The geodesic flow of a nonpositively curved graph manifold

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Abstract

We consider discrete cocompact isometric actions $G \overset{\rho}{\curvearrowright} X$ where X is a locally compact Hadamard space[1], and G belongs to a class of groups ("admissible groups") which includes fundamental groups of 3-dimensional graph manifolds. We identify invariants ("geometric data") of the action ρ which determine, and are determined by, the equivariant homeomorphism type of the action $G \overset{\partial_\infty \rho}{\curvearrowright} \partial_\infty X$ of G on the ideal boundary of X. Moreover, if $G \overset{\rho_i}{\curvearrowright} X_i$ are two actions with the same geometric data and $\Phi : X_1 \to X_2$ is a G-equivariant quasi-isometry, then for every geodesic ray $\gamma_1 : [0, \infty) \to X_1$, there is a geodesic ray $\gamma_2 : [0, \infty) \to X_2$ (unique up to equivalence) so that $\lim_{t\to\infty} \frac{1}{t} d_{X_2}(\Phi \circ \gamma_1(t), \gamma_2([0, \infty))) = 0$. This work was inspired by (and answers) a question of Gromov in [Gro93, p. 136].

1. Introduction

As a consequence of the Morse lemma on quasi-geodesics, geodesic flows are especially simple and well understood in the Gromov hyperbolic case:

a. If $\phi : M_1 \to M_2$ is a homotopy equivalence between closed negatively curved manifolds, then there is an orbit equivalence $\hat{\phi} : SM_1 \to SM_2$ between the unit sphere bundles, which covers ϕ up to homotopy [Gro76].

b. If G is a hyperbolic group, $G \overset{\rho_i}{\curvearrowright} X_i$ is a discrete, cocompact, isometric action on a Hadamard space X_i for $i = 1, 2$, and $\Phi : X_1 \to X_2$ is a G-equivariant quasi-isometry, then Φ maps each geodesic $\gamma_1 \subset X_1$ to a subset at uniformly bounded Hausdorff distance from a geodesic $\gamma_2 \subset X_2$. Moreover, Φ induces an equivariant homeomorphism $\partial_\infty \Phi : \partial_\infty X_1 \to \partial_\infty X_2$ between ideal boundaries, [Gro87].

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[1]Following [Bal95] we will refer to $CAT(0)$ spaces (complete, simply connected length spaces with nonpositive curvature in the sense of Alexandrov) as Hadamard spaces.

c. When $G \overset{\rho}{\curvearrowright} X$ is a discrete, cocompact action of a hyperbolic group on a Hadamard space X, then the induced action $G \overset{\partial_\infty \rho}{\curvearrowright} \partial_\infty X$ of G on the boundary of X is a finitely presented dynamical system, [Gro87, CDP90].

Naturally one may ask if properties b and c hold without the assumption of Gromov hyperbolicity. It turns out that they do not: one can readily produce examples of pairs of discrete, cocompact, isometric actions $G \curvearrowright X_1$, $G \curvearrowright X_2$ where G-equivariant quasi-isometries $X_1 \to X_2$ do not induce boundary homeomorphisms[2] (this was observed independently by Ruane [Rua96]). In [Gro93] Gromov asked whether two actions $G \curvearrowright X_i$ induce G-equivariantly homeomorphic boundary actions $G \curvearrowright \partial_\infty X_i$. The answer to this is also no: S. Buyalo [Buy98] and the authors independently found pairs of actions which induce inequivalent boundary actions[3]. Finally, we remark that the boundary action $G \curvearrowright \partial_\infty X$ is finitely presented if and only if G is hyperbolic[4].

In this paper we examine actions $G \curvearrowright X$ where G belongs to a class of groups which generalize fundamental groups of 3-dimensional graph manifolds. We develop a kind of "coding" for geodesic rays in X, which allows us to understand the boundary action $G \curvearrowright \partial_\infty X$ and the Tits metric on $\partial_\infty X$. Before stating our main result in complete generality, we first formulate it for nonpositively curved 3-dimensional graph manifolds.

By the theorem of [Sch86], when M is a 3-dimensional graph manifold with a nonpositively curved Riemannian metric, then M has the following structure. There is a collection $M_1, \ldots, M_k$ of compact nonpositively curved 3-manifolds with nonempty totally geodesic boundary (the geometric Seifert components of M), and Seifert fibrations $M_i \xrightarrow{p_i} N_i$ where the metric on M_i has local product structure compatible with the fibration p_i, and the N_i are nonpositively curved orbifolds; M is obtained from the disjoint union $\amalg_i M_i$ by gluing boundary components isometrically in pairs via gluing isometries which are incompatible with the boundary fiberings. In what follows we will only consider graph manifolds whose Seifert fibered components have orientable fiber. Note that for each $1 \leq i \leq k$, the universal cover of M_i is isometric to a Riemannian product $\tilde{N}_i \times \mathbb{R}$; the action of $\pi_1(M_i)$ on $\tilde{M}_i$ preserves this product structure and so there is an induced action of $\pi_1(M_i)$ on the $\mathbb{R}$ factor by translations. Hence we get a homomorphism $\tau_i : \pi_1(M_i) \to \mathbb{R}$ for each i. We may also define a class function $MLS_i : \pi_1(M_i) \to \mathbb{R}_+$ by taking the minimum of the displacement function for the induced action $\pi_1(M_i) \curvearrowright \tilde{N}_i$, i.e. $MLS_i(g) = \inf\{d_{\tilde{N}_i}(gx, x) \mid x \in \tilde{N}_i\}$; this corresponds to the marked length spectrum of the nonpositively curved orbifold N_i.

Now suppose M and M' are graph manifolds as above, and $f : M \to M'$ is a homotopy equivalence. Embedded incompressible tori in Haken manifolds are determined up to isotopy by their fundamental groups up to conjugacy [Lau74], so we may assume after isotoping f that it is a homeomorphism which induces homeomorphisms

[2]Let M_1 and M_2 be closed surfaces with nonpositive curvature, and let N_1 and N_2 be the Riemannian products $N_i := M_i \times S^1$. Suppose $f_0 : M_1 \to M_2$ is a homotopy equivalence, $f := f_0 \times id_{S^1} : N_1 \to N_2$ is the corresponding map between the N_i's, and $\hat{f} : \tilde{N}_1 \to \tilde{N}_2$ is a lift of f to a map between the universal covers. Then it turns out that $\hat{f}$ extends continuously to up to the ideal boundary $\partial_\infty N_1$ if and only if f_0 is homotopic to a homothety.

[3]Boundaries can even fail to be (non-equivariantly) homeomorphic: [CK] describes a pair of homeomorphic nonpositively curved 2-complexes whose universal covers have nonhomeomorphic boundary (see also [Wil]).

[4]The action $G \curvearrowright \partial_\infty X$ is expansive if and only if G is hyperbolic.

$f_i : M_i \to M'_i$ from the Seifert components of M to the Seifert components of M' (and hence isomorphisms on the corresponding fundamental groups). We may then use the maps f_i to compare the invariants τ_i, τ'_i and MLS_i, MLS'_i.

Theorem 1.1. *The following are equivalent:*

1. The functions MLS_i and τ_i are preserved up to scale by f_i: for $i = 1, \dots, k$ there are constants a_i and b_i so that $MLS_i = a_i f^*(MLS'_i)$ and $\tau_i = b_i f^*(\tau'_i)$.

2. Any lift $\tilde{f} : \tilde{M} \to \tilde{M}'$ of f extends continuously to a map $\bar{f} : M \cup \partial_\infty M \to M' \cup \partial_\infty M'$ between the standard compactifications.

3. If $\tilde{f} : \tilde{M} \to \tilde{M}'$ is any lift of f, then $\tilde{f}$ maps geodesic rays to geodesic rays, up to uniform sublinear error: there is a function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{r\to\infty} \theta(r) = 0$ so that if $\gamma : [0, \infty) \to \tilde{M}$ is a unit speed geodesic ray, then there is a ray $\gamma' : [0, \infty) \to \tilde{M}'$ where $d(\tilde{f} \circ \gamma(t), \gamma'([0, \infty))) < (1 + t)\theta(t)$.

4. If we identify $\pi_1(M)$ with $\pi_1(M')$ via f, then the induced boundary actions $\pi_1(M) \curvearrowright \partial_\infty \tilde{M}$ and $\pi_1(M) \curvearrowright \partial_\infty \tilde{M}'$ are equivariantly homeomorphic (by a unique equivariant homeomorphism).

If 1 holds and in addition the constants a_i and b_i are independent of i, then the unique equivariant homeomorphism $\partial_\infty \tilde{M} \to \partial_\infty \tilde{M}'$ in 4 is an isometry with respect to Tits metrics.

In general (see Lemma 6.2) the structure of $\pi_1(M)$ forces the a_i's and b_i's in condition 1 to satisfy $\#\{a_1, \dots, a_k, b_1, \dots, b_k\} \leq 2$, and except in special circumstances they all coincide. The condition $MLS_i = a_i f^*(MLS'_i)$ means that the homotopy equivalence $N_i \to N'_i$ induced by f_i preserves the marked length spectrum of the nonpositively curved orbifolds up to the scale factor a_i. Although closed nonpositively curved surfaces with the same marked length spectrum are isometric by [Cro90, Ota90, CFF92], compact nonpositively curved surfaces with geodesic boundary can have the same marked length spectrum without being isometric:

Example 1.2. Let N be a pair of pants with a (constant curvature -1) hyperbolic metric where the boundary components are geodesics with length L, and let $\{c_1, c_2\} \subset N$ be the fixed point set of the order 3 isometry of N. If L is sufficiently large (so that N looks like a bikini) then a closed geodesic in N cannot pass near $\{c_1, c_2\}$. This means that one can change the metric near $\{c_1, c_2\}$ without disturbing the marked length spectrum of N. Note that one can modify this example slightly so that the metric is flat in a neighborhood of the boundary geodesics.

Suppose M is a nonpositively curved graph manifold with a Seifert component M_i isometric to $N \times S^1$, where N is as in the example. One can change the metric on the N factor as in example 1.2 to get a Riemannian manifold M' so that the conditions of Theorem 1.1 hold (with $f = id$), but M' is not isometric to M.

In section 8 we give an example to show that the uniform sublinear divergence estimate in condition 3 cannot be improved to a bounded distance estimate as in the Gromov hyperbolic case.

We now sketch some of the main points in the proof of Theorem 1.1.

First consider a single nonpositively curved graph manifold M with geometric Seifert components $M_1, \dots, M_k$. The universal cover $\tilde{M}_i$ is isometric to $\tilde{N}_i \times \mathbb{R}$ – a nonpositively curved 3-manifold with a countable collection of totally geodesic boundary components isometric to $\mathbb{E}^2$. The universal cover $\tilde{M}$ of M is tiled by a countable collection of copies of the universal covers $\tilde{M}_i$ for $i = 1, \ldots k$; we call these subsets *vertex spaces*. We refer to boundary components of vertex spaces as *edge spaces*. Two vertex spaces are either disjoint, or intersect along an edge space. Let T be the incidence graph for the collection of vertex spaces: T is the graph which has one vertex for each vertex space, and an edge joining two vertices whenever the corresponding vertex spaces intersect. T is isomorphic to the Bass-Serre tree of the graph of groups associated with the decomposition $\amalg_i M_i \to M$ (see section 2.5). If $v \in V := Vertex(T)$ (resp. $e \in E := Edge(T)$) we will use the notation $\tilde{M}_v$ (resp. $\tilde{M}_e$) for the vertex space (resp. edge space) associated with v (resp. e); and we let $\tilde{M}_v \simeq \tilde{N}_v \times \mathbb{R}$ be the Riemannian product decomposition of $\tilde{M}_v$. It is not difficult to check (Lemma 3.23) that if $G_v := Stabilizer(\tilde{M}_v) \subset G \equiv \pi_1(M)$, then the center $Z(G_v)$ of G_v is isomorphic to $\mathbb{Z}$, and the fixed point set of $Z(G_v)$ in $\partial_\infty \tilde{M}$ is just $\partial_\infty \tilde{M}_v$; similarly, if $\tilde{M}_e$ is an edge space then the fixed point set of $\mathbb{Z}^2 \simeq G_e := Stabilizer(\tilde{M}_e) \subset G$ in $\partial_\infty \tilde{M}$ is $\partial_\infty \tilde{M}_e$.

Let $p \in \tilde{M}$ be an interior point of a vertex space, pick $\xi \in \partial_\infty M$, and let $\overline{p\xi}$ denote the geodesic ray starting at p which is asymptotic to ξ. The ray $\overline{p\xi}$ encounters a (possibly finite) sequence of vertex and edge spaces called the *itinerary* of $\overline{p\xi}$. The convexity of vertex and edge spaces forces the itinerary $v_0, e_1, v_1, e_2, \ldots$ of $\overline{p\xi}$ to be the sequence of successive vertices and edges of a geodesic segment or ray in T. In order to understand the rays with itinerary $v_0, e_1, v_1, e_2, \ldots$, we construct a piecewise flat complex $\mathcal{T}$ – a *template* – in $\tilde{M}$ as follows. First let $\gamma_i \subset \tilde{M}_{v_i}$ be a shortest geodesic from $\tilde{M}_{e_i}$ to $\tilde{M}_{e_{i+1}}$ for $i > 0$, and let γ_0 be a shortest path from $p \in \tilde{M}_{v_0}$ to $\tilde{M}_{e_1}$. For $i \geq 0$ define $\mathcal{S}_i \subset \tilde{M}_{v_i}$ to be the flat strip which is the union of the geodesics in $\tilde{M}_{v_i}$ which are parallel to the $\mathbb{R}$-factor of $\tilde{M}_{v_i}$ and which pass through γ_i. We define $\mathcal{T}$ to be the union of the edge spaces $\{\tilde{M}_{e_i}\}$ with the strips $\{\mathcal{S}_i\}$; then $\mathcal{T}$ is a Hadamard space with respect to the induced path metric. A key technical step in the proof of Theorem 1.1 is Theorem 5.1, which shows that for any geodesic ray $\overline{p\zeta}$ in the Hadamard space $\mathcal{T}$, there is a unique geodesic ray $\overline{p\zeta'}$ in $\tilde{M}$ with the property that for all $x \in \overline{p\zeta}$,

$$d_{\tilde{M}}(x, \overline{p\zeta'}) \leq \theta(d_{\mathcal{T}}(x,p))(1 + d_{\mathcal{T}}(x,p))$$

for some function $\theta : [0, \infty) \to [0, \infty)$ with $\lim_{r\to\infty} \theta(r) = 0$ which is independent of the choice of itinerary. Using Theorem 5.1 one finds that the set of boundary points $\zeta \in \partial_\infty \tilde{M}$ for which the ray $\overline{p\zeta}$ has a given infinite itinerary $v_0, e_1, \ldots$ is homeomorphic to the set boundary points $\zeta' \in \mathcal{T}$ so that the $\mathcal{T}$-ray $\overline{p\zeta'}$ passes through $\tilde{M}_{e_i}$ for every i. One sees (Proposition 7.3) that the latter is either a single point or is homeomorphic to a closed interval, depending on the geometry of $\mathcal{T}$ (which depends, in turn, on the choice of itinerary and the geometry of M).

We now consider a second nonpositively curved 3-manifold M', and use primes to denote the vertex spaces, edge spaces, etc for M'. Let $f : M \to M'$ be a homeomorphism as in Theorem 1.1, and identify the deck groups $G := \pi_1(M) \simeq \pi_1(M')$ via a lift $\tilde{f}$ of f. Then $\tilde{f}$ maps vertex (resp. edge) spaces of M homeomorphically to vertex (resp. edge) spaces of M', so we may use $\tilde{f}$ to identify the incidence tree T' with T. Suppose $\phi : \partial_\infty \tilde{M} \to \partial_\infty \tilde{M}'$ is a G-equivariant homeomorphism. Using the remarks about fixed point sets made above, it follows that $\phi(\partial_\infty \tilde{M}_v) = \partial_\infty \tilde{M}'_v$

and $\phi(\partial_\infty \tilde{M}_e) = \partial_\infty \tilde{M}'_e$ for every $v \in V$ and every $e \in E$. Also, if $p \in \tilde{M}_{v_0}$, and $v_0, e_1, v_1, e_2, \ldots$ is an infinite itinerary, then $\phi(S) = S'$ where $S \subset \partial_\infty \tilde{M}$ and $S' \subset \partial_\infty \tilde{M}'$ are the subsets corresponding to the itinerary $v_0, e_1, v_1, e_2, \ldots$ (Corollary 5.28); in particular, either S and S' are both points or they are both intervals. By considering all possible infinite itineraries and exploiting this correlation, we are able to see (section 7) that the invariants MLS_i, MLS'_i and τ_i, τ'_i must agree as in condition 1 of Theorem 1.1. Conversely, if condition 1 holds and $p \in \tilde{M}$, one shows (section 6) that for each itinerary the corresponding templates in $\tilde{M}$ and $\tilde{M}'$ have sufficiently similar geometry that their geodesics are "similar"; and this implies that $\tilde{f}$ extends to the compactifications as in 2 of Theorem 1.1.

Our main result generalizes Theorem 1.1 and applies to *admissible groups*, a class of (fundamental groups of) graphs of groups, see section 3.1 for the precise definition. When an admissible group G acts discretely and cocompactly on a Hadamard space X then we associate geometric data to each vertex group $G_v \subset G$ consisting of a class function $MLS_v : G_v \to \mathbb{R}^+$ and a homomorphism $\tau_v : G_v \to \mathbb{R}$ (see section 3.2).

Theorem 1.3. *Let $G \curvearrowright X$ be a discrete, cocompact, isometric action of an admissible group on a Hadamard space X. Then for every vertex v, MLS_v and τ_v are determined up to scale factors a_v and b_v by the topological conjugacy class of the boundary action $G \curvearrowright \partial_\infty X$, and vice-versa. If $G \curvearrowright X'$ is another such action, then the following are equivalent:*

1. $G \curvearrowright X$ and $G \curvearrowright X'$ have the same geometric data up to scale.

2. G-equivariant quasi-isometries $X \to X'$ extend canonically to the compactifications $X \cup \partial_\infty X \to X' \cup \partial_\infty X'$.

3. The boundary actions $G \curvearrowright \partial_\infty X$ and $G \curvearrowright \partial_\infty X'$ are G-equivariantly homeomorphic (by a unique[5] G-equivariant homeomorphism).

4. If $f : X \to X'$ is a G-equivariant quasi-isometry, then there is a function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{r\to\infty} \theta(r) = 0$ so that for every unit speed geodesic ray $\gamma : [0,\infty) \to X$ there is a ray $\gamma' : [0,\infty) \to X'$ with $d(f \circ \gamma(t), \gamma'(t)) < (1+t)\theta(t)$.

Furthermore, if there is a single scale factor s so that $MLS'_v = sMLS_v$ and $\tau'_v = s\tau_v$ for every vertex v, then the unique G-equivariant homeomorphism $\partial_\infty X \to \partial_\infty X'$ is an isometry with respect to the Tits metrics.

The authors proved theorem 1.3 while attempting to digest the negative answer to Gromov's question about boundary actions. A key factor in our example was the (unanticipated) presence of intervals in the Tits boundary. After the examples and their properties had been announced, similar structure was found in other manifolds, [HS98]. The paper [BS] also contains some discussion of the Tits boundary of universal covers of nonpositively curved graph manifolds.

Open questions. The results in this paper raise a number of questions. First of all, for each group G one may ask for a generalization of Theorem 1.3, where the

[5]It follows from the methods of [Bal95, III.3] that if $G \curvearrowright X$ is a cocompact isometric action on a Hadamard space, $g \in G$ is an axial isometry, and $\gamma_g \subset$ is an axis for g which does not bound a flat half plane, then the orbit of $\partial_\infty \gamma \subset \partial_\infty X$ under the action $G \curvearrowright \partial_\infty X$ is dense in $\partial_\infty X$. Hence the set of points in $\partial_\infty X$ which are the unique attracting fixed point of some element of G is dense in $\partial_\infty X$. Any G-equivariant homeomorphism $\partial_\infty X \to \partial_\infty X$ must fix this dense set pointwise, and must therefore be the identity. The uniqueness statement follows immediately from this.

geometric data MLS_v and τ_v are replaced with suitable substitutes. Our methods actually yield more information about the behavior of geodesics than is stated in Theorem 1.3 alone. We are able to give a good description of all the geodesic rays in the Hadamard space X in terms of concrete geometric information; it seems likely that other classes of groups are amenable to a similar treatment. The fundamental groups of the real-analytic manifolds considered in [HS98] are natural candidates for this, as they have structure similar to graph manifold groups. Here are two other questions:

1. What determines the (non-equivariant) homeomorphism type of $\partial_\infty X$, when X is a Hadamard space with an action $G \curvearrowright X$ by an admissible group G?

2. Does part 4 of Theorem 1.3 have an analog where rays are replaced by complete geodesics? This seems within reach.

Contents

2. Preliminaries

2.1. Coarse geometry

Let X and X' be metric spaces, and let $\Phi : X \to X'$ be a map.

Definition 2.1. 1. Φ is (L, A)-*Lipschitz* if for all $x_1, x_2 \in X$,

$$d(\Phi(x_1), \Phi(x_2)) \le Ld(x_1, x_2) + A.$$

Φ is *coarse Lipschitz* if it is (L, A)-Lipschitz for some L, $A > 0$.

2. Φ is an (L, A)-*quasi-isometric embedding* if it is (L, A)-Lipschitz and for all $x_1, x_2 \in X$,

$$d(\Phi(x_1), \Phi(x_2)) \ge L^{-1}d(x_1, x_2) - A.$$

The constants (L, A) will often be suppressed. A *quasi-geodesic* (respectively *segment/ray*) is a quasi-isometric embedding $\Phi : \mathbb{R} \to X$ (respectively $\Phi : [a, b] \to X$, $\Phi : [0, \infty) \to X$). We sometimes refer to the image of a quasi-geodesic as a quasi-geodesic.

3. Φ is an (L, A)-*quasi-isometry* if it is an (L, A)-quasi-isometric embedding and for all $x' \in X'$, $d(x', \Phi(X)) < A$.

4. Φ is a *D-Hausdorff approximation* if it is a $(1, D)$-quasi-isometry.

We will use the following well-known lemma:

Lemma 2.2. *If $G \curvearrowright X$ is a discrete, cocompact, isometric action of a group G on a length space X, then there is a G-equivariant quasi-isometry $\Phi : Cayley(G) \to X$, where $Cayley(G)$ is any Cayley graph of G.*

2.2. Hadamard spaces

We refer the reader to [Bal95] for the material recalled here.

Geodesics and the boundary. Let X be a locally compact Hadamard space. If p, $q \in X$ then $\overline{pq} \subset X$ denotes the segment from p to q. If $p, x, y \in X$ and $p \notin \{x, y\}$, then $\widetilde{\angle}_p(x, y)$ (respectively $\angle_p(x, y)$) denotes the comparison angle (respectively angle) of the triangle Δpxy at p. We will use $\partial_\infty X$ to denote the set of asymptote classes of geodesic rays in X, with the cone topology. If $p \in X$ and $\xi \in \partial_\infty X$, then $\overline{p\xi}$ denotes the ray leaving p in the asymptote class of ξ. $\bar{X} := X \cup \partial_\infty X$ denotes the usual compactification: a sequence $x_i \in \bar{X}$ converges if and only if for any basepoint $p \in X$ the sequence of geodesic segments/rays $\overline{px_i}$ converges in the compact open topology. We denote the Tits angle between $\xi_1, \xi_2 \in \partial_\infty X$ by $\angle_T(\xi_1, \xi_2)$, and $\partial_T X$ denotes the underlying set of $\partial_\infty X$ equipped with the Tits angle metric (which usually induces a topology different from the one defined above). The metric space $\partial_T X$ is a $CAT(1)$ space with respect to this metric. When $\xi_1, \xi_2 \in \partial_T X$ and $\angle_T(\xi_1, \xi_2) < \pi$ then there is a segment between ξ_1 and ξ_2 in $\partial_T X$, which we denote by $\overline{\xi_1\xi_2} \subset \partial_T X$. This segment is the limit set in $\bar{X}$ of any sequence of segments $\overline{x_1^k x_2^k}$ where x_i^k tends to infinity along a ray asymptotic to ξ_i. We will not use the Tits path metric. We recall that ∂_∞ and ∂_T behave nicely with respect to products: $\partial_\infty(X_1 \times X_2) = \partial_\infty X_1 \circ \partial_\infty X_2$ and $\partial_T(X_1 \times X_2) = \partial_T X_1 \circ \partial_T X_2$ where in the first case $\circ$ represents the topological join and in the second the $\frac{\pi}{2}$-metric join. We will use this in the case where $X_2 = \mathbb{R}$.

We will let $N_R(C)$ be the closed metric tubular neighborhood of radius R of a subset $C \subset X$. A closed convex subset $C \subset X$ is also a locally compact Hadamard space as is $N_R(C)$, since it is also convex.

Standard comparison arguments show the following.

Lemma 2.3. *Let X be a locally compact Hadamard space, and let $C \subset X$ be a closed convex subset. Then for any $R > 0$, $\xi \in \partial_\infty N_R C$, and $z \in C$ we have $\overline{z\xi_\infty} \subset C$. In particular, $\partial_\infty C = \partial_\infty N_R C$.*

One consequence is:

Lemma 2.4. *Let X be a locally compact Hadamard space, and let $C \subset X$ be a closed convex subset. If $p \in X$, $\xi_i \in \partial_\infty X$, and $\xi_i \to \xi_\infty$, $\overline{p\xi_i} \cap C \neq \emptyset$ for all i, then either $\overline{p\xi_\infty} \cap C \neq \emptyset$ or $\xi_\infty \in \partial_\infty C$.*

Proof. Pick $x_i \in \overline{p\xi_i} \cap C$. If $\liminf d(x_i, p) < \infty$ then a subsequence of x_i converges to $x_\infty \in C \cap \overline{p\xi_\infty}$. On the other hand, if $\liminf d(x_i, p) = \infty$ then for some subsequence $\overline{px_i} \to \overline{p\xi_\infty}$. By the convexity of $N_R(C)$, $\overline{p\xi_\infty} \subset N_R(C)$ for $R = d(p, C)$ and hence Lemma 2.3 yields the result. $\square$

Lemma 2.5. *Let $\Phi : X \to X'$ be a quasi-isometric embedding, and assume there is a point $x \in X$ and a function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{r\to\infty} \theta(r) = 0$ so that for every $y \in X$, $z \in \overline{xy}$, we have*

$$d_{X'}(\Phi(z), \overline{\Phi(x)\Phi(y)}) \leq (1 + d_X(z, x))\theta(d_X(z, x)). \tag{2.6}$$

Then there is a unique extension $\bar{\Phi} : \bar{X} \to \bar{X}'$ of Φ which is continuous at $\partial_\infty X$, and $\partial_\infty \Phi := \bar{\Phi}\big|_{\partial_\infty X}$ is a topological embedding.

Proof. Let Φ be an $(L, A)-$quasi isometric embedding, and $\xi \in \partial_\infty X$, and $y_k \in X$ be such that $\overline{xy_k}$ converges to $\overline{x\xi}$. By the convergence we can choose $R_k \to \infty$ so that for all $k \geq n$ we have $d(y_k, x) \geq R_n$ and $y_{kn} := \overline{xy_k} \cap S(x, R_n) \subset N_1(\overline{x\xi})$. Note that the point on ξ closest to y_{kn} lies in $\xi([R_n - 1, R_n + 1])$, hence by the triangle inequality $d(y_{kn}, y_{ln}) \leq 4$ for $k, l \geq n$. Using (2.6), for every $k \geq n$ choose $y'_{kn} \in \overline{\Phi(x)\Phi(y_k)}$ with $d_{X'}(y'_{kn}, \Phi(y_{kn})) \leq (1 + R_n)\theta(R_n)$. Then for every $k, l \geq n$ we have $d_{X'}(\Phi(y_{kn}), \Phi(y_{ln})) \leq 4L + A$, and so $d_{X'}(y'_{kn}, y'_{ln}) \leq 2(1 + R_n)\theta(R_n) + 4L + A$. This, along with the fact that $d(\Phi(x), y'_{kn}) \geq L^{-1}R_n - A - (1 + R_n)\theta(R_n)$, forces $\widetilde{\angle}_{\Phi(x)}(y'_{kn}, y'_{ln})$ to zero as $n \to \infty$. This in turn forces $\overline{\Phi(x)\Phi(y_k)}$ to converge to a ray $\overline{\Phi(x)\Phi(\xi)}$ since for each $R > 0$ we have for large enough k that the sequence $\{\overline{\Phi(x)\Phi(y_k)} \cap S(\Phi(x), R)\}$ is Cauchy and hence converges. This proves that Φ has a unique extension $\bar{\Phi} : \bar{X} \to \bar{X}'$ which is continuous at $\partial_\infty X$. The map $\partial_\infty \Phi := \bar{\Phi}\big|_{\partial_\infty X}$ clearly has the property that for all $\xi \in \partial_\infty X$ and all $y \in x\xi$,

$$d(\Phi(y), \overline{\Phi(x)\partial_\infty\Phi(\xi)}) \leq (1 + d(x, y))\theta(d(x, y)). \tag{2.7}$$

When $\xi_1, \xi_2 \in \partial_\infty X$ are distinct, the rays $\overline{x\xi_i}$ diverge linearly, and hence $\Phi(\overline{x\xi_1})$ and $\Phi(\overline{x\xi_2})$ diverge linearly since Φ is a quasi isometric embedding. Now if $\partial_\infty \Phi(\xi_1) = \partial_\infty \Phi(\xi_2)$ then (2.7) would imply that $\Phi(\overline{x\xi_1})$ and $\Phi(\overline{x\xi_2})$ would each diverge sublinearly from $\overline{\Phi(x)\partial_\infty \Phi(\xi_1)}$ and hence diverge sublinearly from each other. Thus we conclude that $\partial_\infty \Phi(\xi_1) \neq \partial_\infty \Phi(\xi_2)$. □

2.3. Groups acting on Hadamard spaces.

Let X be a Hadamard space. We denote the displacement function of an isometry $g : X \to X$ by d_g, and the infimum of d_g by δ_g. When g is axial, we let $Minset(g)$ denote the convex subset where d_g attains its minimum. We recall that $Minset(g)$ splits as a metric product $C \times \mathbb{R}$ where C is convex and g acts trivially on the C factor and by translation on the $\mathbb{R}$ factor.

