ON THE DIFFERENTIABILITY OF LIPSCHITZ MAPS FROM METRIC MEASURE SPACES TO BANACH SPACES

JEFF CHEEGER AND BRUCE KLEINER

Abstract. We consider metric measure spaces satisfing a doubling condition and a Poincaré inequality in the upper gradient sense. We show that the results of [Che99] on differentiability of real valued Lipschitz functions and the resulting bi-Lipschitz nonembedding theorems for finite dimensional vector space targets extend to Banach space targets having what we term a *good finite dimensional approximation.* This class of targets includes separable dual spaces. We also observe that there is a straightforward extension of Pansu's differentiation theory for Lipschitz maps between Carnot groups, [Pan89], to the most general possible class of Banach space targets, those with the Radon-Nikodym property.

Contents

Date: December 6, 2006.
The first author was partially supported by NSF Grant DMS 0105128 and the second by NSF Grant DMS 0505610.

1. Introduction

A classical theorem of Rademacher states that real valued Lipschitz functions on $\mathbf{R}^n$ are differentiable almost everywhere with respect to Lebesgue measure; [Rad20]. The literature contains numerous extensions of Rademacher's result in which either the domain, the target, or the class of maps is generalized. Classical examples include almost everywhere approximate differentiability of Sobolev functions, [Zie89], analysis of and on rectifiable sets, [Fed69, Mat95], almost everywhere differentiability of quasiconformal homeomorphisms between domains in Euclidean space, [Geh61, Mos73], and differentiability of Lipschitz maps from $\mathbf{R}^k$ into certain Banach spaces.

In many of the recent results in this vein, a significant part of the achievement is to make sense of differentiation in a context where some component of the classical setting is absent e.g. the infinitesimal affine structure on the domain or target, or a good measure on the domain. Included are Pansu's differentiation theorem, [Pan89], for Lipschitz maps between Carnot groups, differentiation theorems for Lipschitz mappings between Banach spaces [BL00, Chap. 6-7], metric differentiation, [Kir94, Pau01], and the differentiation theory developed in [Che99], for Lipschitz functions on metric measure spaces which are doubling and for which a Poincaré inequality holds in the upper gradient sense.

We now discuss in more detail those developments which are most relevant to our main theme.

The Radon-Nikodym property.

While the extension of Rademacher's theorem from real valued functions to $\mathbf{R}^m$ valued functions is essentially immediate, the extension to infinite dimensional Banach space targets is not. In fact, the theorem holds for some Banach space targets and fails for others.

The following example from [Aro76] illustrates the failure when the target is L^1. Let $\mathcal{L}$ denote Lebesgue measure and let $f : (0,1) \to L^1((0,1), \mathcal{L})$ be defined by $f(t) = \chi_{(0,t)}$, where χ_A denotes the characteristic function of A. If f were differentiable at $t \in (0,1)$, then the difference quotients

$$\frac{f(t') - f(t)}{t' - t}$$

would converge in L^1 as $t' \to t$. However, this is clearly not the case, since the difference quotients converge weakly to a delta function supported at t.

A Banach space, V, for which every Lipschitz function, $f : \mathbf{R}^k \to V$, is differentiable almost everywhere, is said to have the *Radon-Nikodym property*. The following facts are taken from Chapter 5 of [BL00], which contains an extensive discussion of the Radon-Nikodym property in its various (not obviously) equivalent formulations. The term, "Radon-Nikodym property", derives from a formulation which posits the validity of the Radon-Nikodym theorem for V-valued measures.

According to an early result of Gelfand, separable dual spaces and reflexive spaces have the Radon-Nikodym property; [Gel38] and Corollary 5.12 of [BL00]. In fact, a dual space, E^*, has the Radon-Nikodym property if and only if every separable subspace of E has a separable dual; see Corollary 5.24 of [BL00]. There does exist a Banach space with the Radon-Nikodym property which is not isomorphic to a subspace of a separable dual space; see Example 5.25 of [BL00]. However, every infinite dimensional Banach space with the Radon-Nikodym property contains an infinite dimensional separable subspace which is isomorphic to a dual space; see [GM84].

Differentiating maps between Carnot groups.

Pansu gave an extension of Rademacher's theorem in which both the domain and the target are generalized. He considered Lipschitz maps between Carnot groups i.e. nilpotent groups with certain left-invariant *sub-riemannian* metrics, of which the Heisenberg group, $\mathbb{H}$, with its so-called Carnot-Cartheodory metric, $d^{\mathbb{H}}$, is the simplest example; see [Pan89] and compare Section 6 below. Pansu showed for such maps, almost everywhere with respect to Haar measure, the function converges under the natural inhomogeneous rescaling to a group homorphism. This assertion makes sense only for Carnot groups.

Differentiating real valued functions on metric measure spaces.

At a minimum, the statement of Rademacher's theorem requires that the domain carries both a metric, so that Lipschitz functions are defined, and a notion of "almost everywhere", which for finite dimensional domains, is typically provided by a measure. (In the case of infinite dimensional Banach space domains, absent an appropriate measure, other notions of "almost everywhere" have been used; see [BL00, Chap. 6-7].)

The paper [Che99] introduced a new notion of differentiability for metric measure spaces and proved a Rademacher type theorem in that setting. To explain this notion of differentiation, we begin with the following definition.

Definition 1.1. Let X denote a metric measure space, $A \subset X$ a measurable subset and $f_1, \ldots, f_k : A \to \mathbf{R}$, real-valued Lipschitz functions. For $x \in A$, we say $\{f_i\}$ *is dependent to first order at* x, if there a linear combination, $\sum_i a_i f_i$, such that for $x' \in A$, as $x' \to x$,

$$\sum_i a_i \left(f_i(x') - f_i(x)\right) \;=\; o(d^X(x', x))\,.$$

We say that f *is dependent on* $\{f_1, \ldots, f_k\}$ *to first order at* x, if $\{f, f_1, \ldots, f_k\}$ is *dependent* to first order at x. *Linear independence* at x is defined similarly.

If the f_i's were smooth functions on a manifold, then dependence to first order at x would be equivalent to the linear dependence of their differentials at x.

A key fact established in [Che99] is that if a metric measure space, (X, d^X, μ), is a PI space (see below for the definition) then there is an integer N such that any collection of Lipschitz functions of cardinality at least $N+1$ is dependent to first order almost everywhere. Intuitively, this says that there is at most an N-dimensional space of possibilities for the differential of any Lipschitz function at almost every point.

Definition 1.2. A countable collection of pairs, $\{(A_\alpha, u_\alpha)\}$, with A_α a measurable set and $u_\alpha : A_\alpha \to \mathbf{R}^{n_\alpha}$ a Lipschitz map, is an *atlas* if $\bigcup_\alpha A_\alpha$ has full measure and:

1) For every Lipschitz function, $f : X \to \mathbf{R}$, and for μ-a.e. $x \in A_\alpha$ and all α, the restriction, $f \big| A_\alpha$, is dependent to first order on the component functions, $u_1, \ldots, u_{n_\alpha}$, of u_α at x.

2) For all α, the functions, $u_1, \ldots, u_{n_\alpha}$, are independent to first order at μ-a.e. $x \in A_\alpha$.

