THE WEAK HYPERBOLIZATION CONJECTURE FOR 3-DIMENSIONAL CAT(0) GROUPS

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1. Introduction

Let G be a 3-dimensional Poincare duality group over a commutative hereditary ring $\mathcal{R}$ with a unit; for instance, G could be the fundamental group of a closed aspherical 3-manifold. Suppose in addition that G is a $CAT(0)$-group, i.e. a group which admits a cocompact isometric properly discontinuous action $G \curvearrowright X$ on a proper $CAT(0)$ space X. The main result of this note is the following:

Theorem 1. *Under the above assumptions either G is Gromov-hyperbolic or G contains a copy of $\mathbb{Z}^2$.*

We note that special cases of this theorem were proven earlier by various people: Buyalo [6] and Schroeder [17] have independently proven that this theorem holds provided that X is the universal cover $\tilde{M}$ of a closed 3-manifold M, the $CAT(0)$-structure on $\tilde{M}$ is Riemannian and $G = \pi_1(M)$ acts on X by deck-transformations. Mosher [15] proved that Theorem 1 holds provided that $X = \tilde{M}$, $G = \pi_1(M)$, and the $CAT(0)$ metric on is obtained by lifting a piecewise-Euclidean $CAT(0)$-cubulation from M. Bridson and Mosher also have an unpublished proof of Theorem 1 under the assumption that $X = \tilde{M}$ has an arbitrary G-invariant $CAT(0)$-structure. Unlike all these proofs, our proof takes place on the ideal boundary of X; this allows us to treat 3-dimensional Poincare duality groups and drop the assumption that the $CAT(0)$-space is a topological manifold.

Despite recent progress towards Thurston's Geometrization Conjecture using the Ricci flow, the following conjecture remains of independent interest, and could be viewed as a possible step toward showing that $PD(3)$ groups (over $\mathbb{Z}$) are fundamental groups of 3-manifolds:

Weak Hyperbolization Conjecture for $PD(3)$ groups: *If G is a 3-dimensional Poincare duality group over a commutative ring $\mathcal{R}$ with unit, then either G is Gromov-hyperbolic or G contains a copy of $\mathbb{Z}^2$.*

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2. Pretrees

In what follows we will need definitions and basic facts about pretrees; the definitions which we give follow [4].

A *pretree* is a set T together with a ternary relation (the *betweenness relation*, "y is between x and z"), to be denoted $\beta(xyz)$, satisfying the following axioms:

(1) $\beta(xyz)$ implies that $x \neq y \neq z$.
(2) $\beta(xyz) \iff \beta(zyx)$.
(3) $\beta(xyz)$ and $\beta(yxz)$ cannot hold simultaneously.
(4) If $w \neq y$ then $\beta(xyz)$ implies that either $\beta(xyw)$ or $\beta(wyz)$.

Given a pretree T one can define closed, open and half-open intervals in T by

$$(x, z) := \{y \in T : \beta(xyz)\}, [x, z] := (x, z) \cup \{x, z\}, \text{etc.}$$

Given an increasing union of intervals

$$[x_1, y_1] \subset [x_2, y_2] \subset ... \subset [x_i, y_i] \subset ...$$

we will also refer to the union of these intervals as a (possibly infinite) interval in T.

We note that β defines a linear order (up to orientation) on each interval in T.

Define a "triangle" in T with vertices a, b, c to be the union of the segments (called "sides" of the triangle) $[a, b], [b, c], [c, a]$.

Lemma 2. *Each triangle Δ in T is 0-thin, i.e. each side of Δ is contained in the union of the two other sides.*

Proof. Follows immediately from Axiom 4. □

Suppose that T is a pretree which is given a measure μ (without atoms) defined on closed intervals in T and the σ-algebra which these intervals generate. Define a function $d(x, y)$ on T by $d(x, y) := \mu([x, y])$.

Lemma 3. *d is a pseudo-metric on T.*

Proof. It is clear that d is symmetric and $d(x, x) = 0$ (since μ has no atoms). The triangle inequality follows because for each triangle with the vertices a, b, c we have (see Lemma 2)

$$[a, b] \subset [a, c] \cup [b, c]. \quad \square$$

We recall that a metric tree is a complete geodesic metric space where each geodesic triangle is isometric to a tripod.