Let $G \curvearrowright X$ be a discrete, cocompact, isometric action of a group G on a Hadamard space X. If $H \subset G$ is a subgroup isomorphic to $\mathbb{Z}^k$, we let $Minset(H) := \cap_{h \in H} Minset(h)$. We recall that $Minset(H) = \cap_{h \in S} Minset(h)$ for any generating set $S \subset H$, and that $Minset(H)$ splits isometrically as a metric product $C \times \mathbb{E}^k$ so that H acts trivially on the C factor, and as a translation lattice on the $\mathbb{E}^k$ factor. The centralizer $Z(H,G)$ of H in G preserves δ_h for every $h \in H$, and hence also $Minset(H)$. If $S \subset H$ is a finite generating set, then the function $\sum_{h \in S} \delta_h : X \to \mathbb{R}$ descends to a proper function on $X/Z(H,G)$; in particular, $Minset(H)/Z(H,G)$ is compact.

2.4. Gromov hyperbolic groups and spaces

For background on the material in this section see [Gro87], [GdlH90], and [CDP90]. Some standard facts that we will use: If a Gromov hyperbolic group G acts cocompactly on a Hadamard space X then (since X is then quasi-isometric to $Cayley(G)$ and Gromov hyperbolicity is a quasi isometry invariant) X is Gromov hyperbolic (i.e. δ-hyperbolic for some δ). Further $\partial_\infty X$ is homeomorphic to $\partial_\infty G$, and all infinite order elements $g \in G$ are axial. The Tits metric $\partial_T X$ is the discrete metric with any two distinct points having distance π.

In this section we make use of the Morse lemma for quasi-geodesic segments (see [Gro87, CDP90]):

Lemma 2.8. *(Morse Lemma) Given $\delta > 0$, $L > 0$ and $A \geq 0$ there is a constant $C = C(\delta, L, A)$ such that if γ_1 and γ_2 are (L, A)-quasi-geodesic segments with the same endpoints sitting in a δ-hyperbolic space, then their Hausdorff distance satisfies $d_{\mathcal{H}}(\gamma_1, \gamma_2) < C$.*

Two geodesics γ_1 and γ_2 in a Hadamard space X are *parallel* if they stay a bounded distance apart. The parallel set $P(\gamma) \subset X$ of a geodesic γ is the union of all geodesics parallel to γ. By the flat strip theorem, $P(\gamma)$ is a convex subset of X, and is isometric to $C_\gamma \times \mathbb{R}$ where $C_\gamma \subset X$ is convex. A bounded convex set C always contains a unique circumcenter: the center of the smallest metric ball containing C.

Lemma 2.9. *Let X be a δ-hyperbolic Hadamard space. Then*

1. If $\gamma \subset X$ is a geodesic and $P(\gamma) \simeq C_\gamma \times \mathbb{R}$ is its parallel set, then $Diam(C_\gamma) < \delta$. In particular $P(\gamma)$ contains a canonical geodesic $z \times \mathbb{R} \subset C_\gamma \times \mathbb{R}$ where $z \in C_\gamma$ is the circumcenter of C_γ.

2. If $\gamma_1, \gamma_2 \subset X$ are geodesics, $x_i \in \gamma_i$, then $\gamma_1 \cup \overline{x_1 x_2} \cup \gamma_2$ is 2δ-quasi-convex[6].

3. Suppose $\gamma_1, \gamma_2 \subset X$ are geodesics with $\partial_\infty \gamma_1 \cap \partial_\infty \gamma_2 = \emptyset$, and let η a minimal geodesic segment between γ_1 and γ_2. Then any geodesic segment running from γ_1 to γ_2 will pass within distance $D = D(\gamma_1, \gamma_2)$ of both endpoints of η; when $d(\gamma_1, \gamma_2) > 4\delta$ then we may take $D = 2\delta$.

Proof. 1 follows from the fact that a δ-hyperbolic Euclidean strip has width at most δ. 2 and 3 follow from repeated application of the δ-thinness property of geodesic triangles. ☐

In the following lemma is a slight variation on results from [Gro87]. It shows that discrete isometric actions on Gromov hyperbolic spaces behave like free group actions on trees.

Lemma 2.10. *Let X be a δ-hyperbolic Hadamard space, and let $\star \in X$. Suppose $(g_i)_{i\in\mathbb{Z}}$ is a periodic sequence of axial isometries of X with period k (i.e. $g_{i+k} = g_i$ for all i; and in particular $g_0 = g_k$ and $g_{-1} = g_{k-1}$), and let the attracting (respectively repelling) fixed point of g_i be $\xi_i^+ \in \partial_\infty X$ (respectively $\xi_i^- \in \partial_\infty X$). If for every i we have $\xi_i^- \neq \xi_{i+1}^+$, then there are constants L, A, N, and D with the following property.*

1. If $(m_i)_{i\in\mathbb{Z}}$ is a sequence with $m_i > N$, then the broken geodesic with vertices

$$\ldots, v_{-2} = (g_{-1}^{-m_{-1}} g_{-2}^{-m_{-2}})(\star),\ v_{-1} = g_{-1}^{-m_{-1}}(\star),\ v_0 = \star,\ v_1 = g_0^{m_0}(\star),\ v_2 = (g_0^{m_0} g_1^{m_1})(\star), \ldots \tag{2.11}$$

is an (L, A) quasi-geodesic, and

$$|d(v_i, v_{i+l}) - \sum_{j=i}^{i+l-1} m_j \delta_{g_j}| < lD \tag{2.12}$$

2. If $(m_i)_{i\in\mathbb{Z}}$ is a sequence with $m_i > N$ and period k, then $g := g_0^{m_0} \ldots g_{k-1}^{m_{k-1}}$ is an axial isometry with an axis γ within Hausdorff distance D of the g-invariant broken geodesic with vertices (2.11), and the minimal displacement of g satisfies

$$|\delta_g - (m_0 \delta_{g_0} + \ldots + m_{k-1} \delta_{g_{k-1}})| < D. \tag{2.13}$$

Furthermore, as $m_0 \to \infty$ (respectively $m_{k-1} \to \infty$), the attracting (respectively repelling) fixed point of g tends to ξ_0^+ (respectively ξ_{k-1}^-).

Proof. Since $\xi_i^- \neq \xi_{i+1}^+$ for all $i \in \mathbb{Z}$, there are constants L_1, A_1, and N_1, so that when $m_\pm > N_1$ then for any i the broken geodesic with vertices $g_i^{-m_-}(\star)$, $\star$, $g_{i+1}^{m_+}\star$ is an (L_1, A_1) quasi-geodesic segment. Let $(m_i)_{i\in\mathbb{Z}}$ be a sequence, and let $\eta : \mathbb{R} \to X$ be the broken geodesic with vertices (2.11). By the local characterization of quasi-geodesics given in [CDP90, Chapitre 3], we get constants $L = L(L_1, A_1, \delta)$, $A = A(L_1, A_1, \delta)$,

[6] A subset $Z \subset X$ is C-quasi-convex if for all $x, y \in Z$ we have $\overline{xy} \subset N_C(Z)$.

and $N_2 = N_2(L_1, A_1, \delta, \{g_i\}_{i\in\mathbb{Z}}) \geq N_1$ so that η is an (L, A) quasi-geodesic provided $m_i \geq N_2$ for all i.

We now assume that $m_i \geq N_2$ for all i. By the Morse lemma there is a $D_1 = D_1(L, A, \delta)$ so that there is a geodesic at Hausdorff distance at most D_1 from $\eta(\mathbb{R})$, and any geodesic $\gamma \subset X$ with $\partial_\infty\gamma = \partial_\infty\eta$ has Hausdorff distance at most D_1 from η. Fix such a geodesic $\gamma \subset X$, and for each $i \in \mathbb{Z}$ let $w_i \in \gamma$ be the point in γ nearest v_i. By the triangle inequality we have

$$|d(v_i, v_{i+1}) - d(w_i, w_{i+1})| \leq 2D_1. \tag{2.14}$$

Choose $c_1 = c_1(\star, \{g_i\})$ so that the distance from $\star$ to the nearest axis of g_i is less than c_1; then for all $i \in \mathbb{Z}$

$$|d(v_i, v_{i+1}) - m_i\delta_{g_i}| = |d(\star, g_i^{m_i}(\star)) - m_i\delta_{g_i}| < 2c_1. \tag{2.15}$$

Since for each i, the broken segment with vertices v_{i-1}, v_i, v_{i+1} is an (L_1, A_1) quasi-geodesic, the Morse Lemma gives

$$d(v_{i-1}, v_{i+1}) \geq d(v_{i-1}, v_i) + d(v_i, v_{i+1}) - 2D_1. \tag{2.16}$$

This gives

$$d(w_{i-1}, w_{i+1}) \geq d(w_{i-1}, w_i) + d(w_i, w_{i+1}) - 8D_1. \tag{2.17}$$

Therefore there is an $N = N(\star, \{g_i\}) \geq N_2$ so that if $m_i \geq N$ then w_i lies between w_{i-1} and w_{i+1} for all i. So when $m_i \geq N$ we have

$$|d(v_i, v_{i+l}) - \sum_{j=i}^{j+l-1} m_j\delta_{g_j}| \leq 2D_1 + |d(w_i, w_{i+l}) - \sum_{j=i}^{j+l-1} m_j\delta_{g_j}|$$

$$\leq 2D_1 + \sum_{j=i}^{j+l-1} |d(w_j, w_{j+1}) - m_j\delta_{g_j}| \leq 2D_1(l+1) + 2lc_1. \tag{2.18}$$

We now set $D := (2k+4)D_1 + 2kc_1$, and note that we have proved 1. When the sequence (m_i) has period k, $m_i \geq N$ for all i, and $g := g_0^{m_0} \dots g_k^{m_{k-1}}$, then we may take γ to be an axis for g. We have $\delta_g = d(w_0, w_k)$, and (2.13) follows from (2.18). The last assertion follows immediately from the fact that as $m_0 \to \infty$ and $m_{k-1} \to \infty$, the segments $\overline{\star g_0^{m_0}(\star)}$ and $\overline{\star g_{k-1}^{-m_{k-1}}(\star)}$ converge to the rays $\overline{\star\xi_0^+}$ and $\overline{\star\xi_{k-1}^-}$ respectively. $\square$

Lemma 2.19. *Let $G \overset{\rho}{\curvearrowright} X$ be a discrete, cocompact isometric action of a hyperbolic group G on a Hadamard space X. There is a constant $D = D(\delta, \rho)$ so that for every $x_1, x_2 \in X$ there is a $g \in G$ and an axis γ for g with $d(x_i, \gamma) < D$ for $i = 1, 2$, and $d(g(x_1), x_2) < D$.*

Proof. If G is elementary, then either G is finite (in which case the result holds trivially) or there is a hyperbolic element $g \in G$ with an axis γ so that $X = N_R(\gamma)$ for some R; this implies 2 in this case. So we may assume that G is nonelementary, and hence G does not fix any $\xi \in \partial_\infty X$.

Pick $\star \in X$ and a finite generating set $\Sigma \subset G$. Fix $\sigma_0 \in \Sigma$, let $\Sigma' = \{\sigma_0\} \cup \{\sigma\sigma_0 | \sigma \in \Sigma\}$, and let $C(\Sigma') = \min\{d(\star, \sigma'(\star)) | \sigma' \in \Sigma\}$.

We note that by the cocompactness of the action it is sufficient to prove the theorem when x_1 is $\star$; for then (with a larger D) if $g_1(x_i)$ is near $\star$ (within the diameter of the fundamental domain) and g is the solution for $\star$ and $g_1(x_2)$ then $g_1^{-1}gg_1$ works for x_1 and x_2 (since $g_1^{-1}(\gamma_g)$ is an axis for $g_1^{-1}gg_1$).

Claim. There are constants L_1, A_1 such that for all $g \in G$, there is a $\sigma' \in \Sigma'$ so that the broken geodesic with vertices $(g\sigma')^{-1})(\star)$, $\star$, $(g\sigma')(\star)$ is an (L_1, A_1)-quasi-geodesic.

Proof of claim. If not, there is a sequence $g_k \in G$ with $d(g_k(\star), \star) \to \infty$, so that for every $\sigma' \in \Sigma'$ the broken geodesic with vertices $(g_k\sigma')^{-1})(\star)$, $\star$, $(g_k\sigma')(\star)$ is not a (k,k)-quasi-geodesic. This clearly implies that after passing to a subsequence (which works for all σ'), the segments $\overline{\star[(g_k\sigma')(\star)]}$, $\overline{\star[(g_k\sigma')^{-1}(\star)]}$ converge to some ray $\overline{\star\xi(\sigma')}$. But since $d(g_k(\star), (g_k\sigma')(\star)) \leq C(\Sigma')$ we see $(g_k)(\star)$ converges to $\xi(\sigma')$, so $\xi(\sigma') = \xi$ is independent of σ'. The fact that $(g_k\sigma')^{-1}(\star)$ converges to ξ tells us (by applying σ' to the sequence) that $g_k^{-1}(\star)$ converges to $\sigma'(\xi) = \xi'$ which is again independent of the choice of σ'. Thus $\sigma'(\xi) = \xi'$ for all $\sigma' \in \Sigma$, and hence for every $\sigma \in \Sigma$, $\sigma(\xi') = (\sigma\sigma_0)\sigma_0^{-1}(\xi') = \xi'$, which is a contradiction. ☐

Proof of Lemma 2.19 continued. By the claim and an application of [CDP90, Chapitre 3] as in the proof of Lemma 2.10, we see that there are constants L, A, and D_1 so that if $g \in G$ and $d(g(\star), \star) > D_1$, then there is a $\sigma' \in \Sigma'$ so that the broken geodesic with i^{th} vertex $(g\sigma')^i(\star)$ is an (L, A)-quasi-geodesic. By the Morse Lemma and Lemma 2.9, $(g\sigma')$ has an axis at distance $< D_2(L, A, \delta)$ from $\star$ and $(g\sigma')(\star)$. Since $d((g\sigma')(\star), g(\star)) = d(\sigma'(\star), \star) \leq C(\Sigma')$, the lemma clearly follows. ☐

Lemma 2.20. *Consider two discrete, cocompact, isometric actions $G \overset{\rho}{\curvearrowright} X$, $G \overset{\rho'}{\curvearrowright} X'$ where X and X' are δ-hyperbolic Hadamard spaces. Assume that the minimum displacement of any $g \in G$ in X is the same as the minimal displacement in X'. Then any G-equivariant (L, A)-quasi-isometry $\Phi : X \to X'$ maps unit speed geodesics γ to within $D = D(L, A, \delta, \rho, \rho')$ of a unit speed geodesic γ', that is $d(\Phi(\gamma(t)), \gamma'(t)) \leq D$.*

Proof. By the Morse lemma on quasi-geodesics, it suffices to show that Φ is a $D_1 = D_1(L, A, \delta, \rho, \rho')$ Hausdorff approximation. Let $D_5 = \max\{D_4(\delta, \rho), D_4(\delta, \rho')\}$ where the D_4's come from Lemma 2.19. For $x_1, x_2 \in X$ (resp. $\Phi(x_1), \Phi(x_2) \in X'$) let $g \in G$ (resp $g' \in G$) be the elements guaranteed by Lemma 2.19. Since x_1 is D_5 close to an axis of g we know that $2D_5 + \delta_g \geq d(x_1, g(x_1)) \geq \delta_g$ and hence $3D_5 + \delta_g \geq d(x_1, x_2) \geq \delta_g - D_5$. Now

$$d(\Phi(x_1), \Phi(x_2)) \geq d(\Phi(x_1), \Phi(g(x_1))) - d(\Phi(g(x_1)), \Phi(x_2)) \geq$$

$$\geq \delta_g - LD_5 - A \geq d(x_1, x_2) - (L+3)D_5 - A$$

and similarly

$$d(x_1, x_2) \geq d(x_1, g'(x_1)) - d(g'(x_1), x_2) \geq$$

$$\geq \delta_{g'} - LD_5 - A \geq d(\Phi(x_1), \Phi(x_2)) - (L+3)D_5 - A.$$

Hence we can take $D_1 = (L+3)D_5 + A$. ☐

Lemma 2.21. *Let $G \overset{\rho}{\curvearrowright} X$ be a discrete, cocompact isometric action on a δ-Hyperbolic Hadamard space X.*

1. If $\{\gamma_g | g \in S \subset G\}$ is a collection of distinct axis of distinct elements $g \in G$ such that $\{\delta_g\}$ is bounded then $\{\gamma_g\}$ forms a discrete set of geodesics.

2. If two axial elements $g_1, g_2 \in G$ have a common fixed point in $\partial_\infty X$ then they have a common axis (i.e. they have both fixed points in common).

Proof. Assume some sequence γ_{g_i} converges to a geodesic γ and let $\star \in \gamma$ then the boundedness of $\{\delta_{g_i}\}$ says that there is a C such that $d(\star, g_i(\star)) < C$ but this cannot be true for infinitely many distinct g_i.

To see the second statement we can assume that the attracting and repelling fixed points satisfy $\xi_1^+ = \xi_2^+$ and $\xi_1^- \neq \xi_2^-$ (the other cases are similar). In this case $\{g_1^{-k} g_2 g_1^k\}$ are distinct since they have distinct fixed point sets $\{\xi_1^+, g_1^{-k}(\xi_2^-)\}$ in $\partial_\infty X$ while the axes converge (after taking a subsequence) to an axis of g_k. But again there is a C such that $d(\star, g_1^{-k} g_2 g_1^k(\star)) < C$ giving the desired contradiction.

□

2.5. Graphs of groups and their Bass-Serre trees

For the remainder of the paper, all group actions on simplicial trees will be assumed to be simplicial actions which do not invert edges, and geodesic segments/rays in simplicial trees will be unions of edges.

References for the material in this section are [Ser80, SW79, DD89].

Definition 2.22. A *graph of groups* is a connected graph $\mathcal{G}$ together with a group G_σ labeling each $\sigma \in Vertex(\mathcal{G}) \cup Edge(\mathcal{G})$, and a monomorphism $G_e \to G_v$ for each pair (e, v) consisting of an oriented edge e entering a vertex v. An isomorphism of two graphs of groups is an isomorphism of labeled graphs which is compatible with edge monomorphisms.

Let $G \overset{\rho}{\curvearrowright} T$ be an action of a group G on a simplicial tree T. We can define an associated graph of groups $\mathcal{G}$ as follows. We let the graph underlying $\mathcal{G}$ be G/T. For each $\sigma \in Vertex(\mathcal{G}) \cup Edge(\mathcal{G})$ we may label σ with the stabilizer of a lift $\hat{\sigma} \subset T$ of σ. An for each pair (e, v), where $e \subset T/G$ is an oriented edge with terminus $v \in T/G$, we can define an edge monomorphism $G_e \to G_v$ by composing the inclusion $G_e := G_{\hat{e}} \to G_{g\hat{v}}$ ($g\hat{v} \in T$ is the terminus of $\hat{e}$) with the isomorphism $G_{g\hat{v}} \to G_{\hat{v}} = G_v$ induced by conjugation by g^{-1}. We refer to this as the *graph of groups associated with the action $G \curvearrowright T$*.

Lemma 2.23. *If $\mathcal{G}$ is a graph of groups, then there is a group G, a simplicial tree T, and an action $G \overset{\rho}{\curvearrowright} T$ so that:*

1. If $\bar{\mathcal{G}}$ is the graph of groups associated with the action $G \overset{\rho}{\curvearrowright} T$, then $\bar{\mathcal{G}} \simeq \mathcal{G}$.

2. If $G' \overset{\rho'}{\curvearrowright} T'$ is another action on a simplicial tree satisfying 1, then there is an isomorphism $G' \simeq G$ so that the actions ρ and ρ' become simplicially isomorphic.

The (isomorphism class of the) group G is the *fundamental group of $\mathcal{G}$*, and the tree T (or really the action $G \overset{\rho}{\curvearrowright} T$) is called the Bass-Serre tree of $\mathcal{G}$. We note that if

v is a vertex of T and G_v is its stabilizer, then the G_v-orbits of $Link(v)$ correspond bijectively to the elements of $Link(\bar{v})$ where $\bar{v} \in \mathcal{G}$ is the corresponding vertex of $G/T \simeq \mathcal{G}$, and the stabilizer of $\xi \in Link(v)$ is just G_e where e is the edge associated with ξ.

Let $\mathcal{G}$ be a graph of groups and let $\bar{e}$ be an edge of $\mathcal{G}$ with endpoints $\bar{v}_1$ and $\bar{v}_2$. We let $\mathcal{G}'$ be the graph of groups determined by $\bar{e}$. The fundamental group of $\mathcal{G}'$ is a free product with amalgamation if $\bar{e}$ is embedded in $\mathcal{G}$ and an HNN extension if $\bar{e}$ is a loop. Choose a lift $e = \overline{v_1 v_2} \subset T$ of $\bar{e}$ to the Bass-Serre tree T. We may identify $G_{\bar{v}_i}$ with G_{v_i} and $G_{\bar{e}}$ with G_e in a fashion compatible with the edge inclusions $G_{\bar{e}} \to G_{\bar{v}_i}$, $G_e \to G_{v_i}$. When $\bar{e}$ is a loop we may choose $t \in G = \pi_1(\mathcal{G})$ so that $t(v_1) = v_2$ and the composition $G_e \to G_{v_1} \stackrel{t(\cdot)t^{-1}}{\longrightarrow} G_{v_2}$ agrees with the edge monomorphism $G_e \to G_{v_2}$. Set $G' := \langle G_{v_1}, G_{v_2} \rangle$ when $\bar{e}$ is embedded and set $G' := \langle G_{v_1}, t \rangle$ when $\bar{e}$ is a loop. Then the orbit $T' := G'(e) \subset T$ is a G'-invariant subtree of T, and the action $G' \curvearrowright T'$ is the Bass-Serre action for $\mathcal{G}'$. When $\bar{e}$ is embedded we choose subsets $\Sigma_i \subset G_{v_i}$ which intersect each right coset of G_e exactly once; then any $g \in G'$ can be written uniquely in the form

$$s_1 \dots s_k r \tag{2.24}$$

where $r \in G_e$, $s_i \notin G_e$, and the s_i's belong alternately to Σ_1 and Σ_2. The combinatorial distance from the edge $g(e)$ to e is k and $e_i = s_1 \dots s_i(e)$ is the sequence of edges along the path from e to $g(e)$. When $\bar{e}$ is a loop, we choose a cross-section $\Sigma_1 \subset G_{v_1}$ (respectively Σ_{-1}) of the right cosets of G_e (respectively $t^{-1}G_e t$). Then any $g \in G'$ can be written uniquely in the form

$$s_1 t^{\epsilon_1} s_2 t^{\epsilon_2} \dots s_k t^{\epsilon_k} r \tag{2.25}$$

where for $i = 1, \dots, k$, $\epsilon_i = \pm 1$, $s_i \in \Sigma_{\epsilon_i}$, $r \in G_{v_1}$, and if $s_i \in G_e$ then $\epsilon_{i-1} = -\epsilon_i$.

3. Graphs of groups and the structure of Hadamard spaces on which they act

3.1. Admissible groups and actions

Definition 3.1. A graph of groups $\mathcal{G}$ is *admissible* if

1. $\mathcal{G}$ is a finite graph with at least one edge.

2. Each vertex group $\bar{G}_v$ has center $Z(\bar{G}_v) \simeq \mathbb{Z}$, $\bar{H}_v := \bar{G}_v / Z(\bar{G}_v)$ is a nonelementary hyperbolic group, and every edge subgroup $\bar{G}_e$ is isomorphic to $\mathbb{Z}^2$.

3. Let e_1, e_2 be distinct directed edges entering a vertex v, and for $i = 1, 2$ let $K_i \subset \bar{G}_v$ be the image of the edge homomorphism $\bar{G}_{e_i} \to \bar{G}_v$. Then for every $g \in \bar{G}_v$, gK_1g^{-1} is not commensurable with K_2, and for every $g \in \bar{G}_v - K_i$, gK_ig^{-1} is not commensurable with K_i.

4. For every edge group $\bar{G}_e$, if $\alpha_i : \bar{G}_e \to \bar{G}_{v_i}$ are the edge monomorphisms, then the subgroup generated by $\alpha_1^{-1}(Z(\bar{G}_{v_1}))$ and $\alpha_2^{-1}(Z(\bar{G}_{v_2}))$ has finite index in $G_e \simeq \mathbb{Z}^2$.

A group G is *admissible* if it is the fundamental group of an admissible graph of groups.

Let G be the fundamental group of an admissible graph of groups $\mathcal{G}$, and let $G \curvearrowright T$ be the action of G on the associated Bass-Serre tree. We let $V := Vertex(T)$ and $E := Edge(T)$ denote the vertex and edge sets of T, and when $\sigma \in V \cup E$ we let $G_\sigma \subset G$ denote corresponding stabilizer. Properties 1-4 of definition 3.1 imply:

Lemma 3.2. *1. T is an unbounded tree with infinite valence at each vertex, and G acts on T with quotient $\mathcal{G}$.*

2. Each vertex group G_v has center $Z(G_v) \simeq \mathbb{Z}$, $H_v := G_v/Z(G_v)$ is a nonelementary hyperbolic group, and every edge subgroup G_e is isomorphic to $\mathbb{Z}^2$.

3. If e_1, e_2 are distinct edges emanating from $v \in V$, then G_{e_1} is not commensurable with G_{e_2}. In particular, $Z(G_v) \subset G_{e_i}$ since $g \in Z(G_v)$ implies $G_{e_i} = gG_{e_i}g^{-1} = G_{ge_i}$ forcing $ge_i = e_i$, i.e. $g \in G_{e_i}$.

4. If $e \in E$ has endpoints v_1, $v_2 \in V$, then $(Z(G_{v_1}) \cup Z(G_{v_2})) \subset G_e$ generates a finite index subgroup of G_e.

Most of the time we will work with the action $G \curvearrowright T$ and ignore the graph of groups that produced it.

Examples of admissible groups:

1. (Graph manifolds) Let M be a 3-dimensional nonpositively curved graph manifold as in Theorem 1.1, and let M_i, $i = 1, \dots k$ be the geometric Seifert components of M. Let $\mathcal{G}$ be the graph of groups which has one vertex labeled with $\pi_1(M_i)$ for each i, and an edge labeled by $\mathbb{Z}^2$ for each pair of totally geodesic boundary tori in the disjoint union $\cup_i M_i$ which are glued to form M. The edge monomorphisms come from the two different embeddings of a gluing torus into Seifert components.

2. (Torus complexes) Let T_0, T_1, T_2 be flat two-dimensional tori. For $i = 1, 2$, we choose primitive closed geodesics $a_i \subset T_0$ and $b_i \subset T_i$ with $length(a_i) = length(b_i)$, and we glue T_i to T_0 by identifying a_i with b_i isometrically. We assume that a_1 and a_2 lie in distinct free homotopy classes, and intersect at an angle $\alpha \in (0, \frac{\pi}{2}]$. Let $\mathcal{G}$ be the graph of groups associated with the decomposition $(T_0 \cup T_1) \amalg (T_0 \cup T_1) \to \cup T_i$. Note that $T_0 \cup T_1$ is homeomorphic to $S^1 \times (S^1 \wedge S^1)$, so $\pi_1(T_0 \cup T_i) = \mathbb{Z} \times F_2$ where F_2 is the free group on two generators.

Lemma 3.3. *1. If e_1, $e_2 \in E$ are distinct edges incident to $v \in V$, then $Z(G_v) \simeq \mathbb{Z}$ is a finite index subgroup of $G_{e_1} \cap G_{e_2}$. In particular $G_{e_1} \cap G_{e_2} \simeq \mathbb{Z}$.*

2. If $v_1, v_2 \in V$ are the endpoints of an edge $e \in E$, then $Z(G_{v_1}) \cap Z(G_{v_2}) = \{id\}$.

Proof. Note that $G_{e_1} \cap G_{e_2}$ has infinite index in each G_{e_i}, for otherwise the $G_{e_i} \simeq \mathbb{Z}^2$ would be commensurable, contradicting 3 of Lemma 3.2. Thus $G_{e_1} \cap G_{e_2} \simeq \mathbb{Z}$. Also by 3 of Lemma 3.2 we have $Z(G_v) \subset G_{e_1} \cap G_{e_2}$, so both are rank 1 free abelian groups and 1 follows.

2 follows immediately from 4 of Lemma 3.2, since the $Z(G_{v_i})$ are rank 1 subgroups of $G_e \simeq \mathbb{Z}^2$ which generate a finite index subgroup of G_e. □

Lemma 3.4. *If v_1, $v_2 \in V$, then*

1. If $d(v_1, v_2) > 2$ then $G_{v_1} \cap G_{v_2} = \{id\}$.

2. If $d(v_1, v_2) = 2$ and $v \in V$ is the vertex between them, then $Z(G_v)$ is a finite index subgroup of $G_{v_1} \cap G_{v_2}$.

3. If $d(v_1, v_2) = 1$, then $G_{v_1} \cap G_{v_2} = G_e$ where $e = \overline{v_1 v_2}$.

Proof. We will prove the assertions in reverse order. Part 3 is immediate since the action $G \curvearrowright T$ does not invert edges, see section 2.5. To prove 2 we let e_i be the edge between v_i and v. Clearly $G_{v_1} \cap G_{v_2} = G_{e_1} \cap G_{e_2}$, which by Lemma 3.3 contains $Z(G_v)$ as a subgroup of finite index. To prove 1, let $e_1, e, e_2 \in E$ be three consecutive edges of the segment $\overline{v_1 v_2}$, and let $w_1, w_2 \in V$ be the endpoints of e. Then by 1 of Lemma 3.3, $Z(G_{w_i})$ ($\simeq \mathbb{Z}$) has finite index in $G_{e_i} \cap G_e$ ($\simeq \mathbb{Z}$), so we have $(G_{e_1} \cap G_e) \cap (G_e \cap G_{e_2}) = \emptyset$ since otherwise $Z(G_{w_1}) \cup Z(G_{w_2})$ would generate a cyclic group in G_e ($\simeq \mathbb{Z}^2$) contradicting 4 of Lemma 3.2. Since

$$G_{v_1} \cap G_{v_2} \subset (G_{e_1} \cap G_e) \cap (G_e \cap G_{e_2})$$

1 follows. $\square$

Lemma 3.5. *For $v \in V$, the fixed point set of $Z(G_v)$ is the closed star $\overline{Star(v)}$. For $e \in E$, the fixed point set of G_e is e. In particular, for any $\sigma \in V \cup E$, G_σ leaves no point in $\partial_\infty T$ fixed. Further, if $\overline{Star(v)}$ is invariant under $Z(G_\sigma)$ then $\sigma \in \overline{Star(v)}$.*

Proof. Fix a vertex v. Since $Z(G_v) \subset G_e$ for all e in the star it is clear that the closed star is in the fixed point set of $Z(G_v)$. On the other hand 1 of Lemma 3.4 says we need only consider vertices v_1 such that $d(v, v_1) = 2$. Let w be the vertex between v and v_1. Now 2 of Lemma 3.4 says that $Z(G_w)$ is a finite index subgroup of $G_v \cap G_{v_1}$, while 2 of Lemma 3.3 says that $Z(G_v) \cap Z(G_w) = \emptyset$. Thus $Z(G_v) \cap G_{v_1} = \emptyset$ and the first statement follows.