We refer to the maps $u_\alpha : A_\alpha \to \mathbf{R}^{n_\alpha}$ as *charts.*

It was shown in [Che99] that any PI space admits an atlas by using the above mentioned bound on the cardinality of set of Lipschitz functions which is independent to first order, and a selection argument (analogous to choosing a basis from a spanning set).

Property 1) leads to a definition of pointwise differentiability with respect to the chart (A_α, u_α).

Definition 1.3. A Lipschitz function is *differentiable at* $x \in A_\alpha$ *, with respect to* (A_α, u_α), if there is a linear function, $\Phi : \mathbf{R}^{n_\alpha} \to \mathbf{R}$, such that for $x' \in A_\alpha$, as $x' \to x$,

$$f(x') = \; f(x) \; + \; \Phi \circ (u_\alpha(x') - u_\alpha(x)) \; + \; o(d^X(x', x))\,. \tag{1.4}$$

Note that (1.4) generalizes the notion of first order Taylor expansion in local coordinates.

It follows readily from properties 1), 2), above, that every Lipschitz function, $f : X \to \mathbf{R}$, is differentiable almost everywhere with respect to every chart in $\{(A_\alpha, u_\alpha)\}$, and that the corresponding linear function, Φ, is uniquely determined almost everywhere, and varies measurably. Furthermore, as in the case of Lipschitz manifolds, on the overlap of any two charts one gets almost everywhere defined bounded measurable transition functions for derivatives (i.e. bounded measurable jacobians) permitting one to define the cotangent bundle T^*X. This is a measurable vector bundle which carries a natural measurable norm. The norm is characterized by the property that every Lipschitz function $f : X \to \mathbf{R}$ defines a measurable section Df whose fiberwise norm is given by its pointwise Lipschitz constant almost everywhere. The *tangent bundle*, TX, is defined as the dual bundle of T^*X, and is equipped with the dual norm.

PI spaces.

We now define the term *PI space*. We note that conditions (1.6), (1.7), below were introduced by Heinonen-Koskela; [HK96].

A measure μ is *locally doubling* there are constants R, β, such that for every point x and every $r \leq R$,

$$\mu(B_{2r}(x)) \leq \beta \cdot \mu(B_r(x)). \tag{1.5}$$

The function, g, is called an *upper gradient* for f if for every rectifiable curve, $c : [0, \ell] \to X$, parameterized by arclength, s,

$$|f(c(\ell)) - f(c(0))| \leq \int_0^\ell g(c(s))\, ds\,. \tag{1.6}$$

Put

$$f_{x,r} = \frac{1}{\mu(B_r(x))} \int_{B_{2r}(x)} d\mu\,.$$

The *local* $(1,p)$-*Poincaré inequality* is the condition that for some $\lambda, R, \tau < \infty$,

(1.7)

$$\frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f - f_{x.r}|\, d\mu \quad \leq \quad r \cdot \tau \cdot \left(\frac{1}{\mu(B_r(x))} \int_{B_{\lambda r}(x)} g^p \, d\mu \right)^{\frac{1}{p}} ,$$

where $x \in X$, $r \leq R$, and f is an L^1 function for which has an upper gradient in L^p.

Definition 1.8. Let (X, d^X) denote a metric space such that closed metric balls are compact, and let μ denote a Radon measure on X which for $r \leq 1$, satisfies a local doubling condition and a local $(1,p)$-Poincaré inequality. The triple, (X, d^X, μ), will be refered to as a *PI space.*

Sometimes we will supress the distance function, d^X, and measure, μ, and just say that X is a PI space.

Examples of PI spaces include Gromov-Hausdorff limits of sequences of Riemannian manifolds with a uniform lower Ricci curvature bound, Bourdon-Pajot spaces, Laakso spaces and Carnot groups such as the Heisenberg group $\mathbb{H}$; see [CC97], [BP99], [Laa00], [Gro96].

Unlike $\mathbf{R}^n$ or Carnot groups, general PI spaces do not have any apriori infinitesimal linear structure. Indeed, the Hausdorff dimension of a PI space need not be an integer and there need not exist any points for which sufficiently small neighborhoods have finite topology.

In addition to the measurable tangent bundle, there is a quite distinct notion of a *tangent cone*, at $x \in X$, which, by definition, is any rescaled (pointed, measured) Gromov-Hausdorff limit space of a sequence of rescaled spaces, $(X, x, r_i^{-1} d^X, \mu_i)$, where $r_i \to 0$ and $\mu_i = \mu/\mu(B_{r_i}(x))$. It follows from the doubling condition and Gromov's compactness theorem that tangent cones exist for all $x \in X$. Although tangent cones at x need not be unique, we use the notation, X_x, for any tangent cone at x (supressing the base point, distance function and measure). The terminology notwithstanding, X_x, need not be a cone in either the metric or topological sense. However, any tangent cone is again a PI space; see Section 9 of [Che99].

As shown in [Che99], for almost every $x \in X$, there exist natural *surjective Lipschitz maps,* $X_x \to T_xX$, for any X_x; hence

$$\dim T_xX \leq \dim X_x \,, \tag{1.9}$$

where dim denotes Hausdorff dimension.

Bi-Lipschitz nonembedding.

Generalized differentiation theories have been used to show that solutions of certain mapping problems are very restricted or even fail to exist. A good example is Mostow's rigidity theorem for hyperbolic space forms, whose proof uses the differentiability of quasi-conformal mappings; see [Mos73]. The same principle is a powerful tool in the bi-Lipschitz classification of Banach spaces [BL00, Chap. 7].

Closer to our topic is the observation of Semmes that the differentiablity theory of Pansu for Lipschitz maps between Carnot groups implies that certain metric spaces do not bi-Lipschitz embed in any finite dimensional Banach space; see [Sem96].

It was shown in [Che99] using the differentiation theory for real valued Lipschitz functions on PI spaces developed there, that if X admits a bi-Lipschitz embedding into a finite dimensional normed space, then there is a full measure set of points $x \in X$ where the tangent space T_xX is defined, and for every tangent cone the canonical map

$$X_x \to T_xX$$

is an isometry; see Section 14 of [Che99] and compare (1.9) above. This result unifies and generalizes earlier nonembedding theorems which were known in particular cases, the proofs of which employed special features of the domains e.g. that of [Sem96]; compare Section 6. The class of spaces to which the result of [Che99] applies includes Bourdon-Pajot spaces, Laakso spaces and Carnot groups.

For example, with respect to the Carnot-Caratheodory distance, the Hausdorff dimension of the Heisenberg group, $\mathbb{H}$, is 4; compare [Gro96]. Since any tangent cone of $\mathbb{H}$ is isometric to $\mathbb{H}$ itself, dim $\mathbb{H}_x = 4$ as well. On the other hand the measurable tangent bundle of $T\mathbb{H}$ can be seen to have dimension 2, so tangent cones are not isometric to tangent spaces.

Summary of results.