We note that if for each interval $[a, b] \subset T$, with $a \neq b$, $\mu(a, b) > 0$ then d is a metric. Moreover, it follows that $(a, b) \neq \emptyset$ for each $a \neq b$. If the restriction of the metric d to each interval $[x, y]$ is complete then $[x, y]$ is order isomorphic to an interval in $\mathbb{R}$ and moreover, $([x, y], d)$ is isometric to an interval in $\mathbb{R}$. We thus get:

Lemma 4. *Suppose that for each interval $[x, y] \subset T$, with $x \neq y$, $\mu[x, y] > 0$, and that the restriction of the metric d to each interval in T is complete. Then (T, d) is a metric tree.*

Proof. It is clear from the above discussion that T is a geodesic metric space. Since each triangle in T is 0-thin, it follows that each triangle in T is isometric to a tripod. Finally, let's check completeness of T: Suppose that $x_i, i \geq 0$, is a Cauchy sequence in T. Then there exists an increasing sequence of intervals $I_i \subset T$ such that

$$\lim_i \mu([x_0, x_i] \cap I_i) = \lim_i d(x_0, x_i).$$

Then completeness of d restricted to the union I of I_i's implies that (x_i) converges to a point in the interval I. □

3. Coarse topology and the ideal boundaries of CAT(0) spaces

Let X be a CAT(0) space. Recall that the *ideal boundary* of X, as a set, is the collection of equivalence classes of geodesic rays in X, where two rays are equivalent (asymptotic) if they are within finite Hausdorff distance from each other. In what follows we will be considering the ideal boundary of the space X equipped with two different structures:

(1) The *visual topology*, in which case we will be using the notation $\partial_\infty X$ for the ideal boundary.

(2) The *Tits boundary*, denoted by $\partial_T X$, where the ideal boundary of X is equipped with the Tits angle metric $\angle_T$. We recall that the topology induced by $\angle_T$ is usually different from the topology on $\partial_\infty X$, but the identity map $\partial_T X \to \partial_\infty X$ is a continuous bijection.

We refer the reader to [1, 5] for a detailed discussion of $\partial_\infty X$ and $\partial_T X$.

Definition 5. Points $\xi_1, \xi_2 \in \partial_T X$ are called *antipodal* if $\angle_T(\xi_1, \xi_2) = \pi$. A *standard circle* in $\partial_T X$ is a subset isometric to the unit circle.

A subset $S \subset \partial_T X$ isometric to a standard (unit) circle also determines a topologically embedded circle in $\partial_\infty X$ (the topology is induced by the Tits metric agrees with the visual topology on S). Note also that if $F \subset X$ is a 2-flat then $\partial_T F \subset \partial_T X$ is a standard circle.

Lemma 6. *The ideal boundary $\partial_\infty X$ of X is homeomorphic to S^2.*

Proof. Let $\overline{X} = X \cup \partial_\infty X$ denote the visual compactification of X. We recall that the action $G \curvearrowright Z = \partial_\infty X$ satisfies the axioms for $\mathcal{Z}$-boundary defined by Bestvina in [2]. Bestvina proves, [2, Theorem 2.8], that if G is a $PD(3)$ group over $\mathcal{R}$, then Z is homeomorphic to S^2. We note that Bestvina proves the latter theorem under more restrictive assumptions than ours (although, his class of groups G includes 3-manifold groups); the following remarks explain why his arguments apply in our situation.

1. Bestvina assumes that the commutative ring $\mathcal{R}$ is a PID. However this assumption is used only to apply the Universal Coefficient Theorem, which works for hereditary rings as well, see [7].

2. Bestvina's definition of an n-dimensional Poincare duality group is more restrictive than the usual one: instead of the FP property he assumes that a group G acts freely, properly discontinuously, cocompactly on a contractible cell complex Y. Note however that Bestvina in his proof uses only the fact that $G \curvearrowright Y^{(i)}$ is cocompact on each i-skeleton of Y. Then existence of such an action for the $CAT(0)$-groups follows from a general construction described in [13]. Namely, if a group G admits a properly discontinuous cocompact action on a contractible space X (e.g. the $CAT(0)$-space in our case) then it also admits a free, properly discontinuous action on a contractible cell complex Y (possibly of infinite dimension) such that $Y^{(i)}/G$ is compact for each i.