For an edge $e = \overline{v_1 v_2}$ the first part (since $G_{v_i} \subset G_e$) says that the fixed point set of G_e is contained in $\overline{Star(v_1)} \cap \overline{Star(v_2)} = e$. So the second statement follows.

The last two statements follow from the first two. $\square$

Lemma 3.6. *If $v \in V$ then the centralizer of $Z(G_v)$ in G is just G_v. If $e \in E$ the centralizer of G_e in G is G_e.*

Proof. By Lemma 3.5, the fixed point set of $Z(G_v)$ in T is just the closed star of v in T. Hence any $g \in G$ which commutes with $Z(G_v)$ must take the star of v to itself and hence fix v.

If $e \in E$ and $e = \overline{v_1 v_2}$, then a finite index subgroup G_e is generated by $Z(G_{v_1}) \cup Z(G_{v_2})$. So the centralizer of G_e in G is a subgroup of the intersection of the centralizers of $Z(G_{v_1})$ and $Z(G_{v_2})$, i.e. $G_{v_1} \cap G_{v_2}$ which is G_e itself. $\square$

Lemma 3.7. *(Uniqueness of decomposition) Let $G \curvearrowright T$ and $G' \curvearrowright T'$ be the Bass-Serre actions associated with two admissible graphs of groups, and suppose $G \xrightarrow{\phi} G'$ is an isomorphism. Then after identifying G with G' via ϕ, the trees T and T' become G-equivariantly isomorphic.*

Proof. We will use primes to denote the vertex and edge set of T'. Pick $v \in V$.

Claim: G_v fixes a unique vertex in T'. Let $g \in Z(G_v)$ be a generator. If $Fix(g, T')$ is empty, then g translates a unique geodesic $\gamma \subset T'$, and since $g \in Z(G_v)$ the whole vertex group G_v must preserve γ, and act on it by translations. The signed translation distance yields a homomorphism $G_v \to \mathbb{Z}$ with nontrivial kernel. But then

$Ker(G_v \to \mathbb{Z})$ fixes γ pointwise, which contradicts 1 of Lemma 3.4. Consequently $Fix(g, T')$ is nonempty, and by 1 of Lemma 3.4 this is a subcomplex of T' with diameter at most 2. So G_v must fix the center of $Fix(g, T')$. It can fix nothing more, since no edge stabilizer can contain the nonabelian G_v. Thus we have proved the claim.

Now consider the G-equivariant map $f : V \to V'$ which assigns to each $v \in V$ the unique vertex in T' fixed by G_v; and define a map $f' : V' \to V$ by reversing the roles of T and T'. For all $v \in V$, G_v fixes $f' \circ f(v)$, so we must have $f' \circ f(v) = v$; similar reasoning applies to $f \circ f'$, and we see that f and f' are inverses. The maps f and f' are adjacency preserving since two vertices are adjacent iff their stabilizers intersect in a subgroup isomorphic to $\mathbb{Z}^2$. It is now straightforward to see that f defines a G-equivariant isomorphism $T \to T'$. $\square$

Lemma 3.7 justifies use of the phrase "$G \overset{\rho}{\curvearrowright} T$ is the Bass-Serre tree of the admissible group G."

3.2. Vertex spaces, edge spaces, and geometric data for admissible actions

Definition 3.8. We say that $G \overset{\rho}{\curvearrowright} X$ is an *admissible action* if G is an admissible group, X is a Hadamard space, and the action is discrete, cocompact, and isometric.

For the remainder of this section $G \curvearrowright X$ will be a fixed admissible action. In particular, all constants depend on $G \curvearrowright X$ (i.e. the group G, the Riemannian manifold X, and the action) in addition to other explicitly mentioned quantities. By Lemma 3.7 there is an essentially unique admissible graph of groups associated with G, and we will let $G \curvearrowright T$ be the corresponding Bass-Serre tree.

We refer the reader to section 2.3 for properties of Minsets that we use here. For each $v \in V$ we let $Y_v := Minset(Z(G_v)) := \cap_{g \in Z(G_v)} Minset(g)$ (this will be the $Minset$ of a generator), and for every $e \in E$ we let $Y_e := Minset(G_e) := \cap_{g \in G_e} Minset(g)$.

$Minset(\alpha)$ is a convex subset of X, invariant under the centralizer of α, which is a metric product of $\mathbb{R}$ with a Hadamard space. If α belongs to a group of isometries that acts cocompactly on X then the centralizer of α acts cocompactly on $Minset(\alpha)$ (see section 2.3). Thus Y_v is the product of $\mathbb{R}$ with a Hadamard space $\bar{Y}_v$. $Z(G_v)$ acts by translation on the $\mathbb{R}$ factor and the induced action of H_v on $\bar{Y}_v$ is discrete and cocompact. Y_e is the product of $\mathbb{R}^2$ with a compact Hadamard space $\bar{Y}_e$, and $G_e = \mathbb{Z}^2$ acts by translations on the $\mathbb{R}^2$ factor (section 2.3).

Note that the assignments $v \mapsto Y_v$ and $e \mapsto Y_e$ are G-equivariant with respect to the natural G actions. The minimal displacement of a generator of $Z(G_v)$ is the same as that of a generator of $Z(G_{g(v)}) = gZ(G_v)g^{-1}$. By the finiteness of $\mathcal{G}$ there is a number C such that for all $v \in V$ the minimal displacement of a generator of $Z(G_v)$ is less than C.

Definition 3.9. Let $G \overset{\rho}{\curvearrowright} X$ be an admissible action, and let T be the Bass-Serre tree for G. For each $v \in V$ we choose a generator $\zeta_v \in Z(G_v)$ in a G-equivariant way. We have an isometric splitting $Y_v \simeq \bar{Y}_v \times \mathbb{R}$, which is preserved by G_v. The choice of generator ζ_v defines an orientation of the $\mathbb{R}$ factor of Y_v. We have a map $MLS_v : G_v \to \mathbb{R}_+$ which assigns to each $g \in G_v$ the minimum displacement of the induced isometry $\bar{Y}_v \to \bar{Y}_v$. MLS_v descends to $G_v/Z(G_v) \simeq H_v$ since $Z(G_v)$ acts trivially on $\bar{Y}_v$. We define a homomorphism $\tau_v : G_v \to \mathbb{R}$ by sending $g \in G_v$ to the signed distance that g translates the $\mathbb{R}$ factor of $Y_v \simeq \bar{Y}_v \times \mathbb{R}$. The collections of functions MLS_v and τ_v constitute the *geometric data* of the action. Both MLS_v and τ_v descend to functions of the vertex groups of the graph of groups $\mathcal{G}$ defining G; we will sometimes find it more convenient to think of the geometric data in this way.

We remark that it follows from the discreteness of the action $H_v \curvearrowright \bar{Y}_v$ that $g \in G_v$ then $MLS_v(g) = 0$ iff g projects to an element of finite order in H_v.

Lemma 3.10. *The collections $\{Y_v\}_{v \in V}$ and $\{Y_e\}_{e \in E}$ are locally finite. More precisely, for every R there is an N so that if $x \in X$ then there are at most N elements $\sigma \in V \cup E$ so that $Y_\sigma \cap B(x, R) \neq \emptyset$.*

Proof. Suppose $v \in V$ and $p \in Y_v$. Then p has displacement $< C$ under the generators of $Z(G_v)$. Therefore if $p \in B(x, R)$, then x has displacement $< 2R + C$ under the generators of $Z(G_v)$. But there are only finitely many $g \in G$ with $d(g \cdot x, x) < 2R+C$; since $Z(G_{v_1}) \cap Z(G_{v_2}) = \{e\}$ when $v_1 \neq v_2$, the local finiteness of $\{Y_v\}_{v \in V}$ follows. Similar reasoning proves the local finiteness of $\{Y_e\}_{e \in E}$. The fact that N can be chosen independent of x follows from the cocompactness of the G action. □

The lemma implies that for any D, the collection of D-tubular neighborhoods of the Y.'s is locally finite. We also have the following consequences:

Lemma 3.11. *For every D there are only finitely many pairs $(\sigma_1, \sigma_2) \in (V \cup E) \times (V \cup E)$ – modulo the diagonal action of G – with $N_D(Y_{\sigma_1}) \cap N_D(Y_{\sigma_2}) \neq \emptyset$.*

Proof. By the finiteness of $[V \cup E]/G$ we need only show that for fixed σ there are only finitely many σ_2 modulo G_σ such that $N_D(Y_\sigma) \cap N_D(Y_{\sigma_2}) \neq \emptyset$. This follows from Lemma 3.10, since G_σ acts cocompactly on $N_D(Y_\sigma)$ and hence for some $g \in G_\sigma$ $N_D(Y_\sigma) \cap (N_D(Y_{g(\sigma_2)})$ intersects a fixed ball. □

Lemma 3.12. *For $\sigma_1, \sigma_2 \in V \cup E$, $G_{\sigma_1} \cap G_{\sigma_2}$ acts cocompactly on the intersection $N_D(Y_{\sigma_1}) \cap N_D(Y_{\sigma_2})$. Thus, in particular, the diameter of $N_D(Y_{\sigma_1}) \cap N_D(Y_{\sigma_2})/[G_{\sigma_1} \cap G_{\sigma_2}]$ is uniformly bounded by a function of D.*

Proof. This follows from the local finiteness of the family $\{Y_\sigma\}_{\sigma \in V \cup E}$ and the discreteness of the cocompact action $G \curvearrowright X$. Pick $D > 0$ and $\sigma_1, \sigma_2 \in V \cup E$. If $x_k \in N_D(Y_{\sigma_1}) \cap N_D(Y_{\sigma_2})$, we may choose a sequence $g_k \in G_{\sigma_1}$ such that $g_k(x_k) \to x_\infty$ for some $x_\infty \in N_D(Y_{\sigma_1})$. Then $g_k(\sigma_2)$ lies in a finite subset of $V \cup E$ (since $g_k(N_D(Y_{\sigma_2}))$ intersects some ball $B(x_\infty, R)$ for all k) so after passing to a subsequence if necessary we may assume that $g_k\sigma_2$ is constant. Then $g_1^{-1}g_k \in G_{\sigma_1} \cap G_{\sigma_2}$ and $(g_1^{-1}g_k)(x_k) \to g_1^{-1}(x_\infty)$. Thus $G_{v_1} \cap G_{v_2}$ acts cocompactly. The second statement now follows from Lemma 3.11.

□

Lemma 3.13. *For every D there is a D' (depending only on D) such that if $\sigma \in V \cup E$ separates $\sigma_1 \in V \cup E$ from $\sigma_2 \in V \cup E$, then $N_D(Y_{\sigma_1}) \cap N_D(Y_{\sigma_2}) \subset N_{D'}(Y_\sigma)$. In particular, if T_1 and T_2 are the closures of distinct connected components of $T - \sigma$ then*

$$[\cup_{\hat{\sigma} \subset T_1} N_D(Y_{\hat{\sigma}})] \cap [\cup_{\hat{\sigma} \subset T_2} N_D(Y_{\hat{\sigma}})] \subset N_{D'}(Y_\sigma).$$

Proof. Pick $D > 0$. Suppose $(\sigma_1, \sigma_2, \sigma)$ is a triple with σ_i, $\sigma \in V \cup E$, σ separates σ_1 from σ_2 in T, and $N_D(Y_{\sigma_1}) \cap N_D(Y_{\sigma_2}) \neq \emptyset$. Then $G_{\sigma_1} \cap G_{\sigma_2} \subset G_\sigma$ and $G_{\sigma_1} \cap G_{\sigma_2}$ acts cocompactly on $N_D(Y_{\sigma_1}) \cap N_D(Y_{\sigma_2})$ by Lemma 3.12; hence $d(Y_\sigma, \cdot)$ is bounded on $N_D(Y_{\sigma_1}) \cap N_D(Y_{\sigma_2})$. By Lemma 3.11 there are only finitely many such triples $(\sigma_1, \sigma_2, \sigma)$ modulo G, so the lemma follows. □

Definition 3.14. Since G acts cocompactly on X we can now fix a D so that $\cup_{v \in V} N_D(Y_v) = \cup_{e \in E} N_D(Y_e) = X$. We define $X_v := N_D(Y_v)$ for all $v \in V$. Let D' denote the constant in the previous lemma, $D'' = \max(D, D')$, and set $X_e := N_{D''}(Y_e)$ for all $e \in E$. We will refer to the X_v's and X_e's as *vertex spaces* and *edge spaces* respectively.

We note that Lemma 2.3 implies that for any $\sigma \in V \cup E$ we have $\partial_\infty X_\sigma = \partial_\infty Y_\sigma$. We summarize the properties of vertex and edge spaces:

Lemma 3.15. *There is a constant C_1 with the following property.*

1. $\cup_{v \in V} X_v = \cup_{e \in E} X_e = X$.
2. *If $\hat{e} \in E$ and T_1 and T_2 are the distinct connected components of $T - Int(\hat{e})$, then $[\cup_{v \in T_1} X_v] - X_{\hat{e}}$ and $[\cup_{v \in T_2} X_v] - X_{\hat{e}}$ are disjoint closed and open subsets of $X - X_{\hat{e}}$; and $\cup_{e \in T_1} X_e - N_{C_1}(X_{\hat{e}})$ and $\cup_{e \in T_2} X_e - N_{C_1}(X_{\hat{e}})$ are disjoint closed and open subsets of $X - N_{C_1}(X_{\hat{e}})$.*
3. *If σ_1, $\sigma_2 \in V \cup E$ and $X_{\sigma_1} \cap X_{\sigma_2} \neq \emptyset$ then $d_T(\sigma_1, \sigma_2) < C_1$.*

Proof. 1 and 2 follow from the definition of vertex/edge spaces and Lemma 3.13. By Lemma 3.11 we can choose C_1 so that 3 holds. □

Corollary 3.16. *For any $v \in V$, $\partial_T X_v$ is isometric to the metric suspension of an uncountable discrete space, and for every $e \in E$, $\partial_T X_e$ is isometric to a standard circle. Pick v_1, $v_2 \in V$.*

1. *If $d(v_1, v_2) > 2$, then $\partial_T X_{v_1} \cap \partial_T X_{v_2} = \emptyset$.*
2. *If $d(v_1, v_2) = 2$ and v is the vertex in between v_1 and v_2, then $\partial_T X_{v_1} \cap \partial_T X_{v_2} = \partial_T \gamma$ where $\gamma \subset Y_v$ is a geodesic of the form $\{p\} \times \mathbb{R} \subset \bar{Y}_v \times \mathbb{R} = Y_v$; i.e. $\partial_T X_{v_1} \cap \partial_T X_{v_2}$ is the pair of suspension points of $\partial_T X_v$.*
3. *If $d(v_1, v_2) = 1$, then $\partial_T X_{v_1} \cap \partial_T X_{v_2} = \partial_T X_e \simeq S^1$, where $e := \overline{v_1 v_2}$.*

Proof. Since $Y_v \simeq \bar{Y}_v \times \mathbb{R}$, we have $\partial_T X_v = \partial_T Y_v = \Sigma(\partial_T \bar{Y}_v)$, and since $\bar{Y}_v$ admits a discrete cocompact action by the non-elementary hyperbolic group $H_v := G_v / Z(G_v)$, $\partial_T Y_v$ is a discrete set with the cardinality of $\mathbb{R}$. For all $e \in E$, $Y_e \simeq \bar{Y}_e \times \mathbb{R}^2$ where $\bar{Y}_e$ is compact, so $\partial_T X_e = \partial_T Y_e \simeq \partial_T \mathbb{R}^2$, and the latter is the standard circle.

Pick v_1, $v_2 \in V$, and choose R large enough that $Z := N_R(X_{v_1}) \cap N_R(X_{v_2}) \neq \emptyset$. Then $\partial_T X_{v_1} \cap \partial_T X_{v_2} = \partial_T Z$. The lemma now follows from Lemmas 3.4 and 3.12. □

Lemma 3.17. *There is a constant C_2 with the following property. Suppose v, $v' \in V$, $e_1, \ldots, e_n \in E$ are the consecutive edges of the segment $\overline{vv'} \subset T$, $x \in X_v$ and $y \in X_{v'}$. Then for $1 \leq i \leq n$ we can find points $z_i \in \overline{xy}$ such that*

1. $d(z_i, X_{e_i}) < C_2$
2. *For all $1 \leq i \leq j \leq n$ we have $d(z_i, x) \leq d(z_j, x)$.*
3. *For every $p \in X$ we have $\#\{z_i \in B(p, 1)\} < C_2$.*

Proof. Pick $v,\, v' \in V$, $x \in X_v$, and $y \in X_{v'}$. Suppose $\hat{e} \in E$ and let T_1 and T_2 be the two connected components of $T - Int(\hat{e})$. If $\overline{xy} \cap N_{C_1}(X_{\hat{e}}) = \emptyset$ (hence in particular $\overline{xy} \cap X_{\hat{e}} = \emptyset$) then by the first part of 2 of Lemma 3.15, $\overline{xy}$ is contained in one of the two disjoint open sets $(\cup_{e \subset T_i} X_e) - N_{C_1}(X_{\hat{e}})$ for $i = 1$ or $i = 2$. It follows that $\overline{xy} \cap N_{C_1}(X_{e_i})$ is nonempty for every $1 \leq i \leq n$. Let $w_i \in \overline{xy}$ be the point in $\overline{xy} \cap N_{C_1}(X_{e_i})$ closest to x. Let $z_1 = w_1$, and let z_i be the element of $\{w_i, \ldots, w_n\}$ closest to x. So we have either $z_i = w_i$, or $z_i = w_{i'}$ for some $i' > i$. In the latter case $\overline{xz_i} \subset \cup_{e \subset T'} X_e$ (hence in particular $z_i = w_{i'} \in \cup_{e \subset T'} X_e$) where $T' \subset T$ is the component of $T - Int(e_i)$ containing e_1, so by Lemma 3.13 we have $z_i \in N_{D'}(X_{e_i})$ where D' depends only on C_1. If $p \in X$ and $z_i \in B(p,1)$, then $X_{e_i} \cap B(p, 1 + D') \neq \emptyset$ and thus $Y_{e_i} \cap B(p, 1 + D' + D'') \neq \emptyset$ so by Lemma 3.10 we have $\#\{z_i \in B(p,1)\} < N$ where N depends only on D'. Setting $C_2 := \max\{D', N\}$, the lemma follows. $\square$

3.3. Itineraries

Our next objective is to associate an itinerary to any ray $\overline{p\xi} \subset X$ which is not contained in a finite tubular neighborhood of a single vertex space; the itinerary of $\overline{p\xi}$ is a ray in T which (roughly speaking) records the sequence of vertex spaces visited by $\overline{p\xi}$.

Let $\rho : X \to V \subset T$ be a G-equivariant coarse Lipschitz map from the Hadamard space X to the vertex set of the tree T with the property that for every $x \in X$ we have $x \in X_{\rho(x)}$. Such a ρ may be constructed as follows. Let $\Sigma \subset X$ be a set theoretic cross-section for the free action $G \curvearrowright X$; define $\rho_0 : \Sigma \to T$ so that $\sigma \in X_{\rho_0(\sigma)}$ for every $\sigma \in \Sigma$, and then extend ρ_0 to an equivariant map $X \to T$. Let L be such that $N_{2D}(Y_{v_1}) \cap N_{2D}(Y_{v_2}) \neq \emptyset$ implies $d(v_1, v_2) < L$, which exists by Corollary 3.11. In particular if $d(x,y) < 2D$ then $d(\rho(x), \rho(y)) \leq L$. In general, by dividing $\overline{xy}$ into less than $\frac{d(x,y)}{2D} + 1$ segments of length less than $2D$ and adding the previous estimates we see that ρ will be coarse Lipschitz; i.e. $d(\rho(x), \rho(y)) \leq \frac{L}{2D}\, d(x,y) + L$.

Lemma 3.18. *If $\gamma : [0,\infty) \to X$ is a geodesic, then $\rho \circ \gamma : [0, \infty) \to T$ has the bounded backtracking property*[7].

Proof. Let $e \in E$ be an edge in T, and let T_1, $T_2 \subset T$ be the connected components of $T - Int(e)$. Suppose $\rho \circ \gamma(t_1) \in T_1$, and $\rho \circ \gamma(t_2) \in T_2 - N_{C_1}(e)$, where $t_2 > t_1$. By Lemma 3.13 we have

$$[\cup_{v \in T_1} X_v] \cap [\cup_{v \in T_2} X_v] \subset X_e.$$

Therefore there is a $t_3 \in [t_1, t_2)$ such that $\gamma(t_3) \in X_e$. Since $d(\rho \circ \gamma(t_2), e) \geq C_1$, the choice of C_1 and lemma 3.15 implies that $\gamma(t_2) \notin X_e$. Hence the convexity of X_e gives $\gamma([t_2, \infty)) \subset X - X_e$, which forces $\rho \circ \gamma([t_2, \infty)) \subset T_2$. This property clearly implies uniformly bounded backtracking. $\square$

Lemma 3.19. *If $\gamma : [0, \infty) \to X$ is a geodesic ray, then one of the following holds:*

1. $\rho \circ \gamma : [0, \infty) \to T$ is unbounded, and $\rho \circ \gamma([0, \infty))$ lies in a uniform tubular neighborhood of a unique geodesic ray, τ, in T starting at $\rho(\gamma(0))$. The geodesic γ intersects X_e for all but finitely many edges e of τ.

[7] A map $c : [0, \infty) \to T$ has the *bounded backtracking property* if for every $r \in (0, \infty)$ there is an $r' \in (0, \infty)$ such that if $t_1 < t_2$, and $d(c(t_1), c(t_2)) > r'$, then $d(c(t), c(t_1)) > r$ for every $t > t_2$.

2. $\rho \circ \gamma : [0, \infty) \to T$ is bounded, and γ eventually lies in $N_{D'}(Y_v)$ (where D' comes from Definition 3.14) for some $v \in V$. In this case there is a subcomplex $T_\gamma \subset T$ defined by the property that for each simplex σ in T, $\sigma \in T_\gamma$ if and only if γ is asymptotic to X_σ . The possibilities for T_γ are: a single vertex $v \in V$, a single edge $e \in E$ along with its vertices, or the closed star $\overline{Star(v)}$ for some $v \in V$.

Proof. Pick $v \in V$. By the convexity of $N_{D'}(Y_v)$, either γ is eventually contained in $N_{D'}(Y_v)$, or γ is eventually contained in $X - N_{D'}(Y_v)$. In the latter case $\rho \circ \gamma$ eventually remains in a unique component of $T - v$, by Lemma 3.13.

If for every $v \in V$ the ray γ eventually lies in $X - N_{D'}(Y_v)$, then clearly $\rho \circ \gamma$ is unbounded and hence it must lie within uniform distance of a ray in T by the bounded backtracking property. So we may assume that γ is eventually contained in $N_{D'}(Y_v)$ for some $v \in V$. We note that if $e \in T_\gamma$ then any vertex v' of e must also be in T_γ since $Y_e \subset Y_{v'}$. Also if $v, v' \in V$ and $d(v, v') > 2$ then part 1 of Lemma 3.4 along with Lemma 3.12 says that for any K, $N_K X_v \cap N_K X_{v'}$ is compact so $\gamma \notin N_K X_v \cap N_K X_{v'}$ and hence at most one of v and v' can be in T_γ.

If there are vertices v_1 and v_2 in T_γ with $d(v_1, v_2) = 2$ and v is the vertex between them then part 1 of Lemma 3.4 along with Lemma 3.12 says that $Z(G_v)$ acts cocompactly on $N_K X_{v_1} \cap N_K X_{v_2}$ which contains γ for some K, and hence there is a K' such that for all $t > 0$ there is a $g_t \in Z(G_v)$ such that $d(\gamma(t), g_t(\gamma(0)) < K'$ and hence γ stays a distance at most $K' + d(\gamma(0), Y_v)$ from a geodesic in the $\mathbb{R}$ direction of $Y_v = \bar{Y}_v \times \mathbb{R}$ (since $Z(G_v)$ translates the $\mathbb{R}$ direction). Thus for every e with v as a vertex we have γ is asymptotic to a geodesic in Y_e and hence $e \in T_\gamma$. Thus $\overline{Star(v)} \subset T_\gamma$. But since vertices in T_γ are at most distance 2 apart we see that $\overline{Star(v)} = T_\gamma$.

The only cases left for T_γ are the two mentioned and the case of two vertices a unit distance apart. But in the final case a similar argument shows that if e is the edge between them then γ stays a bounded distance from Y_e and hence e must also be in T_γ. $\square$

Definition 3.20. Let γ be a geodesic ray in X. If case 1 of Lemma 3.19 applies then we will say that γ has *itinerary* τ, and otherwise we say that the itinerary of γ is the subtree $T_\gamma \subset T$ described in case 2 of the lemma. In either case we denote the itinerary of γ by $\mathcal{I}(\gamma)$.

One immediate consequence of the proof of Lemma 3.19 is

Corollary 3.21. *If $\mathcal{I}(\gamma) = \overline{Star(v)}$ then γ is asymptotic to either the positive or the negative $\mathbb{R}$ direction in the decomposition $Y_v = \bar{Y}_v \times \mathbb{R}$.*

Lemma 3.22. *If $\overline{p_1\xi}$, $\overline{p_2\xi} \subset X$ are asymptotic geodesic rays, then either both $\mathcal{I}(\overline{p_1\xi})$ and $\mathcal{I}(\overline{p_2\xi})$ are finite subtrees, in which case they agree, or both $\mathcal{I}(\overline{p_1\xi})$ and $\mathcal{I}(\overline{p_2\xi})$ are rays, in which case $\partial_\infty \mathcal{I}(\overline{p_1\xi}) = \partial_\infty \mathcal{I}(\overline{p_2\xi})$. In other words, $\mathcal{I}(\overline{p_1\xi})$ is a ray in T if and only if $\mathcal{I}(\overline{p_2\xi})$ is a ray in T asymptotic to $\mathcal{I}(\overline{p_1\xi})$.*

Proof. $\overline{p_1\xi}$ lies in a tubular neighborhood of some Y_v if and only if $\overline{p_2\xi}$ does, thus the case where $\mathcal{I}(\overline{p_1\xi})$ (or $\mathcal{I}(\overline{p_2\xi})$) is finite follows. Thus $\overline{p_1\xi}$ has itinerary a ray τ_1 if and only if $\overline{p_2\xi}$ has itinerary τ_2 for some ray $\tau_2 \subset T$. But the sets $\rho(\overline{p_1\xi})$ and $\rho(\overline{p_2\xi})$ are at finite Hausdorff distance from one another since ρ is coarse Lipschitz; hence the τ_i are asymptotic. $\square$

By the lemma we have a well-defined G-equivariant map from $\partial_\infty X$ to the union

$$\partial_\infty T \cup (\text{finite subsets of } T)$$

which assigns to each $\xi \in \partial_\infty X$ either $\partial_\infty \mathcal{I}(\overline{p\xi})$, $p \in X$ if $\mathcal{I}(\overline{p\xi})$ is a ray or $\mathcal{I}(\overline{p\xi})$ otherwise; we will also denote this map by $\mathcal{I}$. If $\eta \in \partial_\infty T$, we use $\partial_\infty^\eta X$ to denote the corresponding subset: $\partial_\infty^\eta X := \mathcal{I}^{-1}(\eta) \subset \partial_\infty X$. We will say that $\partial_\infty^\eta X$ is *trivial* if $\partial_\infty^\eta X$ is a point or *nontrivial* otherwise; (in the latter case we will see that $\partial_\infty^\eta X$ is homeomorphic to a closed interval and is in fact an interval in the Tits metric.).

In particular $\partial_\infty X = (\cup_{v \in V} \partial_\infty X_v) \cup (\cup_{\eta \in \partial_\infty T} \partial_\infty^\eta X)$, where $\cup_{v \in V} \partial_\infty X_v$ is disjoint from $\cup_{\eta \in \partial_\infty T} \partial_\infty^\eta X$.

The cone topology and Tits metric on $\partial_\infty X_v = \partial_\infty Y_v = \partial_\infty(\bar{Y}_v \times \mathbb{R})$ is described in sections 2.2 and 2.4. We see that in the cone topology $\partial_\infty X_v$ is just the suspension $\Sigma(\partial_\infty H_v)$ and is independent of the metric on X. The Tits metric is just the metric suspension of the discrete metric.

We now study the dynamics of the action of G on $\partial_\infty X$.

Lemma 3.23. *1. For every $v \in V$, the fixed point set of $Z(G_v)$ in $\partial_\infty X$ is $\partial_\infty X_v$; this set is homeomorphic to the suspension of $\partial_\infty H_v$ where H_v is the nonelementary hyperbolic group $G_v/Z(G_v)$.*

2. For every $e \in E$, $Fix(G_e, \partial_\infty X) = \partial_\infty X_e$ which is homeomorphic to a circle.

Proof. Let $\xi \in \partial_\infty X$ be fixed by $Z(G_v)$ and $p \in Y_v$. If $\mathcal{I}(\overline{p\xi})$ is a ray then by Lemma 3.22 $\partial_\infty \mathcal{I}(\overline{p\xi})$ is fixed by $Z(G_v)$. But this can not happen since by Lemma 3.5 $Z(G_v)$ leaves no point in $\partial_\infty T$ fixed. So $\mathcal{I}(\overline{p\xi})$ is a finite subtree which by Lemma 3.22 is invariant under $Z(G_v)$. Thus by Lemma 3.19 and Lemma 3.5 $v \in \mathcal{I}(\overline{p\xi})$. Thus $\overline{p\xi} \subset Y_v$ and hence $\xi \in \partial_\infty X_v = \partial_\infty Y_v$. On the other hand geodesics rays $\overline{p\xi}$ in Y_v are translated by a fixed amount by elements $g \in G_v$, so $g(\overline{p\xi})$ is asymptotic to $\overline{p\xi}$ and so ξ is fixed by g. The rest of part 1 follows from sections 2.2 and 2.4 as above.