The notion of differentiability for real-valued Lipschitz functions on PI spaces introduced in [Che99] extends in a straightforward fashion to Lipschitz functions taking values in a Banach space V. For X a PI space and $\{(A_\alpha, u_\alpha)\}$ an atlas, it reads as follows:

Definition 1.10. A Lipschitz map $f : X \to V$ is *differentiable at* $x \in A_\alpha$ *, with respect to* (A_α, u_α), if there is a linear map, $\Phi : \mathbf{R}^{n_\alpha} \to V$, such that for $x' \in A_\alpha$, as $x' \to x$,

$$f(x') = f(x) + \Phi \circ (u_\alpha(x') - u_\alpha(x)) + o(d^X(x', x))\,. \tag{1.11}$$

With the aim of finding a common generalization of earlier results, one may ask:

Question 1.12. For which pairs, (X, V), with X a PI space and V a Banach space, does one have almost everywhere differentiability for every Lipschitz map $X \to V$?

The main result of this paper gives a partial answer to this question. We show in Theorem 4.1 that if the Banach space V satisfies the GFDA condition (see below), then a Lipschitz map from any PI space into V is differentiable almost everywhere. As a corollary, the criterion, (5.2), of [Che99], also implies the nonexistence of bi-Lipschitz embeddings into GFDA targets, see Theorem 5.1.

Discussion of the proof.

From the theory of [Che99], it immediately follows that every Lipschitz map from a PI space into a Banach space V is weakly differentiable, in the sense that its composition with every $\ell \in V^*$ is differentiable almost everywhere. In general, weak differentiability does not imply differentiability, and is too weak to give rise to bi-Lipschitz nonembedding theorems. This is inevitable, since *any* metric space, X, can be canonically isometrically embedded in $L^\infty(X)$ via the *Kuratowski embedding.*[1]

The following definition identifies a property that is *not* implied by weak differentiability.

[1]The Kuratowski embedding of a metric space X is the map $X \to L^\infty(X)$ which assigns to each $x \in X$ the function $d(x, \cdot) - d(x, x_0)$, where $x_0 \in X$ is a basepoint.

Definition 1.13. A map $f : X \to V$ is *finite dimensional to first order at* $x \in X$, if there is a finite dimensional subspace $F \subset V$ such that

$$\limsup_{r\to 0} \frac{1}{r} \sup_{x'\in B_r(x)} d(f(x') - f(x), F) = 0\,, \tag{1.14}$$

where $d(\,\cdot\,, F)$ denotes the distance in V to the subspace F.

One easily checks that if a Lipschitz map from a PI space to a Banach space is finite dimensional to first order almost everywhere, then weak differentiability can be promoted to differentiability, see Corollary 4.4.

Thus, the proof is reduced to verifying that the GFDA property for the target implies that Lipschitz maps are finite dimensional to first order almost everywhere. This in turn, is proved by combining the GFDA property of the target with weak differentiability.

The GFDA property.

The essential idea of the GFDA property is that a vector, $v \in V$, should be determined up to small error once one knows, up to small error, a *suitable* projection, $\pi(v)$, onto a finite dimensional quotient space, with the property that $\pi(v)$ has nearly the same length as v. A key point is to choose a meaning for "suitable" such that the corresponding GFDA property is strong enough to imply differentiability but weak enough to apply to large class of Banach spaces; see Definition 3.6.

Separable dual spaces have the GFDA property. This follows readily from an elementary, but extremely useful, renorming result of Kadec and Klee; see Proposition 1.b.11 of [LT77] and [Kad59], [Kle61]. It asserts that a separable dual space admits an equivalent norm, for which weak* convergence together with convergence of norms implies strong convergence i.e. $v_i \overset{w^*}{\to} v$ and $\|v_i\| \to \|v\|$, implies $\|v - v_i\| \to 0$.

Examples of separable dual spaces include the classical spaces, L^p, for $1 < p < \infty$, and l^p, for $1 \le p < \infty$. Also, if V^{**} is separable, then we can naturally regard f as taking values in the separable dual space V^{**} and the theory applies. If V is reflexive (but not necessarily separable) then every separable subspace, $\widehat{V} \subset V$ is reflexive and in particular, a dual space; see [BL00]. Thus, the theory applies to all reflexive targets, separable or not.

Radon-Nikodym targets.

It is natural to ask if the differentiation assertion of the main theorem is actually valid for Lipschitz maps into an arbitrary Banach space with the Radon-Nikodym property. This would follow if for example, the GFDA property, which implies the Radon-Nikodym property were actually equivalent to it. At present, we do not know a counter example to this statement. At least for duals, E^*, of separable spaces, E, our result is optimal, since in such a case, as indicated above, E^* has the Radon Nikodym property if and only if it is separable.

In any event, for certain special PI spaces such as Carnot groups, for which there are sufficiently nice curve families, the differentiation and bi-Lipschitz nonembedding theorems do hold for arbitrary Radon-Nikodym targets. Since we do not have a completely general result, we will just illustrate matters by pointing out in Section 6, that in the special case of the Heisenberg group, the proof of the nonembedding theorem of [Sem96] can be quite directly carried over to Radon-Nikodym targets; see Section 6. This was also observed by Lee and Naor, [LN06].

The proof is quite short and does not require any new concepts. In contrast to the main theorems, which are valid for general PI spaces, the argument does make strong use of special properties of the Heisenberg group. Some readers might prefer to read Section 6 before proceeding to the main body of the paper.

Results proved elsewhere; L^1 targets.

The separable space, L^1, is not the dual space of any Banach space. Moreover, the example of [Aro76] demonstrates the failure of differentiatiability. Nonetheless, as we will show elsewhere, by using a different notion of differentiation, the Heisenberg group with its Carnot-Caratheodory metric does not admit a bi-Lipschitz embedding into L^1; see [CK06a].

On the other hand, there is a class of PI spaces which includes Laakso spaces, every member of which has a bi-Lipschitz embedding in L^1; see [CK06b]. In particular, the members of this class are examples of spaces which bi-Lipschitz embed in L^1 but not in ℓ^1.

2. Finite dimensional approximations and weak derivatives

Henceforth V will denote a Banach space and $\{W_i\}$ will denote an inverse system of finite dimensional Banach spaces indexed by the positive integers, whose projection maps, $\theta_i : W_i \to W_{i-1}$, are *quotient maps.*

A sequence, $w_1, w_2, \ldots$, with $w_k \in W_k$ is called *compatible* if $\theta_i(w_i) = w_{i-1}$, for all $i \leq j$. For future reference, we recall that the *inverse limit* Banach space, $\varprojlim W_i$, is defined to be the set of all compatible sequences, $\{w_k\}$, such that

$$\sup_i \|w_i\| < \infty\,,$$

with the obvious vector space structure and norm

$$\|\{w_i\}\| := \lim_{i\to\infty} \|w_i\|.$$

Definition 2.1. A *finite dimensional approximation of V* is a pair, $\{(W_i, \pi_i)\}$, with $\{W_i\}$ an inverse limits system as above and $\pi_i : X \to W_i$ a compatible system of quotient maps, such that for all $v \in V$, the sequence of lengths $\|\pi_i(v)\|$ converges to $\|v\|$.

It is clear that the induced map, $\pi : V \to \varprojlim W_i$, is an isometric embedding.

A finite dimensional approximation is equivalent to a choice of a suitable inverse system $\{V_i\}$ of closed finite codimensional subspaces of V. We will frequently refer to finite dimensional approximations as FDA's.

