}}_{e, e^{\prime}}$ be a strip which is isometric to $\mathcal{S}_{e, e^{\prime}} \subset X$ if $\operatorname{Width}\left(\mathcal{S}_{e, e^{\prime}}\right) \geq 1$, and isometric to $\mathbb{R} \times[0,1]$ otherwise; we let $\phi_{e, e^{\prime}}: \hat{\mathcal{S}}_{e, e^{\prime}} \rightarrow \mathcal{S}_{e, e^{\prime}}$ be an affine map which
respects product structure ( $\phi_{e, e^{\prime}}$ is an isometry if $\operatorname{Width}\left(\mathcal{S}_{e, e^{\prime}}\right) \geq 1$ and compresses the interval otherwise). We construct $\mathcal{T}$ by gluing the strips and walls so that the maps $\phi_{e}$ and $\phi_{e, e^{\prime}}$ descend to continuous maps on the quotient.

The above construction yields
Lemma 4.5. There is a constant $K=K(X)$ such that for every geodesic segment or ray, $\gamma \subset T$, there is a $K$-template for $\gamma$.

There is a $\beta=\beta(X)>0$ such that for any $K$-template the angle function $\alpha$ : Wallo $\rightarrow(0, \pi)$ satisfies $0<\beta \leq \alpha \leq \pi-\beta<\pi$.

Proof. We check that each condition of Definition 4.4 holds for the standard template described above, for sufficiently large $K$.

First, since $d_{\mathcal{H}}\left(F_{e}, X_{e}\right)$ is uniformly bounded and $F_{f(W)}=\phi(W)$ for every $W \in$ $W^{\prime} l_{\mathcal{T}}$, we have $\phi(W) \subset N_{K}\left(X_{f(W)}\right)$ and $X_{f(W)} \subset N_{K}(\phi(W))$ for all $W \in W$ all $\mathcal{T}_{\mathcal{T}}$ for large enough $K$.

Conditions 1 and 3 follow immediately from the description of standard templates.
We now verify condition 2 . Pick adjacent walls $W, W^{\prime} \in W$ all $\mathcal{T}_{\mathcal{T}}$ and set $e:=f(W)$, $e^{\prime}:=f\left(W^{\prime}\right)$, and $v:=e \cap e^{\prime}$. Recall that $F_{e} \cup \mathcal{S}_{e, e^{\prime}} \cup F_{e^{\prime}} \subset Y_{v}$ and $Y_{v}$ splits isometrically as $Y_{v}=\bar{Y}_{v} \times \mathbb{R}$ where $\bar{Y}_{v}$ is Gromov hyperbolic. Furthermore, $\phi$ induces a map $W_{e} \cup \hat{\mathcal{S}}_{e, e^{\prime}} \cup W_{e^{\prime}} \rightarrow F_{e} \cup \mathcal{S}_{e, e^{\prime}} \cup F_{e^{\prime}}$ which is compatible with the product structure. Hence condition 2 follows from part 3 of Lemma 2.9 (and triangle inequalities) when $d\left(F_{e}, F_{e^{\prime}}\right) \geq 4 \delta$ (where $\delta$ is the maximum of the hyperbolicity constants of the $\bar{Y}_{v}$ 's); modulo $G$ there are only finitely many cases when $d\left(F_{e}, F_{e^{\prime}}\right)<4 \delta$ (Lemma 3.11), and each of these is also settled by part 3 of Lemma 2.9.

For any $K$ template $(\mathcal{T}, f, \phi)$ for a $\gamma$ containing an interior edge $e=\overline{v^{\prime} v}$ we claim that the wall $W$ with $f(W)=e$ will have the same angle, up to taking supplements (i.e. $\alpha$ might be replaced by $\pi-\alpha$ ), as the angle $\alpha$ between the $\mathbb{R}$ factors of $Y_{v^{\prime}}=$ $\bar{Y}_{v^{\prime}} \times \mathbb{R}$ and $Y_{v}=\bar{Y}_{v} \times \mathbb{R}$ in $Y_{e}=Y_{v} \cap Y_{v^{\prime}}$. The fact that these angles are positive and the finiteness of edges modulo $G$ will yield the result. We note that $\alpha$ is the Tits angle between the $\mathbb{R}$ factors.

To see this we first note that Property 2 of Definition 4.4 says that the angle for $W$, i.e. the angle between the gluing lines $L^{\prime}$ and $L$, is the same as the comparison angle $\lim _{t \rightarrow \infty} \tilde{Z}_{\phi(o)} \phi\left(L^{\prime}(t)\right), \phi(L(t))$. Also Property 2 of Definition 4.4 says that there are geodesic rays $\sigma=\lim _{i \rightarrow \infty} \overline{p \phi\left(L\left(t_{i}\right)\right)}$ where $t_{i} \rightarrow \infty$ and $p \in Y_{e}$. Since $\phi(W) \subset N_{K} X_{e}$ we see that $\phi(L) \subset N_{K} X_{e}$ and hence $e \in \mathcal{I}(\sigma)$. Now if we let $e^{\prime}$ be the other edge incident to $v$ in $\gamma$ then the intersection of the wall $f^{-1}\left(e^{\prime}\right)$ with the strip between $W$ and $f^{-1}\left(e^{\prime}\right)$, is a line parallel to L and hence, again by 2 of Definition 4.4, $\phi(L)$ stays in a uniform neighborhood of $X_{e^{\prime}}$ so $e^{\prime} \in \mathcal{I}(\sigma)$. but $e, e^{\prime} \in \mathcal{I}(\sigma)$ implies by Lemma 3.19 that $\mathcal{I}(\sigma)=\overline{\operatorname{Star}(v)}$ and hence by Corollary 3.21 any such $\sigma$ is asymptotic to the $\mathbb{R}$ factor of $Y_{v}=\bar{Y}_{v} \times \mathbb{R}$. Since $p \in Y_{e} \subset Y_{v}, \sigma$ is a half line of such an $\mathbb{R}$ and we assume without loss of generality that it points in the positive direction. A similar argument works for $\phi\left(L^{\prime}\right)$. Thus $\angle\left(L^{\prime}, L\right)=\lim _{t \rightarrow \infty} \tilde{L}_{\phi(o)} \phi\left(L^{\prime}(t)\right), \phi(L(t)) \leq \alpha$. Also the same arguments applied to $-L^{\prime}$ and $L$ yield $\angle\left(-L^{\prime}, L\right)=\leq \pi-\alpha$. Thus we get equality and the result.

The next proposition and Proposition 4.8 are technical results that compare template geometry with ambient geometry.

Proposition 4.6. Suppose $K>0$. There is a constant $C_{5}$ depending only on $K$ and the geometry of $X$ with the following property. Suppose $\gamma \subset T$ is a geodesic segment or geodesic ray in $T$ with $i^{\text {th }}$ edge $e_{i}$, and set $Z:=\left[\cup_{e \subset \gamma} X_{e}\right] \cup\left[\cup_{e, e^{\prime} \subset \gamma} \mathcal{S}_{e, e^{\prime}}\right]$. If $(\mathcal{T}, f, \phi)$ is a $K$-template for $\gamma$, and $x, y \in Z$, then there is a continuous map $\alpha: \overline{x y} \rightarrow \mathcal{T}$ so that

1. $d\left(\phi \circ \alpha,\left.i d\right|_{\overline{x y}}\right)<C_{5}$
2. For all $p, q \in \overline{x y}$ we have

$$
\begin{equation*}
d_{X}(p, q)-k C_{5} \leq d_{\mathcal{T}}(\alpha(p), \alpha(q)) \leq \text { length }\left(\left.\alpha\right|_{\overline{p q}}\right) \leq d_{X}(p, q)+k C_{5} \tag{4.7}
\end{equation*}
$$

where the segment $\overline{\alpha(p) \alpha(q)} \subset \mathcal{T}$ intersects at most $k-1$ strips and walls in $\mathcal{T}$. In particular there are constants $(L, A)$ depending only on $K$ and $X$ so that $\phi$ is an $(L, A)$ quasi-isometric embedding for every $K$-template $(\mathcal{T}, f, \phi)$.

Proof. By the standard properties of Hadamard spaces we may reduce to the case that $x \in Y_{e_{x}} \cup \mathcal{S}_{e_{x}, e_{x}^{\prime}} \subset X_{v_{x}}\left(v_{x}=e_{x} \cap e_{x}^{\prime}\right)$ and $y \in \mathcal{S}_{e_{y}, e_{y}^{\prime}} \cup Y_{e_{y}^{\prime}} \subset X_{v_{y}}$ since the original $x$ and $y$ are within a bounded distance of such. Let $W_{i}:=f^{-1}\left(e_{i}\right) \in W$ all $\mathcal{T}_{\mathcal{T}}$ for $e_{i} 1 \leq i \leq n$ the edges between $v_{x}$ and $v_{y}$. We may apply Lemma 3.17 to the pair $x, y$ obtaining points $z_{i} \in \overline{x y}$. We let $z_{0}=x$ and $z_{n+1}=y$. After making a small perturbation of the $z_{i}$ 's if necessary, we may assume that they satisfy $d_{X}\left(z_{i}, x\right)<d\left(z_{j}, x\right)$ when $i<j$. For $1 \leq i \leq n$ pick $w_{i} \in \mathcal{T}$ with $w_{i} \in W_{i} \subset \mathcal{T}$ with $d\left(z_{i}, \phi\left(w_{i}\right)\right) \leq 1+\inf \left\{d\left(z_{i}, \phi(w)\right) \mid\right.$ $\left.w \in W_{i}\right\} \leq C_{2}+1$. By Definition 4.4 part 3 we can also choose $w_{0} \in W_{0} \cup \mathcal{S} \cup W_{1}$ and such that $d_{X}\left(x, \phi\left(w_{0}\right)\right) \leq K$ and similarly choose $w_{n+1}$. Now define $\alpha$ by the condition that $\alpha\left(z_{i}\right)=w_{i}$, and $\alpha$ is a constant speed geodesic on the segment $\overline{z_{i} z_{i+1}}$.

Proof of 1. Apply Definition 4.4 to see that the constant speed parameterization $[0,1] \rightarrow \overline{\phi\left(w_{i}\right) \phi\left(w_{i+1}\right)}$ is at uniformly bounded distance from the composition of the constant speed parameterization $[0,1] \rightarrow \overline{w_{i} w_{i+1}} \subset \mathcal{T}$ with $\phi: \mathcal{T} \rightarrow X$. Since $d\left(\phi\left(w_{i}\right), z_{i}\right)$ is uniformly bounded, we know that the constant speed parameterizations $[0,1] \rightarrow \overline{\phi\left(w_{i}\right) \phi\left(w_{i+1}\right)}$ and $[0,1] \rightarrow \overline{z_{i} z_{i+1}}$ are also at uniformly bounded distance from one another, so there is a constant $c_{1}$ depending on $K$ so that $d\left(\phi \circ \alpha,\left.i d\right|_{\overline{x y}}\right)<c_{1}$.

Proof of 2. Assume $p \in \overline{z_{j-1} z_{j}}-z_{j-1}$ and $q \in \overline{z_{j^{\prime}} z_{j^{\prime}+1}}-z_{j^{\prime}+1}$ for $j \leq j^{\prime}$. By Definition 4.4 we have, for $c_{2}=2 c_{1}+K$

$$
\begin{gathered}
\mid \text { length }\left(\left.\alpha\right|_{\overline{p z_{j}}}\right)-d_{X}\left(p, z_{j}\right) \mid<c_{2} \\
\mid \text { length }\left(\left.\alpha\right|_{\overline{z_{i} z_{i+1}}}\right)-d_{X}\left(z_{i}, z_{i+1}\right) \mid<c_{2} \text { for every } i=1, \ldots n-1 \\
\mid \text { length }\left(\left.\alpha\right|_{\overline{z_{j^{\prime}} q}}\right)-d_{X}\left(z_{j^{\prime}}, q\right) \mid<c_{2} .
\end{gathered}
$$

Hence there is a $c_{3}=c_{3}(K)$ so that

$$
\begin{aligned}
& \text { length }\left(\alpha_{\overline{p q}}\right)=\text { length }\left(\left.\alpha\right|_{\overline{p z_{j}}}\right)+\ldots+\text { length }\left(\left.\alpha\right|_{\overline{\bar{j}^{\prime} q}}\right) \\
& \leq d_{X}\left(p, z_{j}\right)+\ldots+d_{X}\left(z_{j^{\prime}}, q\right)+\left(j^{\prime}-j+2\right) c_{2} \\
& \leq d_{X}(p, q)+k c_{2} .
\end{aligned}
$$

To prove the remaining inequality of (4.7) we break up the $\mathcal{T}$-geodesic $\overline{\alpha(p) \alpha(q)}$ into at most $k$ subsegments $\overline{u_{i} u_{i+1}}$ so that each subsegment lies in $W_{i} \cup \mathcal{S}_{i} \cup W_{i+1}$ for some $i$. Then by definition 4.4 we have $\left|d_{X}\left(\phi\left(u_{i}\right), \phi\left(u_{i+1}\right)\right)-d_{\mathcal{T}}\left(u_{i}, u_{i+1}\right)\right|<K$ so

$$
\begin{gathered}
d_{X}(p, q) \leq 2 c_{1}+d_{X}(\phi \circ \alpha(p), \phi \circ \alpha(q)) \\
\leq 2 c_{1}+\sum d_{X}\left(\phi\left(u_{i}\right), \phi\left(u_{i+1}\right)\right) \\
\leq 2 c_{1}+k c_{2}+d_{\mathcal{T}}(\alpha(p), \alpha(q)) \\
\leq d_{\mathcal{T}}(\alpha(p), \alpha(q))+k c_{3} .
\end{gathered}
$$

where $c_{3}:=2 c_{1}+c_{2}$.
To see the quasi-isometry property of $\phi$, let $x^{\prime}, y^{\prime} \in \mathcal{T}, x=\phi\left(x^{\prime}\right), y=\phi\left(y^{\prime}\right)$, and let $\alpha$ be the map defined above where we choose $w_{0}=x$ and $w_{1}=y$ (i.e $\alpha(x)=x^{\prime}$ and $\left.\alpha(y)=y^{\prime}\right)$. Now 4.7 applied to $p=x$ and $q=y$ along with $k \leq d_{\mathcal{T}}\left(x^{\prime}, y^{\prime}\right)+1$ (since strips have width at least 1 ) and $k \leq$ const $_{1} d_{X}(p, q)+$ const $_{2}$ (as in the proof of the coarse lipschitz property of $\rho$ - see section 3.3) yields the quasi-isometry property of $\phi$. This completes the proof of Proposition 4.6.

Proposition 4.8. Pick $K>0$. There is a constant $C_{6}=C_{6}(K, X)$ so that the following holds. If $(\mathcal{T}, f, \phi)$ is a $K$-template for $\gamma \subset T$, and $x, y \in \mathcal{T}$, then there is a continuous map $\alpha: \overline{x y} \rightarrow X$ where

1. $d\left(\alpha,\left.\phi\right|_{\overline{x y}}\right)<C_{6}$.
2. For all $p, q \in \overline{x y}$ we have

$$
\begin{equation*}
d_{\mathcal{T}}(p, q)-k C_{6} \leq d_{X}(\alpha(p), \alpha(q)) \leq \text { length }\left(\left.\alpha\right|_{\overline{p q}}\right) \leq d_{\mathcal{T}}(p, q)+k C_{6} \tag{4.9}
\end{equation*}
$$

where the segment $\overline{p q} \subset \mathcal{T}$ intersects at most $(k-1)$ strips and walls in $\mathcal{T}$.
Proof. This is similar to the proof of Proposition 4.6, so we omit it.
Corollary 4.10. If $(\mathcal{T}, f, \phi)$ is a $K$-template and $C_{6}=C_{6}(K)$ is the constant from Proposition 4.8, then for any $x, y \in \mathcal{T}$ we have

$$
d_{\mathcal{T}}(x, y)-k C_{6} \leq d_{X}(\phi(x), \phi(y)) \leq d_{\mathcal{T}}(x, y)+k C_{6}
$$

where $\overline{x y} \subset \mathcal{T}$ meets at most $k-1$ strips and walls.

## 5. Shadowing

In this section we show that geodesic segments in a $K$-template are sublinearly shadowed by ambient geodesic segments, and vice-versa.

Theorem 5.1 (Shadowing). There is a function $\theta: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$depending on $K$ and the geometry of $X$ (sometimes denoted $\theta_{(X, K)}$ ) with $\lim _{R \rightarrow \infty} \theta(R)=0$ so that if $(\mathcal{T}, f, \phi)$ is a $K$-template for a geodesic segment/ray $\gamma \subset T$, then the following hold.

1. If $x, y \in \mathcal{T}, z \in \overline{x y}$ and $R:=d(z, x)$, then $d(\phi(z), \overline{\phi(x) \phi(y)}) \leq(1+R) \theta(R)$.
2. If $x, y \in \mathcal{T}, z \in \overline{\phi(x) \phi(y)}$ and $R:=d(z, \phi(x))$ then $d(z, \phi(\overline{x y})) \leq(1+R) \theta(R)$.
3. Let $\overline{\mathcal{T}}:=\mathcal{T} \cup \partial_{\infty} \mathcal{T}$ and $\bar{X}:=X \cup \partial_{\infty} X$ be the usual compactifications. Then there is a unique topological embedding $\partial_{\infty} \phi: \partial_{\infty} \mathcal{T} \rightarrow \partial_{\infty} X$ so that

$$
\bar{\phi}:=\phi \cup \partial_{\infty} \phi: \overline{\mathcal{T}} \rightarrow \bar{X}
$$

is continuous at every $\xi \in \partial_{\infty} \mathcal{T} \subset \overline{\mathcal{T}}$.
4. The image of $\partial_{\infty} \phi$ is

$$
\left[\cup_{W \in W \text { all }}\left(\partial_{\infty} X_{f(W)}\right)\right] \cup\left[\partial_{\infty}^{\partial_{\infty} \gamma} X\right],
$$

and when $\gamma$ is a ray with $\partial_{\infty} \gamma=\eta$ then $\partial_{\infty} \phi\left(\partial_{\infty}^{\infty} \mathcal{T}\right)=\partial_{\infty}^{\eta} X$ (see section 3.3 for the definition of $\left.\partial_{\infty}^{\eta} X\right)$.
5. $\left.\partial_{\infty} \phi\right|_{\partial \infty \mathcal{T}}: \partial_{\infty}^{\infty} \mathcal{T} \rightarrow \partial_{\infty}^{\eta} X$ is an isometric embedding with respect to the Tits metric.

The proof of the theorem breaks up into two pieces. We first show in Proposition 5.6 that a geodesic segment (in a template or in $X$ ) running through a sequence of consecutive walls has to be "close" to any point $p$ which lies close to sufficiently many walls in the sequence. We then show in Theorem 5.7 that a segment in a template (resp. in $X$ ) which doesn't meet too many walls (i.e. encounters at most Const $\log R$ walls in the segment $\overline{p x}, d(p, x)=R)$ is well shadowed by a geodesic segment in $X$ (resp. in the template). These two arguments are combined in section 5.3 to prove Theorem 5.1.

### 5.1. Paths in a template which are close to a cluster of walls

We begin with a result about templates. It estimates the excess length of a path $\eta$ which connects two walls $W, W^{\prime}$ while remaining outside a ball which intersects $W, W^{\prime}$, and all walls between them.

Proposition 5.2. Let $\mathcal{T}$ be a template with angle function $\alpha$ : Wallo $\rightarrow(0, \pi)$ satisfying $0<\beta \leq \alpha \leq \pi-\beta<\pi$. Then there are positive constants $N_{1}=N_{1}(\beta)$, $\left(N_{1} \approx \frac{\text { Const }}{\beta}\right), C_{1}=C_{1}(\beta)$, and $C_{2}=C_{2}(\beta)$ with the following property. Let $W_{n_{0}}, \ldots, W_{n_{1}} \in W$ all $\mathcal{T}$ be a sequence of consecutive walls, and suppose $W_{i} \cap B(p, R) \neq$ $\emptyset$ for some $p \in \mathcal{T}, R>0$ and every $n_{0} \leq i \leq n_{1}$. Then for any $R^{\prime} \geq N_{1} R$ and any path $c:[0,1] \rightarrow \mathcal{T}-B\left(p, R^{\prime}\right)$ with $c(0) \in W_{n_{0}}$ and $c(1) \in W_{n_{1}}$ we have

$$
\text { length }(c) \geq d_{\mathcal{T}}(c(0), c(1))+C_{1}\left(n_{1}-n_{0}-C_{2}\right) R^{\prime}
$$

Proof. Let $\mathcal{S}_{i}$ be the strip incident to $W_{i}$ and $W_{i+1}$ for $i=n_{0}, \ldots, n_{1}-1$, set $L_{i}^{-}:=$ $\mathcal{S}_{i-1} \cap W_{i}$ for $i=n_{0}+1, \ldots n_{1}$, and set $L_{i}^{+}:=\mathcal{S}_{i} \cap W_{i}$ for $i=n_{0}, \ldots, n_{1}-1$.