Let ξ be fixed by G_e and $p \in Y_e$. Again since G_e leaves no point in $\partial_\infty T$ fixed, by Lemma 3.5 $\mathcal{I}(\overline{p\xi})$ is a finite subtree that is invariant under G_e and hence by Lemma 3.19 and Lemma 3.5 must contain e. Hence we see $\overline{p\xi} \subset Y_e$ and $\xi \in \partial_\infty Y_e = \partial_\infty X_e$. Again, since G_e acts by translations on Y_e, we see that if $\xi \in \partial_\infty X_e = \partial_\infty Y_e$ then it is left fixed by G_e. Since $Y_e = \mathbb{R}^2 \times \overline{Y_e}$ where $\overline{Y_e}$ is compact, $\partial_\infty Y_e = \partial_\infty \mathbb{R}^2$ is homeomorphic to a circle. $\square$

4. Templates and the behavior of their geodesics

4.1. Templates

In this section we study "Templates". These are piecewise Euclidean Hadamard spaces (which can be embedded in $\mathbb{R}^3$) which approximate certain subspaces of the spaces we are studying, and carry much of the information about the spaces at infinity.

A *template* is a Hadamard space $\mathcal{T}$ obtained from a disjoint collection of Euclidean planes $\{W\}_{W\in Wall_{\mathcal{T}}}$ (called *walls*) and directed Euclidean strips[8] $\{\mathcal{S}\}_{\mathcal{S}\in Strip_{\mathcal{T}}}$ by isometric gluing[9] subject to the following conditions:

1. The boundary geodesics of each strip $\mathcal{S} \in Strip_{\mathcal{T}}$, which we will refer to as *singular geodesics*, are glued isometrically to distinct walls in $Wall_{\mathcal{T}}$.
2. Each wall $W \in Wall_{\mathcal{T}}$ is glued to at most two strips, and the gluing lines are not parallel.
3. $\mathcal{T}$ is connected.

One can think of $\mathcal{T}$ as sitting in $\mathbb{R}^3$ so that its walls are parallel planes and the strips meet the walls orthogonally. Two walls $W_1, W_2 \in Wall_{\mathcal{T}}$ are *adjacent* if there is a strip $\mathcal{S} \in Strip_{\mathcal{T}}$ with $S \cap W_i \neq \emptyset$. The incidence graph $Graph(\mathcal{T})$ of $\mathcal{T}$ – the graph with vertex set $Wall_{\mathcal{T}}$ and one edge for each pair of incident walls – is a graph isomorphic to a connected subcomplex of $\mathbb{R}$ with the usual triangulation (where the vertices are the integers). A wall is an *interior wall* if it is incident to two strips, and a strip is an *interior strip* if it is incident to two interior walls; $Wall^o_{\mathcal{T}}$ and $Strip^o_{\mathcal{T}}$ denote the interior walls and strips respectively. For every interior wall $W \in Wall^o_{\mathcal{T}}$ we have a distinguished point $o_W := W \cap \mathcal{S}_1 \cap \mathcal{S}_2$, where $\mathcal{S}_i \in Strip_{\mathcal{T}}$, $i = 1, 2$, are the strips incident to W. Let $Strip^+_{\mathcal{T}}$ be the collection of oriented interior strips; an orientation of a strip $\mathcal{S} \in Strip_{\mathcal{T}}$ combines with the direction of $\mathcal{S}$ to give an orientation of the interval factor of $\mathcal{S} \simeq \mathbb{R} \times I$, and also an ordering of the two incident walls. We can define a function $\epsilon : Strip^+_{\mathcal{T}} \to \mathbb{R}$ as follows: if W_-, W_+ are incident to $\mathcal{S}^+ \in Strip^+_{\mathcal{T}}$ and $W_- < W_+$ with respect to the ordering defined by S^+, then $\epsilon(\mathcal{S}^+) \in \mathbb{R}$ is defined to be the signed distance that o_{W_+} lies "above" o_{W_-} in the strip $\mathcal{S}^+$. We also have a strip width function $l : Strip_{\mathcal{T}} \to (0, \infty)$ and an angle function $\alpha : Wall^o_{\mathcal{T}} \to (0, \pi)$ which give the angle between the oriented lines $W \cap \mathcal{S}_i$ where $\mathcal{S}_1$, $\mathcal{S}_2$ are incident to W.

We will sometimes enumerate the consecutive walls and strips of $\mathcal{T}$ so that $Wall_{\mathcal{T}} = \{W_i\}_{a<i<b}$ and $Strip_{\mathcal{T}} = \{\mathcal{S}_i\}_{a<i<b-1}$ where $a \in \{-\infty, 0\}$ and $b \in \mathbb{N} \cup \{\infty\}$. We then define $L^-_i := W_i \cap \mathcal{S}_{i-1}$ for $a+1 < i < b$ and $L^+_i := W_i \cap \mathcal{S}_i$ for $a < i < b-1$.

An *equivalence* between two templates $\mathcal{T}_1$ and $\mathcal{T}_2$ is an isometry $\psi : \mathcal{T}_1 \to \mathcal{T}_2$ which respects strip directions. Two templates are equivalent if and only if there is an incidence preserving bijection $Wall_{\mathcal{T}_1} \cup Strip_{\mathcal{T}_1} \to Wall_{\mathcal{T}_2} \cup Strip_{\mathcal{T}_2}$ which respects the functions l, ϵ, and α. We will call a template *uniform* if there is a $\frac{\pi}{2} \geq \beta > 0$ so that the angle function $\alpha : Wall^o_{\mathcal{T}} \to (0, \pi)$ satisfies $\pi - \beta \geq \alpha \geq \beta$, and if the strip widths are bounded away from zero. We are mostly interested in uniform templates. A template $\mathcal{T}$ is *full* if $Graph(\mathcal{T}) \simeq \mathbb{R}$, *half* if $Graph(\mathcal{T}) \simeq \mathbb{R}_+$, and *finite* if $|Wall_{\mathcal{T}}| < \infty$.

If $W \in Wall^o_{\mathcal{T}}$ and $\mathcal{S}_1$, $\mathcal{S}_2$ are the incident strips, then the oriented lines $W \cap \mathcal{S}_1$ and $W \cap \mathcal{S}_2$ divide the plane W into four sectors which we call *Quarter Planes* and which we label as Q_I, Q_{II}, Q_{III}, and Q_{IV} as usual. If we are given a choice Q_W of quarter planes in W for each $W \in Wall_{\mathcal{T}}$ then there is an isometric immersion $\mathcal{D} : [\cup_{W\in Wall_{\mathcal{T}}} Q_W] \cup [\cup_{\mathcal{S}\in Strip_{\mathcal{T}}} \mathcal{S}] \to \mathbb{R}^2$ (the development) which takes any geodesic ray $\gamma \subset \mathcal{T}$ such that $\gamma \cap W \subset Q_W$, to a Euclidean ray (see Figure 3 in section 7.2 for an example of the developement of a special kind of template).

[8]A direction for a strip $\mathcal{S}$ is an orientation for its $\mathbb{R}$-factor $\mathcal{S} \simeq \mathbb{R} \times I$.

[9]In general one may also have to complete the resulting quotient space to get a Hadamard space.

When $\mathcal{T}$ is a half template we will be primarily interested in geodesic rays $\gamma \subset \mathcal{T}$ that start at a given base point and intersect all but finitely many walls of $\mathcal{T}$. From the separation properties of walls it is clear that such a ray intersects the walls $W \in Wall_{\mathcal{T}}$ in order. We let $\partial_\infty^\infty \mathcal{T}$ (resp. $\partial_T^\infty \mathcal{T}$) denote the corresponding subset of $\partial_\infty \mathcal{T}$ (resp. $\partial_T \mathcal{T}$). In section 7.1 we will show that $\partial_T^\infty \mathcal{T}$ is isometric to either a point, in which case $\mathcal{T}$ is called trivial, or an interval of length $< \pi$.

Remark 4.1. One can show directly that any two half templates such that corresponding angles agree, and both corresponding strip widths and displacements differ by a bounded amount will have ∂_∞^∞'s with the same Tits length. We will only need a weaker version (that will follow from Theorem 5.1) in this paper so we will not digress to prove it here.

4.2. Templates associated with itineraries in T

We now return to the setting of our paper: G is an admissible group with a discrete cocompact isometric action on a Hadamard space X. We now want to associate a template with each geodesic segment/ray in T; these templates capture the asymptotic geometry of geodesic segments/rays in X which pass near the corresponding edge spaces.

We first choose, in a G-equivariant way, a plane $F_e \subset Y_e$ for each edge $e \in E$. Then for every pair of adjacent edges e_1, e_2 we choose, again equivariantly, a minimal geodesic from F_{e_1} to F_{e_2}; by the convexity of $Y_v = \bar{Y}_v \times \mathbb{R}$, $v := e_1 \cap e_2$, this geodesic determines a Euclidean strip[10] $\mathcal{S}_{e_1,e_2} := \gamma_{e_1,e_2} \times \mathbb{R}$ for some geodesic segment $\gamma_{e_1,e_2} \subset \bar{Y}_v$; note that $\mathcal{S}_{e_1,e_2} \cap F_{e_i}$ is an axis of $Z(G_v)$. Hence if e, e_1, $e_2 \in E$, $e_i \cap e = v_i \in V$ are distinct vertices, then the angle between the geodesics $\mathcal{S}_{e_1,e} \cap F_e$ and $\mathcal{S}_{e_2,e} \cap F_e$ is bounded away from zero (since only finitely many angles show up). We also note that Definition 3.14 tells us that $d_H(F_e, X_e)$ is bounded by $D'' + Diam(\bar{Y}_e)$ which, since there are only finitely many e up to the action of G, is uniformly bounded.

The significance of the strips $\mathcal{S}_{e_1,e_2}$ can be seen in the next two lemmas.

Lemma 4.2. *There is a constant C_3 so that if $e_1 = \overline{vv_1}$ and $e_2 = \overline{vv_2}$ are adjacent edges, then $X_{e_1} \cup \mathcal{S}_{e_1,e_2} \cup X_{e_2}$ and $X_{v_1} \cup \mathcal{S}_{e_1,e_2} \cup X_{v_2}$ are C_3-quasi-convex.*

Proof. Since the Hausdorff distance $d_H(X_{\hat{e}}, F_{\hat{e}})$ is uniformly bounded for $\hat{e} \in E$, it suffices to show that there is a constant C so that the unions $F_{e_1} \cup \mathcal{S}_{e_1,e_2} \cup F_{e_2}$ are C-quasi-convex for all pairs of adjacent edges. But if e_1, $e_2 \in E$ are adjacent then $F_{e_1} \cup \mathcal{S}_{e_1,e_2} \cup F_{e_2} \subset Y_v \simeq \bar{Y}_v \times \mathbb{R}$, and we are reduced to showing that $\bar{F}_{e_1} \cup \gamma_{e_1,e_2} \cup \bar{F}_{e_2} \subset \bar{Y}_v$ is uniformly quasi-convex, where $\bar{F}_{e_i}$ is the image of F_{e_i} under the projection $Y_v \simeq \bar{Y}_v \times \mathbb{R} \to \bar{Y}_v$. This follows from Lemma 2.9. This gives the first statement.

The interesting part of the second statement is when we consider $\overline{xy}$ when $x \in X_{v_1}$ and $y \in X_{v_2}$. In this case Lemma 3.17 gives us $z_1 \in N_{C_2} X_{e_1}$ and $z_2 \in N_{C_2} X_{e_2}$ on $\overline{xy}$. Now the first statement along with the convexity of $N_{C_2+D''}(X_{v_i})$ (note $z_i \in N_{C_2} X_{e_i} \subset N_{C_2+D''} X_{v_i}$) yields the second statement.

□

[10] $\mathcal{S}_{e_1,e_2}$ may have width zero.

Lemma 4.3. *There is a constant C_4 so that if $e_1, \dots, e_n \in E$ is a geodesic edge path in T with initial vertex v_1 and terminal vertex v_n, then*

$$Z := X_{e_1} \cup \mathcal{S}_{e_1,e_2} \cup X_{e_2} \cup \ldots \cup X_{e_{n-1}} \cup \mathcal{S}_{e_{n-1},e_n} \cup X_{e_n}$$

and

$$Z' := X_{v_1} \cup \mathcal{S}_{e_1,e_2} \cup X_{e_2} \cup \ldots \cup X_{e_{n-1}} \cup \mathcal{S}_{e_{n-1},e_n} \cup X_{v_n}$$

are C_4-quasi-convex.

Proof. Pick $x, y \in Z$. We may assume without loss of generality that $x \in X_{e_1} \cup \mathcal{S}_{e_1,e_2} \cup X_{e_2} \subset X_v$ where $v = e_1 \cap e_2$, and $y \in X_{e_{n-1}} \cup \mathcal{S}_{e_{n-1},e_n} \cup X_{e_n} \subset X_{v'}$ where $v' = e_{n-1} \cap e_n$. Applying Lemma 3.17 we get points $z_i \in \overline{xy} \cap N_{C_2}(X_{e_i})$ for $2 \le i \le n-1$, with $d(z_i, x) \le d(z_j, x)$ when $i \le j$. If $C_4 := C_2 + C_3$, then by Lemma 4.2 we have $\overline{xz_2} \subset N_{C_4}(X_{e_1} \cup \mathcal{S}_{e_1,e_2} \cup X_{e_2})$, $\overline{z_i z_{i+1}} \subset N_{C_4}(X_{e_i} \cup \mathcal{S}_{e_i,e_{i+1}} \cup X_{e_{i+1}})$ for $i = 2, \dots n-1$, and $\overline{z_{n-1}y} \subset N_{C_4}(X_{e_{n-1}} \cup \mathcal{S}_{e_{n-1},e_n} \cup X_{e_n})$.

We omit the proof that Z' is quasi-convex, as it is similar. □

Lemma 4.3 suggests that we will understand the geodesic geometry of X if the geometry of the sets Z (as in the lemma) can be easily modeled. To this end, we "approximate" Z with a template.

Definition 4.4. Suppose $\gamma \subset T$ is a geodesic segment or ray. Let $\mathcal{T}$ be a template with walls $\{W\}_{W \in Wall_{\mathcal{T}}}$ and strips $\{\mathcal{S}\}_{\mathcal{S} \in Strip_{\mathcal{T}}}$, let $f : Wall_{\mathcal{T}} \to E$ be an adjacency preserving bijection between the walls of $\mathcal{T}$ and the edges of γ, and let $\phi : \mathcal{T} \to X$ be a (not necessarily continuous) map. Then the triple $(\mathcal{T}, f, \phi)$ is a *K-template* for γ if for all $W \in Wall_{\mathcal{T}}$ we have $\phi(W) \subset N_K(X_{f(W)})$ and $X_{f(W)} \subset N_K(\phi(W))$ and the following conditions are met for every $\mathcal{S} \in Strip_{\mathcal{T}}$.

1. $Width(\mathcal{S}) \ge 1$.

2. If $\mathcal{S}$ is incident to $W_1, W_2 \in Wall_{\mathcal{T}}$ and $x, y \in W_1 \cup \mathcal{S} \cup W_2$ then we have $|d_X(\phi(x), \phi(y)) - d_{\mathcal{T}}(x, y)| < K$ and if $\gamma_1 : [0,1] \to \overline{xy}$ and $\gamma_2 : [0,1] \to \overline{\phi(x)\phi(y)}$ are constant speed parameterizations, then $d(\phi \circ \gamma_1(t), \gamma_2(t)) < K$ for all $t \in [0,1]$.

3. If $W_1, W_2 \in Wall_{\mathcal{T}}$ are adjacent to $\mathcal{S}$ then the Hausdorff distance $d_H((X_{f(W_1)} \cup \mathcal{S}_{f(W_1), f(W_2)} \cup X_{f(W_2)}), \phi(W_1 \cup \mathcal{S} \cup W_2)) < K$.

Often when the value of K is not relevant we will refer to the triple $(\mathcal{T}, f, \phi)$ as a template for γ, by which we mean a K-template for some K.

Let $\mathcal{T}'$ be another template such that the angles on corresponding walls agree with those of $\mathcal{T}$ and such that the other data (strip widths and displacements) differ from $\mathcal{T}$ by a bounded amount. Then there is a natural (discontinuous) map $F : \mathcal{T}' \to \mathcal{T}$ which is an isometry on each wall and simply stretches the width of the strips. It is easy to check that using $\phi' = \phi \circ F$ that we get a K' template $(\mathcal{T}', f, \phi')$ for γ (see step 3 of the proof of Lemma 7.14).

For a suitably large K we describe a construction that gives a K-template for any geodesic segment or geodesic ray $\gamma \subset \mathcal{T}$. We will refer to these K-templates as *standard K-templates*. We begin with a disjoint collection of walls W_e and an isometry $\phi_e : W_e \to F_e$ for each edge $e \subset \gamma$. For every pair e, e' of adjacent edges of γ, we let $\hat{\mathcal{S}}_{e,e'}$ be a strip which is isometric to $\mathcal{S}_{e,e'} \subset X$ if $Width(\mathcal{S}_{e,e'}) \ge 1$, and isometric to $\mathbb{R} \times [0,1]$ otherwise; we let $\phi_{e,e'} : \hat{\mathcal{S}}_{e,e'} \to \mathcal{S}_{e,e'}$ be an affine map which respects product structure ($\phi_{e,e'}$ is an isometry if $Width(\mathcal{S}_{e,e'}) \geq 1$ and compresses the interval otherwise). We construct $\mathcal{T}$ by gluing the strips and walls so that the maps ϕ_e and $\phi_{e,e'}$ descend to continuous maps on the quotient.

The above construction yields

Lemma 4.5. *There is a constant $K = K(X)$ such that for every geodesic segment or ray, $\gamma \subset T$, there is a K-template for γ.*

There is a $\beta = \beta(X) > 0$ such that for any K-template the angle function $\alpha : Wall^o_{\mathcal{T}} \to (0, \pi)$ satisfies $0 < \beta \leq \alpha \leq \pi - \beta < \pi$.

Proof. We check that each condition of Definition 4.4 holds for the standard template described above, for sufficiently large K.

First, since $d_{\mathcal{H}}(F_e, X_e)$ is uniformly bounded and $F_{f(W)} = \phi(W)$ for every $W \in Wall_{\mathcal{T}}$, we have $\phi(W) \subset N_K(X_{f(W)})$ and $X_{f(W)} \subset N_K(\phi(W))$ for all $W \in Wall_{\mathcal{T}}$ for large enough K.

Conditions 1 and 3 follow immediately from the description of standard templates.

We now verify condition 2. Pick adjacent walls $W, W' \in Wall_{\mathcal{T}}$ and set $e := f(W)$, $e' := f(W')$, and $v := e \cap e'$. Recall that $F_e \cup \mathcal{S}_{e,e'} \cup F_{e'} \subset Y_v$ and Y_v splits isometrically as $Y_v = \bar{Y}_v \times \mathbb{R}$ where $\bar{Y}_v$ is Gromov hyperbolic. Furthermore, ϕ induces a map $W_e \cup \hat{\mathcal{S}}_{e,e'} \cup W_{e'} \to F_e \cup \mathcal{S}_{e,e'} \cup F_{e'}$ which is compatible with the product structure. Hence condition 2 follows from part 3 of Lemma 2.9 (and triangle inequalities) when $d(F_e, F_{e'}) \geq 4\delta$ (where δ is the maximum of the hyperbolicity constants of the $\bar{Y}_v$'s); modulo G there are only finitely many cases when $d(F_e, F_{e'}) < 4\delta$ (Lemma 3.11), and each of these is also settled by part 3 of Lemma 2.9.

For any K template $(\mathcal{T}, f, \phi)$ for a γ containing an interior edge $e = \overline{v'v}$ we claim that the wall W with $f(W) = e$ will have the same angle, up to taking supplements (i.e. α might be replaced by $\pi - \alpha$), as the angle α between the $\mathbb{R}$ factors of $Y_{v'} = \bar{Y}_{v'} \times \mathbb{R}$ and $Y_v = \bar{Y}_v \times \mathbb{R}$ in $Y_e = Y_v \cap Y_{v'}$. The fact that these angles are positive and the finiteness of edges modulo G will yield the result. We note that α is the Tits angle between the $\mathbb{R}$ factors.

To see this we first note that Property 2 of Definition 4.4 says that the angle for W, i.e. the angle between the gluing lines L' and L, is the same as the comparison angle $\lim_{t\to\infty} \tilde{\angle}_{\phi(o)}\phi(L'(t)), \phi(L(t))$. Also Property 2 of Definition 4.4 says that there are geodesic rays $\sigma = \lim_{i\to\infty} \overline{p\phi(L(t_i))}$ where $t_i \to \infty$ and $p \in Y_e$. Since $\phi(W) \subset N_K X_e$ we see that $\phi(L) \subset N_K X_e$ and hence $e \in \mathcal{I}(\sigma)$. Now if we let e' be the other edge incident to v in γ then the intersection of the wall $f^{-1}(e')$ with the strip between W and $f^{-1}(e')$, is a line parallel to L and hence, again by 2 of Definition 4.4, $\phi(L)$ stays in a uniform neighborhood of $X_{e'}$ so $e' \in \mathcal{I}(\sigma)$. but $e, e' \in \mathcal{I}(\sigma)$ implies by Lemma 3.19 that $\mathcal{I}(\sigma) = \overline{Star(v)}$ and hence by Corollary 3.21 any such σ is asymptotic to the $\mathbb{R}$ factor of $Y_v = \bar{Y}_v \times \mathbb{R}$. Since $p \in Y_e \subset Y_v$, σ is a half line of such an $\mathbb{R}$ and we assume without loss of generality that it points in the positive direction. A similar argument works for $\phi(L')$. Thus $\angle(L', L) = \lim_{t\to\infty} \tilde{\angle}_{\phi(o)}\phi(L'(t)), \phi(L(t)) \leq \alpha$. Also the same arguments applied to $-L'$ and L yield $\angle(-L', L) =\leq \pi - \alpha$. Thus we get equality and the result.

□

The next proposition and Proposition 4.8 are technical results that compare template geometry with ambient geometry.

Proposition 4.6. *Suppose $K > 0$. There is a constant C_5 depending only on K and the geometry of X with the following property. Suppose $\gamma \subset T$ is a geodesic segment or geodesic ray in T with i^{th} edge e_i, and set $Z := [\cup_{e\subset\gamma} X_e] \cup [\cup_{e,e'\subset\gamma} \mathcal{S}_{e,e'}]$. If $(\mathcal{T}, f, \phi)$ is a K-template for γ, and x, $y \in Z$, then there is a continuous map $\alpha : \overline{xy} \to \mathcal{T}$ so that*

1. $d(\phi \circ \alpha, id\big|_{\overline{xy}}) < C_5$
2. *For all p, $q \in \overline{xy}$ we have*

$$d_X(p, q) - kC_5 \leq d_{\mathcal{T}}(\alpha(p), \alpha(q)) \leq length(\alpha\big|_{\overline{pq}}) \leq d_X(p, q) + kC_5 \tag{4.7}$$

where the segment $\overline{\alpha(p)\alpha(q)} \subset \mathcal{T}$ intersects at most $k-1$ strips and walls in $\mathcal{T}$. In particular there are constants (L, A) depending only on K and X so that ϕ is an (L, A) quasi-isometric embedding for every K-template $(\mathcal{T}, f, \phi)$.

Proof. By the standard properties of Hadamard spaces we may reduce to the case that $x \in Y_{e_x} \cup \mathcal{S}_{e_x,e'_x} \subset X_{v_x}$ ($v_x = e_x \cap e'_x$) and $y \in \mathcal{S}_{e_y,e'_y} \cup Y_{e'_y} \subset X_{v_y}$ since the original x and y are within a bounded distance of such. Let $W_i := f^{-1}(e_i) \in Wall_{\mathcal{T}}$ for e_i $1 \leq i \leq n$ the edges between v_x and v_y. We may apply Lemma 3.17 to the pair x, y obtaining points $z_i \in \overline{xy}$. We let $z_0 = x$ and $z_{n+1} = y$. After making a small perturbation of the z_i's if necessary, we may assume that they satisfy $d_X(z_i, x) < d(z_j, x)$ when $i < j$. For $1 \leq i \leq n$ pick $w_i \in \mathcal{T}$ with $w_i \in W_i \subset \mathcal{T}$ with $d(z_i, \phi(w_i)) \leq 1 + \inf\{d(z_i, \phi(w)) \mid w \in W_i\} \leq C_2 + 1$. By Definition 4.4 part 3 we can also choose $w_0 \in W_0 \cup \mathcal{S} \cup W_1$ and such that $d_X(x, \phi(w_0)) \leq K$ and similarly choose w_{n+1}. Now define α by the condition that $\alpha(z_i) = w_i$, and α is a constant speed geodesic on the segment $\overline{z_i z_{i+1}}$.

Proof of 1. Apply Definition 4.4 to see that the constant speed parameterization $[0, 1] \to \overline{\phi(w_i)\phi(w_{i+1})}$ is at uniformly bounded distance from the composition of the constant speed parameterization $[0, 1] \to \overline{w_i w_{i+1}} \subset \mathcal{T}$ with $\phi : \mathcal{T} \to X$. Since $d(\phi(w_i), z_i)$ is uniformly bounded, we know that the constant speed parameterizations $[0, 1] \to \overline{\phi(w_i)\phi(w_{i+1})}$ and $[0, 1] \to \overline{z_i z_{i+1}}$ are also at uniformly bounded distance from one another, so there is a constant c_1 depending on K so that $d(\phi \circ \alpha, id\big|_{\overline{xy}}) < c_1$.

Proof of 2. Assume $p \in \overline{z_{j-1} z_j} - z_{j-1}$ and $q \in \overline{z_{j'} z_{j'+1}} - z_{j'+1}$ for $j \leq j'$. By Definition 4.4 we have, for $c_2 = 2c_1 + K$

$$|length(\alpha\big|_{\overline{pz_j}}) - d_X(p, z_j)| < c_2$$

$$|length(\alpha\big|_{\overline{z_i z_{i+1}}}) - d_X(z_i, z_{i+1})| < c_2 \text{ for every } i = 1, \ldots n-1$$

$$|length(\alpha\big|_{\overline{z_{j'} q}}) - d_X(z_{j'}, q)| < c_2.$$

Hence there is a $c_3 = c_3(K)$ so that

$$length(\alpha_{\overline{pq}}) = length(\alpha\big|_{\overline{pz_j}}) + \ldots + length(\alpha\big|_{\overline{z_{j'} q}})$$

$$\leq d_X(p, z_j) + \ldots + d_X(z_{j'}, q) + (j' - j + 2)c_2$$
$$\leq d_X(p, q) + kc_2.$$

To prove the remaining inequality of (4.7) we break up the $\mathcal{T}$-geodesic $\overline{\alpha(p)\alpha(q)}$ into at most k subsegments $\overline{u_i u_{i+1}}$ so that each subsegment lies in $W_i \cup \mathcal{S}_i \cup W_{i+1}$ for some i. Then by definition 4.4 we have $|d_X(\phi(u_i), \phi(u_{i+1})) - d_{\mathcal{T}}(u_i, u_{i+1})| < K$ so

$$\begin{aligned} d_X(p,q) &\leq 2c_1 + d_X(\phi \circ \alpha(p), \phi \circ \alpha(q)) \\ &\leq 2c_1 + \sum d_X(\phi(u_i), \phi(u_{i+1})) \\ &\leq 2c_1 + kc_2 + d_{\mathcal{T}}(\alpha(p), \alpha(q)) \\ &\leq d_{\mathcal{T}}(\alpha(p), \alpha(q)) + kc_3. \end{aligned}$$

where $c_3 := 2c_1 + c_2$.

To see the quasi-isometry property of ϕ, let $x', y' \in \mathcal{T}$, $x = \phi(x')$, $y = \phi(y')$, and let α be the map defined above where we choose $w_0 = x$ and $w_1 = y$ (i.e $\alpha(x) = x'$ and $\alpha(y) = y'$). Now 4.7 applied to $p = x$ and $q = y$ along with $k \leq d_{\mathcal{T}}(x', y') + 1$ (since strips have width at least 1) and $k \leq const_1 d_X(p,q) + const_2$ (as in the proof of the coarse lipschitz property of ρ - see section 3.3) yields the quasi-isometry property of ϕ. This completes the proof of Proposition 4.6. □

Proposition 4.8. *Pick $K > 0$. There is a constant $C_6 = C_6(K, X)$ so that the following holds. If $(\mathcal{T}, f, \phi)$ is a K-template for $\gamma \subset T$, and x, $y \in \mathcal{T}$, then there is a continuous map $\alpha : \overline{xy} \to X$ where*

1. *$d(\alpha, \phi\big|_{\overline{xy}}) < C_6$.*
2. *For all p, $q \in \overline{xy}$ we have*

$$d_{\mathcal{T}}(p,q) - kC_6 \leq d_X(\alpha(p), \alpha(q)) \leq length(\alpha\big|_{\overline{pq}}) \leq d_{\mathcal{T}}(p,q) + kC_6 \qquad (4.9)$$

where the segment $\overline{pq} \subset \mathcal{T}$ intersects at most $(k-1)$ strips and walls in $\mathcal{T}$.

Proof. This is similar to the proof of Proposition 4.6, so we omit it. □

Corollary 4.10. *If $(\mathcal{T}, f, \phi)$ is a K-template and $C_6 = C_6(K)$ is the constant from Proposition 4.8, then for any x, $y \in \mathcal{T}$ we have*

$$d_{\mathcal{T}}(x,y) - kC_6 \leq d_X(\phi(x), \phi(y)) \leq d_{\mathcal{T}}(x,y) + kC_6$$

where $\overline{xy} \subset \mathcal{T}$ meets at most $k-1$ strips and walls.

5. Shadowing

In this section we show that geodesic segments in a K-template are sublinearly shadowed by ambient geodesic segments, and vice-versa.