Example 2.2. Let $V = \ell^1$, and let $\pi_i : V \to \mathbf{R}^i$ be the map which sends the sequence $(a_j)_{j=1}^{\infty}$ to the finite sequence $(a_j)_{j=1}^{i}$, and let $\theta_i : \mathbf{R}^i \to \mathbf{R}^{i-1}$ be the truncation map. Then the pair, $\{(\mathbf{R}^i, \pi_i)\}$, is an FDA.

Any separable Banach space admits finite dimensional approximations:

Lemma 2.3. *A separable Banach space has a finite dimensional approximation.*

Proof. Take a sequence, v_i, of unit vectors in the separable Banach space, V, which is dense in the unit sphere. For each i, apply the Hahn-Banach theorem to obtain a unit norm linear functional, ℓ_i :

$V \to \mathbf{R}$, such that $\ell_i(v_i) = 1$. Let $V_i \subset X$ denote the intersection of the kernels of $\ell_1, \ldots, \ell_i$, and define W_k to be the quotient Banach space $W_i := V/V_i$. $\square$

Consider a PI space, (X, d^X, μ), and an atlas $\{(A_\alpha, u_\alpha)\}$. For every Lipschitz function, $f : X \to \mathbf{R}$, the differential, Df, is defined on a full measure subset, domain$(Df) \subset X$.

Let $\{(W_i, \pi_i)\}$ denote a finite dimensional approximation of a Banach space V. A Lipschitz map, $f : X \to V$, induces a compatible family of Lipschitz maps $f_i := \pi_i \circ f : X \to W_i$. The differentiation theory of [Che99] for real valued functions extends immediately to maps with finite dimensional targets, and in particular, to each f_i. Thus, for μ-a.e. $x \in X$, we obtain a compatible system of linear maps $D_x f_i : T_x X \to W_i$. For each i, this defines a measurable family of maps which is uniquely determined almost everywhere.

Definition 2.4. The *weak derivative* of f at $x \in X$ with respect to the finite dimensional approximation, $\{(W_i, \pi_i)\}$, is the induced linear mapping $\{D_x f_i\} : T_x X \to \varprojlim W_i$.

The weak derivative is defined for almost every $x \in X$. We denote by domain $(\{Df_i\}) \subset X$, the full measure subset on which the weak derivative, $\{D_x f_i\}$, is defined.

Lemma 2.5. *Weak derivatives have the following properties:*

1) *For μ-a.e. $x \in X$, and every $e \in T_x X$,*

$$\lim_{n\to\infty} \|(D_x f_n(e)\| = \|\{D_x f_i\}(e)\|. \tag{2.6}$$

2) $(x, e) \to \|\{D_x f_i\}(e)\|$ *is a measurable function on TX.*

3) *If $X = [0,1]$, then*

$$\int_0^1 \|\{D_t f\}\| dt = \mathrm{length}(f)\,.$$

Proof.

1) If $x \in$ domain $(\{Df_i\})$, then for every $e \in T_x X$, we obtain a compatible system, $\{D_x f_n(e)\}$, which defines the element $\{D_x f_n\}(e)$. Since the norm of an element in $\varprojlim W_i$ is defined by (2.6), we are done.

2) For all $n < \infty$, the formula $(x, e) \to \|D_x f_n(e)\|$ defines a measurable function on the restriction of TX to domain$(Df_n) \subset X$. This is a monotone increasing family of functions, so its pointwise limit is measurable.

3) For $X = [0, 1]$, it is classical that

$$\text{length}(f_n) = \int_X \|Df_n\| dt\,.$$

If $\mathcal{P} := \{0 = t_0 < t_1 < \ldots < t_k = 1\}$ is a partition of $[0, 1]$, then

$$\begin{aligned}\text{length}(f, \mathcal{P}) = \sum_i \|f(t_i) - f(t_{i-1})\| &= \sum_i \lim_{n\to\infty} \|f_n(t_i) - f_n(t_{i-1})\| \\ = \lim_{n\to\infty} \text{length}(f_n, \mathcal{P}) \quad &\le \limsup_{n\to\infty} \text{length}(f_n) \\ = \limsup_{n\to\infty} \int_{[0,1]} \|Df_n\| \quad &= \int_{[0,1]} \|\{Df_n\}\|\,,\end{aligned}$$

where the last equation follows from the monotonicity of the sequence of functions $\|Df_n\|$. Now, property 3) follows from

$$\text{length}(f) := \sup_{\mathcal{P}} \text{length}(f, \mathcal{P})\,.$$

□

3. Good finite dimensional approximations

In this section we discuss GFDA's, a special class of FDA's with an additional property which is sufficient for proving differentiability and which can be verified for a large class of Banach spaces, including separable dual spaces.

Example 3.1. (Example 2.2 continued.) To motivate the the GFDA property, we examine the FDA from Example 2.2 more closely.

Note that if $v, v' \in \ell^1$, and for some i, the projections, $\pi_i(v)$, $\pi_i(v')$, are close, and their norms are close to the norms of v, v', respectively, then the vectors v and v' themselves are close. Indeed, if $\widehat{\pi}_i : \ell^1 \to \ell^1$ denotes the projection map which drops the first i terms of the the sequence then

$$\begin{aligned}\|v - v'\| &= \|\pi_i(v) - \pi_i(v')\| + \|\widehat{\pi}_i(v) - \widehat{\pi}_i(v')\| \\ &\le \|\pi_i(v) - \pi_i(v')\| + \|\widehat{\pi}_i(v)\| + \|\widehat{\pi}_i(v')\| \\ &\le \|\pi_i(v) - \pi_i(v')\| \; + \; \big|\, \|v\| - \|\pi_i(v)\| \,\big| \; + \; \big|\, \|v'\| - \|\pi_i(v')\| \,\big|\,.\end{aligned}$$

The GFDA property is just a more technical version of this kind of statement; Remark 3.7.

To formulate the GFDA property we need the following:

Definition 3.2. Given a finite dimensional approximation, $(\{W_i\}, \{\pi_i\})$, of V, we call a positive decreasing finite sequence, $1 \geq \rho_1, \ldots, \rho_N$, *$\epsilon$-determining* if the conditions,

$$\|v\| - \|\pi_i(v)\| < \rho_i \cdot \|v\|, \qquad \|v'\| - \|\pi_i(v')\| < \rho_i \cdot \|v'\|, \qquad 1 \leq i \leq N\,, \tag{3.3}$$

and

$$\|\pi_N(v) - \pi_N(v')\| < N^{-1} \cdot \max(\|v\|, \|v'\|)\,, \tag{3.4}$$

imply

$$\|v - v'\| < \epsilon \cdot \max(\|v\|, \|v'\|)\,. \tag{3.5}$$

Observe that by dividing by $\max(\|v\|, \|v'\|)$, it suffices to consider pairs v, v' for which $\max(\|v\|, \|v'\|) = 1$.

Definition 3.6. A finite dimensional approximation $\{(W_i, \pi_i\})$, of a Banach space V is *good* if for every $\epsilon > 0$ and every infinite decreasing sequence, $1 \geq \rho_i \searrow 0$, some finite initial segment, $\rho_1, \ldots, \rho_N$, is ϵ-determining.

A Banach space, V, which admits a good finite dimensional approximation will be called a GFDA.