3. Bestvina assumes that the image of the orientation character χ of the Poincare duality group G is finite (he then passes to a finite index subgroup in G which is the kernel of χ). However this assumption can be omitted from his theorem using *twisting* of the action $G \curvearrowright C_*(Y)$ by the character χ as it is done in [13].

With the above modifications, Bestvina's arguments apply in our case and it follows that $\partial_\infty X$ is homeomorphic to the 2-sphere. □

Lemma 7. *If U is a horoball in X then $\partial_\infty U$ does not separate $\partial_\infty X$.*

Proof. By Alexander duality it suffices to show that $\check{H}^1(\partial_\infty U) = 0$. Let $U = \{b \leq 0\}$ where b is a Busemann function, and for $t \in \mathbb{R}$ let $U_t := \{b \leq t\}$. Similarly to [2] we have $\check{H}^1(\partial_\infty U) \cong H^2_c(U_t)$ for each $t \in \mathbb{R}$, since U_t is convex for all t. The inverse system of cohomology groups $\{H^2_c(U_t)\}_{t\in\mathbb{R}}$ is equivalent to a constant system, in the sense that all restriction mappings are isomorphisms (the nearest point maps define proper homotopy inverses to the inclusions). Using the coarse Alexander duality theorem from [12], we deduce that the inverse system $\{\tilde{H}_0(X \setminus U_t)\}_{t\in\mathbb{R}}$ is pro-isomorphic to a constant system (see [12] for the definitions). But this forces $\{\tilde{H}_0(X \setminus U_t)\}_{t\in\mathbb{R}}$ to be pro-zero: if $x, y \in X \setminus U_t$, then the segment $\overline{xy} \subset X \setminus U_{t'}$ for t' sufficiently small, which means that $x - y$ determines the zero element in $H_0(X \setminus U_{t'})$. □

4. Proof of the main theorem

Scheme of the proof: Let G be as in Theorem 1, and let $G \curvearrowright X$ be a discrete cocompact isometric action of G on a $CAT(0)$-space X. There are three main cases which can occur:

(a) G is a virtually abelian group of rank 3.

(b) X contains a parallel set $\mathcal{P}(l) = \mathbb{R}\times Y$ of a geodesic l, so that the ideal boundary of $\mathcal{P}(l)$ equals the ideal boundary of X. In this case, by considering action of G on a

Gromov-hyperbolic space Y we conclude that G is commensurable to the fundamental group of a Seifert 3-manifold.

(c) The generic case, when we show that X contains a G-invariant collection of *peripheral 2-flats* which "do not cross" each other. This allows us to define an action of G on a pretree, then on an $\mathbb{R}$-tree, so that the stabilizers of segments are virtually cyclic. By applying Rips' theory we conclude that in this case G splits over a virtually abelian subgroup of rank 2.

According to Eberlein's *flat plane theorem* (see [9] in the smooth case and [5, Theorem 9.33] in general), the $CAT(0)$ space X is either Gromov-hyperbolic or it contains a 2-flat F. Since in the former case, G is also Gromov-hyperbolic, we assume that X contains a 2-flat F.

If $l \subset F$ is a geodesic we let $\mathcal{P}(l) \subset X$ denote the parallel set of l, i.e. the union of geodesics in X which are parallel to l. The parallel set $\mathcal{P}(l)$ is a closed convex subset in X which splits isometrically as a direct product $l \times Y$; accordingly, the Tits boundary $\partial_T \mathcal{P}(l)$ is the metric $\frac{\pi}{2}$-join $S^0 \star \partial_\infty Y$. Clearly, if l is contained in a 2-flat F, then $F \subset \mathcal{P}(l)$, and hence F determines circles in $\partial_\infty \mathcal{P}(l)$ and $\partial_T \mathcal{P}(l)$.

Lemma 8. *Either G is virtually abelian and $\partial_T X$ is isometric to the unit 2-sphere or for each parallel set $\mathcal{P}(l) = l \times Y \subset X$, the subspace Y is Gromov-hyperbolic.*

Proof. We remark that proof of the flat plane theorem ([9] and [5, Theorem 9.33]) actually shows that if Z is a proper $CAT(0)$ space which is not Gromov hyperbolic, then there is a sequence of pointed subsets (S_i, z_i) in Z which Gromov-Hausdorff converge to a (pointed) flat plane. The assumption that $\mathrm{Isom}(Z)$ acts cocompactly on Z is only used to arrange that the sets actually converge to a flat in the pointed *Hausdorff* topology.