Theorem 5.1 (Shadowing). *There is a function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ depending on K and the geometry of X (sometimes denoted $\theta_{(X,K)}$) with $\lim_{R\to\infty} \theta(R) = 0$ so that if $(\mathcal{T}, f, \phi)$ is a K-template for a geodesic segment/ray $\gamma \subset T$, then the following hold.*

1. *If x, $y \in \mathcal{T}$, $z \in \overline{xy}$ and $R := d(z,x)$, then $d(\phi(z), \overline{\phi(x)\phi(y)}) \leq (1+R)\theta(R)$.*
2. *If x, $y \in \mathcal{T}$, $z \in \overline{\phi(x)\phi(y)}$ and $R := d(z, \phi(x))$ then $d(z, \phi(\overline{xy})) \leq (1+R)\theta(R)$.*

3. Let $\bar{\mathcal{T}} := \mathcal{T} \cup \partial_\infty \mathcal{T}$ and $\bar{X} := X \cup \partial_\infty X$ be the usual compactifications. Then there is a unique topological embedding $\partial_\infty \phi : \partial_\infty \mathcal{T} \to \partial_\infty X$ so that

$$\bar{\phi} := \phi \cup \partial_\infty \phi : \bar{\mathcal{T}} \to \bar{X}$$

is continuous at every $\xi \in \partial_\infty \mathcal{T} \subset \bar{\mathcal{T}}$.

4. The image of $\partial_\infty \phi$ is

$$[\cup_{W \in Wall_{\mathcal{T}}} (\partial_\infty X_{f(W)})] \cup [\partial_\infty^{\partial_\infty \gamma} X],$$

and when γ is a ray with $\partial_\infty \gamma = \eta$ then $\partial_\infty \phi(\partial_\infty^\infty \mathcal{T}) = \partial_\infty^\eta X$ (see section 3.3 for the definition of $\partial_\infty^\eta X$).

5. $\partial_\infty \phi\big|_{\partial_\infty^\infty \mathcal{T}} : \partial_\infty^\infty \mathcal{T} \to \partial_\infty^\eta X$ is an isometric embedding with respect to the Tits metric.

The proof of the theorem breaks up into two pieces. We first show in Proposition 5.6 that a geodesic segment (in a template or in X) running through a sequence of consecutive walls has to be "close" to any point p which lies close to sufficiently many walls in the sequence. We then show in Theorem 5.7 that a segment in a template (resp. in X) which doesn't meet too many walls (i.e. encounters at most $Const \log R$ walls in the segment $\overline{px}$, $d(p, x) = R$) is well shadowed by a geodesic segment in X (resp. in the template). These two arguments are combined in section 5.3 to prove Theorem 5.1.

5.1. Paths in a template which are close to a cluster of walls

We begin with a result about templates. It estimates the excess length of a path η which connects two walls W, W' while remaining outside a ball which intersects W, W', and all walls between them.

Proposition 5.2. *Let $\mathcal{T}$ be a template with angle function $\alpha : Wall^o_{\mathcal{T}} \to (0, \pi)$ satisfying $0 < \beta \leq \alpha \leq \pi - \beta < \pi$. Then there are positive constants $N_1 = N_1(\beta)$, ($N_1 \approx \frac{Const}{\beta}$), $C_1 = C_1(\beta)$, and $C_2 = C_2(\beta)$ with the following property. Let $W_{n_0}, \ldots, W_{n_1} \in Wall_{\mathcal{T}}$ be a sequence of consecutive walls, and suppose $W_i \cap B(p, R) \neq \emptyset$ for some $p \in \mathcal{T}$, $R > 0$ and every $n_0 \leq i \leq n_1$. Then for any $R' \geq N_1 R$ and any path $c : [0,1] \to \mathcal{T} - B(p, R')$ with $c(0) \in W_{n_0}$ and $c(1) \in W_{n_1}$ we have*

$$length(c) \geq d_{\mathcal{T}}(c(0), c(1)) + C_1(n_1 - n_0 - C_2)R'.$$

Proof. Let $\mathcal{S}_i$ be the strip incident to W_i and W_{i+1} for $i = n_0, \ldots, n_1 - 1$, set $L_i^- := \mathcal{S}_{i-1} \cap W_i$ for $i = n_0 + 1, \ldots n_1$, and set $L_i^+ := \mathcal{S}_i \cap W_i$ for $i = n_0, \ldots, n_1 - 1$.

We prove the proposition with the help of some lemmas.

Lemma 5.3. *There is a constant $c_1 \approx \frac{Const}{\beta}$ so that if $p \in \mathcal{T}$ and $d(p, W_j) < R$ for $j = i \pm 1$ then $d(p, o_i) < c_1 R$.*

Proof. By joining $\overline{x_{i-1}p}$ to $\overline{px_{i+1}}$ for appropriate choices of $x_{i-1} \in W_{i-1}$ and $x_{i+1} \in W_{i+1}$ we get a path $\gamma : [0,1] \to \mathcal{T}$ of length at most $2R$ joining W_{i-1} to W_{i+1}. Therefore there is a segment $[a, b] \subset [0, 1]$ with $\gamma(a) \in L_i^-$ and $\gamma(b) \in L_i^+$. So

$$d(p, o_i) \leq R + \min(d(\gamma(a), o_i), d(\gamma(b), o_i))$$

$$\leq R + \frac{d(\gamma(a),\gamma(b))}{2\sin(\frac{\beta}{2})} \leq c_1 R.$$

$\square$

We now define $N_1 := \max(2c_1, [\frac{\pi}{\beta}] + 2)$.

Consider a path $\eta : [0,1] \to \mathcal{T} - B(p,R')$ where $R' > N_1 R$. The ball $B(p,R') \subset \mathcal{T}$ is convex, so clearly $\mathcal{T} - B(p,R')$ is complete and locally compact with respect to the induced path metric. Therefore we may assume that c is a constant speed minimizing path from $c(0)$ to $c(1)$ in $\mathcal{T} - B(p,R')$. Since $\overline{B(p,R')}$ is a convex subset of the Hadamard space $\mathcal{T}$, the nearest point projection $\mathcal{T} \to \overline{B(p,R')}$ is distance non increasing; it follows that the set $c^{-1}(\overline{B(p,R')})$ is either empty or a closed subinterval $[a,b] \subset [0,1]$. $c\Big|_{[0,a]}$ and $c\Big|_{[b,1]}$ are constant speed geodesic segments in the Hadamard space $\mathcal{T}$, and since $R' > c_1 R$ these segments lie in $\mathcal{T} - \{o_i\}_{n_0<i<n_1}$ by Lemma 5.3.

Lemma 5.4.

$$c([0,a]) \subset \left[\cup_{i=n_0}^{n_0+N_1} W_i\right] \cup \left[\cup_{i=n_0}^{n_0+N_1-1} \mathcal{S}_i\right]$$

and

$$c([b,1]) \subset \left[\cup_{i=n_1-N_1}^{n_1} W_i\right] \cup \left[\cup_{i=n_1-N_1}^{n_1-1} \mathcal{S}_i\right].$$

Proof. We prove the first assertion; the proof of the second is similar. If the lemma were false, we would have $c(t) \in \mathcal{S}_{n_0+N_1}$ for some $t \in [0,a]$. Therefore $c([0,t])$ must cross every strip $\mathcal{S}_i$ for $n_0 \leq i < n_0 + N_1$, and for every $n_0 < i \leq n_0 + N_1$ it must enter W_i through L_i^- and exit through L_i^+. Since $c([0,a])$ is disjoint from $\{o_i\}_{n_0<i<n_1}$ there is a flat convex strip $Y \subset \mathcal{T}$ containing $c([0,a])$ in its interior. Using Y we can define co-orientations for the segments $c([0,a]) \cap \mathcal{S}_i$ and $c([0,a]) \cap W_i$ for $n_0 \leq i \leq n_0 + N_1$. If two of the origins $o_i \in W_i$ for $n_0 < i \leq n_0 + N_1$ lie on opposite sides of the corresponding segments $c([0,a]) \cap W_i$ with respect to the co-orientations then the geodesic between them (of length less than $2c_1R$ by lemma 5.3) will intersect $c([0,a])$ and hence $d(o_i, c([0,a])) < c_1 R$ for some $n_0 < i \leq n_0 + N_1$, and thus $d(p, c([0,a]) < 2c_1 R$. But this cannot happen since $d(p, c([0,a]) \geq R' > 2c_1 R$. Thus all the origins $o_i \in W_i$ for $n_0 < i \leq n_0 + N_1$ lie on the same side of the corresponding segments $c([0,a]) \cap W_i$ with respect to the co-orientations. It follows that the angle between $c([0,a])$ and L_i^- increases by at least β each time $c([0,a])$ passes through a wall. Hence $(N_1 - 1)\beta < \pi$, contradicting the definition of N_1. $\square$

Proof of Proposition 5.2 concluded. Let $[a',b'] \subset [a,b] \subset [0,1]$ be the inverse image of

$$[\cup_{i=n_0+N_1+1}^{n_1-N_1-1} W_i] \cup [\cup_{n_0+N_1}^{n_1-N_1-1} \mathcal{S}_i]$$

under c. We know that $c([a',b'])$ remains in the sphere $S(p,R')$ while it passes through all the walls W_i for $n_0 + N_1 < i < n_1 - N_1$. So for every $n_0 + N_1 < i < n_1 - N_1$, $c([a',b'])$ joins L_i^- to L_i^+ outside $B(p,R') \supset B(o_i, R' - c_1 R)$. Hence $length(c([a',b']) \geq \beta(R' - c_1R)(n_1 - n_0 - (2N_1+2)) \geq \frac{\beta}{2}R'(n_1 - n_0 - (2N_1+2))$ while $d_{\mathcal{T}}(c(a'), c(b')) \leq 2R'$ so $length(c[a',b']) \geq d_{\mathcal{T}}(c(a'),c(b')) + \frac{\beta}{2}(n_1 - n_0 - (2N_1+2) - \frac{4}{\beta})R'$ and hence

$$length(c) \geq d_{\mathcal{T}}(c(0),c(1)) + C_1(n_1 - n_0 - C_2)R'$$

where C_1, C_2 depend only on β. $\square$

Corollary 5.5. *Let $\mathcal{T}$, N_1, C_1, C_2, $W_{n_0}, \ldots, W_{n_1}$, p, R be as in Proposition 5.2. If $n_1 - n_0 > C_2$, then any geodesic segment from W_{n_0} to W_{n_1} must pass through $B(p, N_1R)$.*

The result corresponding to Corollary 5.5 in the space X is:

Proposition 5.6. *There are constants $N_2 = N_2(X)$, $R_0 = R_0(X)$ with the following property. If $n \geq N_2$, $e_1, \ldots e_n \in E$ are the consecutive edges of a geodesic segment, γ, in the tree T, $p \in X$, $R \geq R_0$, and $X_{e_i} \cap B(p, R) \neq \emptyset$ for $1 \leq i \leq n$; then for any $C \geq 0$, and any segment $\overline{xy} \subset X$ with $\overline{xy} \cap N_C(X_{e_i}) \neq \emptyset$ for $i = 1$ and $i = n$, we have $\overline{xy} \cap B(p, N_2R + 2C) \neq \emptyset$.*

Proof. Let $e_1, \ldots, e_n$, p, X_{e_i} be as in the statement of the proposition. If $\overline{xy} \cap N_C(X_{e_i}) \neq \emptyset$ for $i = 1$ and $i = n$, then we have $x_0 \in X_{e_1}$ and $y_0 \in X_{e_n}$ with $d(x_0, \overline{xy}), d(y_0, \overline{xy}) \leq C$. By convexity of the distance function d_X it suffices to show that $\overline{x_0y_0} \cap B(p, N_2R) \neq \emptyset$.

Let K and β be as in Lemma 4.5 and $(\mathcal{T}, f, \phi)$ be a K template for γ whose angles are bounded by β. Let $\alpha : \overline{x_0y_0} \to \mathcal{T}$ be the map guaranteed by Proposition 4.6. So $Length(\alpha) \leq d(x_0, y_0) + nC_5$. By part 3 of definition 4.4 there is a $p' \in \mathcal{T}$ be such that $d(p, \phi(p')) < R + K$ and hence (since $X_{e_i} \subset N_K\phi(W_i)$) $d(\phi(p'), \phi(W_i)) < 2R + 2K$. Since, by Proposition 4.6, ϕ is an (L, A)-quasi isometric embedding we have $B(p', R_2) \cap W_i \neq \emptyset$ for $R_2 = L(2R + 2K) + A$. We will choose R_0 and N_2 large enough so that for $n > N_2$ and $R \geq R_0$ we will have $nC_5 < C_1(n - C_2)N_1R_2$ for the $N_1(\beta)$, $C_1(\beta)$ and $C_2(\beta)$ of Proposition 5.2. Thus Proposition 5.2 forces α to intersect $B(p', N_1R_2)$. So we conclude (again using Proposition 4.6) that $d(\overline{x_0y_0}, p) < C_5 + d(\phi(\alpha), \phi(p')) + R + K \leq C_5 + LN_1R_2 + A + R + K$. The proposition now follows by taking N_2 and R_0 large enough. ◻

5.2. Paths with small length distortion

Proposition 5.7. *Pick $M > 0$ and $\alpha \in (\frac{1}{2}, 1]$. Then there is a constant $C = C(M, \alpha)$ so that if $1 \leq A \leq B$, $\eta : [A, B] \to X$ is a (not necessarily continuous) map to a Hadamard space X, and for all $A \leq t_1 \leq t_2 \leq B$ we have*

$$|d_X(\eta(t_1), \eta(t_2)) - (t_2 - t_1)| \leq M(1 + log(\frac{t_2}{t_1})) \tag{5.8}$$

then

$$d(\eta(t), z) \leq C(1 + t^\alpha) \tag{5.9}$$

where $z \in \overline{\eta(A)\eta(B)}$ is the point with $d(z, \eta(A)) = \frac{t-A}{B-A} d(\eta(A), \eta(B))$. Similarly, if $A \geq 1$ and $\eta : [A, \infty) \to X$ satisfies (5.8) for all $A \leq t_1 \leq t_2$, then there is a unique unit speed geodesic ray $\gamma : [A, \infty) \to X$ with $\gamma(A) = \eta(A)$ such that

$$d_X(\eta(t), \gamma(t)) \leq C(1 + t^\alpha)$$

for all $t \in [A, \infty)$.

Proof. First note that we may assume that $A = 1$, since the map $\eta_1 : [1, B-A+1] \to X$ given by $\eta_1(t) := \eta(t + A - 1)$ will satisfy the hypotheses of the proposition, and the conclusion of the proposition applied to η_1 will imply (5.9) for η.

Step 1: When $1 \le s_1 \le s_2 \le 2s_1 \le B$ and s_1 is sufficiently large then the comparison angle $\widetilde{\angle}_{\eta(1)}(\eta(s_1), \eta(s_2)) \le Const\, s_1^{\alpha-1}$.

We will make use of the following lemma that follows from standard comparisons.

Lemma 5.10. *Let X be a Hadamard space, x, y, $z \in X$. Set $L := d(x, z)$, and the excess $E := d(x, y) + d(y, z) - d(x, z)$. Then*

$$d(y, \overline{xz}) \le \frac{\sqrt{2LE}}{2}\sqrt{1 + \frac{E}{2L}} \tag{5.11}$$

$$\le \sqrt{LE} \text{ if } E \le 2L. \tag{5.12}$$

Proof. Triangle comparison. □

Proof of Proposition 5.7 continued. Take $1 \le s_1 \le s_2 \le B$, and consider the triple $\eta(1)$, $\eta(s_1)$, $\eta(s_2)$. The excess for the triple is

$$\le M(1 + \log s_2) + M(1 + \log s_1) + M(1 + \log(\frac{s_2}{s_1})) = 3M(1 + \log s_2).$$

Since $d(\eta(1), \eta(s_2)) \ge (s_2 - 1) - M(1 + \log s_2)$ when $s_2 > c_1 = c_1(M)$, then the excess is $\le 2d(\eta(1), \eta(s_2))$. Thus since $d(\eta(1), \eta(s_2)) \le (s_2 - 1) + M(1 + \log s_2)$ applying (5.12) we get

$$d(\eta(s_1), \overline{\eta(1)\eta(s_2)}) \le \sqrt{[(s_2 - 1 + M(1 + \log s_2))][3M(1 + \log s_2)]}$$

$$\le c_2(1 + s_2^{\alpha}) \tag{5.13}$$

where $c_2 = c_2(M, \alpha)$. Therefore the comparison angle $\widetilde{\angle}_{\eta(1)}(\eta(s_1), \eta(s_2))$ satisfies

$$\sin(\widetilde{\angle}_{\eta(1)}(\eta(s_1), \eta(s_2))) \le \frac{c_2(1 + s_2^{\alpha})}{d(\eta(1), \eta(s_1))} \le \frac{c_2(1 + s_2^{\alpha})}{[(s_1 - 1) - M(1 + \log s_1)]}.$$

So there are constants $c_3 = c_3(M, \alpha)$ and $c_4 = c_4(M, \alpha)$ so that if $c_3 \le s_1 \le s_2 \le 2s_1 \le B$ then

$$\sin(\widetilde{\angle}_{\eta(1)}(\eta(s_1), \eta(s_2))) \le \frac{2c_2(1 + s_2^{\alpha})}{s_1} \le \frac{c_4}{2} s_2^{\alpha-1} \tag{5.14}$$

and

$$\widetilde{\angle}_{\eta(1)}(\eta(s_1), \eta(s_2)) \le c_4 s_2^{\alpha-1}. \tag{5.15}$$

Step 2: Estimating $d(\eta(t), \overline{\eta(1)\eta(B)})$. Now pick $t_0 \in [1, B]$ with $t_0 \ge c_3$. Let $t_i = 2^i t_0$ for $i = 0, \ldots, n$ where n is the integer part of $\frac{\log B}{\log 2}$, and $t_{n+1} = B$. Then $\frac{B}{t_n} < 2$. Applying the estimate (5.15) with $u_i := \eta(t_i)$ we have for $i = 1, \ldots, n + 1$:

$$\widetilde{\angle}_{\eta(1)}(u_{i-1}, u_i) \le c_4 t_i^{\alpha-1}$$

since $\frac{t_i}{t_{i-1}} \leq 2$ and $t_i \geq c_3$. Either $n = 0$, in which case we have

$$d(\eta(t_0), \overline{\eta(1)\eta(B)}) \leq c_2(1 + t_1^\alpha) \quad \text{by (5.13)}$$
$$\leq c_2(1 + 2^\alpha t_0^\alpha) \leq c_5(1 + t_0^\alpha)$$

where $c_5 = c_5(M, \alpha)$. Otherwise, if $t_0 > c_6 = c_6(M, \alpha)$,

$$d(\eta(1), u_0) \leq (t_0 - 1) + M(1 + \log t_0)$$
$$\leq (t_i - 1) - M(1 + \log t_i) \leq d(\eta(1), u_i).$$

So for $0 \leq i \leq n+1$ we may pick $v_i \in \overline{\eta(1)u_i}$ with $d(\eta(1), v_i) = d(\eta(1), u_0) =: R_0$. By triangle comparison we have

$$d(v_{i-1}, v_i) \leq R_0 \widetilde{\angle}_{\eta(1)}(v_{i-1}, v_i) \leq R_0 \widetilde{\angle}_{\eta(1)}(u_{i-1}, u_i)$$
$$\leq R_0 c_4 t_i^{\alpha-1}$$

so

$$d(u_0, \overline{\eta(1)\eta(B)}) = d(u_0, \overline{\eta(1)u_{n+1}}) \leq d(u_0, v_{n+1})$$
$$\leq \sum_{i=1}^{n+1} d(v_{i-1}, v_i) \leq R_0 c_4 \sum_{i=1}^{n+1} t_i^{\alpha-1}$$
$$\leq R_0 c_7 t_0^{\alpha-1}$$

for $c_7 = c_7(M, \alpha)$ (independent of n). We have $R_0 \leq t_0 - 1 + M(1 + \log t_0) \leq 2t_0$ when t_0 is sufficiently large, so when $t_0 \geq c_8 = c_8(M, \alpha)$

$$d(u_0, \overline{\eta(1)\eta(B)}) \leq c_9 t_0^\alpha$$

where $c_9 = c_9(M, \alpha)$.

Step 3: Estimating $d(\eta(t), z)$ where $z \in \overline{\eta(1)\eta(B)}$ satisfies $d(z, \eta(1)) = \frac{t-1}{B-1} d(\eta(1), \eta(B))$. Let $x \in \overline{\eta(1)\eta(B)}$ be the point nearest $\eta(t)$. Then for $t \geq c_{10}$ we have

$$d(\eta(t), z) \leq d(\eta(t), x) + d(x, z)$$
$$= d(\eta(t), x) + |d(x, \eta(1)) - (t-1)\frac{d(\eta(1), \eta(B))}{B-1}|$$
$$\leq d(\eta(t), x) + [d(x, \eta(t)) + \frac{(t-1)}{B-1} M(1 + \log B) + M(1 + \log t)] \leq 3c_9 t^\alpha.$$

Thus the result holds for large t, but by choosing C large enough we get the result for all t.

For the ray case we redo step 3 above when $d(\eta(1), \eta(B)) > t$ for $z_1 \in \overline{\eta(1)\eta(B)}$ the point which satisfies $d(z_1, \eta(1)) = t$. Let $x \in \overline{\eta(1)\eta(B)}$ be the point nearest $\eta(t)$. Then for $t \geq c_{11}$ we have

$$d(\eta(t), z_1) \leq d(\eta(t), x) + d(x, z_1)$$
$$= d(\eta(t), x) + |d(x, \eta(1)) - (t-1)|$$
$$\leq d(\eta(t), x) + [d(x, \eta(t)) + M(1 + \log t)] \leq 3c_9 t^\alpha.$$

Now choose γ as a limit of a subsequence of $\overline{\eta(1), \eta(B)}$ as B goes to ∞ (the result will in fact imply that the sequence itself converges). Since none of the constants depended on B the above estimate for $d(\eta(t), z_1)$ gives the result for rays. $\square$

5.3. The proof of Theorem 5.1

Let $\beta = \beta(X)$ be the minimum angle between singular geodesics of a template for X defined in Lemma 4.5, and let $C_2(\beta)$, $N_1(\beta)$ be the constants from Corollary 5.5. Let N_2, R_0 be the constants from Proposition 5.6; we will assume that $N_2 \geq \max(C_2(\beta), N_1(\beta))$.

Definition 5.16. For every $\psi > 0$, we will say that $z \in \overline{xy}$ is a *ψ-cluster point* if the segment $\overline{B(z, \psi d(z,x))} \cap \overline{xy}$ intersects at least N_2 walls of $\mathcal{T}$, or if $z \in \{x, y\}$. We will let $P_\psi \subset \overline{xy}$ represent the set of ψ-cluster points.

Proof of 1. Let $(\mathcal{T}, f, \phi)$, x, y, be as in the statement of part 1 of the theorem. We begin with two lemmas.

Lemma 5.17. *There is a constant c_1 depending on K such that if $\overline{xy}$ intersects a wall $W \in Wall_\mathcal{T}$, then $\overline{\phi(x)\phi(y)} \cap N_{c_1}(X_{f(W)}) \neq \emptyset$.*

Proof. If $\{x, y\} \cap W \neq \emptyset$ this is immediate since $\phi(W) \subset N_K(X_{f(W)})$ by definition 4.4. Otherwise x and y must lie in distinct components of $\mathcal{T} - W$ because each component is convex. Say $x \in W_1 \cup \mathcal{S}_1 \cup W_1'$ and $y \in W_2 \cup \mathcal{S}_2 \cup W_2'$ where W_i is adjacent to W_i'. Set $v := f(W_1) \cap f(W_1')$ and $v' := f(W_2) \cap f(W_2')$. Applying Lemma 3.17 the lemma follows. □

Lemma 5.18. *Suppose $p_1, p_2 \in \overline{xy}$, $d(p_1, x) \leq d(p_2, x)$, and $\overline{p_1p_2} \cap P_\psi \subset \{p_1, p_2\}$. Set $A := 1 + d(p_1, x)$, $B := 1 + d(p_2, x)$. Let $\eta_0 : [A, B] \to \mathcal{T}$ be the unit speed parameterization of $\overline{p_1p_2}$. Then the composition $\eta := \phi \circ \eta_0 : [A, B] \to X$ satisfies the hypotheses of Proposition 5.7 with $M = M(K, \psi)$.*

Proof. Pick $A \leq t_1 < t_2 \leq B$. By Corollary 4.10 we have

$$|d_X(\eta(t_1), \eta(t_2)) - (t_2 - t_1)| \leq kC_6 \tag{5.19}$$

where $\overline{\eta_0(t_1)\eta_0(t_2)}$ intersects at most $(k-1)$ walls and strips of $\mathcal{T}$. For each $z \in \overline{\eta_0(t_1)\eta_0(t_2)}$, the segment $\overline{B(z, \psi d(z,x))} \cap \overline{xy}$ intersects at most $N_2 - 1$ walls. We can cover $\overline{\eta_0(t_1)\eta_0(t_2)} - B(x, 1)$ with at most $Const \log(\frac{t_2}{t_1})$ such segments, and $\overline{B(x,1)} \cap \overline{\eta_0(t_1)\eta_0(t_2)}$ intersects at most 2 walls (since strips have width at least 1), so the lemma follows from (5.19). □

Step 1: Estimating $d(\phi(z), \overline{\phi(x)\phi(y)})$ when $z \in P_\psi$. Suppose $z \in P_\psi$. Then $\overline{xy} \cap B(z, \psi R)$ intersects at least N_2 consecutive walls $W_1, \ldots, W_n$ of $\mathcal{T}$. By definition 4.4 and the fact that $\phi : \mathcal{T} \to X$ is a quasi-isometric embedding (Proposition 4.6) there is a constant $c_2 = c_2(K)$ so that $d_X(\phi(z), X_{f(W_i)}) \leq c_2(1 + \psi R)$ for $i = 1, \ldots n$. By Lemma 5.17 and Proposition 5.6, we conclude that $\overline{\phi(x)\phi(y)} \cap B(\phi(z), N_2c_2(1 + \psi R) + 2c_1) \neq \emptyset$ provided $c_2(1 + \psi R) \geq R_0$. So there are positive constants $r_0 = r_0(K, \psi)$, $c_3 = c_3(K, \beta)$ so that if $z \in P_\psi$ and $R := d(z, x) \geq r_0$, then

$$d(\phi(z), \overline{\phi(x)\phi(y)}) \leq c_3\psi R. \tag{5.20}$$

Hence for any $z \in P_\psi$

$$d(\phi(z), \overline{\phi(x)\phi(y)}) \leq c_3\psi R + Lr_0 + A = c_3\psi R + c_4 \tag{5.21}$$

where (L, A) are the quasi-isometric embedding constants of ϕ, and $c_4 := Lr_0 + A = c_4(K, \psi)$.

Step 2: Estimating $d(\phi(z), \overline{\phi(x)\phi(y)})$ when $z \notin P_\psi$. Pick $z \notin P_\psi$. There are points $p_1, p_2 \in \overline{xy}$ so that $z \in \overline{p_1p_2}$, $d(p_1, x) \le d(p_2, x)$, and either $\overline{p_1p_2} \cap P_\psi = \{p_1, p_2\}$ or one or both of $p_1 = x$ or $p_2 = y$ hold. Each step of the argument below holds in the special cases $p_1 = x$ or $p_2 = y$ (often for easier reasons) so we will ignore them. By (5.21) we have

$$d(\phi(p_i), \overline{\phi(x)\phi(y)}) \le c_3\psi d(p_i, x) + c_4 \tag{5.22}$$

for $i = 1, 2$. Since p_1, p_2 satisfy the conditions of Lemma 5.18, we may apply Proposition 5.7 with $\alpha = \frac{3}{4}$ to get that for the $w \in \overline{\phi(p_1)\phi(p_2)}$ with $d(w, \phi(p_1)) = \frac{d(z,p_1)}{d(p_1,p_2)} d(\phi(p_1), \phi(p_2))$ we have

$$d(\phi(z), w) \le C(1 + [d(z, x)]^{\frac{3}{4}}) \le \psi d(z, x) + c_6. \tag{5.23}$$

Where $c_6 = c_6(K, \psi)$.

On the other hand for $\zeta = \frac{d(w,\phi(p_1))}{d(\phi(p_1),\phi(p_2))} = \frac{d(z,p_1)}{d(p_1,p_2)}$ the convexity of the distance function $d(*, \overline{\phi(x)\phi(y)})$ and 5.22 says that

$$d(w, \overline{\phi(x)\phi(y)}) \le c_3\psi((1 - \zeta)d(p_1, x) + \zeta d(p_2, x)) + c_4 = c_3\psi d(z, x) + c_4 \tag{5.24}$$

Combining (5.24) and (5.23) we get

$$d(\phi(z), \overline{\phi(x)\phi(y)}) \le c_7\psi d(z, x) + c_8 \tag{5.25}$$

for $c_7 = c_7(K)$ and $c_8 = c_8(K, \psi)$. Thus (5.21) and (5.25) together imply that for every choice of $\psi > 0$ there are constants $c_9(K, \beta)$ and $c_{10}(K, \psi)$ such that for all z

$$d(\phi(z), \overline{\phi(x)\phi(y)}) \le c_9\psi R + c_{10}.$$

The fact that c_9 does not depend on ψ allows us to pick $\theta(R)$ which decays to 0 as $R \to \infty$ such that for each R there is a $\psi = \psi(R) > 0$ (for example choose $\psi(R)$ decaying to 0 such that $c_{10}(\psi) < R^{-\frac{1}{2}}$) so that

$$c_9\psi R + c_{10}(\psi) \le (1 + R)\theta(R)$$

This implies part 1 of Theorem 5.1.

Proof of 2. We omit this, as it is similar to the proof of 1.

Proof of 3. Part 3 follows directly from part 1 and Lemma 2.5.

Proof of 4. Part 2 along with reasoning similar to the proof of Lemma 2.5 shows that if $x, y_k \in \mathcal{T}$, $\xi \in \partial_\infty X$, and $\overline{\phi(x)\phi(y_k)} \to \overline{x\xi}$, then $\overline{xy_k}$ converges to some ray $\overline{x\xi'} \subset \mathcal{T}$. Hence $\partial_\infty\phi$ is a homeomorphism from $\partial_\infty\mathcal{T}$ onto the limit set of $\phi(\mathcal{T}) \subset X$, which we denote by $\partial_\infty\phi(\mathcal{T})$. It follows immediately from this that $\partial_\infty\phi$ maps $\partial_\infty\mathcal{T} - \partial_\infty^\infty\mathcal{T} = \cup_{W\in Wall_\mathcal{T}}\partial_\infty W$ homeomorphically onto $\cup_{W\in Wall_\mathcal{T}}\partial_\infty X_{f(W)}$. This implies 4 when $\gamma \subset T$ is a geodesic segment, so we now assume that γ is a geodesic ray, and we need only show that $\partial_\infty \phi(\partial_\infty^\infty \mathcal{T}) = \partial_\infty^\eta X$ where $\eta = \partial_\infty \gamma \in \partial_\infty T$. We will let $W_k \in Wall_{\mathcal{T}}$ be the k^{th} wall of $\mathcal{T}$.