Remark 3.7. In the eventual application, relation (3.5), which states that v, v' are close in norm, will be used in proving that for GFDA targets, weak derivatives are actually derivatives. In order to know that relation (3.5) holds, it is necessary to verify relation (3.3). This will follow from Egoroff's theorem, whose conclusion asserts the uniform convergence of a pointwise sequence of functions off subsets of arbitrarily small measure. Note that uniform convergence means convergence no slower than some sequence, $\rho_i \to 0$, which, as in Definition 3.6, is otherwise uncontrolled.

Lemma 3.8. *If $\{(W_i, \pi_i)\}$ is a GFDA of V, then the induced linear transformation, $\pi : V \to \varprojlim W_i$, is an isometry of Banach spaces.*

Proof. We already know that π is an isometric embedding, so it suffices to check that π is surjective.

Pick $\{w_i\} \in \varprojlim W_i$, and choose a decreasing sequence $\rho_i \to 0$. Let

(3.9)

$$C_i := \{v \in V \mid \pi_i(v) = w_i, \text{ and for all } 1 \le j \le i, \|v\| - \|w_j\| \le \rho_j\}.$$

Since π_i is a quotient map, C_i is a nonempty, bounded, closed set for each i. The GFDA property implies that $\operatorname{diam}(C_i) \to 0$. Since the family, $\{C_i\}$, is clearly nested, it follows from completeness that the intersection $\bigcap_i C_i$ is nonempty and consists of a single vector v. Then $\pi(v) = \{w_i\}$. □

Remark 3.10. The above argument does not use the full strength of the GFDA assumption.

Proposition 3.11. *If $V = E^*$ is a separable dual space, then it is isomorphic to a GFDA space.*

Proof. We apply the following renorming procedure to E, and hence E^*.

Lemma 3.12. [LT77, p. 12] *Suppose E is a separable Banach space and $F \subset E^*$ is a separable subspace of its dual. Then E can be renormed so that if the sequence, $e_i^* \in E^*$, weak* converges to $e_\infty^* \in F$ and $\|e_i^*\| \to \|e_\infty^*\|$, then $\|e_i^* - e_\infty^*\| \to 0$.*

To continue with the proof of Proposition 3.11 we construct an inverse system $\{W_i\}$ by taking an increasing family of finite dimensional subspaces $E_1 \subset E_2 \subset \ldots \subset E$ with dense union, and taking π_i to be the restriction mapping $E^* \to W_i := E_i^*$, where W_i is endowed with the norm dual to the norm on E_i. It follows from the Hahn-Banach theorem that $\{(W_i, \pi_i)\}$ is an FDA of $V = E^*$.

Suppose $\{(W_i, \pi_i)\}$ is not a GFDA. Then for some decreasing sequence, $\{\rho_i\} \subset (0, \infty)$, with $\rho_i \to 0$, and some $\epsilon > 0$, there are sequences, $v_k, v_k' \in V$, such that for all $k < \infty$,

$$\|v_k\|,\ \|v_k'\| \le 1, \tag{3.13}$$

$$\max\left(\|v_k\| - \|\pi_i(v_k)\|,\ \|v_k'\| - \|\pi_i(v_k')\|\right) < \rho_i \ \text{ for } 1 \le i \le k, \tag{3.14}$$

$$\|\pi_j(v_k) - \pi_j(v_k')\| < \frac{1}{k}, \tag{3.15}$$

$$\|v_k - v_k'\| \ge \epsilon. \tag{3.16}$$

By the Banach-Alaoglu theorem, we can pass to weak* convergent subsequences, with respective limits v_∞ and v'_∞. Semicontinuity of norm with respect to weak* convergence implies

$$\|v_\infty\| \leq \liminf_{k\to\infty} \|v_k\| \leq 1, \quad \|v'_\infty\| \leq \liminf_{k\to\infty} \|v'_k\| \leq 1 \,.$$

Also, by the definition of π_i and weak* convergence, for all i we have $\pi_i(x_v) \to \pi_i(v_\infty)$, $\pi_i(v'_k) \to \pi_i(v'_\infty)$. Hence, for fixed i,

$$\|v_\infty\| \geq \|\pi_i(v_\infty)\| = \lim_{k\to\infty} \|\pi_i(v_k)\| \geq \limsup_{k\to\infty} \|v_k\| - \rho_i \,, \tag{3.17}$$

which forces $\|v_\infty\| \geq \limsup_{k\to\infty} \|v_k\|$. Similarly $\|v'_\infty\| \geq 1$. Therefore, $\|v_k\| \to \|v_\infty\|$ and $\|v'_k\| \to \|v'_\infty\|$. By Lemma 3.12, this implies that v_k and v'_k converge strongly. However, it is clear from (3.15) that v_k and v'_k have the same weak* limit. Thus, they converge strongly to the same limit, contradicting (3.16). □

Remark 3.18. By the lemma, any separable reflexive space, and ℓ^1 are isomorphic to GFDA spaces. However, $L^1([0,1])$ is not a GFDA space. This follows from the failure of differentiability (see the example from [Aro76] mentioned in the introduction) and Theorem 4.1.

4. Differentiability for GFDA targets

In this section (X, d^X, μ) will denote a fixed PI space, and $f : X \to V$, a Lipschitz map to a Banach space V.

Fix $\{(A_\alpha, u_\alpha)\}$, an atlas for X as in Definition 1.2. Without loss of generality, we may assume that for each α, there exists an $L_\alpha < \infty$, such that the derivative of u_α induces an L_α-bi-Lipschitz fiberwise isomorphism, $D_x u_\alpha : T_x X \to \mathbf{R}^{n_\alpha}$, for each $x \in A_\alpha$, where $T_x X$ is equipped with its canonical norm, and $\mathbf{R}^{n_\alpha}$ is given the usual norm.

Our main theorem is:

Theorem 4.1. *If V satisfies the GFDA property, then the Lipschitz map, $f : X \to V$, is differentiable at μ-a.e. $x \in X$. Moreover, the derivative is uniquely determined at μ-a.e. x and defines a bounded measurable section, Df, of the bundle* $\mathrm{End}(TX, V)$.

As indicated in the introduction, we use the next lemma and its corollary to reduce the proof of Theorem 4.1 to showing that f is finite dimensional to first order μ-a.e..

Lemma 4.2. *Let $\{(W_i, \pi_i)\}$ denote an FDA of a Banach space V. If $f : X \to V$ is finite dimensional to first order at $x \in X$, and x is a point of weak differentiability of f with respect to $\{(W_i, \pi_i)\}$, then f is differentiable at x.*

Proof. Let $F \subset V$ denote a finite dimensional subspace so that

$$\limsup_{r \to 0} \frac{1}{r} \sup_{x' \in B_r(x)} d(f(x') - f(x), F) = 0. \tag{4.3}$$

Since F is finite dimensional, there is a bounded linear projection map $\Psi : V \to F$. Set $\widehat{f} := f(x) + \Psi \circ (f - f(x))$. By (4.3), the maps, f and $\widehat{f}$, agree to first order at x. This has two implications: that $\widehat{f}$ is also weakly differentiable at x, and that it suffices to prove that $\widehat{f}$ is differentiable at x. But $\widehat{f}$ takes values in a finite dimensional subspace, and is weakly differentiable at x, so clearly $\widehat{f}$ is differentiable at x. □

Corollary 4.4. *If a Lipschitz map, $f : X \to V$, from a PI space to an arbitrary Banach space is finite dimensional to first order μ-a.e., then f is differentiable μ-a.e..*

Proof. Since X is separable, the map f takes values in a separable subspace of V. It follows that without loss of generality, we may assume that V itself is separable. By Lemma 2.3, V admits an FDA, $\{(W_i, \pi_i)\}$, with respect to which and the weak derivative of f is defined almost everywhere. Thus, the corollary follows from Lemma 4.2. □

In preparation for the proof of Theorem 4.1, we need a two more definitions.