We prove the proposition with the help of some lemmas.
Lemma 5.3. There is a constant $c_{1} \approx \frac{\text { Const }}{\beta}$ so that if $p \in \mathcal{T}$ and $d\left(p, W_{j}\right)<R$ for $j=i \pm 1$ then $d\left(p, o_{i}\right)<c_{1} R$.

Proof. By joining $\overline{x_{i-1} p}$ to $\overline{p x_{i+1}}$ for appropriate choices of $x_{i-1} \in W_{i-1}$ and $x_{i+1} \in$ $W_{i+1}$ we get a path $\gamma:[0,1] \rightarrow \mathcal{T}$ of length at most $2 R$ joining $W_{i-1}$ to $W_{i+1}$. Therefore there is a segment $[a, b] \subset[0,1]$ with $\gamma(a) \in L_{i}^{-}$and $\gamma(b) \in L_{i}^{+}$. So

$$
d\left(p, o_{i}\right) \leq R+\min \left(d\left(\gamma(a), o_{i}\right), d\left(\gamma(b), o_{i}\right)\right)
$$

$$
\leq R+\frac{d(\gamma(a), \gamma(b))}{2 \sin \left(\frac{\beta}{2}\right)} \leq c_{1} R .
$$

We now define $N_{1}:=\max \left(2 c_{1},\left[\frac{\pi}{\beta}\right]+2\right)$.
Consider a path $\eta:[0,1] \rightarrow \mathcal{T}-B\left(p, R^{\prime}\right)$ where $R^{\prime}>N_{1} R$. The ball $B\left(p, R^{\prime}\right) \subset$ $\mathcal{T}$ is convex, so clearly $\mathcal{T}-B\left(p, R^{\prime}\right)$ is complete and locally compact with respect to the induced path metric. Therefore we may assume that $c$ is a constant speed minimizing path from $c(0)$ to $c(1)$ in $\mathcal{T}-B\left(p, R^{\prime}\right)$. Since $\overline{B\left(p, R^{\prime}\right)}$ is a convex subset of the Hadamard space $\mathcal{T}$, the nearest point projection $\mathcal{T} \rightarrow \overline{B\left(p, R^{\prime}\right)}$ is distance non increasing; it follows that the set $c^{-1}\left(\overline{B\left(p, R^{\prime}\right)}\right)$ is either empty or a closed subinterval $[a, b] \subset[0,1] .\left.c\right|_{[0, a]}$ and $\left.c\right|_{[b, 1]}$ are constant speed geodesic segments in the Hadamard space $\mathcal{T}$, and since $R^{\prime}>c_{1} R$ these segments lie in $\mathcal{T}-\left\{o_{i}\right\}_{n_{0}<i<n_{1}}$ by Lemma 5.3.
Lemma 5.4.

$$
c([0, a]) \subset\left[\cup_{i=n_{0}}^{n_{0}+N_{1}} W_{i}\right] \cup\left[\cup_{i=n_{0}}^{n_{0}+N_{1}-1} \mathcal{S}_{i}\right]
$$

and

$$
c([b, 1]) \subset\left[\cup_{i=n_{1}-N_{1}}^{n_{1}} W_{i}\right] \cup\left[\cup_{i=n_{1}-N_{1}}^{n_{1}-1} \mathcal{S}_{i}\right] .
$$

Proof. We prove the first assertion; the proof of the second is similar. If the lemma were false, we would have $c(t) \in \mathcal{S}_{n_{0}+N_{1}}$ for some $t \in[0, a]$. Therefore $c([0, t])$ must cross every strip $\mathcal{S}_{i}$ for $n_{0} \leq i<n_{0}+N_{1}$, and for every $n_{0}<i \leq n_{0}+N_{1}$ it must enter $W_{i}$ through $L_{i}^{-}$and exit through $L_{i}^{+}$. Since $c([0, a])$ is disjoint from $\left\{o_{i}\right\}_{n_{0}<i<n_{1}}$ there is a flat convex strip $Y \subset \mathcal{T}$ containing $c([0, a])$ in its interior. Using $Y$ we can define co-orientations for the segments $c([0, a]) \cap \mathcal{S}_{i}$ and $c([0, a]) \cap W_{i}$ for $n_{0} \leq i \leq n_{0}+N_{1}$. If two of the origins $o_{i} \in W_{i}$ for $n_{0}<i \leq n_{0}+N_{1}$ lie on opposite sides of the corresponding segments $c([0, a]) \cap W_{i}$ with respect to the coorientations then the geodesic between them (of length less than $2 c_{1} R$ by lemma 5.3) will intersect $c([0, a])$ and hence $d\left(o_{i}, c([0, a])\right)<c_{1} R$ for some $n_{0}<i \leq n_{0}+N_{1}$, and thus $d\left(p, c([0, a])<2 c_{1} R\right.$. But this cannot happen since $d\left(p, c([0, a]) \geq R^{\prime}>2 c_{1} R\right.$. Thus all the origins $o_{i} \in W_{i}$ for $n_{0}<i \leq n_{0}+N_{1}$ lie on the same side of the corresponding segments $c([0, a]) \cap W_{i}$ with respect to the co-orientations. It follows that the angle between $c([0, a])$ and $L_{i}^{-}$increases by at least $\beta$ each time $c([0, a])$ passes through a wall. Hence $\left(N_{1}-1\right) \beta<\pi$, contradicting the definition of $N_{1}$.

Proof of Proposition 5.2 concluded. Let $\left[a^{\prime}, b^{\prime}\right] \subset[a, b] \subset[0,1]$ be the inverse image of

$$
\left[\cup_{i=n_{0}+N_{1}+1}^{n_{1}-N_{1}-1} W_{i}\right] \cup\left[\cup_{n_{0}+N_{1}}^{n_{1}-N_{1}-1} \mathcal{S}_{i}\right]
$$

under $c$. We know that $c\left(\left[a^{\prime}, b^{\prime}\right]\right)$ remains in the sphere $S\left(p, R^{\prime}\right)$ while it passes through all the walls $W_{i}$ for $n_{0}+N_{1}<i<n_{1}-N_{1}$. So for every $n_{0}+N_{1}<i<n_{1}-N_{1}$, $c\left(\left[a^{\prime}, b^{\prime}\right]\right)$ joins $L_{i}^{-}$to $L_{i}^{+}$outside $B\left(p, R^{\prime}\right) \supset B\left(o_{i}, R^{\prime}-c_{1} R\right)$. Hence length $\left(c\left(\left[a^{\prime}, b^{\prime}\right]\right) \geq\right.$ $\beta\left(R^{\prime}-c_{1} R\right)\left(n_{1}-n_{0}-\left(2 N_{1}+2\right)\right) \geq \frac{\beta}{2} R^{\prime}\left(n_{1}-n_{0}-\left(2 N_{1}+2\right)\right)$ while $d_{\mathcal{T}}\left(c\left(a^{\prime}\right), c\left(b^{\prime}\right)\right) \leq 2 R^{\prime}$ so length $\left(c\left[a^{\prime}, b^{\prime}\right]\right) \geq d_{\mathcal{T}}\left(c\left(a^{\prime}\right), c\left(b^{\prime}\right)\right)+\frac{\beta}{2}\left(n_{1}-n_{0}-\left(2 N_{1}+2\right)-\frac{4}{\beta}\right) R^{\prime}$ and hence

$$
\text { length }(c) \geq d_{\mathcal{T}}(c(0), c(1))+C_{1}\left(n_{1}-n_{0}-C_{2}\right) R^{\prime}
$$

where $C_{1}, C_{2}$ depend only on $\beta$.

Corollary 5.5. Let $\mathcal{T}, N_{1}, C_{1}, C_{2}, W_{n_{0}}, \ldots, W_{n_{1}}, p, R$ be as in Proposition 5.2. If $n_{1}-n_{0}>C_{2}$, then any geodesic segment from $W_{n_{0}}$ to $W_{n_{1}}$ must pass through $B\left(p, N_{1} R\right)$.

The result corresponding to Corollary 5.5 in the space $X$ is:
Proposition 5.6. There are constants $N_{2}=N_{2}(X), R_{0}=R_{0}(X)$ with the following property. If $n \geq N_{2}, e_{1}, \ldots e_{n} \in E$ are the consecutive edges of a geodesic segment, $\gamma$, in the tree $T, p \in X, R \geq R_{0}$, and $X_{e_{i}} \cap B(p, R) \neq \emptyset$ for $1 \leq i \leq n$; then for any $C \geq 0$, and any segment $\overline{x y} \subset X$ with $\overline{x y} \cap N_{C}\left(X_{e_{i}}\right) \neq \emptyset$ for $i=1$ and $i=n$, we have $\overline{x y} \cap B\left(p, N_{2} R+2 C\right) \neq \emptyset$.

Proof. Let $e_{1}, \ldots, e_{n}, p, X_{e_{i}}$ be as in the statement of the proposition. If $\overline{x y} \cap$ $N_{C}\left(X_{e_{i}}\right) \neq \emptyset$ for $i=1$ and $i=n$, then we have $x_{0} \in X_{e_{1}}$ and $y_{0} \in X_{e_{n}}$ with $d\left(x_{0}, \overline{x y}\right), d\left(y_{0}, \overline{x y}\right) \leq C$. By convexity of the distance function $d_{X}$ it suffices to show that $\overline{x_{0} y_{0}} \cap B\left(p, N_{2} R\right) \neq \emptyset$.

Let $K$ and $\beta$ be as in Lemma 4.5 and $(\mathcal{T}, f, \phi)$ be a $K$ template for $\gamma$ whose angles are bounded by $\beta$. Let $\alpha: \overline{x_{0} y_{0}} \rightarrow \mathcal{T}$ be the map guaranteed by Proposition 4.6. So Length $(\alpha) \leq d\left(x_{0}, y_{0}\right)+n C_{5}$. By part 3 of definition 4.4 there is a $p^{\prime} \in \mathcal{T}$ be such that $d\left(p, \phi\left(p^{\prime}\right)\right)<R+K$ and hence (since $\left.X_{e_{i}} \subset N_{K} \phi\left(W_{i}\right)\right) d\left(\phi\left(p^{\prime}\right), \phi\left(W_{i}\right)\right)<$ $2 R+2 K$. Since, by Proposition 4.6, $\phi$ is an $(L, A)$-quasi isometric embedding we have $B\left(p^{\prime}, R_{2}\right) \cap W_{i} \neq \emptyset$ for $R_{2}=L(2 R+2 K)+A$. We will choose $R_{0}$ and $N_{2}$ large enough so that for $n>N_{2}$ and $R \geq R_{0}$ we will have $n C_{5}<C_{1}\left(n-C_{2}\right) N_{1} R_{2}$ for the $N_{1}(\beta), C_{1}(\beta)$ and $C_{2}(\beta)$ of Proposition 5.2. Thus Proposition 5.2 forces $\alpha$ to intersect $B\left(p^{\prime}, N_{1} R_{2}\right)$. So we conclude (again using Proposition 4.6) that $d\left(\overline{x_{0} y_{0}}, p\right)<$ $C_{5}+d\left(\phi(\alpha), \phi\left(p^{\prime}\right)\right)+R+K \leq C_{5}+L N_{1} R_{2}+A+R+K$. The proposition now follows by taking $N_{2}$ and $R_{0}$ large enough.

### 5.2. Paths with small length distortion

Proposition 5.7. Pick $M>0$ and $\alpha \in\left(\frac{1}{2}, 1\right]$. Then there is a constant $C=C(M, \alpha)$ so that if $1 \leq A \leq B, \eta:[A, B] \rightarrow X$ is a (not necessarily continuous) map to $a$ Hadamard space $X$, and for all $A \leq t_{1} \leq t_{2} \leq B$ we have

$$
\begin{equation*}
\left|d_{X}\left(\eta\left(t_{1}\right), \eta\left(t_{2}\right)\right)-\left(t_{2}-t_{1}\right)\right| \leq M\left(1+\log \left(\frac{t_{2}}{t_{1}}\right)\right) \tag{5.8}
\end{equation*}
$$

then

$$
\begin{equation*}
d(\eta(t), z) \leq C\left(1+t^{\alpha}\right) \tag{5.9}
\end{equation*}
$$

where $z \in \overline{\eta(A) \eta(B)}$ is the point with $d(z, \eta(A))=\frac{t-A}{B-A} d(\eta(A), \eta(B))$. Similarly, if $A \geq 1$ and $\eta:[A, \infty) \rightarrow X$ satisfies (5.8) for all $A \leq t_{1} \leq t_{2}$, then there is a unique unit speed geodesic ray $\gamma:[A, \infty) \rightarrow X$ with $\gamma(A)=\eta(A)$ such that

$$
d_{X}(\eta(t), \gamma(t)) \leq C\left(1+t^{\alpha}\right)
$$

for all $t \in[A, \infty)$.

Proof. First note that we may assume that $A=1$, since the map $\eta_{1}:[1, B-A+1] \rightarrow$ $X$ given by $\eta_{1}(t):=\eta(t+A-1)$ will satisfy the hypotheses of the proposition, and the conclusion of the proposition applied to $\eta_{1}$ will imply (5.9) for $\eta$.

Step 1: When $1 \leq s_{1} \leq s_{2} \leq 2 s_{1} \leq B$ and $s_{1}$ is sufficiently large then the comparison angle $\widetilde{乙}_{\eta(1)}\left(\eta\left(s_{1}\right), \eta\left(s_{2}\right)\right) \leq$ Const $s_{1}^{\alpha-1}$.

We will make use of the following lemma that follows from standard comparisons.
Lemma 5.10. Let $X$ be a Hadamard space, $x, y, z \in X$. Set $L:=d(x, z)$, and the excess $E:=d(x, y)+d(y, z)-d(x, z)$. Then

$$
\begin{align*}
& d(y, \overline{x z}) \leq \frac{\sqrt{2 L E}}{2} \sqrt{1+\frac{E}{2 L}}  \tag{5.11}\\
& \quad \leq \sqrt{L E} \text { if } E \leq 2 L \tag{5.12}
\end{align*}
$$

Proof. Triangle comparison.
Proof of Proposition 5.7 continued. Take $1 \leq s_{1} \leq s_{2} \leq B$, and consider the triple $\eta(1), \eta\left(s_{1}\right), \eta\left(s_{2}\right)$. The excess for the triple is

$$
\leq M\left(1+\log s_{2}\right)+M\left(1+\log s_{1}\right)+M\left(1+\log \left(\frac{s_{2}}{s_{1}}\right)\right)=3 M\left(1+\log s_{2}\right)
$$

Since $d\left(\eta(1), \eta\left(s_{2}\right)\right) \geq\left(s_{2}-1\right)-M\left(1+\log s_{2}\right)$ when $s_{2}>c_{1}=c_{1}(M)$, then the excess is $\leq 2 d\left(\eta(1), \eta\left(s_{2}\right)\right)$. Thus since $d\left(\eta(1), \eta\left(s_{2}\right)\right) \leq\left(s_{2}-1\right)+M\left(1+\log s_{2}\right)$ applying (5.12) we get

$$
\begin{gather*}
d\left(\eta\left(s_{1}\right), \overline{\eta(1) \eta\left(s_{2}\right)}\right) \leq \sqrt{\left[\left(s_{2}-1+M\left(1+\log s_{2}\right)\right)\right]\left[3 M\left(1+\log s_{2}\right)\right]} \\
\leq c_{2}\left(1+s_{2}^{\alpha}\right) \tag{5.13}
\end{gather*}
$$

where $c_{2}=c_{2}(M, \alpha)$. Therefore the comparison angle $\widetilde{\mathcal{L}}_{\eta(1)}\left(\eta\left(s_{1}\right), \eta\left(s_{2}\right)\right)$ satisfies

$$
\sin \left(\tilde{\zeta}_{\eta(1)}\left(\eta\left(s_{1}\right), \eta\left(s_{2}\right)\right)\right) \leq \frac{c_{2}\left(1+s_{2}^{\alpha}\right)}{d\left(\eta(1), \eta\left(s_{1}\right)\right)} \leq \frac{c_{2}\left(1+s_{2}^{\alpha}\right)}{\left[\left(s_{1}-1\right)-M\left(1+\log s_{1}\right)\right]} .
$$

So there are constants $c_{3}=c_{3}(M, \alpha)$ and $c_{4}=c_{4}(M, \alpha)$ so that if $c_{3} \leq s_{1} \leq s_{2} \leq$ $2 s_{1} \leq B$ then

$$
\begin{equation*}
\sin \left(\tilde{\zeta}_{\eta(1)}\left(\eta\left(s_{1}\right), \eta\left(s_{2}\right)\right)\right) \leq \frac{2 c_{2}\left(1+s_{2}^{\alpha}\right)}{s_{1}} \leq \frac{c_{4}}{2} s_{2}^{\alpha-1} \tag{5.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{\mathcal{L}}_{\eta(1)}\left(\eta\left(s_{1}\right), \eta\left(s_{2}\right)\right) \leq c_{4} s_{2}^{\alpha-1} \tag{5.15}
\end{equation*}
$$

Step 2: Estimating $d(\eta(t), \overline{\eta(1) \eta(B)})$. Now pick $t_{0} \in[1, B]$ with $t_{0} \geq c_{3}$. Let $t_{i}=2^{i} t_{0}$ for $i=0, \ldots, n$ where $n$ is the integer part of $\frac{\log B}{\log 2}$, and $t_{n+1}=B$. Then $\frac{B}{t_{n}}<2$. Applying the estimate (5.15) with $u_{i}:=\eta\left(t_{i}\right)$ we have for $i=1, \ldots, n+1$ :

$$
\widetilde{Z}_{\eta(1)}\left(u_{i-1}, u_{i}\right) \leq c_{4} t_{i}^{\alpha-1}
$$

since $\frac{t_{i}}{t_{i-1}} \leq 2$ and $t_{i} \geq c_{3}$. Either $n=0$, in which case we have

$$
\begin{gathered}
d\left(\eta\left(t_{0}\right), \overline{\eta(1) \eta(B)}\right) \leq c_{2}\left(1+t_{1}^{\alpha}\right) \quad \text { by }(5.13) \\
\leq c_{2}\left(1+2^{\alpha} t_{0}^{\alpha}\right) \leq c_{5}\left(1+t_{0}^{\alpha}\right)
\end{gathered}
$$

where $c_{5}=c_{5}(M, \alpha)$. Otherwise, if $t_{0}>c_{6}=c_{6}(M, \alpha)$,

$$
\begin{gathered}
d\left(\eta(1), u_{0}\right) \leq\left(t_{0}-1\right)+M\left(1+\log t_{0}\right) \\
\leq\left(t_{i}-1\right)-M\left(1+\log t_{i}\right) \leq d\left(\eta(1), u_{i}\right)
\end{gathered}
$$

So for $0 \leq i \leq n+1$ we may pick $v_{i} \in \overline{\eta(1) u_{i}}$ with $d\left(\eta(1), v_{i}\right)=d\left(\eta(1), u_{0}\right)=: R_{0}$. By triangle comparison we have

$$
\begin{gathered}
d\left(v_{i-1}, v_{i}\right) \leq R_{0} \tilde{\llcorner }_{\eta(1)}\left(v_{i-1}, v_{i}\right) \leq R_{0} \tilde{\llcorner }_{\eta(1)}\left(u_{i-1}, u_{i}\right) \\
\leq R_{0} c_{4} t_{i}^{\alpha-1}
\end{gathered}
$$

so

$$
\begin{gathered}
d\left(u_{0}, \overline{\eta(1) \eta(B)}\right)=d\left(u_{0}, \overline{\eta(1) u_{n+1}}\right) \leq d\left(u_{0}, v_{n+1}\right) \\
\leq \sum_{i=1}^{n+1} d\left(v_{i-1}, v_{i}\right) \leq R_{0} c_{4} \sum_{i=1}^{n+1} t_{i}^{\alpha-1} \\
\leq R_{0} c_{7} t_{0}^{\alpha-1}
\end{gathered}
$$

for $c_{7}=c_{7}(M, \alpha)$ (independent of $n$ ). We have $R_{0} \leq t_{0}-1+M\left(1+\log t_{0}\right) \leq 2 t_{0}$ when $t_{0}$ is sufficiently large, so when $t_{0} \geq c_{8}=c_{8}(M, \alpha)$

$$
d\left(u_{0}, \overline{\eta(1) \eta(B)}\right) \leq c_{9} t_{0}^{\alpha}
$$

where $c_{9}=c_{9}(M, \alpha)$.
Step 3: Estimating $d(\eta(t), z)$ where $z \in \overline{\eta(1) \eta(B)}$ satisfies $d(z, \eta(1))=\frac{t-1}{B-1} d(\eta(1), \eta(B))$. Let $x \in \overline{\eta(1) \eta(B)}$ be the point nearest $\eta(t)$. Then for $t \geq c_{10}$ we have

$$
\begin{gathered}
d(\eta(t), z) \leq d(\eta(t), x)+d(x, z) \\
=d(\eta(t), x)+\left|d(x, \eta(1))-(t-1) \frac{d(\eta(1), \eta(B))}{B-1}\right| \\
\leq d(\eta(t), x)+\left[d(x, \eta(t))+\frac{(t-1)}{B-1} M(1+\log B)+M(1+\log t)\right] \leq 3 c_{9} t^{\alpha} .
\end{gathered}
$$

Thus the result holds for large $t$, but by choosing $C$ large enough we get the result for all $t$.