We first show $\partial_\infty \phi(\partial_\infty^\infty \mathcal{T}) \subset \partial_\infty^\eta X$. Suppose $\xi \in \partial_\infty^\infty \mathcal{T}$, $z_k \in \overline{x\xi} \cap W_k$, and $d(z_k, x) \to \infty$. Thus $\overline{\phi(x)\phi(z_k)} \to \overline{\phi(x)\partial_\infty\phi(\xi)}$. By lemma 3.17 part 1, for any l we have $\overline{\phi(x)\phi(z_k)} \cap N_c(X_{f(W_l)}) \neq \emptyset$ for sufficiently large k. So by Lemma 2.4 either $\partial_\infty \phi(\xi) \in \partial_\infty N_c(X_{f(W_l)}) = \partial_\infty X_{f(W_l)}$ or $\overline{\phi(x)\partial_\infty\phi(\xi)} \cap N_c(X_{f(W_l)}) \neq \emptyset$. The former case is impossible since we already know that $(\partial_\infty \phi)^{-1}(\partial_\infty X_{f(W_l)}) = \partial_\infty W_l$. So $\overline{\phi(x)\partial_\infty\phi(\xi)} \cap N_c(X_{f(W_l)}) \neq \emptyset$ for every $W \in Wall_{\mathcal{T}}$, forcing $\mathcal{I}(\overline{\phi(x)\partial_\infty\phi(\xi)}) = \gamma$. So $\partial_\infty \phi(\partial_\infty^\infty \mathcal{T}) \subset \partial_\infty^\eta X$.

We now show $\partial_\infty^\eta X \subset \partial_\infty \phi(\partial_\infty^\infty \mathcal{T})$. Suppose $x \in W_1 \in Wall_{\mathcal{T}}$ and $\xi \in \partial_\infty^\eta X$. Then from the definition of itineraries 3.20 and Lemma 3.19 there are $z_k \in \overline{\phi(x)\xi} \cap N_C(X_{e_k})$ for $e_k \in T$ so that $d_T(e_k, f(W_1)) \to \infty$ and $d_T(e_k, \gamma)$ is uniformly bounded. So for all but finitely many W, $f(W)$ separates e_k from $f(W_1)$ for sufficiently large k, so by Lemma 3.17

$$\overline{\phi(x)z_k} \cap N_c(X_{f(W)}) \neq \emptyset$$

for sufficiently large k. Hence ξ belongs to the limit set of $\cup_{W \in Wall_{\mathcal{T}}} X_{f(W)}$, which is the same as $\partial_\infty(\phi(\mathcal{T})) = \partial_\infty \phi(\partial_\infty \mathcal{T})$. Therefore $\xi \in (\partial_\infty \phi)(\partial_\infty \mathcal{T} - \cup_{W \in Wall_{\mathcal{T}}} \partial_\infty W) = (\partial_\infty \phi)(\partial_\infty^\infty \mathcal{T})$.

Proof of 5. Pick $x \in \mathcal{T}$ and $\xi \in \partial_\infty^\infty \mathcal{T}$. Suppose there is a sequence $z_k \in \overline{x\xi}$ with $\lim_{k\to\infty}(z_k, x) = \infty$ so that z_k is a ψ_k-cluster point $\overline{x\xi}$ where $\psi_k \to 0$. Set $R_k := d(z_k, x)$. Applying Corollary 5.5 to each z_k, we see that if $\xi' \in \partial_\infty^\infty \mathcal{T}$, then for sufficiently large k the intersection $\overline{x\xi'} \cap B(z_k, N_1\psi_k R_k)$ is nonempty, and this clearly forces $\overline{x\xi'} = \overline{x\xi}$. In this case we have $\partial_\infty^\infty \mathcal{T} = \{\xi\}$, and 5 is immediate. So we may assume that for every ξ_1, $\xi_2 \in \partial_\infty^\infty \mathcal{T}$ there is a $\psi > 0$ such that $\overline{x\xi_1}$ and $\overline{x\xi_2}$ contain no ψ-cluster points. Let $\bar{\eta}_i : [1, \infty) \to \mathcal{T}$ be the unit speed parameterization of $\overline{x\xi_i}$, and let $\eta_i : [1, \infty) \to X$ be the composition $\phi \circ \bar{\eta}_i : [1, \infty) \to X$. Then by Lemma 5.18 the η_i satisfy the hypotheses of Proposition 5.7 for a suitable M; so we have unit speed geodesic rays $\gamma_i : [1, \infty) \to X$ with $\gamma_i(1) = \eta_i(1)$ and $d_X(\eta_i(t), \gamma_i(t)) \leq C(1 + t^{\frac{3}{4}})$. Now, for sufficiently large k, choose t_k^i so that $\bar{\eta}_i(t_k^i)$ lies in the k^{th} wall of $\mathcal{T}$. By definition 4.4

$$|d_X(\eta_1(t_k^1), \eta_2(t_k^2)) - d_{\mathcal{T}}(\bar{\eta}_1(t_k^1), \bar{\eta}_2(t_k^2))| < K.$$

Thus

$$\angle_T(\gamma_1, \gamma_2) = \lim_{k\to\infty} \widetilde{\angle}_{\eta_1(1)}(\eta_1(t_k^1), \eta_2(t_k^2)) = \lim_{k\to\infty} \widetilde{\angle}_x(\bar{\eta}_1(t_k^1), \bar{\eta}_2(t_k^2)) = \partial_T(\xi_1, \xi_2).$$

This proves 5. ☐

5.4. Applications of Theorem 5.1

Theorem 5.1 has a number of corollaries:

Corollary 5.26. *Let G be the fundamental group of an admissible graph of groups $\mathcal{G}$, and let $G \curvearrowright T$ be the Bass-Serre tree of $\mathcal{G}$. Let $\gamma \subset T$ be a geodesic ray with i^{th} edge $e_i \subset T$, and set $\eta := \partial_\infty \gamma \in \partial_\infty T$. Then for any admissible action $G \curvearrowright X$, the subset $\partial_\infty^\eta X \subset \partial_\infty X$ defined in section 3.3 is precisely the set of limit points of the sequence of subsets $\partial_\infty X_{e_k} \subset \partial_\infty X$.*

Proof. Let $(\mathcal{T}, f, \phi)$ be a template for $\gamma \subset T$. By parts 3 and 4 of Theorem 5.1, $\partial_\infty \phi : \partial_\infty \mathcal{T} \to \partial_\infty X$ is a homeomorphism onto $(\cup_k \partial_\infty X_{e_k}) \cup (\partial^\eta_\infty X)$. Therefore it suffices to show that $\partial^\infty_\infty \mathcal{T} \subset \partial_\infty \mathcal{T}$ is the set of limit points of the sequence $\partial_\infty f^{-1}(e_k) \subset \partial_\infty \mathcal{T}$. To see this, observe that any geodesic segment $\overline{px} \subset \mathcal{T}$ which arrives at a wall $W \in Wall_{\mathcal{T}}$ via a strip $\mathcal{S} \in Strip_{\mathcal{T}}$ may be prolonged by a geodesic ray contained in W; this implies that every $\xi \in \partial^\infty_\infty \mathcal{T}$ is a limit of a sequence $\xi_k \in \partial_\infty f^{-1}(e_k)$. On the other hand, if $\xi_k \in \partial_\infty f^{-1}(e_k)$ converges to $\xi \in \partial_\infty \mathcal{T}$, then Lemma 2.4 implies that either $\xi \in \partial^\infty_\infty \mathcal{T}$, or ξ belongs to $\partial_\infty f^{-1}(e_k)$ for all sufficiently large k, which is absurd. $\square$

Remark 5.27. Let $G \curvearrowright X$, T, γ, and η be as in Corollary 5.26. When X is a 3-dimensional Hadamard manifold, then $\partial_\infty X \simeq S^2$ and there is an alternate characterization of $\partial^\eta_\infty X$ which uses little more than the definition of itineraries. For each i, $\partial_\infty X_{e_i} \subset \partial_\infty X \simeq S^2$ determines two closed disks by the Jordan separation theorem; let D_i be the one which contains $\partial_\infty X_{e_j}$ for all $j \geq i$. Then $\partial^\eta_\infty X = \cap_i D_i$. To see this note that if $F \subset X$ is a flat totally geodesic plane and $p \in X - F$, then the two components of $\partial_\infty X - \partial_\infty F$ are $\{\xi \in \partial_\infty X \mid \overline{p\xi} \cap F \neq \emptyset\}$ and $\{\xi \in \partial_\infty X \mid \overline{p\xi} \cap F = \emptyset\}$.

Corollary 5.28. *Let $G \curvearrowright X$ and $G \curvearrowright X'$ be admissible actions, let $G \curvearrowright T$ be the Bass-Serre tree of G, and let V, E be the sets of vertices and edges of T, respectively. If $\Phi : \partial_\infty X \to \partial_\infty X'$ is any G-equivariant homeomorphism, then*

1. *Φ maps $\partial_\infty X_\sigma$ homeomorphically to $\partial_\infty X'_\sigma$ for all $\sigma \in V \cup E$.*
2. *Φ maps $\partial^\eta_\infty X$ homeomorphically to $\partial^\eta_\infty X'$ for all $\eta \in \partial_\infty T$.*

Proof. Part 1 follows from the characterization of $\partial_\infty X_\sigma$ as a fixed point set which is stated in Lemma 3.23. Part 2 follows from part 1 and Corollary 5.26. $\square$

Corollary 5.29. *Let $G \curvearrowright X$ be an admissible action, and $G \curvearrowright T$ be the Bass-Serre action for G. Then*

1. *The union $\cup_{v\in V} \partial_T X_v \subset \partial_T X$ is a $CAT(1)$ space with respect to the induced metric, and may be described metrically as follows. First, $\partial_T X_v$ is a metric suspension of an uncountable discrete $CAT(1)$ space for each $v \in V$, and $\partial_T X_e$ is isometric to the standard circle for every $e \in E$. Take the disjoint union $\amalg_{v\in V} \partial_T X_v$, and for each edge $e = \overline{v_1 v_2} \in E$, glue $\partial_T X_{v_1}$ to $\partial_T X_{v_2}$ isometrically by identifying the copies of $\partial_T X_e \subset \partial_T X_{v_i}$; the result is isometric to $\cup_{v\in V} \partial_T X_v \subset \partial_T X$.*
2. *The union $\cup_{v\in V} \partial_T X_v$ forms a connected component of $\partial_T X$. The remaining components are contained in the subsets $\partial^\eta_T X$ for $\eta \in \partial_\infty T$. We will show in Lemma 7.3 that each $\partial^\eta_T X$ is either a point or isometric to an interval of length $< \pi$.*

Proof. To prove 1, we first observe that if $e_1, \dots, e_n$ is an edge path in T with initial vertex v_1 and terminal vertex v_n, $\xi_1 \in \partial_T X_{v_1}$, $\xi_n \in \partial_T X_{v_n}$, and $\angle_T(\xi_1, \xi_n) < \pi$, then the Tits segment $\overline{\xi_1 \xi_n} \subset \partial_T X$ is contained in $\cup_{i=1}^n \partial_T X_{v_i}$. To see this, pick $\xi \in \overline{\xi_1 \xi_n}$. Recall that for a given base point x, $\overline{x\xi}$ may be obtained as the limit of a sequence $\overline{xy_k}$ where y_k lies on a segment $\overline{x_1^k x_2^k}$ and $x_i^k \in \overline{x\xi_i}$ is a sequence tending to infinity. The quasi-convexity property of Lemma 4.3 (applied in sucession to $\overline{x\xi_i}$, $\overline{x_1^k x_2^k}$, and then $\overline{xy_k}$) implies that $\overline{xy_k} \subset N_C(\cup_{i=1}^n X_{v_i})$ for some C, and the convexity of the $N_C(X_{v_i})$'s implies that $\xi \in \partial_T X_{v_i}$ for some $i \in \{1, \dots n\}$. Part 1 now follows from Corollary 3.16.

Before proving 2, we recall that open balls of radius $\frac{\pi}{2}$ in $CAT(1)$ spaces are geodesically convex, so two points in a $CAT(1)$ space belong to the same connected component iff they can be joined by a unit speed path.

Suppose $\eta \in \partial_\infty T$ and $\xi \in \partial_T^\eta X$. Fix $v \in V$, $p \in X_v$, and let $e_k \in E$ denote the k^{th} edge of the ray $\overline{v\eta} \subset T$. Part 1 of Lemma 3.19 implies that $\overline{p\xi} \cap X_{e_k} \neq \emptyset$ for all but finitely many k. If $c : [0, L] \to \partial_T X$ is a unit speed path starting at ξ, then by Lemma 2.4, we find that either $c([0, L]) \subset \partial_T^\eta X$ or for all sufficiently large k there is a $t_k \in [0, L]$ so that $\overline{pc(t_k)} \in \partial_T X_{e_k}$. But part 1 shows that when e, $e' \in E$ and $d(e, e') \geq 2$, then $d(\partial_T X_e, \partial_T X_{e'})$ is bounded away from 0, which gives a contradiction. Therefore $c([0, L]) \subset \partial_T^\eta X$ and we have shown that the connected component of ξ is contained in $\partial_T^\eta X$.

Finally, we note that $\cup_{v\in V} \partial_T X_v$ is connected: if $v_1, \ldots, v_k$ are the consecutive vertices of a geodesic segment in T, then $\partial_T X_{v_i} \cap \partial_T X_{v_{i+1}} = \partial_T X_{\overline{v_i v_{i+1}}} \neq \emptyset$. $\square$

6. Geometric data and equivariant quasi-isometries

Throughout this section, $G \overset{\rho}{\curvearrowright} X$ and $G \overset{\rho'}{\curvearrowright} X'$ will denote admissible actions of an admissible group G on Hadamard spaces X and X', and we let MLS_v, τ_v and MLS'_v, τ'_v denote their respective geometric data (see Definition 3.9). The main result in this section is Theorem 6.5, which shows that a G-equivariant quasi-isometry $\Phi : X \to X'$ induces an equivariant homeomorphism $\partial_\infty \Phi : \partial_\infty X \to \partial_\infty X'$ provided ρ and ρ' have *equivalent* geometric data.

Definition 6.1. We say that ρ and ρ' have *equivalent* geometric data if there are functions $\lambda : V \to \mathbb{R}_+$ and $\mu : V \to \mathbb{R}_+$ so that for every $v \in V$

$$MLS'_v = \lambda(v) MLS_v \quad \text{and} \quad \tau'_v = \mu(v)\tau_v.$$

It follows from the G-invariance of the geometric data that λ and μ will be G-invariant.

The structure of G strongly restricts the possibilities for the functions λ and μ:

Lemma 6.2. *Suppose the geometric data for the actions $G \curvearrowright X$ and $G \curvearrowright X'$ are equivalent, and let $\lambda : V \to \mathbb{R}_+$ and $\mu : V \to \mathbb{R}_+$ be as in Definition 6.1. Then either*

1. $\lambda \equiv \mu \equiv a$, for some $a > 0$.

or

2. There are constants a and b, $a \neq b$, so that $\lambda(V) = \{a, b\} = \mu(V)$. Moreover, for any pair v_1, v_2 of adjacent vertices in T, $\lambda(v_1) = \mu(v_2)$, $\lambda(v_2) = \mu(v_1)$, and the $\mathbb{R}$ directions of $Y_{v_i} \simeq \bar{Y}_{v_i} \times \mathbb{R}$ determine orthogonal directions in Y_e, where $e := \overline{v_1 v_2}$. In particular, there is a G-equivariant 2-coloring of V (i.e. a two coloring of the finite graph $\mathcal{G} = T/G$) such that λ and μ are functions of the vertex color.

Proof. Pick $e = \overline{v_1 v_2} \in E$, and consider $G_e \otimes \mathbb{R} \simeq \mathbb{Z}^2 \otimes \mathbb{R} \simeq \mathbb{R}^2$. For $i = 1, 2$ we have subspaces $Z_i := (Z(G_{v_i}) \cap G_e) \otimes \mathbb{R} \simeq \mathbb{Z} \otimes \mathbb{R} \simeq \mathbb{R}$ determined by the centers of the G_{v_i}'s. The action $G_e \curvearrowright Y_e$ induces an inner product $\langle \cdot, \cdot \rangle$ on $G_e \otimes \mathbb{R}$ by letting, for $g \in G_e$, $\langle g, g \rangle = \delta_g^2$. (For $p \in Y_e$ there is an embedded Euclidean plane $\mathbb{R}^2 \subset Y_e$ invariant under the action of $G_e \curvearrowright Y_e$ on which G_e acts by translations of δ_g. So our metric is naturally related to the metric on this $\mathbb{R}^2$). We can extend the maps

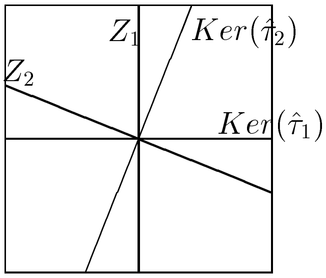

Figure 1: On $G_e \otimes \mathbb{R} \simeq \mathbb{R}^2$ we have two metrics $< \cdot, \cdot >$ and $< \cdot, \cdot >'$. $Ker(\hat{\tau}_i)$ (which is the same as $Ker(\hat{\tau}'_i)$) for $i \in \{1, 2\}$ is perpendicular to Z_i with respect to both metrics.

$\tau_i : G_e \to R$ to linear maps $\hat{\tau}_i : G_e \otimes \mathbb{R} \to R$. For $g \in G_e$ we see that $\tau_i(g)$ is just the change in the i-vertical component from y to $g(y)$. So for $x \in G_e \otimes \mathbb{R}$, $\hat{\tau}_i(x)$ is just the length of the orthogonal projection of x onto Z_i. In particular, for $x \in Z_i$ we have $\langle x, x \rangle = \hat{\tau}_i(x)^2$, and $Ker(\hat{\tau}_i)$ is $\langle \cdot, \cdot \rangle$ perpendicular to Z_i. Similarly $\hat{MLS}_i(x)$ corresponds to the length of the projection of x perpendicular to Z_i (i.e. to $Ker(\tau_i)$). In general Z_1 and Z_2 are not perpendicular but they are always linearly independent (see figure 1.

Similarly, using the action $G_e \curvearrowright Y'_e$, we get an induced inner product $\langle \cdot, \cdot \rangle'$ and linear maps $\hat{\tau}'_1$ and $\hat{\tau}'_2$. By assumption $\hat{\tau}'_i = \mu(v_i)\hat{\tau}_i$, hence $Ker(\hat{\tau}_i) = Ker(\hat{\tau}'_i)$. So the space perpendicular to Z_1 (resp. Z_2) with respect to $\langle \cdot, \cdot \rangle$ is the same as that with respect to $\langle \cdot, \cdot \rangle'$. Hence, by the independence of Z_1 and Z_2, either $\langle \cdot, \cdot \rangle' = a \langle \cdot, \cdot \rangle$ for some $a > 0$ or Z_1 and Z_2 are perpendicular in both metrics so $Ker(\hat{\tau}_1) = Ker(\hat{\tau}'_1) = Z_2$ and $Ker(\hat{\tau}_2) = Ker(\hat{\tau}'_2) = Z_1$. In the latter case, choosing $x \in Z_2$ we have $\langle x, x \rangle' = (\hat{MLS}_1(x))^2 = \mu(v_2)^2(\hat{MLS}_1(x))^2 = \mu(v_2)^2\langle x, x \rangle$ and $\langle x, x \rangle' = (\tau'_2(x))^2 = \lambda(v_1)^2(\tau_2(x))^2 = \lambda(v_1)^2\langle x, x \rangle$ so $\lambda(v_1) = \mu(v_2)$; similarly $\lambda(v_2) = \mu(v_1)$. $\square$

We now fix a G-equivariant (L, A)-quasi-isometry $\Phi : X \to X'$ for the rest of this section. The constants defined will depend on the geometry of $G \overset{\rho}{\curvearrowright} X$ and $G \overset{\rho'}{\curvearrowright} X'$ as well as any other explicitly stated quantities.

Lemma 6.3. *There is a constant $D_1 = D_1(L, A)$ so that for every $\sigma \in V \cup E$, the Hausdorff distance $d_H(\Phi(X_\sigma), X'_\sigma)$ is at most D_1.*

Proof. For $g \in G$ let d_g and d'_g denote the displacement functions for g in X and X' respectively. In particular $d'_g(x) \leq Ld_g(x) + A$.

For any $v \in V$, if $g \in Z(G_v)$ is a generator of the center $Z(G_v)$ then (see section 2.3) d_g is proper on $X/Z(Z(G_v), G) = X/G_v$ (by Lemma 3.6) and hence d_g (resp d'_g) grows with the distance from X_v (resp X'_v). This (along with the fact that there are only finitely many σ modulo G) implies the lemma when $\sigma \in V$. If $e \in E$, and g_1, $g_2 \in G_e$ are a basis for G_e then $\max\{d_{g_1}, d_{g_2}\}$ (resp $\max\{d'_{g_1}, d'_{g_2}\}$) grows with the distance from X_e (resp X'_e). This implies the lemma when $\sigma \in E$. $\square$

We now assume for the remainder of this section that ρ and ρ' have equivalent geometric data, and we let $\lambda : V \to \mathbb{R}_+$ and $\mu : V \to \mathbb{R}_+$ be as in Definition 6.1. For each $v \in V$, we have nearest point projections $p_v : X \to Y_v$ and $p'_v : X' \to Y'_v$. Modulo renormalization of the metrics on the spaces Y_v, the equivariant quasi-isometry Φ restricts to a Hausdorff approximation:

Lemma 6.4. *For every $v \in V$, the G_v-equivariant quasi-isometry $\Phi_v := p'_v \circ \Phi\big|_{Y_v} : Y_v \to Y'_v$ has the following properties:*

1. It is at distance $< D_2 = D_2(L, A)$ from a G_v-equivariant map $\Psi_v : Y_v \to Y'_v$ which respects the product structures $Y_v \simeq \bar{Y}_v \times \mathbb{R}$ and $Y'_v \simeq \bar{Y}'_v \times \mathbb{R}$.

2. If we stretch the metric on the $\bar{Y}_v$ factor of Y_v by $\lambda(v)$, and the metric on the $\mathbb{R}$ factor by $\mu(v)$, then Φ_v becomes a $D_3 = D_3(L, A)$-Hausdorff approximation, and maps unit speed geodesic segments to within D_3 of unit speed geodesic segments.

Proof. We first prove part 1. The $\mathbb{R}$ fibers of Y_v and Y'_v are within uniform Hausdorff distance of $Z(G_v)$-orbits, so G_v-equivariance implies that Φ_v takes $\mathbb{R}$ fibers of Y_v to within uniform Hausdorff distance (say C_1) of $\mathbb{R}$ fibers of Y'_v. The $\bar{Y}_v$ fibers of Y_v are within uniform Hausdorff distance of sets of the form $\{g(p) \mid p \in Y_v, g \in G_v,\ |\tau_v(g)| < C\}$ for sufficiently large C, and a similar characterization of the $\bar{Y}'_v$ fibers of Y'_v holds. We now define the product map $\Psi_v = \bar{\Psi}_v \times \Psi_v^{\mathbb{R}} : \bar{Y}_v \times \mathbb{R} \to \bar{Y}'_v \times \mathbb{R}$. Fix a basepoint $p \in Y_v$. We may assume that the $\mathbb{R}$ factor of p and $\Phi_v(p)$ are 0 and take $\Psi_v^{\mathbb{R}}(t) = \mu(v)t$. We let $S \subset \bar{Y}_v$ be a (set theoretical) cross section for the H_v action. For $s \in S$ choose a $g \in G_v$ such that $|\tau_v(g)| < C$ and $d(g(p), (s, 0)) \le C_1$. Now, since $|\tau'_v(g)| < \mu(v)C$, we can choose a point $\bar{\Psi}_v(s)$ such that $d(g(\Phi(p)), (\bar{\Psi}_v(s), 0)) \le \mu(v)C$. We note that

$$d(\Psi((s,0)), \Phi((s,0))) \le d((\bar{\Psi}_v(s), 0), \Phi(g(p))) + LC_1 + A \le \mu(v)C + LC_1 + A$$

Extend this to an H_v equivariant map $\bar{\Psi}_v : \bar{Y}_v \to \bar{Y}'_v$. Thus, along with the fact that $\tau'(g) = \mu(v)\tau(g)$, we see that Ψ_v is a G_v-equivariant map. Now for every $q \in Y_v$ there is a $g \in G_v$ such that $g(q) = (s, t)$ for some $s \in S$ and some $-C < t < C$. Now $d(\Psi(q), \Phi(q)) =$

$$d(\Psi((s,t)), \Phi((s,t))) \le 2C\mu(v) + d(\Psi((s,0)), \Phi((s,0))) \le 3\mu(v)C + LC_1 + A$$

This proves 1.

Part 2 follows from part 1 if we can show that $\bar{\Psi}_v$ and $\Psi_v^{\mathbb{R}}$ carry unit speed geodesics to within uniform distance of unit speed geodesics. The map $\Psi_v^{\mathbb{R}}$ clearly does, because the translation distance in the $\mathbb{R}$-direction is measured by τ_v (resp. τ'_v) and these have ratio $\mu(v)$. Recall that $H_v := G_v/Z(G_v)$ acts discretely and cocompactly on the hyperbolic metric spaces $\bar{Y}_v$ and $\bar{Y}'_v$. Since $MLS'_v = \lambda(v)MLS_v$, if we renormalize the metric on $\bar{Y}_v$ by $\lambda(v)$ we can apply Lemma 2.20 to see that $\bar{\Psi}_v : \bar{Y}_v \to \bar{Y}'_v$ preserves unit speed geodesics up to uniform error. This proves 2. ☐

We may now use our quasi-isometry $\Phi : X \to X'$ to transport standard K-templates for X to templates for X' (see section 4.2). Start with a standard K-template $(\mathcal{T}, f, \phi)$ for some segment or ray $\gamma \subset T$. Recall that the walls of $\mathcal{T}$ come from flats $F_e \subset Y_e \subset X_e$. To produce the new template $\mathcal{T}'$ distort the metric on $\mathcal{T}$ by an affine change as follows. For each $v \in V$, we think of scaling the metric on $\bar{Y}_v$ by $\lambda(v)$ and the $\mathbb{R}$-factor of Y_v by $\mu(v)$; and then we distort the flats $F_e = F_{\overline{vv'}} \subset F_v \cap F_{v'}$ and strips $\mathcal{S}_{ee'} \subset Y_{e \cap e'}$ used to build $\mathcal{T}$ accordingly. Lemmas 6.3 and 6.4 imply that $(\mathcal{T}', f, \Phi \circ \phi)$ is a K'-template for γ where $K' = K'(L, A)$. Notice (using Lemma 6.2) that the identity map $\mathcal{T} \to \mathcal{T}'$ is an affine map (i.e. maps constant speed geodesics to constant speed geodesics), and is a homothety when $\lambda = \mu = a \in \mathbb{R}$.

Theorem 6.5. *Let $G \overset{\rho}{\curvearrowright} X$ and $G \overset{\rho'}{\curvearrowright} X'$ be admissible actions of an admissible group G on Hadamard spaces X and X' such that ρ and ρ' have equivalent geometric data, and let $\Phi : X \to X'$ be a G-equivariant (L, A)-quasi-isometry. Then there is a function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ (depending on K,L, A and the geometry of X and X') with $\lim_{r\to\infty} \theta(r) = 0$ so that for every $x, y \in X$, $z \in \overline{xy}$, we have*

$$d_{X'}(\Phi(z), \overline{\Phi(x)\Phi(y)}) \leq (1 + d_X(z, x))\theta(d_X(z, x)). \tag{6.6}$$

Consequently, by Lemma 2.5, Φ extends to a unique map $\bar{\Phi} : \bar{X} \to \bar{X}'$ which is continuous at $\partial_\infty X \subset \bar{X}$. Setting $\partial_\infty \Phi := \bar{\Phi}\big|_{\partial_\infty X} : \partial_\infty X \to \partial_\infty X'$, we obtain a G-equivariant homeomorphism. If $\lambda = \mu = a \in \mathbb{R}$, then $\partial_\infty \Phi$ is an isometry with respect to Tits metrics.

Proof. Pick $x, y \in X$, and then find a segment $\gamma \subset T$ so that $x \in X_{e_1}$, $y \in X_{e_n}$, and the i^{th} edge of γ is e_i. Now let $(\mathcal{T}, f, \phi)$ be the standard K-template for γ, and define the K'-template $(\mathcal{T}', f, \phi')$ as in the paragraph preceding the statement of the Theorem. We may assume, after moving x and y a uniformly bounded distance if necessary (see part 3 of definition 4.4), that $x = \phi(x_1)$, $y = \phi(y_1)$ for some x_1, $y_1 \in \mathcal{T}$. We get (6.6) by applying Theorem 5.1 twice – once to $(\mathcal{T}, f, \phi)$ and once to $(\mathcal{T}', f, \phi')$. Specifically, Since $z \in \overline{\phi(x_1)\phi(y_1)}$ an application of Theorem 5.1 part 2 gives

$$d_X(z, \phi(\overline{x_1 y_1})) \leq (1 + d_X(z, x))\theta_{(X,K)}(d_X(z, x)).$$

So there is a $z_1 \in \overline{x_1 y_1}$ with $d_X(z, \phi(z_1)) \leq (1 + d_X(z, x))\theta_{(X,K)}(d_X(z, x))$. Hence we see

$$d_{X'}(\Phi(z), \phi'(z_1)) \leq L(1 + d_X(z, x))\theta_{(X,K)}(d_X(z, x)) + A.$$

Now an application of Theorem 5.1 part 1 to ϕ' gives

$$d_{X'}(\phi'(z_1), \overline{\Phi(x)\Phi(y)}) = d_{X'}(\phi'(z_1), \overline{\phi'(x_1)\phi'(y_1)}) \leq (1+d_{\mathcal{T}'}(x_1, z_1))\theta_{(X',K')}(d_{\mathcal{T}'}(x_1, z_1)).$$

We thus need to bound $d_{\mathcal{T}'}(x_1, z_1)$ linearly from above by $d_X(z, x)$. But this follows since $d_{\mathcal{T}'}(x_1, z_1) \leq L' d_{X'}(\phi'(x_1), \phi'(z_1)) + A'$ (where L' and A' come from Proposition 4.6 and depend only on K' and the geometry of X'), $d_{X'}(\phi'(x_1), \phi'(z_1)) \leq L d_X(\phi(x_1), \phi(z_1)) + A$, and $d_X(\phi(x_1), \phi(z_1)) \leq d_X(x, z) + d_X(z, \phi(z_1)) \leq d_X(x, z) + (1 + d_X(z, x))\theta_{(X,K)}(d_X(z, x))$. Equation (6.6) will thus follow for an appropriate choice $\theta_{(K,L,A,X,X')}$ that will depend only on L, A, K, and the geometry of X and X'. Lemma 2.5 then applies to Φ, so we get an induced embedding $\partial_\infty \Phi : \partial_\infty X \to \partial_\infty X'$ which is automatically G-equivariant.