Definition 4.5. If F is a finite dimensional normed space and $h : X \to F$ is a Lipschitz map, then x *is an approximate continuity point of Dh* if the point, x, is a density point of A_α, and $Dh \,\big|\, A_\alpha$ is approximately continuous at x, with respect to the trivialization, $TX \,\big|\, A_\alpha \to A_\alpha \times \mathbf{R}^{n_\alpha}$, induced by u_α.

Given $\{(W_i, \pi_i)\}$, we set $f_i := \pi_i \circ f$. We recall that domain$(\{Df_i\}) \subset X$ denotes the domain of definition of the weak derivative of f; see Definition 2.4.

Definition 4.6. A point $x \in$ domain$(\{Df_i\})$ is a *weak approximate continuity point of* $\{Df_i\}$ if x is an approximate continuity point of Df_i for all i.

Clearly, the set of weak approximate continuity points of $\{Df_i\}$ has full measure.

For each $x \in \text{domain}(\{Df_i\})$, and every i, define the semi-norm, $\|\cdot\|_i$, on T_xX, to be the pull-back,

$$\|\cdot\|_i = Df_i^*(\|\cdot\|_{W_i})\,.$$

This defines a measurable Finsler pseudo-metric on TX (also denoted $\|\cdot\|_i$) which is a pointwise nondecreasing function of i, and for which we have $\|\cdot\|_i \leq C\cdot\|\cdot\|_{TX}$, for some constant C. We take the pointwise limit of this sequence and put

$$\begin{aligned}\|\cdot\|_\infty &:= \lim_{i\to\infty}\|\cdot\|_i \\ &\leq C\cdot\|\cdot\|_{TX}\,.\end{aligned}$$

For each $x \in \text{domain}\,(\{Df_i\})$ and $i < \infty$, set

$$\nu(x,i) := \sup\{\|\xi\|_\infty - \|\xi\|_i \;\mid\; \xi \in T_xX,\ \|\xi\| \leq 1\}. \tag{4.7}$$

Then $\nu(\,\cdot\,,i)$ is uniformly bounded, measurable, and converges pointwise to zero almost everywhere.

Proof of Theorem 4.1. Since $\{(W_i,\pi_i)\}$ is assumed to be a GFDA, by Lemma 3.8, the map $V \xrightarrow{\pi} \underleftarrow{\lim}\, W_i$ is an isometry and hence invertible.

By Corollary 4.4, it suffices to show that f is finite dimensional to first order almost everywhere. In fact, we will show that for μ-a.e. $x \in X$, relation (1.14) holds, with $F = \text{image}\,(\{D_xf_i\})$, the image of the weak derivative. Here we identify $\underleftarrow{\lim}\, W_i$ with V, using the isomorphism $\underleftarrow{\lim}\,\pi_i$.

Suppose f is L-Lipschitz. It will be convenient to assume that for every α, every $x \in A_\alpha \cap \text{domain}\,(Du_\alpha)$, and every $\xi \in \mathbf{R}^{n_\alpha}$, with $\|\xi\| \leq 1$, we have

$$\|(D_xu_\alpha)^{-1}(\xi)\| \leq \min(1, \frac{1}{L})\,.$$

This can be arranged by rescaling the maps u_α. In particular, if $x \in A_\alpha \cap \text{domain}\,(\{Df_i\}) \cap \text{domain}\,(Du_\alpha)$, then

$$\|(\{D_xf_i\})(D_xu_\alpha)^{-1}(\xi)\| \leq \|\{D_xf_i\}\|\left(\frac{1}{L}\right) \leq 1\,. \tag{4.8}$$

Pick $\epsilon, \epsilon_1, \epsilon_2, \epsilon_3 > 0$, to be further constrained later.

Choose a subset $X' \subset X$ with finite μ-measure. By Egoroff's theorem, there is a subset $S_1 \subset X'$ such that $\mu(X' \setminus S_1) < \epsilon_1$, and the quantity, $\nu(\,\cdot\,, i)$, of (4.7) converges to zero uniformly on S_1. Otherwise put, there is a sequence, $\rho_i \to 0$, such that for all $x \in S_1$,

$$\nu(x, i) < \rho_i\,. \tag{4.9}$$

Let $S_2 \subset \text{domain}\,(\{Df_i\})$ denote the full measure set of weak approximate continuity points of $\{Df_i\}$, and let $S_3 := S_1 \cap S_2$.

Pick $x \in S_3$. Using the GFDA property, there exists a number N which is ϵ_2-determining for the sequence $\{\rho_i\}$. Since x is an approximate continuity point of Df_j, certainly $x \in A_\alpha$ for some α. Moreover, there exists $r_0 > 0$, such that for all $r < r_0$, there is a subset, $\Theta_r \subset B_r(x) \cap A_\alpha$, with density,

$$\frac{\mu(\Theta_r)}{\mu(B_r(x))} > 1 - \epsilon_3\,,$$

such that for all $x' \in \Theta_r$, $\xi \in \mathbf{R}^{n_\alpha}$, with $\|\xi\| \leq 1$, we have

$$\|(D_{x'} f_j)((D_{x'} u_\alpha)^{-1}(\xi) - (D_x f_j)((D_x u_\alpha)^{-1}(\xi)\| < \frac{1}{N}\,. \tag{4.10}$$

By the GFDA property of $\{(W_i, \pi_i)\}$ and the choice of N, one gets for $\xi \in \mathbf{R}^{n_\alpha}$, with $\|\xi\| \leq 1$, $x' \in \Theta_r$,

$$\|\{D_{x'} f_i\} \circ (D_{x'} u_\alpha)^{-1}(\xi) - \{D_x f_i\} \circ (D_x u_\alpha)^{-1}(\xi)\| < \epsilon_2\,. \tag{4.11}$$

In particular,

$$d(\{D_{x'} f_i\} \circ (D_{x'} u_\alpha)^{-1}(\xi), \text{image}\,(\{D_x f_i\})) < \epsilon_2. \tag{4.12}$$

Pick $k < \infty$, and $\ell \in W_k^*$, with $\|\ell\| \leq 1$, such that ℓ annihilates $\text{image}\,(Df_k) \subset W_k$. Then by (4.11), for every $x' \in \Theta_r$ and every $\xi \in \mathbf{R}^{n_\alpha}$, with $\|\xi\| \leq 1$, there holds

$$|D_{x'}(\ell \circ f_k)(D_{x'} u_\alpha)^{-1}(\xi)| = |\ell\left((D_{x'} f_k)(D_{x'} u_\alpha)^{-1}(\xi)\right)| < \epsilon_2. \tag{4.13}$$