For the ray case we redo step 3 above when $d(\eta(1), \eta(B))>t$ for $z_{1} \in \overline{\eta(1) \eta(B)}$ the point which satisfies $d\left(z_{1}, \eta(1)\right)=t$. Let $x \in \overline{\eta(1) \eta(B)}$ be the point nearest $\eta(t)$. Then for $t \geq c_{11}$ we have

$$
\begin{gathered}
d\left(\eta(t), z_{1}\right) \leq d(\eta(t), x)+d\left(x, z_{1}\right) \\
=d(\eta(t), x)+|d(x, \eta(1))-(t-1)| \\
\leq d(\eta(t), x)+[d(x, \eta(t))+M(1+\log t)] \leq 3 c_{9} t^{\alpha} .
\end{gathered}
$$

Now choose $\gamma$ as a limit of a subsequence of $\overline{\eta(1), \eta(B)}$ as $B$ goes to $\infty$ (the result will in fact imply that the sequence itself converges). Since none of the constants depended on $B$ the above estimate for $d\left(\eta(t), z_{1}\right)$ gives the result for rays.

### 5.3. The proof of Theorem 5.1

Let $\beta=\beta(X)$ be the minimum angle between singular geodesics of a template for $X$ defined in Lemma 4.5, and let $C_{2}(\beta), N_{1}(\beta)$ be the constants from Corollary 5.5. Let $N_{2}, R_{0}$ be the constants from Proposition 5.6; we will assume that $N_{2} \geq \max \left(C_{2}(\beta), N_{1}(\beta)\right)$.
Definition 5.16. For every $\psi>0$, we will say that $z \in \overline{x y}$ is a $\psi$-cluster point if the segment $\overline{B(z, \psi d(z, x))} \cap \overline{x y}$ intersects at least $N_{2}$ walls of $\mathcal{T}$, or if $z \in\{x, y\}$. We will let $P_{\psi} \subset \overline{x y}$ represent the set of $\psi$-cluster points.

Proof of 1 . Let $(\mathcal{T}, f, \phi), x, y$, be as in the statement of part 1 of the theorem. We begin with two lemmas.

Lemma 5.17. There is a constant $c_{1}$ depending on $K$ such that if $\overline{x y}$ intersects $a$ wall $W \in$ Wall $_{\mathcal{T}}$, then $\overline{\phi(x) \phi(y)} \cap N_{c_{1}}\left(X_{f(W)}\right) \neq \emptyset$.

Proof. If $\{x, y\} \cap W \neq \emptyset$ this is immediate since $\phi(W) \subset N_{K}\left(X_{f(W)}\right)$ by definition 4.4. Otherwise $x$ and $y$ must lie in distinct components of $\mathcal{T}-W$ because each component is convex. Say $x \in W_{1} \cup \mathcal{S}_{1} \cup W_{1}^{\prime}$ and $y \in W_{2} \cup \mathcal{S}_{2} \cup W_{2}^{\prime}$ where $W_{i}$ is adjacent to $W_{i}^{\prime}$. Set $v:=f\left(W_{1}\right) \cap f\left(W_{1}^{\prime}\right)$ and $v^{\prime}:=f\left(W_{2}\right) \cap f\left(W_{2}^{\prime}\right)$. Applying Lemma 3.17 the lemma follows.

Lemma 5.18. Suppose $p_{1}, p_{2} \in \overline{x y}, d\left(p_{1}, x\right) \leq d\left(p_{2}, x\right)$, and $\overline{p_{1} p_{2}} \cap P_{\psi} \subset\left\{p_{1}, p_{2}\right\}$. Set $A:=1+d\left(p_{1}, x\right), B:=1+d\left(p_{2}, x\right)$. Let $\eta_{0}:[A, B] \rightarrow \mathcal{T}$ be the unit speed parameterization of $\overline{p_{1} p_{2}}$. Then the composition $\eta:=\phi \circ \eta_{0}:[A, B] \rightarrow X$ satisfies the hypotheses of Proposition 5.7 with $M=M(K, \psi)$.

Proof. Pick $A \leq t_{1}<t_{2} \leq B$. By Corollary 4.10 we have

$$
\begin{equation*}
\left|d_{X}\left(\eta\left(t_{1}\right), \eta\left(t_{2}\right)\right)-\left(t_{2}-t_{1}\right)\right| \leq k C_{6} \tag{5.19}
\end{equation*}
$$

where $\overline{\eta_{0}\left(t_{1}\right) \eta_{0}\left(t_{2}\right)}$ intersects at most $(k-1)$ walls and strips of $\mathcal{T}$. For each $z \in$ $\overline{\eta_{0}\left(t_{1}\right) \eta_{0}\left(t_{2}\right)}$, the segment $\overline{B(z, \psi d(z, x))} \cap \overline{x y}$ intersects at most $N_{2}-1$ walls. We can cover $\overline{\eta_{0}\left(t_{1}\right) \eta_{0}\left(t_{2}\right)}-B(x, 1)$ with at most Const $\log \left(\frac{t_{2}}{t_{1}}\right)$ such segments, and $\overline{B(x, 1)} \cap$ $\overline{\eta_{0}\left(t_{1}\right) \eta_{0}\left(t_{2}\right)}$ intersects at most 2 walls (since strips have width at least 1 ), so the lemma follows from (5.19).

Step 1: Estimating $d(\phi(z), \overline{\phi(x) \phi(y)})$ when $z \in P_{\psi}$. Suppose $z \in P_{\psi}$. Then $\overline{x y} \cap B(z, \psi R)$ intersects at least $N_{2}$ consecutive walls $W_{1}, \ldots, W_{n}$ of $\mathcal{T}$. By definition 4.4 and the fact that $\phi: \mathcal{T} \rightarrow X$ is a quasi-isometric embedding (Proposition 4.6) there is a constant $c_{2}=c_{2}(K)$ so that $d_{X}\left(\phi(z), X_{f\left(W_{i}\right)}\right) \leq c_{2}(1+\psi R)$ for $i=1, \ldots n$. By Lemma 5.17 and Proposition 5.6, we conclude that $\overline{\phi(x) \phi(y)} \cap B\left(\phi(z), N_{2} c_{2}(1+\right.$ $\left.\psi R)+2 c_{1}\right) \neq \emptyset$ provided $c_{2}(1+\psi R) \geq R_{0}$. So there are positive constants $r_{0}=$ $r_{0}(K, \psi), c_{3}=c_{3}(K, \beta)$ so that if $z \in P_{\psi}$ and $R:=d(z, x) \geq r_{0}$, then

$$
\begin{equation*}
d(\phi(z), \overline{\phi(x) \phi(y)}) \leq c_{3} \psi R \tag{5.20}
\end{equation*}
$$

Hence for any $z \in P_{\psi}$

$$
\begin{equation*}
d(\phi(z), \overline{\phi(x) \phi(y)}) \leq c_{3} \psi R+L r_{0}+A=c_{3} \psi R+c_{4} \tag{5.21}
\end{equation*}
$$

where $(L, A)$ are the quasi-isometric embedding constants of $\phi$, and $c_{4}:=L r_{0}+A=$ $c_{4}(K, \psi)$.

Step 2: Estimating $d(\phi(z), \overline{\phi(x) \phi(y)})$ when $z \notin P_{\psi}$. Pick $z \notin P_{\psi}$. There are points $p_{1}, p_{2} \in \overline{x y}$ so that $z \in \overline{p_{1} p_{2}}, d\left(p_{1}, x\right) \leq d\left(p_{2}, x\right)$, and either $\overline{p_{1} p_{2}} \cap P_{\psi}=\left\{p_{1}, p_{2}\right\}$ or one or both of $p_{1}=x$ or $p_{2}=y$ hold. Each step of the argument below holds in the special cases $p_{1}=x$ or $p_{2}=y$ (often for easier reasons) so we will ignore them. By (5.21) we have

$$
\begin{equation*}
d\left(\phi\left(p_{i}\right), \overline{\phi(x) \phi(y)}\right) \leq c_{3} \psi d\left(p_{i}, x\right)+c_{4} \tag{5.22}
\end{equation*}
$$

for $i=1,2$. Since $p_{1}, p_{2}$ satisfy the conditions of Lemma 5.18, we may apply Proposition 5.7 with $\alpha=\frac{3}{4}$ to get that for the $w \in \overline{\phi\left(p_{1}\right) \phi\left(p_{2}\right)}$ with $d\left(w, \phi\left(p_{1}\right)\right)=$ $\frac{d\left(z, p_{1}\right)}{d\left(p_{1}, p_{2}\right)} d\left(\phi\left(p_{1}\right), \phi\left(p_{2}\right)\right)$ we have

$$
\begin{equation*}
d(\phi(z), w) \leq C\left(1+[d(z, x)]^{\frac{3}{4}}\right) \leq \psi d(z, x)+c_{6} . \tag{5.23}
\end{equation*}
$$

Where $c_{6}=c_{6}(K, \psi)$.
On the other hand for $\zeta=\frac{d\left(w, \phi\left(p_{1}\right)\right)}{d\left(\phi\left(p_{1}\right), \phi\left(p_{2}\right)\right)}=\frac{d\left(z, p_{1}\right)}{d\left(p_{1}, p_{2}\right)}$ the convexity of the distance function $d(*, \overline{\phi(x) \phi(y)})$ and 5.22 says that

$$
\begin{equation*}
d(w, \overline{\phi(x) \phi(y)}) \leq c_{3} \psi\left((1-\zeta) d\left(p_{1}, x\right)+\zeta d\left(p_{2}, x\right)\right)+c_{4}=c_{3} \psi d(z, x)+c_{4} \tag{5.24}
\end{equation*}
$$

Combining (5.24) and (5.23) we get

$$
\begin{equation*}
d(\phi(z), \overline{\phi(x) \phi(y)}) \leq c_{7} \psi d(z, x)+c_{8} \tag{5.25}
\end{equation*}
$$

for $c_{7}=c_{7}(K)$ and $c_{8}=c_{8}(K, \psi)$. Thus (5.21) and (5.25) together imply that for every choice of $\psi>0$ there are constants $c_{9}(K, \beta)$ and $c_{10}(K, \psi)$ such that for all $z$

$$
d(\phi(z), \overline{\phi(x) \phi(y)}) \leq c_{9} \psi R+c_{10} .
$$

The fact that $c_{9}$ does not depend on $\psi$ allows us to pick $\theta(R)$ which decays to 0 as $R \rightarrow \infty$ such that for each $R$ there is a $\psi=\psi(R)>0$ (for example choose $\psi(R)$ decaying to 0 such that $\left.c_{10}(\psi)<R^{-\frac{1}{2}}\right)$ so that

$$
c_{9} \psi R+c_{10}(\psi) \leq(1+R) \theta(R)
$$

This implies part 1 of Theorem 5.1.
Proof of 2. We omit this, as it is similar to the proof of 1.
Proof of 3. Part 3 follows directly from part 1 and Lemma 2.5.
Proof of 4. Part 2 along with reasoning similar to the proof of Lemma 2.5 shows that if $x, y_{k} \in \mathcal{T}, \xi \in \partial_{\infty} X$, and $\overline{\phi(x) \phi\left(y_{k}\right)} \rightarrow \overline{x \xi}$, then $\overline{x y_{k}}$ converges to some ray $\overline{x \xi^{\prime}} \subset \mathcal{T}$. Hence $\partial_{\infty} \phi$ is a homeomorphism from $\partial_{\infty} \mathcal{T}$ onto the limit set of $\phi(\mathcal{T}) \subset X$, which we denote by $\partial_{\infty} \phi(\mathcal{T})$. It follows immediately from this that $\partial_{\infty} \phi$ maps $\partial_{\infty} \mathcal{T}-\partial_{\infty}^{\infty} \mathcal{T}=\cup_{W \in \text { Wall } \mathcal{T}} \partial_{\infty} W$ homeomorphically onto $\cup_{W \in W \text { all }}^{\mathcal{T}} \partial_{\infty} X_{f(W)}$. This
implies 4 when $\gamma \subset T$ is a geodesic segment, so we now assume that $\gamma$ is a geodesic ray, and we need only show that $\partial_{\infty} \phi\left(\partial_{\infty}^{\infty} \mathcal{T}\right)=\partial_{\infty}^{\eta} X$ where $\eta=\partial_{\infty} \gamma \in \partial_{\infty} T$. We will let $W_{k} \in W$ all $\mathcal{T}_{\mathcal{T}}$ be the $k^{\text {th }}$ wall of $\mathcal{T}$.

We first show $\partial_{\infty} \phi\left(\partial_{\infty}^{\infty} \mathcal{T}\right) \subset \partial_{\infty}^{\eta} X$. Suppose $\xi \in \partial_{\infty}^{\infty} \mathcal{T}, z_{k} \in \overline{x \xi} \cap W_{k}$, and $d\left(z_{k}, x\right) \rightarrow \infty$. Thus $\overline{\phi(x) \phi\left(z_{k}\right)} \rightarrow \overline{\phi(x) \partial_{\infty} \phi(\xi)}$. By lemma 3.17 part 1, for any $l$ we have $\overline{\phi(x) \phi\left(z_{k}\right)} \cap N_{c}\left(X_{f\left(W_{l}\right)}\right) \neq \emptyset$ for sufficiently large $k$. So by Lemma 2.4 either $\partial_{\infty} \phi(\xi) \in \partial_{\infty} N_{c}\left(X_{f\left(W_{l}\right)}\right)=\partial_{\infty} X_{f\left(W_{l}\right)}$ or $\overline{\phi(x) \partial_{\infty} \phi(\xi)} \cap N_{c}\left(X_{f\left(W_{l}\right)}\right) \neq \emptyset$. The former case is impossible since we already know that $\left(\partial_{\infty} \phi\right)^{-1}\left(\partial_{\infty} X_{f\left(W_{l}\right)}\right)=\partial_{\infty} W_{l}$. So $\overline{\phi(x) \partial_{\infty} \phi(\xi)} \cap N_{c}\left(X_{f\left(W_{l}\right)}\right) \neq \emptyset$ for every $W \in W$ all $\mathcal{T}_{\mathcal{T}}$, forcing $\mathcal{I}\left(\overline{\phi(x) \partial_{\infty} \phi(\xi)}\right)=\gamma$. So $\partial_{\infty} \phi\left(\partial_{\infty}^{\infty} \mathcal{T}\right) \subset \partial_{\infty}^{\eta} X$.

We now show $\partial_{\infty}^{\eta} X \subset \partial_{\infty} \phi\left(\partial_{\infty}^{\infty} \mathcal{T}\right)$. Suppose $x \in W_{1} \in W$ all $\mathcal{T}$ and $\xi \in \partial_{\infty}^{\eta} X$. Then from the definition of itineraries 3.20 and Lemma 3.19 there are $z_{k} \in \overline{\phi(x) \xi} \cap N_{C}\left(X_{e_{k}}\right)$ for $e_{k} \in T$ so that $d_{T}\left(e_{k}, f\left(W_{1}\right)\right) \rightarrow \infty$ and $d_{T}\left(e_{k}, \gamma\right)$ is uniformly bounded. So for all but finitely many $W, f(W)$ separates $e_{k}$ from $f\left(W_{1}\right)$ for sufficiently large $k$, so by Lemma 3.17

$$
\overline{\phi(x) z_{k}} \cap N_{c}\left(X_{f(W)}\right) \neq \emptyset
$$

for sufficiently large $k$. Hence $\xi$ belongs to the limit set of $\cup_{W \in W_{\text {all }}^{\mathcal{T}}} X_{f(W)}$, which is the same as $\partial_{\infty}(\phi(\mathcal{T}))=\partial_{\infty} \phi\left(\partial_{\infty} \mathcal{T}\right)$. Therefore $\xi \in\left(\partial_{\infty} \phi\right)\left(\partial_{\infty} \mathcal{T}-\cup_{W \in W \text { all }}^{\mathcal{T}}\right.$ $\left.\partial_{\infty} W\right)=$ $\left(\partial_{\infty} \phi\right)\left(\partial_{\infty}^{\infty} \mathcal{T}\right)$.

Proof of 5. Pick $x \in \mathcal{T}$ and $\xi \in \partial_{\infty}^{\infty} \mathcal{T}$. Suppose there is a sequence $z_{k} \in \overline{x \xi}$ with $\lim _{k \rightarrow \infty}\left(z_{k}, x\right)=\infty$ so that $z_{k}$ is a $\psi_{k}$-cluster point $\overline{x \xi}$ where $\psi_{k} \rightarrow 0$. Set $R_{k}:=d\left(z_{k}, x\right)$. Applying Corollary 5.5 to each $z_{k}$, we see that if $\xi^{\prime} \in \partial_{\infty}^{\infty} \mathcal{T}$, then for sufficiently large $k$ the intersection $\overline{x \xi^{\prime}} \cap B\left(z_{k}, N_{1} \psi_{k} R_{k}\right)$ is nonempty, and this clearly forces $\overline{x \xi^{\prime}}=\overline{x \xi}$. In this case we have $\partial_{\infty}^{\infty} \mathcal{T}=\{\xi\}$, and 5 is immediate. So we may assume that for every $\xi_{1}, \xi_{2} \in \partial_{\infty}^{\infty} \mathcal{T}$ there is a $\psi>0$ such that $\overline{x \xi_{1}}$ and $\overline{x \xi_{2}}$ contain no $\psi$-cluster points. Let $\bar{\eta}_{i}:[1, \infty) \rightarrow \mathcal{T}$ be the unit speed parameterization of $\overline{x \xi_{i}}$, and let $\eta_{i}:[1, \infty) \rightarrow X$ be the composition $\phi \circ \bar{\eta}_{i}:[1, \infty) \rightarrow X$. Then by Lemma 5.18 the $\eta_{i}$ satisfy the hypotheses of Proposition 5.7 for a suitable $M$; so we have unit speed geodesic rays $\gamma_{i}:[1, \infty) \rightarrow X$ with $\gamma_{i}(1)=\eta_{i}(1)$ and $d_{X}\left(\eta_{i}(t), \gamma_{i}(t)\right) \leq C\left(1+t^{\frac{3}{4}}\right)$. Now, for sufficiently large $k$, choose $t_{k}^{i}$ so that $\bar{\eta}_{i}\left(t_{k}^{i}\right)$ lies in the $k^{\text {th }}$ wall of $\mathcal{T}$. By definition 4.4

$$
\left|d_{X}\left(\eta_{1}\left(t_{k}^{1}\right), \eta_{2}\left(t_{k}^{2}\right)\right)-d_{\mathcal{T}}\left(\bar{\eta}_{1}\left(t_{k}^{1}\right), \bar{\eta}_{2}\left(t_{k}^{2}\right)\right)\right|<K .
$$

Thus

$$
\angle_{T}\left(\gamma_{1}, \gamma_{2}\right)=\lim _{k \rightarrow \infty} \tilde{\angle}_{\eta_{1}(1)}\left(\eta_{1}\left(t_{k}^{1}\right), \eta_{2}\left(t_{k}^{2}\right)\right)=\lim _{k \rightarrow \infty} \tilde{Z}_{x}\left(\bar{\eta}_{1}\left(t_{k}^{1}\right), \bar{\eta}_{2}\left(t_{k}^{2}\right)\right)=\partial_{T}\left(\xi_{1}, \xi_{2}\right)
$$

This proves 5.

### 5.4. Applications of Theorem 5.1

Theorem 5.1 has a number of corollaries:
Corollary 5.26. Let $G$ be the fundamental group of an admissible graph of groups $\mathcal{G}$, and let $G \curvearrowright T$ be the Bass-Serre tree of $\mathcal{G}$. Let $\gamma \subset T$ be a geodesic ray with $i^{\text {th }}$ edge $e_{i} \subset T$, and set $\eta:=\partial_{\infty} \gamma \in \partial_{\infty} T$. Then for any admissible action $G \curvearrowright X$, the subset $\partial_{\infty}^{\eta} X \subset \partial_{\infty} X$ defined in section 3.3 is precisely the set of limit points of the sequence of subsets $\partial_{\infty} X_{e_{k}} \subset \partial_{\infty} X$.