Now suppose $\lambda = \mu = a$. Pick $\eta \in \partial_\infty T$, and a geodesic ray $\gamma \subset T$ with $\partial_\infty \gamma = \eta$. Then the identity map $\mathcal{T} \to \mathcal{T}'$ between the associated templates $(\mathcal{T}, f, \phi)$ and $(\mathcal{T}', f, \phi')$ constructed as above is a homothety, and so part 5 of Theorem 5.1 applied to $\partial_\infty \phi\big|_{\partial_\infty^\eta \mathcal{T}}$ and $\partial_\infty \phi'\big|_{\partial_\infty^\eta \mathcal{T}'}$ then shows that $\partial_\infty \Phi\big|_{\partial_\infty^\eta X}$ induces an isometry $\partial_T^\eta X \to \partial_T^\eta X'$. By Corollary 5.29 it remains only to show that $\partial_\infty \Phi$ induces an isometry $\partial_T X_v \to \partial_T X'_v$. But Lemma 6.4 part 2 implies that $\Phi_v : Y_v \to Y'_v$ is at finite distance from a product of Hausdorff approximations (up to rescaling of X' by $\frac{1}{a}$) and so Φ_v induces an isometry $\partial_T X_v \to \partial_T X'_v$. ∎

7. Recovering the geometric data from the action on the ideal boundary

In this section we will prove the remaining implication of Theorem 1.3: for any admissible action $G \curvearrowright X$, the topological conjugacy class of the action $G \curvearrowright \partial_\infty X$ determines the functions $MLS_v : G_v \to \mathbb{R}_+$ and $\tau_v : G_v \to \mathbb{R}$ up to a multiplicative factor, for every vertex $v \in \mathcal{G}$. Our strategy for proving this is as follows. Using Lemma 3.23 and Corollary 5.26, for any ideal boundary point η of the Bass-Serre tree T, we may detect the subset $\partial_\infty^\eta X \subset \partial_\infty X$; specifically, the action $G \curvearrowright \partial_\infty X$ determines the set of boundary points $\eta \in \partial_\infty T$ for which $|\partial_\infty^\eta X| = 1$. To extract useful information from this, we consider a special class of geodesic rays $\gamma \subset T$ (see Definition 7.11) which admit templates $(\mathcal{T}, f, \phi)$ where $\mathcal{T}$ is asymptotically self-similar: there is a (non-surjective) map $\mathcal{T} \to \mathcal{T}$ which stretches distances by a factor of 2. These templates have two key properties: their geometry relates directly to the geometric data MLS_v and τ_v, and at the same time we can tell explicitly when (in terms of the geometry) we have $|\partial_\infty^{\partial_\infty \gamma} \mathcal{T}| = 1$ (see section 7.2). Putting all this together we able to recover the geometric data from the action $G \curvearrowright \partial_\infty X$.

7.1. $\partial_T^\infty \mathcal{T}$ is either a point or an interval

Lemma 7.1. *Let X be a Hadamard space, $p \in X$, and $\gamma_i \subset X$ a sequence of geodesics with $\lim_{i\to\infty} d(p, \gamma_i) = \infty$. Set*

$$S := \{\xi \in \partial_T X \mid \overline{p\xi} \cap \gamma_i \neq \emptyset \text{ for all } i\}.$$

Then S embeds isometrically in the interval $[0, \pi]$.

Proof. Pick x, y, $z \in S$, and for each i choose $x_i \in \overline{px} \cap \gamma_i$, $y_i \in \overline{py} \cap \gamma_i$, $z_i \in \overline{pz} \cap \gamma_i$. After passing to a subsequence and reordering x, y, z we may assume that y_i lies between x_i and z_i on γ_i, or that it coincides with x_i or z_i. Then it follows that

$$\widetilde{\angle}_p(x_i, y_i) + \widetilde{\angle}_p(y_i, z_i) \leq \widetilde{\angle}_p(x_i, z_i).$$

Taking the limit as $i \to \infty$, we get

$$\angle_T(x, y) + \angle_T(y, z) \leq \angle_T(x, z).$$

Hence $(\{x, y, z\}, \angle_T) \subset \partial_T X$ embeds isometrically in $[0, \pi]$.

If $|S| = 1$ the lemma is immediate, so assume $|S| \geq 2$, and construct a map $f : S \to \mathbb{R}$ as follows. Pick distinct points x_0, $x_1 \in S$ and for $i = 0, 1$ choose $f(x_i) \in \mathbb{R}$ so that $d(x_0, x_1) = d(f(x_0), f(x_1))$. Now define f uniquely by the condition that $d(f(s), f(x_i)) = d(s, x_i)$ for all $s \in S$ and $i = 0, 1$. Clearly f is an isometric embedding, and its image lies in an interval of length at most π. □

Lemma 7.2. *Let $\mathcal{T}$ be a half template with walls $W_0, W_1, \ldots$. For $i \geq 1$ set $\alpha_i := \alpha(W_i)$ where $\alpha : Wall_{\mathcal{T}}^0 \to (0, \pi)$ is the angle function, and assume $\min\{\alpha_i, \pi - \alpha_i\} \geq \beta$ for all $i \geq 1$. Then*

1. *The diameter of $\partial_T^\infty \mathcal{T}$ with respect to the Tits metric is at most $\pi - \beta$.*

2. *For $p \in W_0$ and every $i > 0$ there is an R_i, depending only on β and $d(o_i, p)$, so that if $q \in L_i^+$ then $\overline{pq} \cap L_i^- \subset B(o_i, R_i)$.*

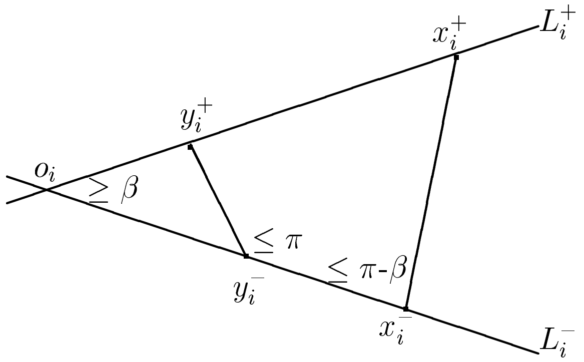

Figure 2: On the wall W_i the angle between L_i^- and L_i^+ is $\geq \beta$ hence the angle at x_i^- is $\leq \pi - \beta$. The angle at y_i^- is always $\leq \pi$ (no mater where y_i^+ lies on L_i^+).

Proof. Pick $p \in W_0 \subset \mathcal{T}$, and distinct points $x, y \in \partial_T^\infty \mathcal{T}$. Let $x_i^- \in L_i^-$ (resp. $x_i^+ \in L_i^+$) be the point where the ray $\overline{px}$ enters (resp. exits) the wall W_i; define y_i^-, y_i^+ similarly. Since $x \neq y$, there is an $i_0 > 0$ so that $x_i^- \neq y_i^-$ when $i \geq i_0$. Pick $i \geq i_0$, and assume that $d(x_i^-, o_i) \geq d(y_i^-, o_i)$ (the case when $d(y_i^-, o_i) \geq d(x_i^-, o_i)$ is similar). Clearly (see figure 2) we have

$$\angle_{x_i^-}(x_i^+, o_i) = \angle_{x_i^-}(x_i^+, y_i^-) \leq \max\{\alpha_i, \pi - \alpha_i\} \leq \pi - \beta.$$

Now this implies 1 since by standard properties of Tits angles

$$\angle_T(x, y) = \left[\lim_{i\to\infty}(\angle_{x_i^-}(y_i^-, x) + \angle_{y_i^-}(x_i^-, y))\right] - \pi \leq \max\{\alpha_i, \pi - \alpha_i\} \leq \pi - \beta.$$

It also implies 2 by applying triangle comparison to the angle at x_i of the triangle with vertices p, o_i, and x_i. □

Proposition 7.3. *Let $\mathcal{T}$ be a uniform half template with* $\min\{\alpha_i, \pi - \alpha_i\} \geq \beta$ *for all i. Then $\partial_T^\infty \mathcal{T}$ is isometric to an interval* $[0, \theta]$ *where* $\theta \in [0, \beta]$.

Proof. Applying Lemma 7.1 with $\gamma_i = L_i^-$ and Lemma 7.2 we get that $\partial_T^\infty \mathcal{T}$ is isometric to a subset of $[0, \beta]$. We need only show that $\partial_T^\infty \mathcal{T}$ is connected. Suppose $x, y \in \partial_T^\infty \mathcal{T}$, and pick a point z lying on the Tits interval $\overline{xy} \subset \partial_T \mathcal{T}$. Let p, x_i^-, y_i^- be as in the proof of Lemma 7.2. Then $\lim_{i\to\infty} \widetilde{\angle}_p(x_i^-, y_i^-) = \angle_T(x, y)$, so we can choose a sequence $z_i \in \overline{x_i^- y_i^-} \subset L_i^-$ so that $\lim_{i\to\infty} \widetilde{\angle}_p(x_i^-, z_i) = \angle_T(x, z)$ and $\lim_{i\to\infty} \widetilde{\angle}_p(z_i^-, y_i) = \angle_T(z, y)$. The segments $\overline{pz_i}$ converge to the ray $\overline{pz}$. Since for any $j > 0$ the segment $\overline{pz_i}$ crosses L_j^+ for sufficiently large i, by Lemma 7.2 part 2 we get

$$\overline{pz_i} \cap L_j^- \subset B(o_j, R_j)$$

and we conclude that $\overline{pz} \cap L_j^- \neq \emptyset$. Hence $z \in \partial_T^\infty \mathcal{T}$. □

The θ in the above proposition is referred to as the Tits angle of $\mathcal{T}$ and is denoted $\theta(\mathcal{T})$.

7.2. Self-similar Templates

In this section we study a special class of full templates called *self-similar templates.*

Definition 7.4. Let $\mathcal{T}$ be a full template with $Wall_{\mathcal{T}} = \{W_i\}_{i\in\mathbb{Z}}$ and $Strip_{\mathcal{T}} = \{\mathcal{S}_i\}_{i\in\mathbb{Z}}$, and set $\alpha_i := \alpha(W_i)$, $l_i := l(\mathcal{S}_i)$, and $\epsilon_i := \epsilon(\mathcal{S}_i)$ (we define ϵ using the strip orientation compatible with the strip directions and the usual ordering on $\mathbb{Z}$). Then $\mathcal{T}$ is a *self-similar template* if for all $i, j \in \mathbb{Z}$ we have $\alpha_i = \alpha_j$, $l_{i+2j} = 2^j l_i$, and $\epsilon_{i+2j} = 2^j \epsilon_i$. In this case we say that $\mathcal{T}$ has data $(\beta; l_0, \epsilon_0, l_1, \epsilon_1)$ where $\beta = \alpha_i$ for all $i \in \mathbb{Z}$.

Note that by the definition, if we rescale the metric on a self-similar template $\mathcal{T}$ by a factor of 2, then we get a template equivalent to $\mathcal{T}$, i.e. there is a homothety $\Phi : \mathcal{T} \to \mathcal{T}$ which preserves strip directions, stretches distances by a factor of 2, and which shifts wall and strip indices by 2. A self-similar template is determined up to equivalence by β and the data $\{l_0, \epsilon_0, l_1, \epsilon_1\}$.

$\mathcal{T}$ contains one point v which is not on any wall or strip (since the union of the walls and strips is not complete). The point v is the limit of the Cauchy sequence $\{o_{-i} | i = 0, 1, 2, ...\}$. On the other hand, for each $j \in \mathbb{Z}$ the half template, $\mathcal{T}_j$, given by the union of planes W_i and strips $\mathcal{S}_i$ for $i \geq j$ is uniform (and complete). The images of the embeddings $\partial_\infty^\infty \mathcal{T}_i \to \partial_\infty \mathcal{T}$ and $\partial_T^\infty \mathcal{T}_i \to \partial_T \mathcal{T}$ are independent of i, and we use $\partial_\infty^\infty \mathcal{T}$ (respectively $\partial_T^\infty \mathcal{T}$) to denote this common subspace. We will say that $\mathcal{T}$ is *trivial* if $|\partial_\infty^\infty \mathcal{T}| = 1$ and *nontrivial* if $|\partial_\infty^\infty \mathcal{T}| > 1$.

We note that if r is a geodesic ray parameterized by arclength then $\Phi \circ r$ is a geodesic ray parameterized by twice arclength. So Φ and Φ^{-1} take rays to rays. Since Φ is a homothety it preserves the Tits angles between rays.

If $\mathcal{R}_i$ represents the space of geodesic rays starting at o_i and intersecting W_j for all $j \geq i$, then the above shows $\Phi^{-1}(\mathcal{R}_i) = \mathcal{R}_{i-2}$ and that Φ^{-1} acts as a Tits isometry on $\partial_T^\infty \mathcal{T}$. In particular since $\partial_T^\infty \mathcal{T}$ is isometric to an interval, Φ leaves the midpoint fixed, and Φ^2 acts as the identity. Thus by repeated applications of Φ^{-1} we see that we can represent $\partial_T^\infty \mathcal{T}$ as the set rays, $\mathcal{R}_{-\infty}$, that start at v and are invariant under Φ^2. Further, the middle ray is invariant under Φ.

We will show in the lemma below that all rays in $\mathcal{R}_{-\infty}$ are Φ invariant. In particular any such ray that intersects an o_i must intersect all o_{i+2n} for $n \in \mathbb{Z}$, and hence there are at most two such rays, r_{even} and r_{odd}, in $\mathcal{R}_{-\infty}$.

Each choice $N \in \{I, II, III, IV\}$ determines a choice of quarter planes $Q_i^N \subset W_i$ as follows: for all $k \in \mathbb{Z}$, $Q_{2k}^N = Q_N \subset W_{2k}$ while $Q_{2k+1}^N = -Q_N \subset W_{2k+1}$. Straightforward Euclidean geometry shows that the corresponding development map, $\mathcal{D}_N$, with $\mathcal{D}_N(v) = 0$ has the property that $\mathcal{D}_N \circ \Phi \circ \mathcal{D}_N^{-1}$ is multiplication by 2 wherever (and however) it is defined (see Figure 3). (The easiest way to see this is to first develop Q_0^N, $\mathcal{S}_0$, Q_1^N, $\mathcal{S}_1$, and Q_2^N into the plane, then shift the origin $(0, 0)$ so that it lies on the line through $\mathcal{D}(o_0)$ and $\mathcal{D}(o_2)$ and such that $\mathcal{D}(o_0)$ lies between $(0, 0)$ and $\mathcal{D}(o_2)$ and such that the distance from $(0, 0)$ to $\mathcal{D}(o_2)$ is twice the distance to $\mathcal{D}(o_0)$. We note that $\mathcal{D}(Q_2^N) = 2\mathcal{D}(Q_0^N)$. We can now define a map $\mathcal{D}$ uniquely so that $\mathcal{D} \circ \Phi \circ \mathcal{D}^{-1}$ is multiplication by 2. It is easy to check that this map is a development and hence by uniqueness is $\mathcal{D}_N$ - up to an element of $O(2)$.) There are rays r_{even} and r_{odd} from the origin such that $\mathcal{D}_N(o_{2i}) \in r_{even}$ and $\mathcal{D}_N(o_{2i+2}) \in r_{odd}$. (We can fix $\mathcal{D}_N$ completely if desired by taking r_{even} to be the positive x-axis and to make r_{odd} point in the upper half plane.)

Lemma 7.5. *Let $\mathcal{T}$ be a nontrivial self-similar template. Then there is a choice*

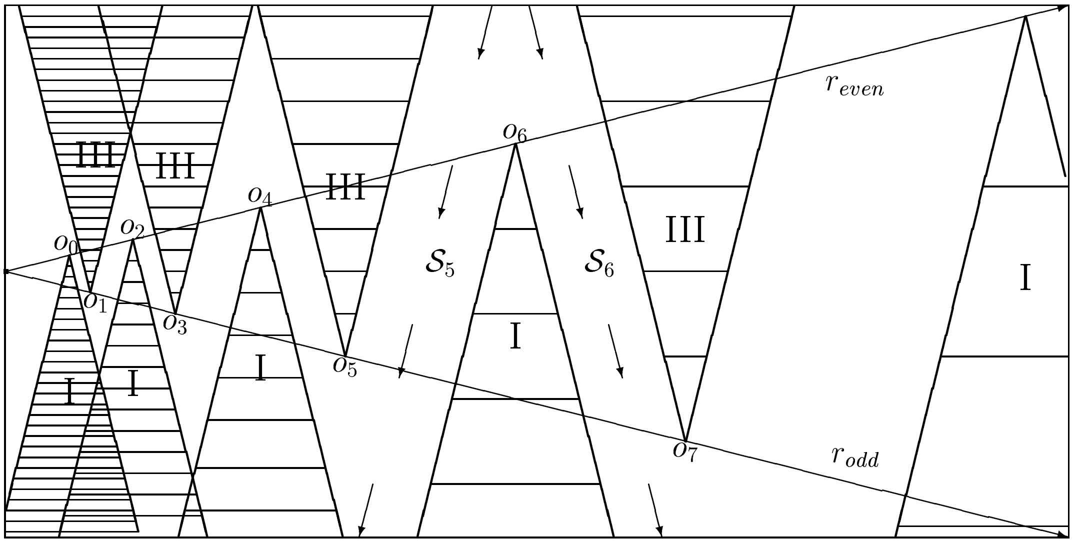

Figure 3: Development of a self-similar template from quarter planes Q_i^I. Developements of quadrants for nonnegative integers are shaded and labeled with their type, while those with negative indices are not shown. The direction of $\mathcal{S}_5$ and $\mathcal{S}_6$ are shown via arrows. The interval of rays between r_{even} and r_{odd} correspond to rays in the template.

$N \in \{I, II, III, IV\}$ such that for $r \in \mathcal{R}_{-\infty}$, $r \cap W_i \subset Q_i^N$. Each $r \in \mathcal{R}_{-\infty}$ is Φ invariant, and r_{even} and r_{odd} are the $\mathcal{D}_N$ images of the boundary rays of $\mathcal{R}_{-\infty}$ as a closed interval. In particular the set of $r \in \mathcal{R}_{-\infty}$ span a space isometric, via $\mathcal{D}_N$ to a convex cone in R^2.

Proof. The fact that $\mathcal{R}_{-\infty}$ is a closed interval follows from Proposition 7.3. If $r_1, r_2 \in \mathcal{R}_{-\infty}$ then their invariance under the homothety Φ^2 implies that $\angle_T(r_1, r_2) = \angle_v(r_1, r_2)$, and hence the rays between r_1 and r_2 span a space isometric to a convex cone in R^2. In particular, no ray between r_1 and r_2 can intersect an origin o_i. Thus the middle (Φ invariant) ray r_m determines $N, M \in \{I, II, III, IV\}$ such that $r_m \cap W_{2n} \subset Q_N$, and $r_m \cap W_{2n+1} \subset Q_M$. The above says that for any interior ray, r, $r \cap W_{2n} \subset Q_N$, and $r \cap W_{2n+1} \subset Q_M$. Consider the development map $\mathcal{D}$ determined by this choice of quarter planes. We will assume that $\mathcal{D}(v)$ is the origin, so in particular $\mathcal{D}(Q_i)$ and $\mathcal{D}(\mathcal{S}_i)$ miss the origin. Now $\mathcal{D} \circ \Phi^2 \circ \mathcal{D}^{-1}$ is defined when restricted to the image of each strip or quarter plane. It is clearly just multiplication by 4, since that is what it does to each ray in the $\mathcal{D}$ image of $\mathcal{R}_{-\infty}$.

We assume that $N = I$ (the other cases are similar) and must prove that $M = III$. If $M = II$ (resp. $M = IV$) then for $i = 1$ (resp. $i = 0$), $\mathcal{D}(Q_i) \cup \mathcal{D}(\mathcal{S}_i) \cup \mathcal{D}(Q_{i+1})$ will contain a half plane which in turn contains one of $\mathcal{D}(Q_i)$ or $\mathcal{D}(Q_{i+1})$. But this cannot happen since any ray entering that half plane will never leave it. If $M = I$ then for $k \geq max\{\frac{\pi}{\beta}, \frac{\pi}{\pi-\beta}\}$ we have $\cup_{i=1}^k(\mathcal{D}(Q_i) \cup \mathcal{D}\mathcal{S}_i)$ contains a half plane which in turn contains one of the quarter planes and the same argument works. Thus we conclude that $M = III$, and $Q_i = Q_i^N$.

We know that $\mathcal{D}_N \circ \Phi \circ \mathcal{D}_N^{-1}$ is just multiplication by 2. This means in particular that all rays in $\mathcal{R}_{-\infty}$ are invariant under Φ. Now since at least one ray between r_{even}

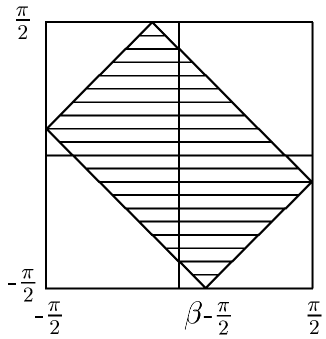

Figure 4: The graph of A_β for $\beta = \frac{3\pi}{5}$.

and r_{odd} hits every $\mathcal{D}_N(Q_i)$ this is true of all such rays, hence all are $\mathcal{D}_N$ of a ray in $\mathcal{T}$. This gives a 1-1 correspondence between rays in $\mathcal{T}$ and rays in the plane between r_{even} and r_{odd} completing the lemma.

□

The argument in the last proof shows that a self-similar template is nontrivial if and only if there is an $N \in \{I, II, III, IV\}$ such that some (and hence any) ray between the corresponding r_{even} and r_{odd} intersects every $\mathcal{D}_N(Q_i^N)$. But by self-similarity this will be true if and only if the line segment from $\mathcal{D}_N(o_0)$ to $\mathcal{D}_N(o_2)$ intersects $\mathcal{D}_N(Q_1^N)$. This leads to the following lemma:

We will use the notation $\mathbb{R}_0^4 = \{(x_1, x_2, x_3, x_4) \subset \mathbb{R}^4 \mid x_1,\, x_3 > 0\}$.

Let A_β be:

$$\{(x,y) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \mid x + \beta \geq y \geq x - \beta \text{ and } \; -x + (\pi - \beta) \geq y \geq -x - (\pi - \beta)\}.$$

See Figure 4.

Lemma 7.6. *Fix $\pi > \beta > 0$ and let $\mathcal{A}_\beta$ be the set of $(l_0, \epsilon_0, l_1, \epsilon_1) \subset \mathbb{R}_0^4$ such that the self-similar template with data $(\beta; l_0, \epsilon_0, l_1, \epsilon_1)$ is trivial. Then*

$$\mathcal{A}_\beta = \{(l_0, \epsilon_0, l_1, \epsilon_1) \in \mathbb{R}_0^4 \mid (arctan(\frac{\epsilon_0}{l_0}), arctan(\frac{\epsilon_1}{l_1})) \in A_\beta\}$$

Proof. Let $\psi_i = arctan(\frac{\epsilon_i}{l_i})$. The proof follows once we show that $R^4 - \mathcal{A}_\beta$ consists of 4 components given in order for N=I, II, III, and IV by:

$$\psi_0 > \psi_1 + \beta, \quad -\psi_1 - (\pi - \beta) > \psi_0, \quad \psi_1 - \beta > \psi_0, \quad \text{and} \quad \psi_0 > -\psi_1 + (\pi - \beta).$$

Here we do the case $N = I$, i.e. the line segment from $\mathcal{D}_I(o_0)$ to $\mathcal{D}_I(o_2)$ intersects $\mathcal{D}_I(Q_1^I)$. The cases $N = II, III,$ and IV are similar.

Let $\pi > \theta_0 > 0$ be the angle between the line segment from $\mathcal{D}_I(o_1)$ to $\mathcal{D}_I(o_0)$ and the "incoming" edge of $\mathcal{D}_I(Q_1^I)$ (see Figure 5). Here the "incoming" edge is the $\mathcal{D}_I$ image of the negative half line of L_1^- (since Q_1^I is of type III). Similarly let θ_2 be the angle between the line segment from $\mathcal{D}_I(o_1)$ to $\mathcal{D}_I(o_2)$ and the "outgoing" edge of $\mathcal{D}_I(Q_1^I)$. θ_1 will be the angle at $\mathcal{D}_I(o_1)$ of the sector $\mathcal{D}_I(Q_1^I)$ (which in our case is just β). It is easy to see that our condition is equivalent to

$$\pi > \theta_0 + \theta_1 + \theta_2.$$

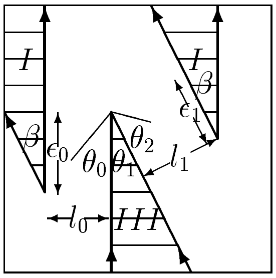

Figure 5: Developement in case I; $\epsilon_0 > 0$, $\epsilon_1 < 0$.

The rest of the argument is just that the definitions (being careful about how the sign of ϵ_i and the orientation of the strips interact) give us, when $N = I$,

$$\theta_0 = \frac{\pi}{2} - arctan(\frac{\epsilon_0}{l_0}) \quad \text{and} \quad \theta_2 = \frac{\pi}{2} - arctan(\frac{-\epsilon_1}{l_1})$$

which completes the argument. □

We extract the information we need with the following elementary (but somewhat non-trivial):

Corollary 7.7. *Pick $a_i > 0$ for $i = 1, \ldots, 4$, $b_1 > 0$, $b_2 \in \mathbb{R}$, and a subset $\mathcal{B} \subset \mathbb{R}^4_0$. Suppose there is a $\beta \in (0, \pi)$ so that $\mathcal{B}$ is precisely the set of $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4_0$ for which*

$$(a_1x_1 + b_1, a_2x_2 + b_2, a_3x_3, a_4x_4) \in \mathcal{A}_\beta.$$

Then β, $\frac{b_1}{a_1}$, and $\frac{b_2}{a_2}$ are uniquely determined by $\mathcal{B}$. If this unique β is not $\frac{\pi}{2}$ then $\frac{b_1}{a_2}$ and $\frac{b_2}{a_1}$ are also determined by $\mathcal{B}$.

Proof. We consider the map

$$\Psi(x_1, x_2, x_3, x_4) = (arctan(\frac{a_2x_2 + b_2}{a_1x_1 + b_1}), arctan(\frac{a_4x_4}{a_3x_3})) =: (\psi_0(x_1, x_2), \psi_1(x_3, x_4)).$$

the previous lemma along with our assumption says that for $a_1x_1 + b_1 > 0$ and $x_3 > 0$ $(x_1, x_2, x_3, x_4) \in \mathcal{B} \iff \Psi(x_1, x_2, x_3, x_4) \in A_\beta$. Ψ maps $\{(x_1, x_2, x_3, x_4)|a_1x_1 + b_1 > 0 \text{ and } x_3 > 0\}$ onto $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$.

Step 1: β is determined by $\mathcal{B}$. We notice that $(x, y^0) \in A_\beta$ for all $\frac{\pi}{2} - \epsilon < x < \frac{\pi}{2}$ if and only if $y^0 = \frac{\pi}{2} - \beta$. Similarly $(x^1, y) \in A$ for all large $\frac{\pi}{2} - \epsilon < y < \frac{\pi}{2}$ if and only if $x^1 = \frac{\pi}{2} - \beta$. Thus $(1, x_2, x_3^0, x_4^0) \in \mathcal{B}$ for all sufficiently large x_2 if and only if $\psi_1((x_3^0, x_4^0)) = \frac{\pi}{2} - \beta$, while $(x_1^1, x_2^1, 1, x_4) \in \mathcal{B}$ for all sufficiently large x_4 if and only if $\psi_0((x_1^1, x_2^1)) = \frac{\pi}{2} - \beta$. Fix such an (x_1^1, x_2^1), then, if $\beta \neq \frac{\pi}{2}$, there is a unique x_4^1 such that for all $x_4 \geq x_4^1$, $(x_1^1, x_2^1, 1, x_4) \in \mathcal{B}$. If $\beta < \frac{\pi}{2}$ we have $\psi_1(1, x_4^1) = \frac{\pi}{2} - 2\beta$, while if $\beta > \frac{\pi}{2}$ we have $\psi_1(1, x_4^1) = 2\beta - \frac{3\pi}{2}$.

Let $\bar{a_i}$, $\bar{b_1}, \bar{b_2}$, and $\bar{\beta}$ be another choice of parameters that work and $\bar{\Psi}$ and $\bar{\psi_i}$ the corresponding functions. Then $\tan(\bar{\psi_1}) = c\tan(\psi_1)$ where $c = \frac{\bar{a}_4a_3}{\bar{a}_3a_4} > 0$. Plugging in (x_3^0, x_4^0) from the previous paragraph allows us to conclude $\tan(\frac{\pi}{2} - \bar{\beta}) = c\tan(\frac{\pi}{2} - \beta)$, (i.e. $\tan(\bar{\beta}) = \frac{1}{c}\tan(\beta)$ if $\beta \neq \frac{\pi}{2}$). Note in particular that the sign of $\frac{\pi}{2} - \bar{\beta}$ is the same as the sign of $\frac{\pi}{2} - \beta$. Plugging in $(1, x_4^1)$ from the above paragraph we get $\tan(\frac{\pi}{2} - 2\bar{\beta}) = c\tan(\frac{\pi}{2} - 2\beta)$ if $\beta < \frac{\pi}{2}$ and $\tan(2\bar{\beta} - \frac{3\pi}{2}) = c\tan(2\beta - \frac{3\pi}{2})$ if

$\beta > \frac{\pi}{2}$. However this can only happen if c=1 and $\beta = \bar{\beta}$. Thus we conclude that β is determined by $\mathcal{B}$ and that either $\beta = \frac{\pi}{2}$ or else $c = 1$ and $\psi_1 = \bar{\psi}_1$.