Thus, by choosing ϵ_2, ϵ_3 small enough, we can arrange that $\ell \circ f_k$ has small derivative on a subset of $B_r(x)$ with density as close to 1 as we like. By the Poincaré inequality and the fact that f is L-Lipschitz, we can choose ϵ_2, ϵ_3 small enough (independent of k, ℓ) such that when $r \leq c \cdot r_0$, for a suitable constant, c, we have

$$\max_{B_r(x)} \ell \circ f_k - \min_{B_r(x)} \ell \circ f_k < \frac{\epsilon r}{2}. \tag{4.14}$$

It follows that for all $x' \in B_r(x)$,

$$\begin{aligned} d(f(x') - f(x), \text{image}(\{D_z f_i\})) &= \sup_{k,\ell} \{\ell \circ f_k(x') - \ell \circ f_k(x)\} \\ &\leq \frac{\epsilon r}{2}\,. \end{aligned} \tag{4.15}$$

Since ϵ is arbitrary, (4.15) implies that f is finite dimensional to first order at points $x \in S_1$. Since $\mu(X' \setminus S_1) < \epsilon_1$ and ϵ_1 is arbitrary, this implies that f is differentiable on a full measure subset of X'. Since X' was arbitrary, it follows that f is differentiable on a full measure subset of X.

Uniqueness of the differential.

We claim that for each α, the equation (1.11) has a unique solution for almost every $x \in A_\alpha$.

By composing both sides with π_i, we get for $x' \in A_\alpha$, as $x' \to 0$,

$$f_i(x') = \; f_i(x) \; + \; (\pi_i \circ \Phi) \circ (u_\alpha(x') - u_\alpha(x)) \; + \; o(d^X(x', x))\,.$$

Now, from the almost everywhere uniqueness of the differential for maps into finite dimensional targets, it follows that $\pi_i \circ \Phi$ is uniquely determined almost everywhere. Since this holds for all i, it follows that Φ itself is uniquely determined almost everywhere.

Measurability of the differential Df.

From the proof above and the proof of Lemma 4.2, it follows that the differential Df agrees with the composition, $\{\pi_i\}^{-1} \circ \{D_x f_i\} : T_x X \to V$, almost everywhere, so it suffices to verify the measurability of this composition. This follows in a straightforward fashion from the measurability of Df_i, for all i, and the GFDA property of V.

□

5. Bi-Lipschitz (non)embedding for GFDA targets

We indicate briefly how the discussion of Section 14 of [Che99], which contains nonembedding theorems for finite dimensional targets, extends directly to GFDA targets.

Let $f : X \to V$ be a Lipschitz map from a PI space into a Banach space V, which is differentiable almost everywhere; in particular, by Theorem 4.1, this is true for any Lipschitz map when V is a GFDA space. Then for μ-a.e. $x \in X$, blow-ups converge to a map $X_x \to \text{image}\,(D_x f) \subset V$ which factors through a surjective map $X_x \to T_x X$.

Now suppose f is a bi-Lipschitz embedding. Then each A_α contains a full measure subset $\widehat{A}_\alpha$ whose image under f has the property that blow-ups are equal to linear subspaces of dimension equal to n_α. This implies:

1) For almost every $x \in A_\alpha$, every tangent cone at x is bi-Lipschitz equivalent to $\mathbf{R}^{n_\alpha}$.

2) By a straightforward modification of a standard result in the finite dimensional case [Fed69], the set $f(\widehat{A}_\alpha) \subset V$ is rectifiable; in particular, the Hausdorff dimension of $\widehat{A}_\alpha$ is at most n_α.

Assertion 1) yields:

Theorem 5.1. *If X contains a positive μ-measure set of points, x, such that the strict inequality*

$$\dim(T_x X) < \dim X_x\,, \tag{5.2}$$

holds for some tangent X_x, then X does not admit a bi-Lipschitz embedding in any space bi-Lipschitz homeomorphic to a GFDA space.

It follows in particular that nontrivial Carnot groups, Bourdon-Pajot spaces, and Laakso spaces do not bi-Lipschitz embed in a GFDA space.

Relation (5.2) should be compared to (1.9).

We remark that as far as we know, the n_α-dimensional Hausdorff measure of this rectifiable part $\widehat{A}_\alpha$ could be zero, and the μ-null set being discarded could have Hausdorff dimension $> n_\alpha$.

We note that for any doubling space, X, there exist injective Lipschitz maps, $f : X \to \ell_1$, for which f^{-1} has better regularity than C^α, for all $\alpha < 1$.

6. Appendix; Carnot groups and Radon-Nikodym targets

Here we consider the special case of the 3-dimensional Heisenberg group, $\mathbb{H}$, and general Radon Nikodym targets.

We recall that $\mathbb{H}$, is the simply connected nilpotent Lie group whose Lie algebra has a basis P, Q, Z, with the bracket relations

$$[P,Q] = Z, \quad [P,Z] = [Q,Z] = 0.$$

Let Δ denote the two dimensional distribution on $\mathbb{H}$ spanned by the left invariant vector fields P, Q. We define a riemannian metric on Δ by stipulating that P, Q are pointwise orthonormal. Then the *Carnot-Caratheodory distance* between two points $x_1, , x_2 \in \mathbb{H}$ is defined to be the infimum of the lengths of the C^1 paths joining x_1 to x_2, which are everywhere tangent to Δ. We denote the Carnot-Caratheodory distance function by $d^{\mathbb{H}}$.

Theorem 6.1. *The Heisenberg group with the Carnot-Caratheodory metric does not admit a bi-Lipschitz embedding into any Banach space satisfying the Radon-Nikodym property.*

Proof. The proof is almost identical to the proof in [Sem96]. The hypothesis on the target space is invoked only once, to deduce the almost everywhere directional differentiability of the map.

Let V denote a Banach space with the Radon-Nikodym property and let $f : \mathbb{H} \to V$ denote an L-Lipschitz map.

Step 1. The map, f, has directional derivatives $P(f)$ and $Q(f)$, almost everywhere.

It is well known that the (images of) integral curves of P and Q are isometric copies of the real line. Thus if $\gamma : \mathbf{R} \to \mathbb{H}$ is such an integral curve, the composition $f \circ \gamma : \mathbf{R} \to V$ is a Lipschitz mapping, and since V satisfies the Radon-Nikodym property, $f \circ \gamma$ is differentiable almost everywhere. Thus the set of points where the directional derivative $P(f)$ is not defined, which is easily seen to be a measurable set, intersects (the image of) each integral curve of P in a set of measure zero; as the integral curves define a smooth 1-dimensional foliation of $\mathbb{H}$, it follows from Fubini's theorem that the directional derivative $P(f)$ is defined almost everywhere. It follows by standard reasoning that $P(f)$ is measurable (it agrees almost everywhere with a pointwise limit of a sequence of measurable difference quotients). The Lipschitz condition on f implies that $\|P(f)\| \leq L$ almost everywhere. In particular, $P(f) \in L^\infty(\mathbb{H}, V)$, and similarly, $Q(f) \in L^\infty(\mathbb{H}, V)$.

Step 2. The directional derivatives $P(f)$ and $Q(f)$ are approximately continuous almost everywhere.