Proof. Let $(\mathcal{T}, f, \phi)$ be a template for $\gamma \subset T$. By parts 3 and 4 of Theorem 5.1, $\partial_{\infty} \phi$ : $\partial_{\infty} \mathcal{T} \rightarrow \partial_{\infty} X$ is a homeomorphism onto $\left(\cup_{k} \partial_{\infty} X_{e_{k}}\right) \cup\left(\partial_{\infty}^{\eta} X\right)$. Therefore it suffices to show that $\partial_{\infty}^{\infty} \mathcal{T} \subset \partial_{\infty} \mathcal{T}$ is the set of limit points of the sequence $\partial_{\infty} f^{-1}\left(e_{k}\right) \subset \partial_{\infty} \mathcal{T}$. To see this, observe that any geodesic segment $\overline{p x} \subset \mathcal{T}$ which arrives at a wall $W \in W a l_{\mathcal{T}}$ via a $\operatorname{strip} \mathcal{S} \in \operatorname{Strip}_{\mathcal{T}}$ may be prolonged by a geodesic ray contained in $W$; this implies that every $\xi \in \partial_{\infty}^{\infty} \mathcal{T}$ is a limit of a sequence $\xi_{k} \in \partial_{\infty} f^{-1}\left(e_{k}\right)$. On the other hand, if $\xi_{k} \in \partial_{\infty} f^{-1}\left(e_{k}\right)$ converges to $\xi \in \partial_{\infty} \mathcal{T}$, then Lemma 2.4 implies that either $\xi \in \partial_{\infty}^{\infty} \mathcal{T}$, or $\xi$ belongs to $\partial_{\infty} f^{-1}\left(e_{k}\right)$ for all sufficiently large $k$, which is absurd.

Remark 5.27. Let $G \curvearrowright X, T, \gamma$, and $\eta$ be as in Corollary 5.26. When $X$ is a 3-dimensional Hadamard manifold, then $\partial_{\infty} X \simeq S^{2}$ and there is an alternate characterization of $\partial_{\infty}^{\eta} X$ which uses little more than the definition of itineraries. For each $i$, $\partial_{\infty} X_{e_{i}} \subset \partial_{\infty} X \simeq S^{2}$ determines two closed disks by the Jordan separation theorem; let $D_{i}$ be the one which contains $\partial_{\infty} X_{e_{j}}$ for all $j \geq i$. Then $\partial_{\infty}^{\eta} X=\cap_{i} D_{i}$. To see this note that if $F \subset X$ is a flat totally geodesic plane and $p \in X-F$, then the two components of $\partial_{\infty} X-\partial_{\infty} F$ are $\left\{\xi \in \partial_{\infty} X \mid \overline{p \xi} \cap F \neq \emptyset\right\}$ and $\left\{\xi \in \partial_{\infty} X \mid \overline{p \xi} \cap F=\emptyset\right\}$.
Corollary 5.28. Let $G \curvearrowright X$ and $G \curvearrowright X^{\prime}$ be admissible actions, let $G \curvearrowright T$ be the Bass-Serre tree of $G$, and let $V, E$ be the sets of vertices and edges of $T$, respectively. If $\Phi: \partial_{\infty} X \rightarrow \partial_{\infty} X^{\prime}$ is any $G$-equivariant homeomorphism, then

1. $\Phi$ maps $\partial_{\infty} X_{\sigma}$ homeomorphically to $\partial_{\infty} X_{\sigma}^{\prime}$ for all $\sigma \in V \cup E$.
2. $\Phi$ maps $\partial_{\infty}^{\eta} X$ homeomorphically to $\partial_{\infty}^{\eta} X^{\prime}$ for all $\eta \in \partial_{\infty} T$.

Proof. Part 1 follows from the characterization of $\partial_{\infty} X_{\sigma}$ as a fixed point set which is stated in Lemma 3.23. Part 2 follows from part 1 and Corollary 5.26.
Corollary 5.29. Let $G \curvearrowright X$ be an admissible action, and $G \curvearrowright T$ be the Bass-Serre action for $G$. Then

1. The union $\cup_{v \in V} \partial_{T} X_{v} \subset \partial_{T} X$ is a $C A T(1)$ space with respect to the induced metric, and may be described metrically as follows. First, $\partial_{T} X_{v}$ is a metric suspension of an uncountable discrete $C A T(1)$ space for each $v \in V$, and $\partial_{T} X_{e}$ is isometric to the standard circle for every $e \in E$. Take the disjoint union $\amalg_{v \in V} \partial_{T} X_{v}$, and for each edge $e=\overline{v_{1} v_{2}} \in E$, glue $\partial_{T} X_{v_{1}}$ to $\partial_{T} X_{v_{2}}$ isometrically by identifying the copies of $\partial_{T} X_{e} \subset \partial_{T} X_{v_{i}}$; the result is isometric to $\cup_{v \in V} \partial_{T} X_{v} \subset \partial_{T} X$.
2. The union $\cup_{v \in V} \partial_{T} X_{v}$ forms a connected component of $\partial_{T} X$. The remaining components are contained in the subsets $\partial_{T}^{\eta} X$ for $\eta \in \partial_{\infty} T$. We will show in Lemma 7.3 that each $\partial_{T}^{\eta} X$ is either a point or isometric to an interval of length $<\pi$.

Proof. To prove 1, we first observe that if $e_{1}, \ldots, e_{n}$ is an edge path in $T$ with initial vertex $v_{1}$ and terminal vertex $v_{n}, \xi_{1} \in \partial_{T} X_{v_{1}}, \xi_{n} \in \partial_{T} X_{v_{n}}$, and $\angle_{T}\left(\xi_{1}, \xi_{n}\right)<\pi$, then the Tits segment $\overline{\xi_{1} \xi_{n}} \subset \partial_{T} X$ is contained in $\cup_{i=1}^{n} \partial_{T} X_{v_{i}}$. To see this, pick $\xi \in \overline{\xi_{1} \xi_{n}}$. Recall that for a given base point $x, \overline{x \xi}$ may be obtained as the limit of a sequence $\overline{x y_{k}}$ where $y_{k}$ lies on a segment $\overline{x_{1}^{k} x_{2}^{k}}$ and $x_{i}^{k} \in \overline{x \xi_{i}}$ is a sequence tending to infinity. The quasi-convexity property of Lemma 4.3 (applied in sucession to $\overline{x \xi_{i}}, \overline{x_{1}^{k} x_{2}^{k}}$, and then $\left.\overline{x y_{k}}\right)$ implies that $\overline{x y_{k}} \subset N_{C}\left(\cup_{i=1}^{n} X_{v_{i}}\right)$ for some $C$, and the convexity of the $N_{C}\left(X_{v_{i}}\right)$ 's implies that $\xi \in \partial_{T} X_{v_{i}}$ for some $i \in\{1, \ldots n\}$. Part 1 now follows from Corollary 3.16.

Before proving 2, we recall that open balls of radius $\frac{\pi}{2}$ in $C A T(1)$ spaces are geodesically convex, so two points in a $C A T(1)$ space belong to the same connected component iff they can be joined by a unit speed path.

Suppose $\eta \in \partial_{\infty} T$ and $\xi \in \partial_{T}^{\eta} X$. Fix $v \in V, p \in X_{v}$, and let $e_{k} \in E$ denote the $k^{t h}$ edge of the ray $\overline{v \eta} \subset T$. Part 1 of Lemma 3.19 implies that $\overline{p \xi} \cap X_{e_{k}} \neq \emptyset$ for all but finitely many $k$. If $c:[0, L] \rightarrow \partial_{T} X$ is a unit speed path starting at $\xi$, then by Lemma 2.4, we find that either $c([0, L]) \subset \partial_{T}^{\eta} X$ or for all sufficiently large $k$ there is a $t_{k} \in[0, L]$ so that $\overline{p c\left(t_{k}\right)} \in \partial_{T} X_{e_{k}}$. But part 1 shows that when $e, e^{\prime} \in E$ and $d\left(e, e^{\prime}\right) \geq 2$, then $d\left(\partial_{T} X_{e}, \partial_{T} X_{e^{\prime}}\right)$ is bounded away from 0 , which gives a contradiction. Therefore $c([0, L]) \subset \partial_{T}^{\eta} X$ and we have shown that the connected component of $\xi$ is contained in $\partial_{T}^{\eta} X$.

Finally, we note that $\cup_{v \in V} \partial_{T} X_{v}$ is connected: if $v_{1}, \ldots, v_{k}$ are the consecutive vertices of a geodesic segment in $T$, then $\partial_{T} X_{v_{i}} \cap \partial_{T} X_{v_{i+1}}=\partial_{T} X_{\overline{v_{i} v_{i+1}}} \neq \emptyset$.

## 6. Geometric data and equivariant quasi-isometries

Throughout this section, $G \stackrel{\rho}{\curvearrowright} X$ and $G \stackrel{\rho^{\prime}}{\curvearrowright} X^{\prime}$ will denote admissible actions of an admissible group $G$ on Hadamard spaces $X$ and $X^{\prime}$, and we let $M L S_{v}, \tau_{v}$ and $M L S_{v}^{\prime}, \tau_{v}^{\prime}$ denote their respective geometric data (see Definition 3.9). The main result in this section is Theorem 6.5, which shows that a $G$-equivariant quasi-isometry $\Phi: X \rightarrow X^{\prime}$ induces an equivariant homeomorphism $\partial_{\infty} \Phi: \partial_{\infty} X \rightarrow \partial_{\infty} X^{\prime}$ provided $\rho$ and $\rho^{\prime}$ have equivalent geometric data.
Definition 6.1. We say that $\rho$ and $\rho^{\prime}$ have equivalent geometric data if there are functions $\lambda: V \rightarrow \mathbb{R}_{+}$and $\mu: V \rightarrow \mathbb{R}_{+}$so that for every $v \in V$

$$
M L S_{v}^{\prime}=\lambda(v) M L S_{v} \quad \text { and } \quad \tau_{v}^{\prime}=\mu(v) \tau_{v}
$$

It follows from the $G$-invariance of the geometric data that $\lambda$ and $\mu$ will be $G$-invariant.
The structure of $G$ strongly restricts the possibilities for the functions $\lambda$ and $\mu$ :
Lemma 6.2. Suppose the geometric data for the actions $G \curvearrowright X$ and $G \curvearrowright X^{\prime}$ are equivalent, and let $\lambda: V \rightarrow \mathbb{R}_{+}$and $\mu: V \rightarrow \mathbb{R}_{+}$be as in Definition 6.1. Then either 1. $\lambda \equiv \mu \equiv a$, for some $a>0$.
or
2. There are constants $a$ and $b, a \neq b$, so that $\lambda(V)=\{a, b\}=\mu(V)$. Moreover, for any pair $v_{1}, v_{2}$ of adjacent vertices in $T, \lambda\left(v_{1}\right)=\mu\left(v_{2}\right), \lambda\left(v_{2}\right)=\mu\left(v_{1}\right)$, and the $\mathbb{R}$ directions of $Y_{v_{i}} \simeq \bar{Y}_{v_{i}} \times \mathbb{R}$ determine orthogonal directions in $Y_{e}$, where $e:=\overline{v_{1} v_{2}}$. In particular, there is a G-equivariant 2-coloring of $V$ (i.e. a two coloring of the finite graph $\mathcal{G}=T / G)$ such that $\lambda$ and $\mu$ are functions of the vertex color.

Proof. Pick $e=\overline{v_{1} v_{2}} \in E$, and consider $G_{e} \otimes \mathbb{R} \simeq \mathbb{Z}^{2} \otimes \mathbb{R} \simeq \mathbb{R}^{2}$. For $i=1,2$ we have subspaces $Z_{i}:=\left(Z\left(G_{v_{i}}\right) \cap G_{e}\right) \otimes \mathbb{R} \simeq \mathbb{Z} \otimes \mathbb{R} \simeq \mathbb{R}$ determined by the centers of the $G_{v_{i}}$ 's. The action $G_{e} \curvearrowright Y_{e}$ induces an inner product $\langle\cdot, \cdot\rangle$ on $G_{e} \otimes \mathbb{R}$ by letting, for $g \in G_{e},\langle g, g\rangle=\delta_{g}^{2}$. (For $p \in Y_{e}$ there is an embedded Euclidean plane $\mathbb{R}^{2} \subset Y_{e}$ invariant under the action of $G_{e} \curvearrowright Y_{e}$ on which $G_{e}$ acts by translations of $\delta_{g}$. So our metric is naturally related to the metric on this $\mathbb{R}^{2}$ ). We can extend the maps


Figure 1: On $G_{e} \otimes \mathbb{R} \simeq \mathbb{R}^{2}$ we have two metrics $<\cdot, \cdot>$ and $<\cdot, \cdot>^{\prime}$. $\operatorname{Ker}\left(\hat{\tau}_{i}\right)$ (which is the same as $\left.\operatorname{Ker}\left(\hat{\tau}_{i}^{\prime}\right)\right)$ for $i \in\{1,2\}$ is perpendicular to $Z_{i}$ with respect to both metrics.
$\tau_{i}: G_{e} \rightarrow R$ to linear maps $\hat{\tau}_{i}: G_{e} \otimes \mathbb{R} \rightarrow R$. For $g \in G_{e}$ we see that $\tau_{i}(g)$ is just the change in the $i$-vertical component from $y$ to $g(y)$. So for $x \in G_{e} \otimes \mathbb{R}, \hat{\tau}_{i}(x)$ is just the length of the orthogonal projection of $x$ onto $Z_{i}$. In particular, for $x \in Z_{i}$ we have $\langle x, x\rangle=\hat{\tau}_{i}(x)^{2}$, and $\operatorname{Ker}\left(\hat{\tau}_{i}\right)$ is $\langle\cdot, \cdot\rangle$ perpendicular to $Z_{i}$. Similarly $M L S_{i}(x)$ corresponds to the length of the projection of $x$ perpendicular to $Z_{i}$ (i.e. to $\operatorname{Ker}\left(\tau_{i}\right)$ ). In general $Z_{1}$ and $Z_{2}$ are not perpendicular but they are always linearly independent (see figure 1.

Similarly, using the action $G_{e} \curvearrowright Y_{e}^{\prime}$, we get an induced inner product $\langle\cdot, \cdot \cdot\rangle^{\prime}$ and linear maps $\hat{\tau}_{1}^{\prime}$ and $\hat{\tau}_{2}^{\prime}$. By assumption $\hat{\tau}_{i}^{\prime}=\mu\left(v_{i}\right) \hat{\tau}_{i}$, hence $\operatorname{Ker}\left(\hat{\tau}_{i}\right)=\operatorname{Ker}\left(\hat{\tau}_{i}^{\prime}\right)$. So the space perpendicular to $Z_{1}$ (resp. $Z_{2}$ ) with respect to $\langle\cdot, \cdot\rangle$ is the same as that with respect to $\langle\cdot, \cdot\rangle^{\prime}$. Hence, by the independence of $Z_{1}$ and $Z_{2}$, either $\langle\cdot, \cdot\rangle^{\prime}=a\langle\cdot, \cdot\rangle$ for some $a>0$ or $Z_{1}$ and $Z_{2}$ are perpendicular in both metrics so $\operatorname{Ker}\left(\hat{\tau}_{1}\right)=\operatorname{Ker}\left(\hat{\tau}_{1}^{\prime}\right)=$ $Z_{2}$ and $\operatorname{Ker}\left(\hat{\tau}_{2}\right)=\operatorname{Ker}\left(\hat{\tau}_{2}^{\prime}\right)=Z_{1}$. In the latter case, choosing $x \in Z_{2}$ we have $\langle x, x\rangle^{\prime}=\left(M L S_{1}(x)\right)^{2}=\mu\left(v_{2}\right)^{2}\left(M \hat{L} S_{1}(x)\right)^{2}=\mu\left(v_{2}\right)^{2}\langle x, x\rangle$ and $\langle x, x\rangle^{\prime}=\left(\tau_{2}^{\prime}(x)\right)^{2}=$ $\lambda\left(v_{1}\right)^{2}\left(\tau_{2}(x)\right)^{2}=\lambda\left(v_{1}\right)^{2}\langle x, x\rangle$ so $\lambda\left(v_{1}\right)=\mu\left(v_{2}\right) ;$ similarly $\lambda\left(v_{2}\right)=\mu\left(v_{1}\right)$.

We now fix a $G$-equivariant $(L, A)$-quasi-isometry $\Phi: X \rightarrow X^{\prime}$ for the rest of this section. The constants defined will depend on the geometry of $G \stackrel{\rho}{\curvearrowright} X$ and $G \stackrel{\rho^{\prime}}{\curvearrowright} X^{\prime}$ as well as any other explicitly stated quantities.
Lemma 6.3. There is a constant $D_{1}=D_{1}(L, A)$ so that for every $\sigma \in V \cup E$, the Hausdorff distance $d_{H}\left(\Phi\left(X_{\sigma}\right), X_{\sigma}^{\prime}\right)$ is at most $D_{1}$.

Proof. For $g \in G$ let $d_{g}$ and $d_{g}^{\prime}$ denote the displacement functions for $g$ in $X$ and $X^{\prime}$ respectively. In particular $d_{g}^{\prime}(x) \leq L d_{g}(x)+A$.

For any $v \in V$, if $g \in Z\left(G_{v}\right)$ is a generator of the center $Z\left(G_{v}\right)$ then (see section 2.3) $d_{g}$ is proper on $X / Z\left(Z\left(G_{v}\right), G\right)=X / G_{v}$ (by Lemma 3.6) and hence $d_{g}$ (resp $\left.d_{g}^{\prime}\right)$ grows with the distance from $X_{v}\left(\operatorname{resp} X_{v}^{\prime}\right)$. This (along with the fact that there are only finitely many $\sigma$ modulo $G$ ) implies the lemma when $\sigma \in V$. If $e \in E$, and $g_{1}, g_{2} \in G_{e}$ are a basis for $G_{e}$ then $\max \left\{d_{g_{1}}, d_{g_{2}}\right\}$ (resp $\max \left\{d_{g_{1}}^{\prime}, d_{g_{2}}^{\prime}\right\}$ ) grows with the distance from $X_{e}\left(\operatorname{resp} X_{e}^{\prime}\right)$. This implies the lemma when $\sigma \in E$.

We now assume for the remainder of this section that $\rho$ and $\rho^{\prime}$ have equivalent geometric data, and we let $\lambda: V \rightarrow \mathbb{R}_{+}$and $\mu: V \rightarrow \mathbb{R}_{+}$be as in Definition 6.1. For each $v \in V$, we have nearest point projections $p_{v}: X \rightarrow Y_{v}$ and $p_{v}^{\prime}: X^{\prime} \rightarrow Y_{v}^{\prime}$. Modulo
renormalization of the metrics on the spaces $Y_{v}$, the equivariant quasi-isometry $\Phi$ restricts to a Hausdorff approximation:
Lemma 6.4. For every $v \in V$, the $G_{v}$-equivariant quasi-isometry $\Phi_{v}:=\left.p_{v}^{\prime} \circ \Phi\right|_{Y_{v}}$ : $Y_{v} \rightarrow Y_{v}^{\prime}$ has the following properties:

1. It is at distance $<D_{2}=D_{2}(L, A)$ from a $G_{v}$-equivariant map $\Psi_{v}: Y_{v} \rightarrow Y_{v}^{\prime}$ which respects the product structures $Y_{v} \simeq \bar{Y}_{v} \times \mathbb{R}$ and $Y_{v}^{\prime} \simeq \bar{Y}_{v}^{\prime} \times \mathbb{R}$.
2. If we stretch the metric on the $\bar{Y}_{v}$ factor of $Y_{v}$ by $\lambda(v)$, and the metric on the $\mathbb{R}$ factor by $\mu(v)$, then $\Phi_{v}$ becomes a $D_{3}=D_{3}(L, A)$-Hausdorff approximation, and maps unit speed geodesic segments to within $D_{3}$ of unit speed geodesic segments.