Suppose Y is not Gromov-hyperbolic. Then we have a sequence (S_i, y_i) of pointed subsets of Y such that (S_i, y_i) Gromov-Hausdorff converges to a flat plane. Let $l_i \subset \mathcal{P}(l)$ be the geodesic in $\mathcal{P}(l)$ parallel to l which passes through y_i. We may choose a sequence $g_i \in G$ so that the sequence $g_i(y_i)$ is bounded, and then, after passing to a subsequence, we may assume that $(g_i(S_i), g_i(y_i))$ converges in the pointed Hausdorff topology to a pointed flat plane $(F, y_\infty) \subset X$, l_i converges to a geodesic l_∞, and $(g_i(\mathcal{P}(l)), g(y_i))$ converges to a convex subset $\mathcal{P}_\infty$ which splits isometrically as a product $\mathcal{P}_\infty = l_\infty \times Y_\infty$, where $F \subset Y_\infty$. Hence X contains the 3-flat $F' := F \times l_\infty$.

We next claim that F' is Hausdorff-close to X. Otherwise we would have a sequence $x_i \in X$ such that the nearest points $y_i \in F'$ satisfied $d(y_i, x_i) \to \infty$; applying groups elements to this configuration and passing to a subsequence, we find another 3-flat $F'' \subset X$ as a limit, and a ray ρ leaving F' orthogonally. Since $\partial_\infty F'' \cong S^2, \partial_\infty X \cong S^2$, it follows that $\partial_\infty F'' = \partial_\infty X$. This is a contradiction since $\partial_\infty \rho \notin \partial_\infty F''$.

Therefore the parallel set of F' is G-invariant, convex, and splits as a product $\mathcal{P}(F') = F' \times K$, where K is a compact $CAT(0)$ space. The induced action of G on

K has a fixed point $k \in K$, and hence G preserves the 3-flat $F' \times \{k\} \subset \mathcal{P}(F')$, and must be virtually free abelian of rank 3. $\square$

We will assume from now on that G is not virtually abelian, so that the second alternative in the lemma holds for every parallel set.

Corollary 9. *For each parallel set $P = l \times Y$ as above, the Tits boundary $\partial_T Y$ is infinite and totally disconnected, i.e. each pair of distinct points in $\partial_T Y$ is antipodal.*

Proposition 10. *Suppose that for some geodesic $l \subset X$ we have $\partial_\infty \mathcal{P}(l) = \partial_\infty X$. Then:*

1. $\partial_\infty l$ is preserved by G, and hence G preserves the parallel set $\mathcal{P}(l)$, as well as its product structure $\mathcal{P}(l) = l \times Y$.

2. G is commensurable to the fundamental group of a 3-dimensional Seifert manifold. In particular, G contains $\mathbb{Z}^2$.

Proof. 1. By our assumption we have $\mathcal{P}(l) = l \times Y$ where $\partial_T Y$ is a discrete space, and hence $\partial_T \mathcal{P}(l) = \partial_T X$ is a metric suspension of a discrete space. Since $|\partial_T Y| = \infty$, this means that the only points with unique antipodes are the suspension points, and hence the subset $\partial_\infty l$ is preserved by G. But then for all $g \in G$, the geodesic $g(l)$ is parallel to l, which means that $g(\mathcal{P}(l)) = \mathcal{P}(l)$. Hence $\mathcal{P}(l)$ is G-invariant, and we may assume that $X = \mathcal{P}(l)$. The action of G preserves the product structure $\mathcal{P}(l) = l \times Y$, so we get an induced action ρ of G on the Gromov hyperbolic space Y.