We recall that a measurable mapping $u : X \to Y$ from a metric measure space, (X, d^X, μ), to a metric space, Y, is *approximately continuous* at $x \in X$, if for all $\epsilon > 0$, the density of

$$S_\epsilon := \{x' \in B_r(x) \mid d(u(x'), u(x)) < \epsilon\}$$

satisfies

$$\lim_{r\to 0} \frac{\mu(S_\epsilon)}{\mu(B_r(x))} = 1\,.$$

We observe that the Lipschitz map f takes values in a closed separable subspace $V' \subset V$. Hence, $P(f)$ and $Q(f)$ take values in V' as well. Therefore, the almost everywhere approximately continuity of $P(f)$ and $Q(f)$ follows from:

Lemma 6.2. *Let (X, d^X, μ) be a doubling metric measure space, and let $u : X \to Y$ be a measurable mapping from X into a separable metric space Y. Then u is approximately continuous for μ-a.e. $x \in X$.*

Proof. Pick $\epsilon > 0$, and let $\{U_i\}$ be a countable cover of Y by open sets of diameter $< \epsilon$. The the countable collection $\{\Omega_i\}$, where $\Omega_i := u^{-1}(U_i) \subset X$, defines a countable cover of X by measurable sets. By a standard covering argument, the set of density points $\widetilde{\Omega}_i$ in Ω_i has full measure in Ω_i, so $\bigcup_i \widetilde{\Omega}_i$ has full measure in X. By considering a countable sequence, $\epsilon_j \to 0$, this clearly suffices to complete the proof. □

Step 3. If $x \in \mathbb{H}$ is an approximate continuity point of $P(f)$ and $Q(f)$, then the quantity, $d(f(x'), f(x))$ is $o(d(x', x))$, when x' lies on the integral curve $\{x \exp(tZ)\}_{t\in\mathbf{R}}$.

Pick $\epsilon > 0$. We will assume that x' is of the form $x' = x \exp t^2 Z$; the case when $x' = x \exp t^2(-Z)$ is similar.

We write, $\xi_1, \xi_2, \xi_3, \xi_4$, for P, Q, $-P$, $-Q$. It is a standard fact that there is an open quadrilateral $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ starting at x and ending at x', where $\gamma_i : [0, t] \to G$ is an integral curve of ξ_i of elapsed time t. Furthermore, $d^{\mathbb{H}}(x', x) \geq Ct$, for some universal constant $C > 0$.

By using Fubini's theorem, and the fact that x is an approximate continuity point of the L^∞ mappings $P(f)$ and $Q(f)$, it follows that when t is sufficiently small, there exists an integral curve, $\widehat{\gamma}_i$, of ξ_i, with elapsed time t, such that $d(\widehat{\gamma}_i(s), \gamma_i(s)) < \epsilon t$, for all $s \in [0, t]$, and

$$\frac{1}{t}\int_{\widehat{\gamma}_i} \|\xi_i(f) - \xi_i(f)(x)\|\, dt < \epsilon. \tag{6.3}$$

Therefore, since f is L-Lipschitz,

$$
\begin{aligned}
\|f(x') - f(x)\| &= \|f \circ \gamma_4(t) - f \circ \gamma_1(0)\| \\
&\leq \ \|f \circ \widehat{\gamma}_4(t) - f \circ \widehat{\gamma}_1(0)\| + 2L\epsilon t \\
&\leq \| \sum_i \left(f \circ \widehat{\gamma}_i(t) - f \circ \widehat{\gamma}_i(0)\right) \| + 6L\epsilon t \\
&\leq \| \sum_i (\xi_i(f))(x)\, t\| + 4\epsilon t + 5L\epsilon t \\
&\leq 4\epsilon t + 6L\epsilon t \\
&= o(d^X(x', x)),
\end{aligned}
$$

where in the next to last inequality we have used (6.3) and in the last, $\xi_3 = -\xi_1$ and $\xi_4 = -\xi_2$.

□

References

[Aro76] N. Aronszajn. Differentiability of Lipschitzian mappings between Banach spaces. *Studia Math.*, 57(2):147–190, 1976.

[BL00] Y. Benyamini and J. Lindenstrauss. *Geometric nonlinear functional analysis. Vol. 1*, volume 48 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2000.

[BP99] M. Bourdon and H. Pajot. Poincaré inequalities and quasiconformal structure on the boundary of some hyperbolic buildings. *Proc. Amer. Math. Soc.*, 127(8):2315–2324, 1999.

[CC97] J. Cheeger and T. Colding. On the structure of spaces with Ricci curvature bounded below. *J. Diff. Geom.*, 46:406–480, 1997.

[Che99] J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.*, 9(3):428–517, 1999.

[CK06a] J. Cheeger and B. Kleiner. Differentiating maps to L^1 and the geometry of BV functions. Preprint, 2006.

[CK06b] J. Cheeger and B. Kleiner. Embedding Laakso spaces in L^1. Preprint, 2006.

[Fed69] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[Geh61] F. W. Gehring. Rings and quasiconformal mappings in space. *Proc. Nat. Acad. Sci. U.S.A.*, 47:98–105, 1961.

[Gel38] I. M. Gelfand. Abstracte Functionen und lineare Operatoren. *Mat. Sbornik*, 46(4):235–284, 1938.

[GM84] N. Ghoussoub and B. Maurey. Counterexamples to several problems concerning G_δ-embeddings. *Proc. Amer. Math. Soc.*, 92(3):409–412, 1984.

[Gro96] M. Gromov. Carnot-Caratheodory spaces seen from within. In *Subriemannian geometry*, Progr. in Math., pages 79–323. Birkhäuser, Basel, 1996.

[HK96] J. Heinonen and P. Koskela. From local to global in quasiconformal structures. *Proc. Nat. Acad. Sci. USA*, 93:554–556, 1996.

[Kad59] M. Ĭ. Kadec′. On the connection between weak and strong convergence. *Dopovidi Akad. Nauk Ukraïn. RSR*, 1959, 1959.

[Kir94] B. Kirchheim. Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. *Proc. Amer. Math. Soc.*, 121(1):113–123, 1994.

[Kle61] V. Klee. Mappings into normed linear spaces. *Fund. Math.*, 49:25–34, 1960/1961.

[Laa00] T. J. Laakso. Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality. *Geom. Funct. Anal.*, 10(1):111–123, 2000.

[LN06] J. Lee and A. Naor. L^p metrics on the Heisenberg group and the Goemans-Linial conjecture. Preprint, 2006.

[LT77] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces. I.* Springer-Verlag, Berlin, 1977. Sequence spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92.

[Mat95] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[Mos73] G. D. Mostow. *Strong rigidity of locally symmetric spaces*. Princeton University Press, Princeton, N.J., 1973. Annals of Mathematics Studies, No. 78.

[Pan89] P. Pansu. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)*, 129(1):1–60, 1989.

[Pau01] S. D. Pauls. The large scale geometry of nilpotent Lie groups. *Comm. Anal. Geom.*, 9(5):951–982, 2001.

[Rad20] H. Rademacher. Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variabeln. II. *Math. Ann.*, 81(1):52–63, 1920.

[Sem96] S. Semmes. On the nonexistence of bi-Lipschitz parameterizations and geometric problems about A_∞-weights. *Rev. Mat. Iberoamericana*, 12(2):337–410, 1996.

[Zie89] W. P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.

J.C. : Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012

B.K. : Mathematics Department, Yale University, New Haven, CT 06520