Proof. We first prove part 1. The $\mathbb{R}$ fibers of $Y_{v}$ and $Y_{v}^{\prime}$ are within uniform Hausdorff distance of $Z\left(G_{v}\right)$-orbits, so $G_{v}$-equivariance implies that $\Phi_{v}$ takes $\mathbb{R}$ fibers of $Y_{v}$ to within uniform Hausdorff distance (say $C_{1}$ ) of $\mathbb{R}$ fibers of $Y_{v}^{\prime}$. The $\bar{Y}_{v}$ fibers of $Y_{v}$ are within uniform Hausdorff distance of sets of the form $\left\{g(p)\left|p \in Y_{v}, g \in G_{v},\left|\tau_{v}(g)\right|<C\right\}\right.$ for sufficiently large $C$, and a similar characterization of the $\bar{Y}_{v}^{\prime}$ fibers of $Y_{v}^{\prime}$ holds. We now define the product $\operatorname{map} \Psi_{v}=\bar{\Psi}_{v} \times \Psi_{v}^{\mathbb{R}}: \bar{Y}_{v} \times \mathbb{R} \rightarrow \bar{Y}_{v}^{\prime} \times \mathbb{R}$. Fix a basepoint $p \in Y_{v}$. We may assume that the $\mathbb{R}$ factor of $p$ and $\Phi_{v}(p)$ are 0 and take $\Psi_{v}^{\mathbb{R}}(t)=\mu(v) t$. We let $S \subset \bar{Y}_{v}$ be a (set theoretical) cross section for the $H_{v}$ action. For $s \in S$ choose a $g \in G_{v}$ such that $\left|\tau_{v}(g)\right|<C$ and $d(g(p),(s, 0)) \leq C_{1}$. Now, since $\left|\tau_{v}^{\prime}(g)\right|<\mu(v) C$, we can choose a point $\bar{\Psi}_{v}(s)$ such that $d\left(g(\Phi(p)),\left(\bar{\Psi}_{v}(s), 0\right)\right) \leq \mu(v) C$. We note that

$$
d(\Psi((s, 0)), \Phi((s, 0))) \leq d\left(\left(\bar{\Psi}_{v}(s), 0\right), \Phi(g(p))\right)+L C_{1}+A \leq \mu(v) C+L C_{1}+A
$$

Extend this to an $H_{v}$ equivariant map $\bar{\Psi}_{v}: \bar{Y}_{v} \rightarrow \bar{Y}_{v}^{\prime}$. Thus, along with the fact that $\tau^{\prime}(g)=\mu(v) \tau(g)$, we see that $\Psi_{v}$ is a $G_{v}$-equivariant map. Now for every $q \in Y_{v}$ there is a $g \in G_{v}$ such that $g(q)=(s, t)$ for some $s \in S$ and some $-C<t<C$. Now $d(\Psi(q), \Phi(q))=$

$$
d(\Psi((s, t)), \Phi((s, t))) \leq 2 C \mu(v)+d(\Psi((s, 0)), \Phi((s, 0))) \leq 3 \mu(v) C+L C_{1}+A
$$

This proves 1.
Part 2 follows from part 1 if we can show that $\bar{\Psi}_{v}$ and $\Psi_{v}^{\mathbb{R}}$ carry unit speed geodesics to within uniform distance of unit speed geodesics. The map $\Psi_{v}^{\mathbb{R}}$ clearly does, because the translation distance in the $\mathbb{R}$-direction is measured by $\tau_{v}$ (resp. $\left.\tau_{v}^{\prime}\right)$ and these have ratio $\mu(v)$. Recall that $H_{v}:=G_{v} / Z\left(G_{v}\right)$ acts discretely and cocompactly on the hyperbolic metric spaces $\bar{Y}_{v}$ and $\bar{Y}_{v}^{\prime}$. Since $M L S_{v}^{\prime}=\lambda(v) M L S_{v}$, if we renormalize the metric on $\bar{Y}_{v}$ by $\lambda(v)$ we can apply Lemma 2.20 to see that $\bar{\Psi}_{v}: \bar{Y}_{v} \rightarrow \bar{Y}_{v}^{\prime}$ preserves unit speed geodesics up to uniform error. This proves 2.

We may now use our quasi-isometry $\Phi: X \rightarrow X^{\prime}$ to transport standard $K$ templates for $X$ to templates for $X^{\prime}$ (see section 4.2). Start with a standard $K$ template $(\mathcal{T}, f, \phi)$ for some segment or ray $\gamma \subset T$. Recall that the walls of $\mathcal{T}$ come from flats $F_{e} \subset Y_{e} \subset X_{e}$. To produce the new template $\mathcal{T}^{\prime}$ distort the metric on $\mathcal{T}$ by an affine change as follows. For each $v \in V$, we think of scaling the metric on $\bar{Y}_{v}$ by $\lambda(v)$ and the $\mathbb{R}$-factor of $Y_{v}$ by $\mu(v)$; and then we distort the flats $F_{e}=F_{\overline{v v^{\prime}}} \subset F_{v} \cap F_{v^{\prime}}$ and strips $\mathcal{S}_{e e^{\prime}} \subset Y_{e \cap e^{\prime}}$ used to build $\mathcal{T}$ accordingly. Lemmas 6.3 and 6.4 imply that $\left(\mathcal{T}^{\prime}, f, \Phi \circ \phi\right)$ is a $K^{\prime}$-template for $\gamma$ where $K^{\prime}=K^{\prime}(L, A)$. Notice (using Lemma 6.2) that the identity map $\mathcal{T} \rightarrow \mathcal{T}^{\prime}$ is an affine map (i.e. maps constant speed geodesics to constant speed geodesics), and is a homothety when $\lambda=\mu=a \in \mathbb{R}$.

Theorem 6.5. Let $G \stackrel{\rho}{\curvearrowright} X$ and $G \stackrel{\rho^{\prime}}{\curvearrowright} X^{\prime}$ be admissible actions of an admissible group $G$ on Hadamard spaces $X$ and $X^{\prime}$ such that $\rho$ and $\rho^{\prime}$ have equivalent geometric data, and let $\Phi: X \rightarrow X^{\prime}$ be a $G$-equivariant $(L, A)$-quasi-isometry. Then there is a function $\theta: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$(depending on $K, L, A$ and the geometry of $X$ and $X$ ') with $\lim _{r \rightarrow \infty} \theta(r)=0$ so that for every $x, y \in X, z \in \overline{x y}$, we have

$$
\begin{equation*}
d_{X^{\prime}}(\Phi(z), \overline{\Phi(x) \Phi(y)}) \leq\left(1+d_{X}(z, x)\right) \theta\left(d_{X}(z, x)\right) \tag{6.6}
\end{equation*}
$$

Consequently, by Lemma 2.5, $\Phi$ extends to a unique map $\bar{\Phi}: \bar{X} \rightarrow \bar{X}^{\prime}$ which is continuous at $\partial_{\infty} X \subset \bar{X}$. Setting $\partial_{\infty} \Phi:=\left.\bar{\Phi}\right|_{\partial_{\infty} X}: \partial_{\infty} X \rightarrow \partial_{\infty} X^{\prime}$, we obtain a $G$ equivariant homeomorphism. If $\lambda=\mu=a \in \mathbb{R}$, then $\partial_{\infty} \Phi$ is an isometry with respect to Tits metrics.

Proof. Pick $x, y \in X$, and then find a segment $\gamma \subset T$ so that $x \in X_{e_{1}}, y \in X_{e_{n}}$, and the $i^{\text {th }}$ edge of $\gamma$ is $e_{i}$. Now let $(\mathcal{T}, f, \phi)$ be the standard $K$-template for $\gamma$, and define the $K^{\prime}$-template ( $\mathcal{T}^{\prime}, f, \phi^{\prime}$ ) as in the paragraph preceding the statement of the Theorem. We may assume, after moving $x$ and $y$ a uniformly bounded distance if necessary (see part 3 of definition 4.4), that $x=\phi\left(x_{1}\right), y=\phi\left(y_{1}\right)$ for some $x_{1}, y_{1} \in \mathcal{T}$. We get (6.6) by applying Theorem 5.1 twice - once to ( $\mathcal{T}, f, \phi$ ) and once to ( $\mathcal{T}^{\prime}, f, \phi^{\prime}$ ). Specifically, Since $z \in \overline{\phi\left(x_{1}\right) \phi\left(y_{1}\right)}$ an application of Theorem 5.1 part 2 gives

$$
d_{X}\left(z, \phi\left(\overline{x_{1} y_{1}}\right)\right) \leq\left(1+d_{X}(z, x)\right) \theta_{(X, K)}\left(d_{X}(z, x)\right)
$$

So there is a $z_{1} \in \overline{x_{1} y_{1}}$ with $d_{X}\left(z, \phi\left(z_{1}\right)\right) \leq\left(1+d_{X}(z, x)\right) \theta_{(X, K)}\left(d_{X}(z, x)\right)$. Hence we see

$$
d_{X^{\prime}}\left(\Phi(z), \phi^{\prime}\left(z_{1}\right)\right) \leq L\left(1+d_{X}(z, x)\right) \theta_{(X, K)}\left(d_{X}(z, x)\right)+A .
$$

Now an application of Theorem 5.1 part 1 to $\phi^{\prime}$ gives
$d_{X^{\prime}}\left(\phi^{\prime}\left(z_{1}\right), \overline{\Phi(x) \Phi(y)}\right)=d_{X^{\prime}}\left(\phi^{\prime}\left(z_{1}\right), \overline{\phi^{\prime}\left(x_{1}\right) \phi^{\prime}\left(y_{1}\right)}\right) \leq\left(1+d_{\mathcal{T}^{\prime}}\left(x_{1}, z_{1}\right)\right) \theta_{\left(X^{\prime}, K^{\prime}\right)}\left(d_{\mathcal{T}^{\prime}}\left(x_{1}, z_{1}\right)\right)$.
We thus need to bound $d_{\mathcal{T}^{\prime}}\left(x_{1}, z_{1}\right)$ linearly from above by $d_{X}(z, x)$. But this follows since $d_{\mathcal{T}^{\prime}}\left(x_{1}, z_{1}\right) \leq L^{\prime} d_{X^{\prime}}\left(\phi^{\prime}\left(x_{1}\right), \phi^{\prime}\left(z_{1}\right)\right)+A^{\prime}$ (where $L^{\prime}$ and $A^{\prime}$ come from Proposition 4.6 and depend only on $K^{\prime}$ and the geometry of $X^{\prime}$, $d_{X^{\prime}}\left(\phi^{\prime}\left(x_{1}\right), \phi^{\prime}\left(z_{1}\right)\right) \leq$ $L d_{X}\left(\phi\left(x_{1}\right), \phi\left(z_{1}\right)\right)+A$, and $d_{X}\left(\phi\left(x_{1}\right), \phi\left(z_{1}\right)\right) \leq d_{X}(x, z)+d_{X}\left(z, \phi\left(z_{1}\right)\right) \leq d_{X}(x, z)+$ $\left(1+d_{X}(z, x)\right) \theta_{(X, K)}\left(d_{X}(z, x)\right)$. Equation (6.6) will thus follow for an appropriate choice $\theta_{\left(K, L, A, X, X^{\prime}\right)}$ that will depend only on $L, A, K$, and the geometry of $X$ and $X^{\prime}$. Lemma 2.5 then applies to $\Phi$, so we get an induced embedding $\partial_{\infty} \Phi: \partial_{\infty} X \rightarrow \partial_{\infty} X^{\prime}$ which is automatically $G$-equivariant.

Now suppose $\lambda=\mu=a$. Pick $\eta \in \partial_{\infty} T$, and a geodesic ray $\gamma \subset T$ with $\partial_{\infty} \gamma=\eta$. Then the identity map $\mathcal{T} \rightarrow \mathcal{T}^{\prime}$ between the associated templates $(\mathcal{T}, f, \phi)$ and $\left(\mathcal{T}^{\prime}, f, \phi^{\prime}\right)$ constructed as above is a homothety, and so part 5 of Theorem 5.1 applied to $\left.\partial_{\infty} \phi\right|_{\partial_{\infty}^{\infty} \mathcal{T}}$ and $\left.\partial_{\infty} \phi^{\prime}\right|_{\partial_{\infty}^{\infty} \mathcal{T}^{\prime}}$ then shows that $\left.\partial_{\infty} \Phi\right|_{\partial_{\infty}^{n} X}$ induces an isometry $\partial_{T}^{\eta} X \rightarrow \partial_{T}^{\eta} X^{\prime}$. By Corollary 5.29 it remains only to show that $\partial_{\infty} \Phi$ induces an isometry $\partial_{T} X_{v} \rightarrow \partial_{T} X_{v}^{\prime}$. But Lemma 6.4 part 2 implies that $\Phi_{v}: Y_{v} \rightarrow Y_{v}^{\prime}$ is at finite distance from a product of Hausdorff approximations (up to rescaling of $X^{\prime}$ by $\frac{1}{a}$ ) and so $\Phi_{v}$ induces an isometry $\partial_{T} X_{v} \rightarrow \partial_{T} X_{v}^{\prime}$.

## 7. Recovering the geometric data from the action on the ideal boundary

In this section we will prove the remaining implication of Theorem 1.3: for any admissible action $G \curvearrowright X$, the topological conjugacy class of the action $G \curvearrowright \partial_{\infty} X$ determines the functions $M L S_{v}: G_{v} \rightarrow \mathbb{R}_{+}$and $\tau_{v}: G_{v} \rightarrow \mathbb{R}$ up to a multiplicative factor, for every vertex $v \in \mathcal{G}$. Our strategy for proving this is as follows. Using Lemma 3.23 and Corollary 5.26, for any ideal boundary point $\eta$ of the Bass-Serre tree $T$, we may detect the subset $\partial_{\infty}^{\eta} X \subset \partial_{\infty} X$; specifically, the action $G \curvearrowright \partial_{\infty} X$ determines the set of boundary points $\eta \in \partial_{\infty} T$ for which $\left|\partial_{\infty}^{\eta} X\right|=1$. To extract useful information from this, we consider a special class of geodesic rays $\gamma \subset T$ (see Definition 7.11 ) which admit templates $(\mathcal{T}, f, \phi)$ where $\mathcal{T}$ is asymptotically selfsimilar: there is a (non-surjective) map $\mathcal{T} \rightarrow \mathcal{T}$ which stretches distances by a factor of 2 . These templates have two key properties: their geometry relates directly to the geometric data $M L S_{v}$ and $\tau_{v}$, and at the same time we can tell explicitly when (in terms of the geometry) we have $\left|\partial_{\infty}^{\partial_{\infty} \gamma} \mathcal{T}\right|=1$ (see section 7.2). Putting all this together we able to recover the geometric data from the action $G \curvearrowright \partial_{\infty} X$.

## 7.1. $\partial_{T}^{\infty} \mathcal{T}$ is either a point or an interval

Lemma 7.1. Let $X$ be a Hadamard space, $p \in X$, and $\gamma_{i} \subset X$ a sequence of geodesics with $\lim _{i \rightarrow \infty} d\left(p, \gamma_{i}\right)=\infty$. Set

$$
S:=\left\{\xi \in \partial_{T} X \mid \overline{p \xi} \cap \gamma_{i} \neq \emptyset \text { for all } i\right\}
$$

Then $S$ embeds isometrically in the interval $[0, \pi]$.
Proof. Pick $x, y, z \in S$, and for each $i$ choose $x_{i} \in \overline{p x} \cap \gamma_{i}, y_{i} \in \overline{p y} \cap \gamma_{i}, z_{i} \in \overline{p z} \cap \gamma_{i}$. After passing to a subsequence and reordering $x, y, z$ we may assume that $y_{i}$ lies between $x_{i}$ and $z_{i}$ on $\gamma_{i}$, or that it coincides with $x_{i}$ or $z_{i}$. Then it follows that

$$
\tilde{乙}_{p}\left(x_{i}, y_{i}\right)+\tilde{乙}_{p}\left(y_{i}, z_{i}\right) \leq \tilde{Z}_{p}\left(x_{i}, z_{i}\right) .
$$

Taking the limit as $i \rightarrow \infty$, we get

$$
\angle_{T}(x, y)+\angle_{T}(y, z) \leq \angle_{T}(x, z) .
$$

Hence $\left(\{x, y, z\}, \angle_{T}\right) \subset \partial_{T} X$ embeds isometrically in $[0, \pi]$.
If $|S|=1$ the lemma is immediate, so assume $|S| \geq 2$, and construct a map $f: S \rightarrow \mathbb{R}$ as follows. Pick distinct points $x_{0}, x_{1} \in S$ and for $i=0,1$ choose $f\left(x_{i}\right) \in \mathbb{R}$ so that $d\left(x_{0}, x_{1}\right)=d\left(f\left(x_{0}\right), f\left(x_{1}\right)\right)$. Now define $f$ uniquely by the condition that $d\left(f(s), f\left(x_{i}\right)\right)=d\left(s, x_{i}\right)$ for all $s \in S$ and $i=0,1$. Clearly $f$ is an isometric embedding, and its image lies in an interval of length at most $\pi$.

Lemma 7.2. Let $\mathcal{T}$ be a half template with walls $W_{0}, W_{1}, \ldots$ For $i \geq 1$ set $\alpha_{i}:=$ $\alpha\left(W_{i}\right)$ where $\alpha: W$ all $\mathcal{T}_{\mathcal{T}}^{0} \rightarrow(0, \pi)$ is the angle function, and assume $\min \left\{\alpha_{i}, \pi-\alpha_{i}\right\} \geq$ $\beta$ for all $i \geq 1$. Then

1. The diameter of $\partial_{T}^{\infty} \mathcal{T}$ with respect to the Tits metric is at most $\pi-\beta$.
2. For $p \in W_{0}$ and every $i>0$ there is an $R_{i}$, depending only on $\beta$ and $d\left(o_{i}, p\right)$, so that if $q \in L_{i}^{+}$then $\overline{p q} \cap L_{i}^{-} \subset B\left(o_{i}, R_{i}\right)$.


Figure 2: On the wall $W_{i}$ the angle between $L_{i}^{-}$and $L_{i}^{+}$is $\geq \beta$ hence the angle at $x_{i}^{-}$ is $\leq \pi-\beta$. The angle at $y_{i}^{-}$is always $\leq \pi$ (no mater where $y_{i}^{+}$lies on $L_{i}^{+}$).

Proof. Pick $p \in W_{0} \subset \mathcal{T}$, and distinct points $x, y \in \partial_{T}^{\infty} \mathcal{T}$. Let $x_{i}^{-} \in L_{i}^{-}$(resp. $x_{i}^{+} \in L_{i}^{+}$) be the point where the ray $\overline{p x}$ enters (resp. exits) the wall $W_{i}$; define $y_{i}^{-}, y_{i}^{+}$similarly. Since $x \neq y$, there is an $i_{0}>0$ so that $x_{i}^{-} \neq y_{i}^{-}$when $i \geq i_{0}$. Pick $i \geq i_{0}$, and assume that $d\left(x_{i}^{-}, o_{i}\right) \geq d\left(y_{i}^{-}, o_{i}\right)$ (the case when $d\left(y_{i}^{-}, o_{i}\right) \geq d\left(x_{i}^{-}, o_{i}\right)$ is similar). Clearly (see figure 2) we have

$$
\angle_{x_{i}^{-}}\left(x_{i}^{+}, o_{i}\right)=\angle_{x_{i}^{-}}\left(x_{i}^{+}, y_{i}^{-}\right) \leq \max \left\{\alpha_{i}, \pi-\alpha_{i}\right\} \leq \pi-\beta .
$$

Now this implies 1 since by standard properties of Tits angles

$$
\angle_{T}(x, y)=\left[\lim _{i \rightarrow \infty}\left(\angle_{x_{i}^{-}}\left(y_{i}^{-}, x\right)+\angle_{y_{i}^{-}}\left(x_{i}^{-}, y\right)\right)\right]-\pi \leq \max \left\{\alpha_{i}, \pi-\alpha_{i}\right\} \leq \pi-\beta .
$$

It also implies 2 by applying triangle comparison to the angle at $x_{i}$ of the triangle with vertices $p, o_{i}$, and $x_{i}$.

Proposition 7.3. Let $\mathcal{T}$ be a uniform half template with $\min \left\{\alpha_{i}, \pi-\alpha_{i}\right\} \geq \beta$ for all i. Then $\partial_{T}^{\infty} \mathcal{T}$ is isometric to an interval $[0, \theta]$ where $\theta \in[0, \beta]$.