Step 2: If $\beta \neq \frac{\pi}{2}$ then $\psi_0 = \bar{\psi}_0$, while if $\beta = \frac{\pi}{2}$ then $\tan(\bar{\psi}_0) = \frac{1}{c}\tan(\psi_0)$. We will first show that given $\mathcal{B}$, β and ψ_1 then there is at most one choice for ψ_0. To see this fix (x_1, x_2) and consider $S \subset R^2$ such that $\mathcal{B} \cap ((x_1, x_2) \times R^2) = (x_1, x_2) \times S$. Then $\psi_1(S)$ is precisely the interval such that $A_\beta \cap (\psi_0(x_1, x_2) \times R) = (\psi_0(x_1, x_2) \times \psi_1(S))$. However the shape of A_β is such that $\psi_1(S)$ thus determines $\psi_0(x_1, x_2)$, unless $\beta = \frac{\pi}{2}$ in which case it determines $\psi_0(x_1, x_2)$ up to sign. The sign is determined by continuity and the fact that for x_2 large and positive (resp. negative) then $\psi_0(x_1, x_2)$ is positive (resp. negative).

Thus if $\beta \neq \frac{\pi}{2}$ then $\psi_1 = \bar{\psi}_1$ and we see that $\bar{\psi}_0 = \psi_0$ is the unique solution. If $\beta = \frac{\pi}{2}$ then $\tan(\bar{\psi}_1) = c\tan(\psi_1)$ and $A_\beta = \{(x, y) \in R^2 | -x + \frac{\pi}{2} \geq y \geq x - \frac{\pi}{2}$ when $x > 0$ and $x + \frac{\pi}{2} \geq y \geq -x - \frac{\pi}{2}$ when $x < 0\}$. Thus for $c > 0$ the map $(x, y) \to (arctan(\frac{1}{c}\tan(x)), arctan(c\tan(y))$ preserves A_β and step 2 follows.

The rest of the proof follows from the following equation that holds for all positive x_1 and all x_2:

$$\frac{\bar{a}_2 x_2 + \bar{b}_2}{\bar{a}_1 x_1 + \bar{b}_1} = \frac{1}{c}\frac{a_2 x_2 + b_2}{a_1 x_1 + b_1}$$

and the fact that $c = 1$ when $\beta \neq \frac{\pi}{2}$.

□

7.3. Recovering the data

In this section $\mathcal{G}$ will denote a fixed admissible graph of groups, $G := \pi_1(\mathcal{G})$ the fundamental group of $\mathcal{G}$, and $G \curvearrowright T$ the Bass-Serre action for $\mathcal{G}$. For every vertex v of $\mathcal{G}$ we choose a generator $\zeta_v \in Z(G_v)$ for the center of $Z(G_v)$ as in Definition 3.9. Also, we will fix an admissible action $G \curvearrowright X$. We will use the template notation from section 4.1. Recall that when $\mathcal{T}$ is a half-template, then $\partial_\infty^\infty \mathcal{T} \subset \partial_\infty \mathcal{T}$ denotes the set of boundary points corresponding to rays which intersect all but finitely many walls.

The goal of this section is to prove the remaining half of Theorem 1.3: the topological conjugacy class of the action $G \curvearrowright \partial_\infty X$ determines the functions $MLS_v : G_v \to \mathbb{R}_+$ and $\tau_v : G_v \to \mathbb{R}$ up to a multiplicative factor, for every vertex $v \in \mathcal{G}$.

Definition 7.8. An element g of a vertex group G_v is *restricted* if g acts on $\bar{Y}_v$ (see section 3.2) as an axial isometry and its fixed points in $\partial_\infty \bar{Y}_v$ are distinct from the fixed points of G_e where e is any edge incident to v.

It will be convenient to choose, for each vertex v of $\mathcal{G}$, a restricted element $\delta_v \in G_v$ which lies in the commutator subgroup $[G_v, G_v] \subset G_v$:

Lemma 7.9. *For every vertex v, the commutator subgroup $[G_v, G_v]$ contains restricted elements.*

Proof. We first recall that $H_v := G_v/Z(G_v)$ is a nonelementary hyperbolic group, and the induced action $H_v \curvearrowright \bar{Y}_v$ is discrete and cocompact.

Choose a free nonabelian subgroup $S \subset H_v$ [Gro87, p. 212], and elements $\bar{g}_1$, $\bar{g}_2 \in S$ which belong to a free basis for the commutator subgroup $[S, S] \subset [H_v, H_v]$, and let

$g_i \in G_v$ be lift of $\bar{g}_i$ under the projection $G_v \to H_v$. Then g_i acts axially on $\bar{Y}_v$ since $\bar{g}_i$ has infinite order in H_v and H_v acts discretely on $\bar{Y}_v$. Note that $Fix(g_1, \partial_\infty \bar{Y}_v) \cap Fix(g_2, \partial_\infty \bar{Y}_v) = \emptyset$, since otherwise by Lemma 2.21 we would have $Fix(g_1, \partial_\infty \bar{Y}_v) = Fix(g_2, \partial_\infty \bar{Y}_v)$, forcing $\langle g_1, g_2 \rangle$ to be virtually cyclic, which is absurd. Set $h_n := g_1^n g_2^n$. Lemma 2.10 tells us that h_n is axial for large n and $Fix(h_n, \partial_\infty \bar{Y}_v)$ converges to $\{\xi_1, \xi_2\} \subset \partial_\infty \bar{Y}_v$ where $\xi_i \in Fix(g_i, \partial_\infty \bar{Y}_v)$. The induced action of G_e on $\bar{Y}_v$ translates a geodesic $\gamma_e \subset \bar{Y}_v$. By the finiteness of $\mathcal{G}$ we can choose elements $g_e \in G_e$ such that the induced translation of g_e is nonzero but uniformly bounded. Thus Lemma 2.21 says that subsets $Fix(G_e, \partial_\infty \bar{Y}_v) = Fix(g_e, \partial_\infty \bar{Y}_v)$ define a discrete subset of $(\partial_\infty \bar{Y}_v \times \partial_\infty \bar{Y}_v)/\mathbb{Z}_2$, so either

a) $Fix(h_n, \partial_\infty \bar{Y}_v) \cap Fix(G_e, \partial_\infty \bar{Y}_v) = \emptyset$ for all edges e incident to v when n is large, or

b) There is a subsequence h_{n_i} and an edge e incident to v so that $Fix(h_{n_i}, \partial_\infty \bar{Y}_v) = Fix(G_e, \partial_\infty \bar{Y}_v)$.

But if b) held then we would have $Fix(G_e, \partial_\infty \bar{Y}_v) = \{\xi_1, \xi_2\}$, which, by Lemma 2.21, would force the absurd conclusion that $Fix(g_1, \partial_\infty \bar{Y}_v) = Fix(g_e, \partial_\infty \bar{Y}_v) = Fix(g_2, \partial_\infty \bar{Y}_v)$. Hence case a) holds and the lemma is proved. □

Notice that for every v, $MLS_v(\delta_v) \neq 0$ (since δ_v acts on $\bar{Y}_v$ as an axial isometry), $\tau_v(\delta_v) = 0$ since $\delta_v \in [G_v, G_v]$, and $\tau_v(\zeta_v) \neq 0$.

Lemma 7.10. *In order to determine MLS_v and τ_v up to a multiplicative factor, it suffices to determine the ratios*

$$\frac{MLS_v(\sigma)}{MLS_v(\delta_v)} \quad \text{and} \quad \frac{\tau_v(\sigma)}{\tau_v(\zeta_v)}$$

for every restricted element $\sigma \in G_v$ whose fixed point set in $\partial_\infty \bar{Y}_v$ is disjoint from $Fix(\delta_v, \partial_\infty \bar{Y}_v)$.

Proof. Choose an arbitrary $\sigma \in G_v$.

We first discuss MLS_v. Note that $MLS_v(\sigma) = 0$ iff σ projects to an element of finite order in H_v; hence we may assume that σ acts on $\bar{Y}_v$ as an axial isometry. First assume that $Fix(\sigma, \partial_\infty \bar{Y}_v) = Fix(\delta_v, \partial_\infty \bar{Y}_v)$. Let $\bar{\sigma}, \bar{\delta}_v \in H_v$ be the projections to H_v. Then $\bar{\sigma}$ and $\bar{\delta}_v$ generate a virtually cyclic subgroup S because they have a common axis. Hence there is a finite subset $\{s_1, \dots, s_k\} \subset G_v$ so that for any n we have $\sigma^n = s_{i_n} \delta_v^{j_n} \zeta_v^{k_n}$ for suitable i_n, j_n, k_n. Then

$$MLS_v(\sigma) = \lim_{n\to\infty} \frac{1}{n} MLS_v(\sigma^n) = \lim_{n\to\infty} MLS_v(\delta_v^{j_n})$$

so we can recover the ratio above in this case.

Now assume $Fix(\sigma, \partial_\infty \bar{Y}_v) \cap Fix(\delta_v, \partial_\infty \bar{Y}_v) = \emptyset$. Setting $h_k := \delta_v^k \sigma^{k^2}$, we argue as in the proof of Lemma 7.9 to see that for large k, h_k is restricted, $Fix(h_k, \partial_\infty \bar{Y}_v) \cap Fix(\delta_v, \partial_\infty \bar{Y}_v) = \emptyset$, and $|MLS_v(h_k) - kMLS_v(\delta_v) - k^2 MLS_v(\sigma)|$ is uniformly bounded (by Lemma 2.10). We may then recover the desired ratios from the formula

$$MLS_v(\sigma) = \lim_{k\to\infty} \frac{1}{k^2} MLS_v(h_k).$$

We now consider the behavior of τ_v. Suppose σ projects to an element of finite order in H_v, or $Fix(\sigma, \partial_\infty \bar{Y}_v) \cap Fix(\delta_v, \partial_\infty \bar{Y}_v) \neq \emptyset$. In either case we have a finite set $\{s_1, \dots, s_k\} \subset G_v$ so that $\sigma^n = s_{i_n} \delta_v^{j_n} \zeta_v^{k_n}$ for suitable i_n, j_n, k_n. Then

$$\tau_v(\sigma) = \lim_{n\to\infty} \frac{1}{n} \tau_v(\sigma^n) = \lim_{n\to\infty} [\frac{j_n}{n} \tau_v(\delta_v) + \frac{k_n}{n} \tau_v(\zeta_v)].$$

For the case $Fix(\sigma, \partial_\infty \bar{Y}_v) \cap Fix(\delta_v, \partial_\infty \bar{Y}_v) = \emptyset$, use the same h_k as above for large k since $\tau_v(h_k) = k^2 \tau(\sigma)$.

□

We now focus our attention on a vertex $\bar{v}_1$ of $\mathcal{G}$. Choose an edge $\bar{e}$ of $\mathcal{G}$ incident to $\bar{v}_1$, and lift $\bar{e}$ to an edge e of the Bass-Serre tree. We adopt the notation from section 2.5 for the associated graph of groups $\mathcal{G}'$, $G' := \pi_1(\mathcal{G}')$, $T' \subset T$, etc. We fix some restricted element $\sigma \in G_{\bar{v}_1}$ with $Fix(\sigma, \partial_\infty \bar{Y}_{v_1}) \cap Fix(\delta_{v_1}, \partial_\infty \bar{Y}_{v_1}) = \emptyset$.

Definition 7.11. (Special rays) For every $(p, q, r, s) \in \mathbb{R}_0^4$ we define a geodesic ray $\gamma \subset T'$ and sequences $\hat{l}_i = l_i(p, q, r, s)$, $\hat{\epsilon}_i = \hat{\epsilon}_i(p, q, r, s)$ as follows.

Case 1: $\bar{e} \subset \mathcal{G}$ is embedded. Let $e = \overline{v_1 v_2}$. For each $i \in \mathbb{N}$ we set $s_{2i-1} := \sigma^{2^{i-1}} \delta_{v_1}^{[p 2^{i-1}]} \zeta_{v_1}^{[q 2^{i-1}]}$ and $s_{2i} := \delta_{v_2}^{[r 2^{i-1}]} \zeta_{v_2}^{[s 2^{i-1}]}$, where $[x]$ denote the integer part of $x \in \mathbb{R}$. Then we let $\gamma \subset T$ be the geodesic ray with successive edges

$$e,\ s_1 e,\ s_1 s_2 e, \dots, s_1 \dots s_k e, \dots, \tag{7.12}$$

and define, for all $i \in \mathbb{N}$, $\hat{l}_{2i-1} = 2^{i-1}(MLS_{v_1}(\sigma) + |p| MLS_{v_1}(\delta_{v_1}))$, $\hat{l}_{2i} = 2^{i-1} |r| MLS_{v_2}(\delta_{v_2})$; and $\hat{\epsilon}_{2i+1} := 2^i(\tau_{v_1}(\sigma) + q\tau_{v_1}(\zeta_{v_1}))$, and $\hat{\epsilon}_{2i} := 2^{i-1} s \tau_{v_2}(\zeta_{v_2})$.

Case 2: $\bar{e} \subset \mathcal{G}$ is a loop. Let t be as in section 2.5. For each $i \in \mathbb{N}$ we set $s_{2i-1} := \sigma^{2^{i-1}} \delta_{v_1}^{[p 2^{i-1}]} \zeta_{v_1}^{[q 2^{i-1}]}$ and $s_{2i} := \delta_{v_1}^{[r 2^{i-1}]} \zeta_{v_1}^{[s 2^{i-1}]}$; then we let $\gamma \subset T$ be the geodesic with successive vertices

$$v_1,\ t v_1,\ t s_1 t^{-1} v_1,\ t s_1 t^{-1} s_2 t v_1, \dots t s_1 t^{-1} s_2 t \dots s_{2k} t v_1, \dots, \tag{7.13}$$

and define, for all $i \in \mathbb{N}$, $\hat{l}_{2i-1} = 2^{i-1}(MLS_{v_1}(\sigma) + |p| MLS_{v_1}(\delta_{v_1}))$, $\hat{l}_{2i} = 2^{i-1} |r| MLS_{v_1}(\delta_{v_1})$; and $\hat{\epsilon}_{2i+1} := 2^i(\tau_{v_1}(\sigma) + q\tau_{v_1}(\zeta_{v_1}))$, and $\hat{\epsilon}_{2i} := 2^{i-1} s \tau_{v_1}(\zeta_{v_1})$.

These rays are useful because they admit templates that are asymptotically self-similar:

Lemma 7.14. *There is a half template $(\mathcal{T}, f, \phi)$ for γ with walls $Wall_{\mathcal{T}} = \{W_i\}_{i=1}^\infty$ and strips $Strip_{\mathcal{T}} = \{\mathcal{S}_i\}_{i=1}^\infty$, so that for i sufficiently large, $l(\mathcal{S}_i) = \hat{l}_i$, $\epsilon(\mathcal{S}_i) = \hat{\epsilon}_i$, and $\alpha(W_i) = \beta$, where β is the Tits angle between the (positively oriented) $\mathbb{R}$-factors of Y_{v_1} and Y_{v_2}.*

Proof. We will treat the case when $\bar{e}$ is embedded; the other case is similar. To prove the lemma we will show that the desired template may be obtained from a standard template for γ by changing the strip widths and strip gluings by a bounded amount. Thus by "uniformly" we will just mean independent of i, but possibly dependent on all other choices.

For $i \geq 1$ we set $e_i := s_1 \dots s_{i-1} e$ and $v_i := e_i \cap e_{i+1}$. Recall (section 4.2) that the standard template $(\mathcal{T}, f, \phi)$ for γ is constructed using flats $F_{e_i} \subset Y_{e_i}$ and flat strips

$\mathcal{S}_{e_i,e_{i+1}} \subset Y_{v_i}$, where $\mathcal{S}_{e_i,e_{i+1}} = \gamma_{e_i,e_{i+1}} \times \mathbb{R} \subset \bar{Y}_{v_i} \times \mathbb{R} = Y_{v_i}$. We choose $x_1 \in F_{e_1}$ and set $x_i := s_1 \dots s_{i-1} x_1 \in F_{e_i}$.

Step 1: There is a constant c_1 so that $d(x_i, \mathcal{S}_{e_i,e_{i+1}}) < c_1$, $d(x_{i+1}, \mathcal{S}_{e_i,e_{i+1}}) < c_1$ and $|Width(\mathcal{S}_{e_i,e_{i+1}}) - \hat{l}_i| < c_1$. We will do the case when $i = 2j-1$ is odd; the even case is similar. Let $\pi : Y_{v_1} = \bar{Y}_{v_1} \times \mathbb{R} \to \bar{Y}_{v_1}$ be the projection map, and set $\bar{x}_1 := \pi(x_1)$, $\bar{F}_e := \pi(F_e)$, $\bar{F}_{s_ie} := \pi(\bar{s}_i F_e) = \pi(F_{s_ie})$, and $\gamma_{e,s_ie} := \pi(\mathcal{S}_{e,s_ie})$. We apply $(s_1 \dots s_{i-1})^{-1}$ and then π to everything, and are thereby reduced to showing that there is a c_1 so that $d(\bar{x}_1, \gamma_{e,s_ie}) < c_1$, $d(s_i\bar{x}_1, \gamma_{e,s_ie}) < c_1$, and $|d(s_i\bar{x}_1, \bar{x}_1) - \hat{l}_i| < c_1$.

¿From the definition of s_i we have $s_i\bar{x}_1 = \sigma^{2^{j-1}} \delta_{v_1}^{[p2^{j-1}]} \bar{x}_1$ since ζ_{v_1} acts trivially on $\bar{Y}_{v_1}$. Since σ, $\delta_{v_1} \in G_{v_1}$ are restricted elements, the sets $\partial_\infty \bar{F}_e \subset \partial_\infty \bar{Y}_{v_1}$ and $\partial_\infty \bar{F}_{s_ie} \subset \partial_\infty \bar{Y}_{v_1}$ are disjoint from the fixed point sets of σ and δ_{v_1}; the latter two sets are disjoint by assumption. Therefore we may apply Lemma 2.10 to conclude that when j is sufficiently large,

a) $s_i : \bar{Y}_{v_1} \to \bar{Y}_{v_1}$ is an axial isometry with an axis $\gamma_i \subset \bar{Y}_{v_1}$ at uniformly bounded distance from $\bar{x}_1$ and $s_i\bar{x}_1$.

b) $|d(s_i\bar{x}_1, \bar{x}_1) - \hat{l}_i|$ is uniformly bounded.

c) The attracting (resp. repelling) fixed point of s_i in $\partial_\infty \bar{Y}_{v_1}$ is close to the attracting fixed point, ξ^+, of σ (resp. repelling fixed point, ξ^-, of $\delta_{v_1}^{sign(p)}$).

Let γ_{e,s_ie} be the shortest path from $\bar{F}_e$ to $\bar{F}_{s_ie}$ with endpoints $\bar{z}_i \in \bar{F}_e$ and $\bar{w}_i \in \bar{F}_{s_ie}$ Let γ be a geodesic with endpoints ξ^- and ξ^+. The Gromov hyperbolicity of $\bar{Y}_{v_1}$ and Lemma 2.9 part 3 imply that for t fixed and large and i large the axis of s_i comes within a uniform distance of $\gamma_{e,s_ie}(t)$ and hence $\gamma_{e,s_ie}(t)$ stay a uniform distance from γ. Thus $\bar{z}_i$, which is the point on $\bar{F}_e$ closest to $\gamma_{e,s_ie}(t)$, must stay a uniform distance from the set of points on $\bar{F}_e$ closest (in a Buseman function sense) to ξ^+ (which we see by taking t large) and hence stay uniformly close to $\bar{x}_1$. Similarly $s_i^{-1}(\bar{w}_i)$ approaches the set of points on $\bar{F}_e$ closest to ξ^- and hence $\bar{w}_i$ stays uniformly close to $s_i(\bar{x}_1)$.

Step 2: There is a c_2 so that the standard template $(\mathcal{T}, f, \phi)$ satisfies $|\epsilon(\mathcal{S}_i) - \hat{\epsilon}_i| < c_2$ and $\alpha(W_i) = \beta$ for all $i > 1$. The assertion that $\alpha(W_i) = \beta$ is clear from the definition of $\beta(W_i)$ and the construction of standard templates. Step 1 then implies that there is a c_3 so that the origin $o_i \in W_i$ maps under $f : \mathcal{T} \to X$ to within distance c_3 of x_i, for $i > 1$. Hence from the definition of $\epsilon : Strip^o_{\mathcal{T}} \to \mathbb{R}$ we see that $\epsilon(\mathcal{S}_i)$ agrees with $\tau_{v_i}((s_1 \dots s_{i-1}) s_i (s_1 \dots s_{i-1})^{-1})$ to within $2c_3$. The conjugacy invariance of τ_v then gives $|\epsilon(\mathcal{S}_i) - \hat{\epsilon}_i| < c_2$ for a suitable c_2.

Step 3: Adjusting $(\mathcal{T}, f, \phi)$. In steps 1 and 2 we have shown that the standard template satisfies conditions of Lemma 7.14 to within bounded error. So we now modify the construction of $\mathcal{T}$ by changing the metric[11] on $\hat{\mathcal{S}}_{e_i,e_{i+1}}$so that $Width(\hat{\mathcal{S}}_{e_i,e_{i+1}}) = \hat{l}_i$ if $l_i \geq 1$, and leaving $\hat{\mathcal{S}}_{e_i,e_{i+1}}$ untouched otherwise, and by modifying the gluings $\partial\hat{\mathcal{S}}_{e_i,e_{i+1}} \to W_{e_i} \amalg W_{e_{i+1}}$ by a bounded amount so that $\epsilon(\mathcal{S}_i) = \hat{\epsilon}_i$. Finally, if we redefine $f : \mathcal{T} \to X$ to agree with the original f on $\amalg_e W_e$ and on $\mathcal{T} - (\amalg_e W_e)$, then we get the desired template for γ. $\square$

Proof the Theorem 1.3 concluded. Consider the subset $\mathcal{B} \subset \mathbb{R}^4_0$ of 4-tuples (p, q, r, s) for which the geodesic ray $\gamma \subset T$ defined above gives a trivial subset $\partial_\infty^{\partial_\infty \gamma} X$ (i.e. a

[11]We do this in the simplest way: we start with the metric product decomposition $\hat{\mathcal{S}}_{e_i,e_{i+1}} \simeq I \times \mathbb{R}$ and then scale the metric on the I factor.

single point); by Lemma 3.23 and Corollary 5.26 the subset $\partial_\infty^{\partial_\infty\gamma}X$ can be detected just using the action $G \curvearrowright \partial_\infty X$, and so $\mathcal{B}$ is also determined by the action $G \curvearrowright \partial_\infty X$. On the other hand, by Theorem 5.1, $\partial_\infty^{\partial_\infty\gamma}X$ is trivial iff $\partial_\infty^\infty\mathcal{T}$ is trivial (i.e. a single point) where $(\mathcal{T}, f, \phi)$ is any template for γ. Using Lemma 7.14 we arrive at the following: the subset $\mathcal{B}$ of $(p,q,r,s) \in \mathbb{R}_0^4$ so that any template $\mathcal{T}$ with $l(\mathcal{S}_i) = \hat{l}_i(p,q,r,s)$, $\epsilon(\mathcal{S}_i) = \hat{\epsilon}_i(p,q,r,s)$, and $\alpha(W_i) = \beta$ (for i sufficiently large) is trivial, is determined by the action $G \curvearrowright \partial_\infty X$. But since a template $\mathcal{T}$ with $l(\mathcal{S}_i) = \hat{l}_i(p,q,r,s)$, $\epsilon(\mathcal{S}_i) = \hat{\epsilon}_i(p,q,r,s)$, and $\alpha(W_i) = \beta$ (for i sufficiently large) is trivial iff the self-similar template with data $\{\beta; \hat{l}_3, \hat{\epsilon}_3, \hat{l}_4, \hat{\epsilon}_4\}$ is trivial, we may apply Corollary 7.7 to conclude that the ratios

$$\frac{MLS_{v_1}(\sigma)}{MLS_{v_1}(\delta_{v_1})} \quad \text{and} \quad \frac{\tau_{v_1}(\sigma)}{\tau_{v_1}(\zeta_{v_1})}$$

as well as β are determined by the action $G \curvearrowright \partial_\infty X$. Moreover, unless $\beta = \frac{\pi}{2}$ then

$$\frac{MLS_{v_1}(\sigma)}{\tau_{v_1}(\zeta_{v_1})} \quad \text{and} \quad \frac{\tau_{v_1}(\sigma)}{MLS_{v_1}(\delta_{v_1})}$$

are also determined.

8. Examples

In this section we construct the example mentioned in the introduction: we will find two locally compact Hadamard spaces X_0 and X_r on which an admissible group G acts discretely and cocompactly with the same geometric data (i.e. the induced actions of G on $\partial_\infty X_0$ and $\partial_\infty X_r$ are topologically conjugate), an equivariant quasi-isometry $\tilde{F} : X_0 \to X_r$, and a geodesic ray γ of X_0 such that $\tilde{F}(\gamma)$ does not lie within a bounded distance of a geodesic ray in X_r.

To do that we consider for each (small) real r a complex M_r, built out of four flat square tori T_i. On each T_i we use standard angle coordinates $(s,t)_i$ with $(s+2n\pi, t+2m\pi)_i = (s,t)_i$ for any integers n and m. We let $M_r = T_1 \cup T_2 \cup T_3 \cup T_4/\sim_r$, where $(0,t)_1 \sim_r (t,0)_2$, $(\pi,t)_2 \sim_r (t,\pi)_3$, $(\pi,t)_3 \sim (t,\pi)_4$ and $(0,t)_4 \sim_r (t,r)_1$. We will let X_r represent the universal cover.

The X_r are Hadamard spaces with admissible fundamental groups which all have the same geometric data. This is easiest to see by considering M_r as the union of four spaces $T_1 \cup T_2$, $T_2 \cup T_3$, $T_3 \cup T_4$, and $T_4 \cup T_1$ each of which is isometric to a figure eight cross a circle, where both circles in the figure eight and the product circle have length 2π. Each of the three spaces is glued to adjacent spaces along (product) boundary tori (reversing the factors). We note that the underlying finite graph of G is a square with four edges e_i corresponding naturally to T_i.

It is easy to see that the M_r are homeomorphic. In fact for small r the fundamental groups are identified in a natural way. However, we will find it more useful to consider the map $F_r : M_0 \to M_r$ defined by $F(s,t)_i = (s,t)_i$, except that $F((0,t)_4) = (t,0)_1$. This is not continuous along the closed geodesic $(0,t)_4$, but the induced equivariant quasi-isometry $\tilde{F} : X_0 \to X_r$ is relatively easy to study.

We now want to choose a geodesic ray γ in X_0. For these spaces geodesics are just geodesics in templates. It is easier to first choose the degenerate half template $\mathcal{T}$ that it will lie in. By degenerate template we mean a template where we alow the strip widths to be 0. In fact in our case all the strip widths will be 0, all the angles will be $\frac{\pi}{2}$, and the displacements will all be odd multiples of π. We will choose $\mathcal{T}$ so that the edges of the ray in the Bass-Serre tree project to the finite graph periodically in the order e_1, e_2, e_3, e_4, $e_1 \ldots$. Subject to this constraint we can still choose the itinerary such that the displacements in $\mathcal{T}$ are any odd multiples of π we please. Hence we may choose the itinerary (e.g. make it nontrivial) so that there is a geodesic γ in $\mathcal{T}$ which misses all the vertices such that $\gamma \cap W_i$ gets arbitrarily long. (The easiest way to see this is to consider the developement $\mathcal{D}$. First choose a ray r that you want to be $\mathcal{D}(\gamma)$. The choice of itinerary at each step amounts to a choice of quarter planes among those shifted by $2n\pi$. We can thus make the choice so that the quarter plane intersects r in increasingly long intervals).

Of course γ is also a geodesic ray in X_0. $\tilde{F}$ will take $\mathcal{T}$ to a corresponding template $\mathcal{T}_r$, and the geodesic ray corresponding to $F(\gamma)$ must be a geodesic in this template. $\tilde{F}$ will be discontinuous exactly where walls corresponding to e_4 and e_1 are glued. The discontinuity is a translation by r perpendicular to the gluing line.

We develop $\mathcal{T}$ (and $\mathcal{T}_r$) to the plane in such a way that γ goes to a ray with angle $\frac{\pi}{2} > \theta > 0$, the quarter planes map to planes of type II and IV where walls of type e_1 and e_3 yield quarter planes of type II and walls of type e_2 and e_4 yield quarter planes of type IV. $\tilde{F}$ will induce a map of developments which is discontinuous precisely on the horizontal lines where the quarter planes coming from e_4 meet those coming from e_1. The discontinuity will be precisely a vertical shift by r. Thus the image of γ consists of arbitrarily long line segments of slope θ with infinitely many vertical jumps of size r. Since the segments get arbitrarily long, the only rays that can stay a bounded distance from $\tilde{F}(\gamma)$ must also have slope θ. But then, because of the infinitely many vertical jumps of size r in $\tilde{F}(\gamma)$, no such ray can stay a bounded distance from $\tilde{F}(\gamma)$.

One can make similar examples on singular piecewise Euclidean graph manifolds. We construct such examples by gluing together two pieces. Each piece is topologically a twice punctured torus cross S^1. The boundary of each piece will consist of two totally geodesic square flat two-tori. The first space is constructed by gluing both of the corresponding boundary tori together flipping the coordinates. The second space is similar except that for one of the boundary tori the gluing map is coordinate flipping composed with a small translation.

The metric of each piece is a product metric where the circle has length 1 and where the metric on the torus is the completion of the flat square torus minus two line segments (slits) of length $\frac{1}{2}$. The torus with the slits is a compact flat singular space with boundary being two closed geodesics of length 1 (i.e. going "around a slit").

The argument that this gives an example is very similar to the above since geodesics in the space are in fact geodesics in the corresponding templates and since the induced map on the development of appropriate templates has properties similar to the above example.

We also suspect there are such examples on smooth graph manifolds. In fact one may be able to construct such an example by smooth approximations to the above example. To do this carefully it would be necessary to be careful with how closely template geodesics shadow actual geodesics in this case.

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