Proof. Applying Lemma 7.1 with $\gamma_{i}=L_{i}^{-}$and Lemma 7.2 we get that $\partial_{T}^{\infty} \mathcal{T}$ is isometric to a subset of $[0, \beta]$. We need only show that $\partial_{T}^{\infty} \mathcal{T}$ is connected. Suppose $x, y \in \partial_{T}^{\infty} \mathcal{T}$, and pick a point $z$ lying on the Tits interval $\overline{x y} \subset \partial_{T} \mathcal{T}$. Let $p, x_{i}^{-}, y_{i}^{-}$be as in the proof of Lemma 7.2. Then $\lim _{i \rightarrow \infty}{\widetilde{L_{p}}}_{p}\left(x_{i}^{-}, y_{i}^{-}\right)=\angle_{T}(x, y)$, so we can choose a sequence $z_{i} \in \overline{x_{i}^{-} y_{i}^{-}} \subset L_{i}^{-}$so that $\lim _{i \rightarrow \infty} \widetilde{L}_{p}\left(x_{i}^{-}, z_{i}\right)=\angle_{T}(x, z)$ and $\lim _{i \rightarrow \infty} \widetilde{Z}_{p}\left(z_{i}^{-}, y_{i}\right)=\angle_{T}(z, y)$. The segments $\overline{p z_{i}}$ converge to the ray $\overline{p z}$. Since for any $j>0$ the segment $\overline{p z_{i}}$ crosses $L_{j}^{+}$for sufficiently large $i$, by Lemma 7.2 part 2 we get

$$
\overline{p z_{i}} \cap L_{j}^{-} \subset B\left(o_{j}, R_{j}\right)
$$

and we conclude that $\overline{p z} \cap L_{j}^{-} \neq \emptyset$. Hence $z \in \partial_{T}^{\infty} \mathcal{T}$.
The $\theta$ in the above proposition is referred to as the Tits angle of $\mathcal{T}$ and is denoted $\theta(\mathcal{T})$.

### 7.2. Self-similar Templates

In this section we study a special class of full templates called self-similar templates.

Definition 7.4. Let $\mathcal{T}$ be a full template with $W a \operatorname{ll}_{\mathcal{T}}=\left\{W_{i}\right\}_{i \in \mathbb{Z}}$ and $\operatorname{Strip}_{\mathcal{T}}=$ $\left\{\mathcal{S}_{i}\right\}_{i \in \mathbb{Z}}$, and set $\alpha_{i}:=\alpha\left(W_{i}\right), l_{i}:=l\left(\mathcal{S}_{i}\right)$, and $\epsilon_{i}:=\epsilon\left(\mathcal{S}_{i}\right)$ (we define $\epsilon$ using the strip orientation compatible with the strip directions and the usual ordering on $\mathbb{Z}$ ). Then $\mathcal{T}$ is a self-similar template if for all $i, j \in \mathbb{Z}$ we have $\alpha_{i}=\alpha_{j}, l_{i+2 j}=2^{j} l_{i}$, and $\epsilon_{i+2 j}=2^{j} \epsilon_{i}$. In this case we say that $\mathcal{T}$ has data $\left(\beta ; l_{0}, \epsilon_{0}, l_{1}, \epsilon_{1}\right)$ where $\beta=\alpha_{i}$ for all $i \in \mathbb{Z}$.

Note that by the definition, if we rescale the metric on a self-similar template $\mathcal{T}$ by a factor of 2 , then we get a template equivalent to $\mathcal{T}$, i.e. there is a homothety $\Phi: \mathcal{T} \rightarrow \mathcal{T}$ which preserves strip directions, stretches distances by a factor of 2 , and which shifts wall and strip indices by 2 . A self-similar template is determined up to equivalence by $\beta$ and the data $\left\{l_{0}, \epsilon_{0}, l_{1}, \epsilon_{1}\right\}$.
$\mathcal{T}$ contains one point $v$ which is not on any wall or strip (since the union of the walls and strips is not complete). The point $v$ is the limit of the Cauchy sequence $\left\{o_{-i} \mid i=0,1,2, \ldots\right\}$. On the other hand, for each $j \in \mathbb{Z}$ the half template, $\mathcal{T}_{j}$, given by the union of planes $W_{i}$ and strips $\mathcal{S}_{i}$ for $i \geq j$ is uniform (and complete). The images of the embeddings $\partial_{\infty}^{\infty} \mathcal{T}_{i} \rightarrow \partial_{\infty} \mathcal{T}$ and $\partial_{T}^{\infty} \mathcal{T}_{i} \rightarrow \partial_{T} \mathcal{T}$ are independent of $i$, and we use $\partial_{\infty}^{\infty} \mathcal{T}$ (respectively $\partial_{T}^{\infty} \mathcal{T}$ ) to denote this common subspace. We will say that $\mathcal{T}$ is trivial if $\left|\partial_{\infty}^{\infty} \mathcal{T}\right|=1$ and nontrivial if $\left|\partial_{\infty}^{\infty} \mathcal{T}\right|>1$.

We note that if $r$ is a geodesic ray parameterized by arclength then $\Phi \circ r$ is a geodesic ray parameterized by twice arclength. So $\Phi$ and $\Phi^{-1}$ take rays to rays. Since $\Phi$ is a homothety it preserves the Tits angles between rays.

If $\mathcal{R}_{i}$ represents the space of geodesic rays starting at $o_{i}$ and intersecting $W_{j}$ for all $j \geq i$, then the above shows $\Phi^{-1}\left(\mathcal{R}_{i}\right)=\mathcal{R}_{i-2}$ and that $\Phi^{-1}$ acts as a Tits isometry on $\partial_{T}^{\infty} \mathcal{T}$. In particular since $\partial_{T}^{\infty} \mathcal{T}$ is isometric to an interval, $\Phi$ leaves the midpoint fixed, and $\Phi^{2}$ acts as the identity. Thus by repeated applications of $\Phi^{-1}$ we see that we can represent $\partial_{T}^{\infty} \mathcal{T}$ as the set rays, $\mathcal{R}_{-\infty}$, that start at $v$ and are invariant under $\Phi^{2}$. Further, the middle ray is invariant under $\Phi$.

We will show in the lemma below that all rays in $\mathcal{R}_{-\infty}$ are $\Phi$ invariant. In particular any such ray that intersects an $o_{i}$ must intersect all $o_{i+2 n}$ for $n \in \mathbb{Z}$, and hence there are at most two such rays, $r_{\text {even }}$ and $r_{\text {odd }}$, in $\mathcal{R}_{-\infty}$.

Each choice $N \in\{I, I I, I I I, I V\}$ determines a choice of quarter planes $Q_{i}^{N} \subset$ $W_{i}$ as follows: for all $k \in \mathbb{Z}, Q_{2 k}^{N}=Q_{N} \subset W_{2 k}$ while $Q_{2 k+1}^{N}=-Q_{N} \subset W_{2 k+1}$. Straightforward Euclidean geometry shows that the corresponding development map, $\mathcal{D}_{N}$, with $\mathcal{D}_{N}(v)=0$ has the property that $\mathcal{D}_{N} \circ \Phi \circ \mathcal{D}_{N}^{-1}$ is multiplication by 2 wherever (and however) it is defined (see Figure 3). (The easiest way to see this is to first develop $Q_{0}^{N}, \mathcal{S}_{0}, Q_{1}^{N}, \mathcal{S}_{1}$, and $Q_{2}^{N}$ into the plane, then shift the origin $(0,0)$ so that it lies on the line through $\mathcal{D}\left(o_{0}\right)$ and $\mathcal{D}\left(o_{2}\right)$ and such that $\mathcal{D}\left(o_{0}\right)$ lies between $(0,0)$ and $\mathcal{D}\left(o_{2}\right)$ and such that the distance from $(0,0)$ to $\mathcal{D}\left(o_{2}\right)$ is twice the distance to $\mathcal{D}\left(o_{0}\right)$. We note that $\mathcal{D}\left(Q_{2}^{N}\right)=2 \mathcal{D}\left(Q_{0}^{N}\right)$. We can now define a map $\mathcal{D}$ uniquely so that $\mathcal{D} \circ \Phi \circ \mathcal{D}^{-1}$ is multiplication by 2 . It is easy to check that this map is a development and hence by uniqueness is $\mathcal{D}_{N}$ - up to an element of $O(2)$.) There are rays $r_{\text {even }}$ and $r_{\text {odd }}$ from the origin such that $\mathcal{D}_{N}\left(o_{2 i}\right) \in r_{\text {even }}$ and $\mathcal{D}_{N}\left(o_{2 i+2}\right) \in r_{\text {odd }}$. (We can fix $\mathcal{D}_{N}$ completely if desired by taking $r_{\text {even }}$ to be the positive $x$-axis and to make $r_{\text {odd }}$ point in the upper half plane.)
Lemma 7.5. Let $\mathcal{T}$ be a nontrivial self-similar template. Then there is a choice


Figure 3: Development of a self-similar template from quarter planes $Q_{i}^{I}$. Developements of quadrants for nonnegative integers are shaded and labeled with their type, while those with negative indices are not shown. The direction of $\mathcal{S}_{5}$ and $\mathcal{S}_{6}$ are shown via arrows. The interval of rays between $r_{\text {even }}$ and $r_{\text {odd }}$ correspond to rays in the template.
$N \in\{I, I I, I I I, I V\}$ such that for $r \in \mathcal{R}_{-\infty}, r \cap W_{i} \subset Q_{i}^{N}$. Each $r \in \mathcal{R}_{-\infty}$ is $\Phi$ invariant, and $r_{\text {even }}$ and $r_{\text {odd }}$ are the $\mathcal{D}_{N}$ images of the boundary rays of $\mathcal{R}_{-\infty}$ as a closed interval. In particular the set of $r \in \mathcal{R}_{-\infty}$ span a space isometric, via $\mathcal{D}_{N}$ to a convex cone in $R^{2}$.

Proof. The fact that $\mathcal{R}_{-\infty}$ is a closed interval follows from Proposition 7.3. If $r_{1}, r_{2} \in$ $\mathcal{R}_{-\infty}$ then their invariance under the homothety $\Phi^{2}$ implies that $\angle_{T}\left(r_{1}, r_{2}\right)=\angle_{v}\left(r_{1}, r_{2}\right)$, and hence the rays between $r_{1}$ and $r_{2}$ span a space isometric to a convex cone in $R^{2}$. In particular, no ray between $r_{1}$ and $r_{2}$ can intersect an origin $o_{i}$. Thus the middle ( $\Phi$ invariant) ray $r_{m}$ determines $N, M \in\{I, I I, I I I, I V\}$ such that $r_{m} \cap W_{2 n} \subset Q_{N}$, and $r_{m} \cap W_{2 n+1} \subset Q_{M}$. The above says that for any interior ray, $r, r \cap W_{2 n} \subset Q_{N}$, and $r \cap W_{2 n+1} \subset Q_{M}$. Consider the development map $\mathcal{D}$ determined by this choice of quarter planes. We will assume that $\mathcal{D}(v)$ is the origin, so in particular $\mathcal{D}\left(Q_{i}\right)$ and $\mathcal{D}\left(\mathcal{S}_{i}\right)$ miss the origin. Now $\mathcal{D} \circ \Phi^{2} \circ \mathcal{D}^{-1}$ is defined when restricted to the image of each strip or quarter plane. It is clearly just multiplication by 4 , since that is what it does to each ray in the $\mathcal{D}$ image of $\mathcal{R}_{-\infty}$.

We assume that $N=I$ (the other cases are similar) and must prove that $M=I I I$. If $M=I I$ (resp. $M=I V$ ) then for $i=1$ (resp. $i=0$ ), $\mathcal{D}\left(Q_{i}\right) \cup \mathcal{D}\left(\mathcal{S}_{i}\right) \cup \mathcal{D}\left(Q_{i+1}\right)$ will contain a half plane which in turn contains one of $\mathcal{D}\left(Q_{i}\right)$ or $\mathcal{D}\left(Q_{i+1}\right)$. But this cannot happen since any ray entering that half plane will never leave it. If $M=I$ then for $k \geq \max \left\{\frac{\pi}{\beta}, \frac{\pi}{\pi-\beta}\right\}$ we have $\cup_{i=1}^{k}\left(\mathcal{D}\left(Q_{i}\right) \cup \mathcal{D} \mathcal{S}_{i}\right)$ contains a half plane which in turn contains one of the quarter planes and the same argument works. Thus we conclude that $M=I I I$, and $Q_{i}=Q_{i}^{N}$.

We know that $\mathcal{D}_{N} \circ \Phi \circ \mathcal{D}_{N}^{-1}$ is just multiplication by 2. This means in particular that all rays in $\mathcal{R}_{-\infty}$ are invariant under $\Phi$. Now since at least one ray between $r_{\text {even }}$


Figure 4: The graph of $A_{\beta}$ for $\beta=\frac{3 \pi}{5}$.
and $r_{\text {odd }}$ hits every $\mathcal{D}_{N}\left(Q_{i}\right)$ this is true of all such rays, hence all are $\mathcal{D}_{N}$ of a ray in $\mathcal{T}$. This gives a 1-1 correspondence between rays in $\mathcal{T}$ and rays in the plane between $r_{\text {even }}$ and $r_{\text {odd }}$ completing the lemma.

The argument in the last proof shows that a self-similar template is nontrivial if and only if there is an $N \in\{I, I I, I I I, I V\}$ such that some (and hence any) ray between the corresponding $r_{\text {even }}$ and $r_{\text {odd }}$ intersects every $\mathcal{D}_{N}\left(Q_{i}^{N}\right)$. But by selfsimilarity this will be true if and only if the line segment from $\mathcal{D}_{N}\left(o_{0}\right)$ to $\mathcal{D}_{N}\left(o_{2}\right)$ intersects $\mathcal{D}_{N}\left(Q_{1}^{N}\right)$. This leads to the following lemma:

We will use the notation $\mathbb{R}_{0}^{4}=\left\{\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \subset \mathbb{R}^{4} \mid x_{1}, x_{3}>0\right\}$.
Let $A_{\beta}$ be:
$\left\{\left.(x, y) \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\rvert\, x+\beta \geq y \geq x-\beta\right.$ and $\left.-x+(\pi-\beta) \geq y \geq-x-(\pi-\beta)\right\}$.
See Figure 4.
Lemma 7.6. Fix $\pi>\beta>0$ and let $\mathcal{A}_{\beta}$ be the set of $\left(l_{0}, \epsilon_{0}, l_{1}, \epsilon_{1}\right) \subset \mathbb{R}_{0}^{4}$ such that the self-similar template with data $\left(\beta ; l_{0}, \epsilon_{0}, l_{1}, \epsilon_{1}\right)$ is trivial. Then

$$
\mathcal{A}_{\beta}=\left\{\left(l_{0}, \epsilon_{0}, l_{1}, \epsilon_{1}\right) \in \mathbb{R}_{0}^{4} \left\lvert\,\left(\arctan \left(\frac{\epsilon_{0}}{l_{0}}\right), \arctan \left(\frac{\epsilon_{1}}{l_{1}}\right)\right) \in A_{\beta}\right.\right\}
$$

Proof. Let $\psi_{i}=\arctan \left(\frac{\epsilon_{i}}{l_{i}}\right)$. The proof follows once we show that $R^{4}-\mathcal{A}_{\beta}$ consists of 4 components given in order for $N=I, I I, I I I$, and $I V$ by:

$$
\psi_{0}>\psi_{1}+\beta, \quad-\psi_{1}-(\pi-\beta)>\psi_{0}, \quad \psi_{1}-\beta>\psi_{0}, \quad \text { and } \quad \psi_{0}>-\psi_{1}+(\pi-\beta) .
$$

Here we do the case $N=I$, i.e. the line segment from $\mathcal{D}_{I}\left(o_{0}\right)$ to $\mathcal{D}_{I}\left(o_{2}\right)$ intersects $\mathcal{D}_{I}\left(Q_{1}^{I}\right)$. The cases $N=I I, I I I$, and $I V$ are similar.

Let $\pi>\theta_{0}>0$ be the angle between the line segment from $\mathcal{D}_{I}\left(o_{1}\right)$ to $\mathcal{D}_{I}\left(o_{0}\right)$ and the "incoming" edge of $\mathcal{D}_{I}\left(Q_{1}^{I}\right)$ (see Figure 5). Here the "incoming" edge is the $\mathcal{D}_{I}$ image of the negative half line of $L_{1}^{-}$(since $Q_{1}^{I}$ is of type $\left.I I I\right)$. Similarly let $\theta_{2}$ be the angle between the line segment from $\mathcal{D}_{I}\left(o_{1}\right)$ to $\mathcal{D}_{I}\left(o_{2}\right)$ and the "outgoing" edge of $\mathcal{D}_{I}\left(Q_{1}^{I}\right)$. $\theta_{1}$ will be the angle at $\mathcal{D}_{I}\left(o_{1}\right)$ of the sector $\mathcal{D}_{I}\left(Q_{1}^{I}\right)$ (which in our case is just $\beta$ ). It is easy to see that our condition is equivalent to

$$
\pi>\theta_{0}+\theta_{1}+\theta_{2} .
$$



Figure 5: Developement in case I; $\epsilon_{0}>0, \epsilon_{1}<0$.

The rest of the argument is just that the definitions (being careful about how the sign of $\epsilon_{i}$ and the orientation of the strips interact) give us, when $N=I$,

$$
\theta_{0}=\frac{\pi}{2}-\arctan \left(\frac{\epsilon_{0}}{l_{0}}\right) \quad \text { and } \quad \theta_{2}=\frac{\pi}{2}-\arctan \left(\frac{-\epsilon_{1}}{l_{1}}\right)
$$

which completes the argument.
We extract the information we need with the following elementary (but somewhat non-trivial):

Corollary 7.7. Pick $a_{i}>0$ for $i=1, \ldots, 4, b_{1}>0, b_{2} \in \mathbb{R}$, and a subset $\mathcal{B} \subset \mathbb{R}_{0}^{4}$. Suppose there is a $\beta \in(0, \pi)$ so that $\mathcal{B}$ is precisely the set of $\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \in \mathbb{R}_{0}^{4}$ for which

$$
\left(a_{1} x_{1}+b_{1}, a_{2} x_{2}+b_{2}, a_{3} x_{3}, a_{4} x_{4}\right) \in \mathcal{A}_{\beta}
$$

Then $\beta, \frac{b_{1}}{a_{1}}$, and $\frac{b_{2}}{a_{2}}$ are uniquely determined by $\mathcal{B}$. If this unique $\beta$ is not $\frac{\pi}{2}$ then $\frac{b_{1}}{a_{2}}$ and $\frac{b_{2}}{a_{1}}$ are also determined by $\mathcal{B}$.

Proof. We consider the map

$$
\Psi\left(x_{1}, x_{2}, x_{3}, x_{4}\right)=\left(\arctan \left(\frac{a_{2} x_{2}+b_{2}}{a_{1} x_{1}+b_{1}}\right), \arctan \left(\frac{a_{4} x_{4}}{a_{3} x_{3}}\right)\right)=:\left(\psi_{0}\left(x_{1}, x_{2}\right), \psi_{1}\left(x_{3}, x_{4}\right)\right) .
$$

the previous lemma along with our assumption says that for $a_{1} x_{1}+b_{1}>0$ and $x_{3}>0$ $\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \in \mathcal{B} \Longleftrightarrow \Psi\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \in A_{\beta} . \Psi \operatorname{maps}\left\{\left(x_{1}, x_{2}, x_{3}, x_{4}\right) \mid a_{1} x_{1}+b_{1}>\right.$ 0 and $\left.x_{3}>0\right\}$ onto $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Step 1: $\beta$ is determined by $\mathcal{B}$. We notice that $\left(x, y^{0}\right) \in A_{\beta}$ for all $\frac{\pi}{2}-\epsilon<x<\frac{\pi}{2}$ if and only if $y^{0}=\frac{\pi}{2}-\beta$. Similarly $\left(x^{1}, y\right) \in A$ for all large $\frac{\pi}{2}-\epsilon<y<\frac{\pi}{2}$ if and only if $x^{1}=\frac{\pi}{2}-\beta$. Thus $\left(1, x_{2}, x_{3}^{0}, x_{4}^{0}\right) \in \mathcal{B}$ for all sufficiently large $x_{2}$ if and only if $\psi_{1}\left(\left(x_{3}^{0}, x_{4}^{0}\right)\right)=\frac{\pi}{2}-\beta$, while $\left(x_{1}^{1}, x_{2}^{1}, 1, x_{4}\right) \in \mathcal{B}$ for all sufficiently large $x_{4}$ if and only if $\psi_{0}\left(\left(x_{1}^{1}, x_{2}^{1}\right)\right)=\frac{\pi}{2}-\beta$. Fix such an $\left(x_{1}^{1}, x_{2}^{1}\right)$, then, if $\beta \neq \frac{\pi}{2}$, there is a unique $x_{4}^{1}$ such that for all $x_{4} \geq x_{4}^{1},\left(x_{1}^{1}, x_{2}^{1}, 1, x_{4}\right) \in \mathcal{B}$. If $\beta<\frac{\pi}{2}$ we have $\psi_{1}\left(1, x_{4}^{1}\right)=\frac{\pi}{2}-2 \beta$, while if $\beta>\frac{\pi}{2}$ we have $\psi_{1}\left(1, x_{4}^{1}\right)=2 \beta-\frac{3 \pi}{2}$.

Let $\overline{a_{i}}, \overline{b_{1}}, \overline{b_{2}}$, and $\bar{\beta}$ be another choice of parameters that work and $\bar{\Psi}$ and $\bar{\psi}_{i}$ the corresponding functions. Then $\tan \left(\bar{\psi}_{1}\right)=c \tan \left(\psi_{1}\right)$ where $c=\frac{\bar{a}_{4} a_{3}}{\bar{a}_{3} a_{4}}>0$. Plugging in $\left(x_{3}^{0}, x_{4}^{0}\right)$ from the previous paragraph allows us to conclude $\tan \left(\frac{\pi}{2}-\bar{\beta}\right)=c \tan \left(\frac{\pi}{2}-\beta\right)$, (i.e. $\tan (\bar{\beta})=\frac{1}{c} \tan (\beta)$ if $\beta \neq \frac{\pi}{2}$ ). Note in particular that the sign of $\frac{\pi}{2}-\bar{\beta}$ is the same as the sign of $\frac{\pi}{2}-\beta$. Plugging in $\left(1, x_{4}^{1}\right)$ from the above paragraph we get $\tan \left(\frac{\pi}{2}-2 \bar{\beta}\right)=c \tan \left(\frac{\pi}{2}-2 \beta\right)$ if $\beta<\frac{\pi}{2}$ and $\tan \left(2 \bar{\beta}-\frac{3 \pi}{2}\right)=c \tan \left(2 \beta-\frac{3 \pi}{2}\right)$ if
$\beta>\frac{\pi}{2}$. However this can only happen if $\mathrm{c}=1$ and $\beta=\bar{\beta}$. Thus we conclude that $\beta$ is determined by $\mathcal{B}$ and that either $\beta=\frac{\pi}{2}$ or else $c=1$ and $\psi_{1}=\bar{\psi}_{1}$.

Step 2: If $\beta \neq \frac{\pi}{2}$ then $\psi_{0}=\bar{\psi}_{0}$, while if $\beta=\frac{\pi}{2}$ then $\tan \left(\bar{\psi}_{0}\right)=\frac{1}{c} \tan \left(\psi_{0}\right)$. We will first show that given $\mathcal{B}, \beta$ and $\psi_{1}$ then there is at most one choice for $\psi_{0}$. To see this fix $\left(x_{1}, x_{2}\right)$ and consider $S \subset R^{2}$ such that $\mathcal{B} \cap\left(\left(x_{1}, x_{2}\right) \times R^{2}\right)=\left(x_{1}, x_{2}\right) \times S$. Then $\psi_{1}(S)$ is precisely the interval such that $A_{\beta} \cap\left(\psi_{0}\left(x_{1}, x_{2}\right) \times R\right)=\left(\psi_{0}\left(x_{1}, x_{2}\right) \times \psi_{1}(S)\right)$. However the shape of $A_{\beta}$ is such that $\psi_{1}(S)$ thus determines $\psi_{0}\left(x_{1}, x_{2}\right)$, unless $\beta=\frac{\pi}{2}$ in which case it determines $\psi_{0}\left(x_{1}, x_{2}\right)$ up to sign. The sign is determined by continuity and the fact that for $x_{2}$ large and positive (resp. negative) then $\psi_{0}\left(x_{1}, x_{2}\right)$ is positive (resp. negative).

Thus if $\beta \neq \frac{\pi}{2}$ then $\psi_{1}=\bar{\psi}_{1}$ and we see that $\bar{\psi}_{0}=\psi_{0}$ is the unique solution. If $\beta=\frac{\pi}{2}$ then $\tan \left(\bar{\psi}_{1}\right)=c \tan \left(\psi_{1}\right)$ and $A_{\beta}=\left\{(x, y) \in R^{2} \left\lvert\,-x+\frac{\pi}{2} \geq y \geq x-\frac{\pi}{2}\right.\right.$ when $x>0$ and $x+\frac{\pi}{2} \geq y \geq-x-\frac{\pi}{2}$ when $\left.x<0\right\}$. Thus for $c>0$ the map $(x, y) \rightarrow\left(\arctan \left(\frac{1}{c} \tan (x)\right), \arctan (c \tan (y))\right.$ preserves $A_{\beta}$ and step 2 follows.

The rest of the proof follows from the following equation that holds for all positive $x_{1}$ and all $x_{2}$ :

$$
\frac{\bar{a}_{2} x_{2}+\bar{b}_{2}}{\bar{a}_{1} x_{1}+\bar{b}_{1}}=\frac{1}{c} \frac{a_{2} x_{2}+b_{2}}{a_{1} x_{1}+b_{1}}
$$

and the fact that $c=1$ when $\beta \neq \frac{\pi}{2}$.

### 7.3. Recovering the data

In this section $\mathcal{G}$ will denote a fixed admissible graph of groups, $G:=\pi_{1}(\mathcal{G})$ the fundamental group of $\mathcal{G}$, and $G \curvearrowright T$ the Bass-Serre action for $\mathcal{G}$. For every vertex $v$ of $\mathcal{G}$ we choose a generator $\zeta_{v} \in Z\left(G_{v}\right)$ for the center of $Z\left(G_{v}\right)$ as in Definition 3.9. Also, we will fix an admissible action $G \curvearrowright X$. We will use the template notation from section 4.1. Recall that when $\mathcal{T}$ is a half-template, then $\partial_{\infty}^{\infty} \mathcal{T} \subset \partial_{\infty} \mathcal{T}$ denotes the set of boundary points corresponding to rays which intersect all but finitely many walls.

The goal of this section is to prove the remaining half of Theorem 1.3: the topological conjugacy class of the action $G \curvearrowright \partial_{\infty} X$ determines the functions $M L S_{v}$ : $G_{v} \rightarrow \mathbb{R}_{+}$and $\tau_{v}: G_{v} \rightarrow \mathbb{R}$ up to a multiplicative factor, for every vertex $v \in \mathcal{G}$.
Definition 7.8. An element $g$ of a vertex group $G_{v}$ is restricted if $g$ acts on $\bar{Y}_{v}$ (see section 3.2) as an axial isometry and its fixed points in $\partial_{\infty} \bar{Y}_{v}$ are distinct from the fixed points of $G_{e}$ where $e$ is any edge incident to $v$.

It will be convenient to choose, for each vertex $v$ of $\mathcal{G}$, a restricted element $\delta_{v} \in G_{v}$ which lies in the commutator subgroup $\left[G_{v}, G_{v}\right] \subset G_{v}$ :

Lemma 7.9. For every vertex $v$, the commutator subgroup $\left[G_{v}, G_{v}\right]$ contains restricted elements.

Proof. We first recall that $H_{v}:=G_{v} / Z\left(G_{v}\right)$ is a nonelementary hyperbolic group, and the induced action $H_{v} \curvearrowright \bar{Y}_{v}$ is discrete and cocompact.

Choose a free nonabelian subgroup $S \subset H_{v}$ [Gro87, p. 212], and elements $\bar{g}_{1}, \bar{g}_{2} \in$ $S$ which belong to a free basis for the commutator subgroup $[S, S] \subset\left[H_{v}, H_{v}\right]$, and let
$g_{i} \in G_{v}$ be lift of $\bar{g}_{i}$ under the projection $G_{v} \rightarrow H_{v}$. Then $g_{i}$ acts axially on $\bar{Y}_{v}$ since $\bar{g}_{i}$ has infinite order in $H_{v}$ and $H_{v}$ acts discretely on $\bar{Y}_{v}$. Note that Fix $\left(g_{1}, \partial_{\infty} \bar{Y}_{v}\right) \cap$ $\operatorname{Fix}\left(g_{2}, \partial_{\infty} \bar{Y}_{v}\right)=\emptyset$, since otherwise by Lemma 2.21 we would have $\operatorname{Fix}\left(g_{1}, \partial_{\infty} \bar{Y}_{v}\right)=$ Fix $\left(g_{2}, \partial_{\infty} \bar{Y}_{v}\right)$, forcing $\left\langle g_{1}, g_{2}\right\rangle$ to be virtually cyclic, which is absurd. Set $h_{n}:=g_{1}^{n} g_{2}^{n}$. Lemma 2.10 tells us that $h_{n}$ is axial for large $n$ and $\operatorname{Fix}\left(h_{n}, \partial_{\infty} \bar{Y}_{v}\right)$ converges to $\left\{\xi_{1}, \xi_{2}\right\} \subset \partial_{\infty} \bar{Y}_{v}$ where $\xi_{i} \in \operatorname{Fix}\left(g_{i}, \partial_{\infty} \bar{Y}_{v}\right)$. The induced action of $G_{e}$ on $\bar{Y}_{v}$ translates a geodesic $\gamma_{e} \subset \bar{Y}_{v}$. By the finiteness of $\mathcal{G}$ we can choose elements $g_{e} \in G_{e}$ such that the induced translation of $g_{e}$ is nonzero but uniformly bounded. Thus Lemma 2.21 says that subsets $\operatorname{Fix}\left(G_{e}, \partial_{\infty} \bar{Y}_{v}\right)=\operatorname{Fix}\left(g_{e}, \partial_{\infty} \bar{Y}_{v}\right)$ define a discrete subset of $\left(\partial_{\infty} \bar{Y}_{v} \times \partial_{\infty} \bar{Y}_{v}\right) / \mathbb{Z}_{2}$, so either
a) Fix $\left(h_{n}, \partial_{\infty} \bar{Y}_{v}\right) \cap \operatorname{Fix}\left(G_{e}, \partial_{\infty} \bar{Y}_{v}\right)=\emptyset$ for all edges $e$ incident to $v$ when $n$ is large, or
b) There is a subsequence $h_{n_{i}}$ and an edge $e$ incident to $v$ so that $\operatorname{Fix}\left(h_{n_{i}}, \partial_{\infty} \bar{Y}_{v}\right)=$ $\operatorname{Fix}\left(G_{e}, \partial_{\infty} \bar{Y}_{v}\right)$.

But if b) held then we would have $\operatorname{Fix}\left(G_{e}, \partial_{\infty} \bar{Y}_{v}\right)=\left\{\xi_{1}, \xi_{2}\right\}$, which, by Lemma 2.21, would force the absurd conclusion that Fix $\left(g_{1}, \partial_{\infty} \bar{Y}_{v}\right)=\operatorname{Fix}\left(g_{e}, \partial_{\infty} \bar{Y}_{v}\right)=$ Fix $\left(g_{2}, \partial_{\infty} \bar{Y}_{v}\right)$. Hence case a) holds and the lemma is proved.

Notice that for every $v, M L S_{v}\left(\delta_{v}\right) \neq 0$ (since $\delta_{v}$ acts on $\bar{Y}_{v}$ as an axial isometry), $\tau_{v}\left(\delta_{v}\right)=0$ since $\delta_{v} \in\left[G_{v}, G_{v}\right]$, and $\tau_{v}\left(\zeta_{v}\right) \neq 0$.
Lemma 7.10. In order to determine $M L S_{v}$ and $\tau_{v}$ up to a multiplicative factor, it suffices to determine the ratios

$$
\frac{M L S_{v}(\sigma)}{M L S_{v}\left(\delta_{v}\right)} \quad \text { and } \quad \frac{\tau_{v}(\sigma)}{\tau_{v}\left(\zeta_{v}\right)}
$$

for every restricted element $\sigma \in G_{v}$ whose fixed point set in $\partial_{\infty} \bar{Y}_{v}$ is disjoint from $\operatorname{Fix}\left(\delta_{v}, \partial_{\infty} \bar{Y}_{v}\right)$.

Proof. Choose an arbitrary $\sigma \in G_{v}$.
We first discuss $M L S_{v}$. Note that $M L S_{v}(\sigma)=0$ iff $\sigma$ projects to an element of finite order in $H_{v}$; hence we may assume that $\sigma$ acts on $\bar{Y}_{v}$ as an axial isometry. First assume that $\operatorname{Fix}\left(\sigma, \partial_{\infty} \bar{Y}_{v}\right)=\operatorname{Fix}\left(\delta_{v}, \partial_{\infty} \bar{Y}_{v}\right)$. Let $\bar{\sigma}, \bar{\delta}_{v} \in H_{v}$ be the projections to $H_{v}$. Then $\bar{\sigma}$ and $\bar{\delta}_{v}$ generate a virtually cyclic subgroup $S$ because they have a common axis. Hence there is a finite subset $\left\{s_{1}, \ldots, s_{k}\right\} \subset G_{v}$ so that for any $n$ we have $\sigma^{n}=s_{i_{n}} \delta_{v}^{j_{n}} \zeta_{v}^{k_{n}}$ for suitable $i_{n}, j_{n}, k_{n}$. Then

$$
M L S_{v}(\sigma)=\lim _{n \rightarrow \infty} \frac{1}{n} M L S_{v}\left(\sigma^{n}\right)=\lim _{n \rightarrow \infty} M L S_{v}\left(\delta_{v}^{j_{n}}\right)
$$

so we can recover the ratio above in this case.
Now assume $\operatorname{Fix}\left(\sigma, \partial_{\infty} \bar{Y}_{v}\right) \cap \operatorname{Fix}\left(\delta_{v}, \partial_{\infty} \bar{Y}_{v}\right)=\emptyset$. Setting $h_{k}:=\delta_{v}^{k} \sigma^{k^{2}}$, we argue as in the proof of Lemma 7.9 to see that for large $k, h_{k}$ is restricted, $F i x\left(h_{k}, \partial_{\infty} \bar{Y}_{v}\right) \cap$ $F i x\left(\delta_{v}, \partial_{\infty} \bar{Y}_{v}\right)=\emptyset$, and $\left|M L S_{v}\left(h_{k}\right)-k M L S_{v}\left(\delta_{v}\right)-k^{2} M L S_{v}(\sigma)\right|$ is uniformly bounded (by Lemma 2.10). We may then recover the desired ratios from the formula

$$
M L S_{v}(\sigma)=\lim _{k \rightarrow \infty} \frac{1}{k^{2}} M L S_{v}\left(h_{k}\right)
$$

We now consider the behavior of $\tau_{v}$. Suppose $\sigma$ projects to an element of finite order in $H_{v}$, or $\operatorname{Fix}\left(\sigma, \partial_{\infty} \bar{Y}_{v}\right) \cap \operatorname{Fix}\left(\delta_{v}, \partial_{\infty} \bar{Y}_{v}\right) \neq \emptyset$. In either case we have a finite set $\left\{s_{1}, \ldots, s_{k}\right\} \subset G_{v}$ so that $\sigma^{n}=s_{i_{n}} \delta_{v}^{j_{n}} \zeta_{v}^{k_{n}}$ for suitable $i_{n}, j_{n}, k_{n}$. Then

$$
\tau_{v}(\sigma)=\lim _{n \rightarrow \infty} \frac{1}{n} \tau_{v}\left(\sigma^{n}\right)=\lim _{n \rightarrow \infty}\left[\frac{j_{n}}{n} \tau_{v}\left(\delta_{v}\right)+\frac{k_{n}}{n} \tau_{v}\left(\zeta_{v}\right)\right] .
$$

For the case $\operatorname{Fix}\left(\sigma, \partial_{\infty} \bar{Y}_{v}\right) \cap \operatorname{Fix}\left(\delta_{v}, \partial_{\infty} \bar{Y}_{v}\right)=\emptyset$, use the same $h_{k}$ as above for large $k$ since $\tau_{v}\left(h_{k}\right)=k^{2} \tau(\sigma)$.

We now focus our attention on a vertex $\bar{v}_{1}$ of $\mathcal{G}$. Choose an edge $\bar{e}$ of $\mathcal{G}$ incident to $\bar{v}_{1}$, and lift $\bar{e}$ to an edge $e$ of the Bass-Serre tree. We adopt the notation from section 2.5 for the associated graph of groups $\mathcal{G}^{\prime}, G^{\prime}:=\pi_{1}\left(\mathcal{G}^{\prime}\right), T^{\prime} \subset T$, etc. We fix some restricted element $\sigma \in G_{\bar{v}_{1}}$ with $\operatorname{Fix}\left(\sigma, \partial_{\infty} \bar{Y}_{v_{1}}\right) \cap \operatorname{Fix}\left(\delta_{v_{1}}, \partial_{\infty} \bar{Y}_{v_{1}}\right)=\emptyset$.
Definition 7.11. (Special rays) For every $(p, q, r, s) \in \mathbb{R}_{0}^{4}$ we define a geodesic ray $\gamma \subset T^{\prime}$ and sequences $\hat{l}_{i}=l_{i}(p, q, r, s), \hat{\epsilon}_{i}=\hat{\epsilon}_{i}(p, q, r, s)$ as follows.

Case 1: $\bar{e} \subset \mathcal{G}$ is embedded. Let $e=\overline{v_{1} v_{2}}$. For each $i \in \mathbb{N}$ we set $s_{2 i-1}:=$ $\sigma^{2^{i-1}} \delta_{v_{1}}^{\left[p 2^{i-1}\right]} \zeta_{v_{1}}^{\left[q^{i-1}\right]}$ and $s_{2 i}:=\delta_{v_{2}}^{\left[r 2^{i-1}\right]} \zeta_{v_{2}}^{\left[s 2^{i-1}\right]}$, where $[x]$ denote the integer part of $x \in \mathbb{R}$. Then we let $\gamma \subset T$ be the geodesic ray with successive edges

$$
\begin{equation*}
e, s_{1} e, s_{1} s_{2} e, \ldots, s_{1} \ldots s_{k} e, \ldots \tag{7.12}
\end{equation*}
$$

and define, for all $i \in \mathbb{N}, \hat{l}_{2 i-1}=2^{i-1}\left(M L S_{v_{1}}(\sigma)+|p| M L S_{v_{1}}\left(\delta_{v_{1}}\right)\right), \hat{l}_{2 i}=2^{i-1}|r| M L S_{v_{2}}\left(\delta_{v_{2}}\right)$; and $\hat{\epsilon}_{2 i+1}:=2^{i}\left(\tau_{v_{1}}(\sigma)+q \tau_{v_{1}}\left(\zeta_{v_{1}}\right)\right)$, and $\hat{\epsilon}_{2 i}:=2^{i-1} s \tau_{v_{2}}\left(\zeta_{v_{2}}\right)$.

Case 2: $\bar{e} \subset \mathcal{G}$ is a loop. Let $t$ be as in section 2.5. For each $i \in \mathbb{N}$ we set $s_{2 i-1}:=\sigma^{2^{i-1}} \delta_{v_{1}}^{\left[p 2^{i-1}\right]} \zeta_{v_{1}}^{\left[2^{i-1}\right]}$ and $s_{2 i}:=\delta_{v_{1}}^{\left[r r^{i-1}\right]} \zeta_{v_{1}}^{\left[s i^{2-1}\right]}$; then we let $\gamma \subset T$ be the geodesic with successive vertices

$$
\begin{equation*}
v_{1}, t v_{1}, t s_{1} t^{-1} v_{1}, t s_{1} t^{-1} s_{2} t v_{1}, \ldots t s_{1} t^{-1} s_{2} t \ldots s_{2 k} t v_{1}, \ldots \tag{7.13}
\end{equation*}
$$

and define, for all $i \in \mathbb{N}, \hat{l}_{2 i-1}=2^{i-1}\left(M L S_{v_{1}}(\sigma)+|p| M L S_{v_{1}}\left(\delta_{v_{1}}\right)\right), \hat{l}_{2 i}=2^{i-1}|r| M L S_{v_{1}}\left(\delta_{v_{1}}\right)$; and $\hat{\epsilon}_{2 i+1}:=2^{i}\left(\tau_{v_{1}}(\sigma)+q \tau_{v_{1}}\left(\zeta_{v_{1}}\right)\right)$, and $\hat{\epsilon}_{2 i}:=2^{i-1} s \tau_{v_{1}}\left(\zeta_{v_{1}}\right)$.

These rays are useful because they admit templates that are asymptotically selfsimilar:

Lemma 7.14. There is a half template $(\mathcal{T}, f, \phi)$ for $\gamma$ with walls $W_{\text {all }}^{\mathcal{T}}=\left\{W_{i}\right\}_{i=1}^{\infty}$ and strips Strip $_{\mathcal{T}}=\left\{\mathcal{S}_{i}\right\}_{i=1}^{\infty}$, so that for $i$ sufficiently large, $l\left(\mathcal{S}_{i}\right)=\hat{l}_{i}, \epsilon\left(\mathcal{S}_{i}\right)=\hat{\epsilon}_{i}$, and $\alpha\left(W_{i}\right)=\beta$, where $\beta$ is the Tits angle between the (positively oriented) $\mathbb{R}$-factors of $Y_{v_{1}}$ and $Y_{v_{2}}$.

Proof. We will treat the case when $\bar{e}$ is embedded; the other case is similar. To prove the lemma we will show that the desired template may be obtained from a standard template for $\gamma$ by changing the strip widths and strip gluings by a bounded amount. Thus by "uniformly" we will just mean independent of $i$, but possibly dependent on all other choices.

For $i \geq 1$ we set $e_{i}:=s_{1} \ldots s_{i-1} e$ and $v_{i}:=e_{i} \cap e_{i+1}$. Recall (section 4.2) that the standard template $(\mathcal{T}, f, \phi)$ for $\gamma$ is constructed using flats $F_{e_{i}} \subset Y_{e_{i}}$ and flat strips
$\mathcal{S}_{e_{i}, e_{i+1}} \subset Y_{v_{i}}$, where $\mathcal{S}_{e_{i}, e_{i+1}}=\gamma_{e_{i}, e_{i+1}} \times \mathbb{R} \subset \bar{Y}_{v_{i}} \times \mathbb{R}=Y_{v_{i}}$. We choose $x_{1} \in F_{e_{1}}$ and set $x_{i}:=s_{1} \ldots s_{i-1} x_{1} \in F_{e_{i}}$.

Step 1: There is a constant $c_{1}$ so that $d\left(x_{i}, \mathcal{S}_{e_{i}, e_{i+1}}\right)<c_{1}, d\left(x_{i+1}, \mathcal{S}_{e_{i}, e_{i+1}}\right)<c_{1}$ and $\left|\operatorname{Width}\left(\mathcal{S}_{e_{i}, e_{i+1}}\right)-\hat{l}_{i}\right|<c_{1}$. We will do the case when $i=2 j-1$ is odd; the even case is similar. Let $\pi: Y_{v_{1}}=\bar{Y}_{v_{1}} \times \mathbb{R} \rightarrow \bar{Y}_{v_{1}}$ be the projection map, and set $\bar{x}_{1}:=\pi\left(x_{1}\right)$, $\bar{F}_{e}:=\pi\left(F_{e}\right), \bar{F}_{s_{i} e}:=\pi\left(\bar{s}_{i} F_{e}\right)=\pi\left(F_{s_{i} e}\right)$, and $\gamma_{e, s_{i} e}:=\pi\left(\mathcal{S}_{e, s_{i} e}\right)$. We apply $\left(s_{1} \ldots s_{i-1}\right)^{-1}$ and then $\pi$ to everything, and are thereby reduced to showing that there is a $c_{1}$ so that $d\left(\bar{x}_{1}, \gamma_{e, s_{i} e}\right)<c_{1}, d\left(s_{i} \bar{x}_{1}, \gamma_{e, s_{i} e}\right)<c_{1}$, and $\left|d\left(s_{i} \bar{x}_{1}, \bar{x}_{1}\right)-\hat{l}_{i}\right|<c_{1}$.
¿From the definition of $s_{i}$ we have $s_{i} \bar{x}_{1}=\sigma^{2^{j-1}} \delta_{v_{1}}^{\left[p 2^{j-1}\right]} \bar{x}_{1}$ since $\zeta_{v_{1}}$ acts trivially on $\bar{Y}_{v_{1}}$. Since $\sigma, \delta_{v_{1}} \in G_{v_{1}}$ are restricted elements, the sets $\partial_{\infty} \bar{F}_{e} \subset \partial_{\infty} \bar{Y}_{v_{1}}$ and $\partial_{\infty} \bar{F}_{s_{i} e} \subset \partial_{\infty} \bar{Y}_{v_{1}}$ are disjoint from the fixed point sets of $\sigma$ and $\delta_{v_{1}}$; the latter two sets are disjoint by assumption. Therefore we may apply Lemma 2.10 to conclude that when $j$ is sufficiently large,
a) $s_{i}: \bar{Y}_{v_{1}} \rightarrow \bar{Y}_{v_{1}}$ is an axial isometry with an axis $\gamma_{i} \subset \bar{Y}_{v_{1}}$ at uniformly bounded distance from $\bar{x}_{1}$ and $s_{i} \bar{x}_{1}$.
b) $\left|d\left(s_{i} \bar{x}_{1}, \bar{x}_{1}\right)-\hat{l}_{i}\right|$ is uniformly bounded.
c) The attracting (resp. repelling) fixed point of $s_{i}$ in $\partial_{\infty} \bar{Y}_{v_{1}}$ is close to the attracting fixed point, $\xi^{+}$, of $\sigma$ (resp. repelling fixed point, $\xi^{-}$, of $\delta_{v_{1}}^{s i g n(p)}$ ).
Let $\gamma_{e, s_{i} e}$ be the shortest path from $\bar{F}_{e}$ to $\bar{F}_{s_{i} e}$ with endpoints $\bar{z}_{i} \in \bar{F}_{e}$ and $\bar{w}_{i} \in \bar{F}_{s_{i} e}$ Let $\gamma$ be a geodesic with endpoints $\xi^{-}$and $\xi^{+}$. The Gromov hyperbolicity of $Y_{v_{1}}$ and Lemma 2.9 part 3 imply that for $t$ fixed and large and $i$ large the axis of $s_{i}$ comes within a uniform distance of $\gamma_{e, s_{i} e}(t)$ and hence $\gamma_{e, s_{i} e}(t)$ stay a uniform distance from $\gamma$. Thus $\bar{z}_{i}$, which is the point on $\bar{F}_{e}$ closest to $\gamma_{e, s_{i} e} e(t)$, must stay a uniform distance from the set of points on $\bar{F}_{e}$ closest (in a Buseman function sense) to $\xi^{+}$(which we see by taking $t$ large) and hence stay uniformly close to $\bar{x}_{1}$. Similarly $s_{i}^{-1}\left(\bar{w}_{i}\right)$ approaches the set of points on $\bar{F}_{e}$ closest to $\xi^{-}$and hence $\bar{w}_{i}$ stays uniformly close to $s_{i}\left(\bar{x}_{1}\right)$.

Step 2: There is a $c_{2}$ so that the standard template $(\mathcal{T}, f, \phi)$ satisfies $\left|\epsilon\left(\mathcal{S}_{i}\right)-\hat{\epsilon}_{i}\right|<$ $c_{2}$ and $\alpha\left(W_{i}\right)=\beta$ for all $i>1$. The assertion that $\alpha\left(W_{i}\right)=\beta$ is clear from the definition of $\beta\left(W_{i}\right)$ and the construction of standard templates. Step 1 then implies that there is a $c_{3}$ so that the origin $o_{i} \in W_{i}$ maps under $f: \mathcal{T} \rightarrow X$ to within distance $c_{3}$ of $x_{i}$, for $i>1$. Hence from the definition of $\epsilon: \operatorname{Strip}_{\mathcal{T}}^{o} \rightarrow \mathbb{R}$ we see that $\epsilon\left(\mathcal{S}_{i}\right)$ agrees with $\tau_{v_{i}}\left(\left(s_{1} \ldots s_{i-1}\right) s_{i}\left(s_{1} \ldots s_{i-1}\right)^{-1}\right)$ to within $2 c_{3}$. The conjugacy invariance of $\tau_{v}$ then gives $\left|\epsilon\left(\mathcal{S}_{i}\right)-\hat{\epsilon}_{i}\right|<c_{2}$ for a suitable $c_{2}$.

Step 3: Adjusting $(\mathcal{T}, f, \phi)$. In steps 1 and 2 we have shown that the standard template satisfies conditions of Lemma 7.14 to within bounded error. So we now modify the construction of $\mathcal{T}$ by changing the metric ${ }^{11}$ on $\hat{\mathcal{S}}_{e_{i}, e_{i+1}}$ so that $\operatorname{Width}\left(\hat{\mathcal{S}}_{e_{i}, e_{i+1}}\right)=\hat{l}_{i}$ if $l_{i} \geq 1$, and leaving $\hat{\mathcal{S}}_{e_{i}, e_{i+1}}$ untouched otherwise, and by modifying the gluings $\partial \hat{\mathcal{S}}_{e_{i}, e_{i+1}} \rightarrow W_{e_{i}} \amalg W_{e_{i+1}}$ by a bounded amount so that $\epsilon\left(\mathcal{S}_{i}\right)=\hat{\epsilon}_{i}$. Finally, if we redefine $f: \mathcal{T} \rightarrow X$ to agree with the original $f$ on $\amalg_{e} W_{e}$ and on $\mathcal{T}-\left(\amalg_{e} W_{e}\right)$, then we get the desired template for $\gamma$.

Proof the Theorem 1.3 concluded. Consider the subset $\mathcal{B} \subset \mathbb{R}_{0}^{4}$ of 4-tuples $(p, q, r, s)$ for which the geodesic ray $\gamma \subset T$ defined above gives a trivial subset $\partial_{\infty}^{\partial_{\infty} \gamma} X$ (i.e. a

[^7]single point); by Lemma 3.23 and Corollary 5.26 the subset $\partial_{\infty}^{\partial_{\infty} \gamma} X$ can be detected just using the action $G \curvearrowright \partial_{\infty} X$, and so $\mathcal{B}$ is also determined by the action $G \curvearrowright \partial_{\infty} X$. On the other hand, by Theorem 5.1, $\partial_{\infty}^{\partial_{\infty} \gamma} X$ is trivial iff $\partial_{\infty}^{\infty} \mathcal{T}$ is trivial (i.e. a single point) where ( $\mathcal{T}, f, \phi$ ) is any template for $\gamma$. Using Lemma 7.14 we arrive at the following: the subset $\mathcal{B}$ of $(p, q, r, s) \in \mathbb{R}_{0}^{4}$ so that any template $\mathcal{T}$ with $l\left(\mathcal{S}_{i}\right)=$ $\hat{l}_{i}(p, q, r, s), \epsilon\left(\mathcal{S}_{i}\right)=\hat{\epsilon}_{i}(p, q, r, s)$, and $\alpha\left(W_{i}\right)=\beta$ (for $i$ sufficiently large) is trivial, is determined by the action $G \curvearrowright \partial_{\infty} X$. But since a template $\mathcal{T}$ with $l\left(\mathcal{S}_{i}\right)=\hat{l}_{i}(p, q, r, s)$, $\epsilon\left(\mathcal{S}_{i}\right)=\hat{\epsilon}_{i}(p, q, r, s)$, and $\alpha\left(W_{i}\right)=\beta$ (for $i$ sufficiently large) is trivial iff the self-similar template with data $\left\{\beta ; \hat{l}_{3}, \hat{\epsilon}_{3}, \hat{l}_{4}, \hat{\epsilon}_{4}\right\}$ is trivial, we may apply Corollary 7.7 to conclude that the ratios
$$
\frac{M L S_{v_{1}}(\sigma)}{M L S_{v_{1}}\left(\delta_{v_{1}}\right)} \quad \text { and } \quad \frac{\tau_{v_{1}}(\sigma)}{\tau_{v_{1}}\left(\zeta_{v_{1}}\right)}
$$
as well as $\beta$ are determined by the action $G \curvearrowright \partial_{\infty} X$. Moreover, unless $\beta=\frac{\pi}{2}$ then
$$
\frac{M L S_{v_{1}}(\sigma)}{\tau_{v_{1}}\left(\zeta_{v_{1}}\right)} \quad \text { and } \quad \frac{\tau_{v_{1}}(\sigma)}{M L S_{v_{1}}\left(\delta_{v_{1}}\right)}
$$
are also determined.

## 8. Examples

In this section we construct the example mentioned in the introduction: we will find two locally compact Hadamard spaces $X_{0}$ and $X_{r}$ on which an admissible group $G$ acts discretely and cocompactly with the same geometric data (i.e. the induced actions of $G$ on $\partial_{\infty} X_{0}$ and $\partial_{\infty} X_{r}$ are topologically conjugate), an equivariant quasi-isometry $\tilde{F}: X_{0} \rightarrow X_{r}$, and a geodesic ray $\gamma$ of $X_{0}$ such that $\tilde{F}(\gamma)$ does not lie within a bounded distance of a geodesic ray in $X_{r}$.

To do that we consider for each (small) real $r$ a complex $M_{r}$, built out of four flat square tori $T_{i}$. On each $T_{i}$ we use standard angle coordinates $(s, t)_{i}$ with $(s+2 n \pi, t+$ $2 m \pi)_{i}=(s, t)_{i}$ for any integers $n$ and $m$. We let $M_{r}=T_{1} \cup T_{2} \cup T_{3} \cup T_{4} / \sim_{r}$, where $(0, t)_{1} \sim_{r}(t, 0)_{2},(\pi, t)_{2} \sim_{r}(t, \pi)_{3},(\pi, t)_{3} \sim(t, \pi)_{4}$ and $(0, t)_{4} \sim_{r}(t, r)_{1}$. We will let $X_{r}$ represent the universal cover.

The $X_{r}$ are Hadamard spaces with admissible fundamental groups which all have the same geometric data. This is easiest to see by considering $M_{r}$ as the union of four spaces $T_{1} \cup T_{2}, T_{2} \cup T_{3}, T_{3} \cup T_{4}$, and $T_{4} \cup T_{1}$ each of which is isometric to a figure eight cross a circle, where both circles in the figure eight and the product circle have length $2 \pi$. Each of the three spaces is glued to adjacent spaces along (product) boundary tori (reversing the factors). We note that the underlying finite graph of G is a square with four edges $e_{i}$ corresponding naturally to $T_{i}$.

It is easy to see that the $M_{r}$ are homeomorphic. In fact for small $r$ the fundamental groups are identified in a natural way. However, we will find it more useful to consider the map $F_{r}: M_{0} \rightarrow M_{r}$ defined by $F(s, t)_{i}=(s, t)_{i}$, except that $F\left((0, t)_{4}\right)=(t, 0)_{1}$. This is not continuous along the closed geodesic $(0, t)_{4}$, but the induced equivariant quasi-isometry $\tilde{F}: X_{0} \rightarrow X_{r}$ is relatively easy to study.

We now want to choose a geodesic ray $\gamma$ in $X_{0}$. For these spaces geodesics are just geodesics in templates. It is easier to first choose the degenerate half template $\mathcal{T}$ that it will lie in. By degenerate template we mean a template where we alow the strip
widths to be 0 . In fact in our case all the strip widths will be 0 , all the angles will be $\frac{\pi}{2}$, and the displacements will all be odd multiples of $\pi$. We will choose $\mathcal{T}$ so that the edges of the ray in the Bass-Serre tree project to the finite graph periodically in the order $e_{1}, e_{2}, e_{3}, e_{4}, e_{1} \ldots$ Subject to this constraint we can still choose the itinerary such that the displacements in $\mathcal{T}$ are any odd multiples of $\pi$ we please. Hence we may choose the itinerary (e.g. make it nontrivial) so that there is a geodesic $\gamma$ in $\mathcal{T}$ which misses all the vertices such that $\gamma \cap W_{i}$ gets arbitrarily long. (The easiest way to see this is to consider the developement $\mathcal{D}$. First choose a ray $r$ that you want to be $\mathcal{D}(\gamma)$. The choice of itinerary at each step amounts to a choice of quarter planes among those shifted by $2 n \pi$. We can thus make the choice so that the quarter plane intersects $r$ in increasingly long intervals).

Of course $\gamma$ is also a geodesic ray in $X_{0} . \tilde{F}$ will take $\mathcal{T}$ to a corresponding template $\mathcal{T}_{r}$, and the geodesic ray corresponding to $F(\gamma)$ must be a geodesic in this template. $\tilde{F}$ will be discontinuous exactly where walls corresponding to $e_{4}$ and $e_{1}$ are glued. The discontinuity is a translation by $r$ perpendicular to the gluing line.

We develop $\mathcal{T}$ (and $\mathcal{T}_{r}$ ) to the plane in such a way that $\gamma$ goes to a ray with angle $\frac{\pi}{2}>\theta>0$, the quarter planes map to planes of type II and IV where walls of type $e_{1}$ and $e_{3}$ yield quarter planes of type II and walls of type $e_{2}$ and $e_{4}$ yield quarter planes of type IV. $\tilde{F}$ will induce a map of developments which is discontinuous precisely on the horizontal lines where the quarter planes coming from $e_{4}$ meet those coming from $e_{1}$. The discontinuity will be precisely a vertical shift by $r$. Thus the image of $\gamma$ consists of arbitrarily long line segments of slope $\theta$ with infinitely many vertical jumps of size $r$. Since the segments get arbitrarily long, the only rays that can stay a bounded distance from $\tilde{F}(\gamma)$ must also have slope $\theta$. But then, because of the infinitely many vertical jumps of size $r$ in $\tilde{F}(\gamma)$, no such ray can stay a bounded distance from $\tilde{F}(\gamma)$.

One can make similar examples on singular piecewise Euclidean graph manifolds. We construct such examples by gluing together two pieces. Each piece is topologically a twice punctured torus cross $S^{1}$. The boundary of each piece will consist of two totally geodesic square flat two-tori. The first space is constructed by gluing both of the corresponding boundary tori together flipping the coordinates. The second space is similar except that for one of the boundary tori the gluing map is coordinate flipping composed with a small translation.

The metric of each piece is a product metric where the circle has length 1 and where the metric on the torus is the completion of the flat square torus minus two line segments (slits) of length $\frac{1}{2}$. The torus with the slits is a compact flat singular space with boundary being two closed geodesics of length 1 (i.e. going "around a slit").

The argument that this gives an example is very similar to the above since geodesics in the space are in fact geodesics in the corresponding templates and since the induced map on the development of appropriate templates has properties similar to the above example.

We also suspect there are such examples on smooth graph manifolds. In fact one may be able to construct such an example by smooth approximations to the above example. To do this carefully it would be necessary to be careful with how closely template geodesics shadow actual geodesics in this case.

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    ${ }^{1}$ Following [Bal95] we will refer to $C A T(0)$ spaces (complete, simply connected length spaces with nonpositive curvature in the sense of Alexandrov) as Hadamard spaces.

[^1]:    ${ }^{2}$ Let $M_{1}$ and $M_{2}$ be closed surfaces with nonpositive curvature, and let $N_{1}$ and $N_{2}$ be the Riemannian products $N_{i}:=M_{i} \times S^{1}$. Suppose $f_{0}: M_{1} \rightarrow M_{2}$ is a homotopy equivalence, $f:=$ $f_{0} \times i d_{S^{1}}: N_{1} \rightarrow N_{2}$ is the corresponding map between the $N_{i}$ 's, and $\hat{f}: \tilde{N}_{1} \rightarrow \tilde{N}_{2}$ is a lift of $f$ to a map between the universal covers. Then it turns out that $\hat{f}$ extends continuously to up to the ideal boundary $\partial_{\infty} N_{1}$ if and only if $f_{0}$ is homotopic to a homothety.
    ${ }^{3}$ Boundaries can even fail to be (non-equivariantly) homeomorphic: [CK] describes a pair of homeomorphic nonpositively curved 2 -complexes whose universal covers have nonhomeomorphic boundary (see also [Wil]).
    ${ }^{4}$ The action $G \curvearrowright \partial_{\infty} X$ is expansive if and only if $G$ is hyperbolic.

[^2]:    ${ }^{5}$ It follows from the methods of [Bal95, III.3] that if $G \curvearrowright X$ is a cocompact isometric action on a Hadamard space, $g \in G$ is an axial isometry, and $\gamma_{g} \subset$ is an axis for $g$ which does not bound a flat half plane, then the orbit of $\partial_{\infty} \gamma \subset \partial_{\infty} X$ under the action $G \curvearrowright \partial_{\infty} X$ is dense in $\partial_{\infty} X$. Hence the set of points in $\partial_{\infty} X$ which are the unique attracting fixed point of some element of $G$ is dense in $\partial_{\infty} X$. Any $G$-equivariant homeomorphism $\partial_{\infty} X \rightarrow \partial_{\infty} X$ must fix this dense set pointwise, and must therefore be the identity. The uniqueness statement follows immediately from this.

[^3]:    ${ }^{6}$ A subset $Z \subset X$ is $C$-quasi-convex if for all $x, y \in Z$ we have $\overline{x y} \subset N_{C}(Z)$.

[^4]:    ${ }^{7} \mathrm{~A}$ map $c:[0, \infty) \rightarrow T$ has the bounded backtracking property if for every $r \in(0, \infty)$ there is an $r^{\prime} \in(0, \infty)$ such that if $t_{1}<t_{2}$, and $d\left(c\left(t_{1}\right), c\left(t_{2}\right)\right)>r^{\prime}$, then $d\left(c(t), c\left(t_{1}\right)\right)>r$ for every $t>t_{2}$.

[^5]:    ${ }^{8}$ A direction for a strip $\mathcal{S}$ is an orientation for its $\mathbb{R}$-factor $\mathcal{S} \simeq \mathbb{R} \times I$.
    ${ }^{9}$ In general one may also have to complete the resulting quotient space to get a Hadamard space.

[^6]:    ${ }^{10} \mathcal{S}_{e_{1}, e_{2}}$ may have width zero.

[^7]:    ${ }^{11}$ We do this in the simplest way: we start with the metric product decomposition $\hat{\mathcal{S}}_{e_{i}, e_{i+1}} \simeq I \times \mathbb{R}$ and then scale the metric on the $I$ factor.