2. Since the suspension of $\partial_\infty Y$ is homeomorphic to the 2-sphere $\partial_\infty X$, the ideal boundary of Y is homeomorphic to S^1. Thus the cocompact isometric action $\rho : G \curvearrowright Y$ extends to a uniform (topological) convergence action $G \curvearrowright \partial_\infty Y = S^1$. Therefore, according to [8, 10, 11, 18], the action $G \curvearrowright S^1$ is topologically conjugate to a Moebius action ρ'. Note that the kernel of ρ acts with bounded orbit on Y, and hence has a fixed point $y_0 \in Y$. Therefore $\mathrm{Ker}(\rho) = \mathrm{Ker}(\rho')$ acts isometrically on the geodesic $l \times \{y_0\} \subset \mathcal{P}(l)$ and is either finite or a virtually infinite cyclic subgroup of G.

Lemma 11. *The action $G \overset{\rho'}{\curvearrowright} S^1$ factors through the action of a uniform lattice in* $\mathrm{Isom}(\mathbb{H}^2)$ *on $S^1 = \partial_\infty \mathbb{H}^2$.*

Proof. The action $\rho'(G) \curvearrowright \mathbb{H}^2$ is cocompact, therefore we have the following possibilities:

(a) $\rho'(G)$ is a cocompact discrete subgroup in $\mathrm{Isom}(\mathbb{H}^2)$; in this case we are done.

(b) $\rho'(G)$ is a solvable subgroup in $\mathrm{Isom}(\mathbb{H}^2)$, which fixes a point in S^1. Then $\rho'(G)$ is not virtually abelian which contradicts the fact that G is a CAT(0) group.

(c) $\rho'(G)$ is dense in $PSL(2, \mathbb{R})$. Then, the group $\rho'(G)$ contains a nontrivial elliptic element $\hat{g}$ and it also contains a sequence of elements $\hat{h}_i$ which converge to $1 \in PSL(2, \mathbb{R})$. Let $g, h_i \in G$ be elements which map (via ρ') to $\hat{g}$ and $\hat{h}_i$ respectively. Clearly, $\rho(g) \in \mathrm{Isom}(Y)$ is elliptic as well, let $y \in Y$ be its fixed point. By taking conjugates $g_i := h_i g h_i^{-1}$, we get an infinite collection of distinct elements $\{g_i : i \in \mathbb{N}\}$ of G so that for each $n \in \mathbb{Z}$, $g_i(y \times \mathbb{R})$ is contained in $N_R(y \times \mathbb{R})$ where $R \in \mathbb{R}_+$ is independent of i. We note that since all g_i are pariwise conjugate, there exists $C < \infty$ such that $d(x, g_i(x)) < C$ for each $x \in y \times \mathbb{R}$ and $i \in \mathbb{N}$. This contradicts discreteness of the action of G on X. □

The above lemma implies that the kernel of ρ is commensurable to $\mathbb{Z}$ and the quotient $\rho(G)$ is commensurable to the fundamental group of a 2-dimensional hyperbolic surface. Therefore G is commensurable to the fundamental group of a 3-dimensional Seifert manifold and hence G contains $\mathbb{Z} \times \mathbb{Z}$. □

In view of the proposition, we will henceforth assume that no parallel set $\mathcal{P}(l)$ has the same ideal boundary as X.

We now review some properties of the space $\partial_T X$. Recall that a subset $C \subset Z$ is called *convex* if for any two non-antipodal points $x, y \in \partial_T X$, the geodesic segment $\overline{xy}$ connecting x to y is contained in C. The intersection of two convex subsets of $\partial_T X$ is convex. If $Y \subset X$ is a convex of the $CAT(0)$ space X, then $\partial_T Y \subset \partial_T X$ is convex as well. Thus, if $F, F' \subset X$ are 2-flats then the intersection $\partial_T F \cap \partial_T F' \subset \partial_T X$ is convex and either consists of two antipodal points, a circular arc in $\partial_T F$ of the length $\leq \pi$, or we have $\partial_T F = \partial_T F'$.

Definition 12. Given two standard circles $S, S' \subset \partial_T X$, we say that S and S' *cross* if S contains points from each component of $\partial_\infty X \setminus S'$ (note that we are using the visual topology). We say that 2-flats $F, F' \subset X$ *cross* if $S = \partial_\infty F$ and $S' = \partial_\infty F'$ cross.

We will say that the ideal boundaries of two distinct parallel sets $\mathcal{P}, \mathcal{P}'$ *cross* if at least one circle in $\partial_T \mathcal{P}$ crosses a circle in $\partial_T \mathcal{P}'$.

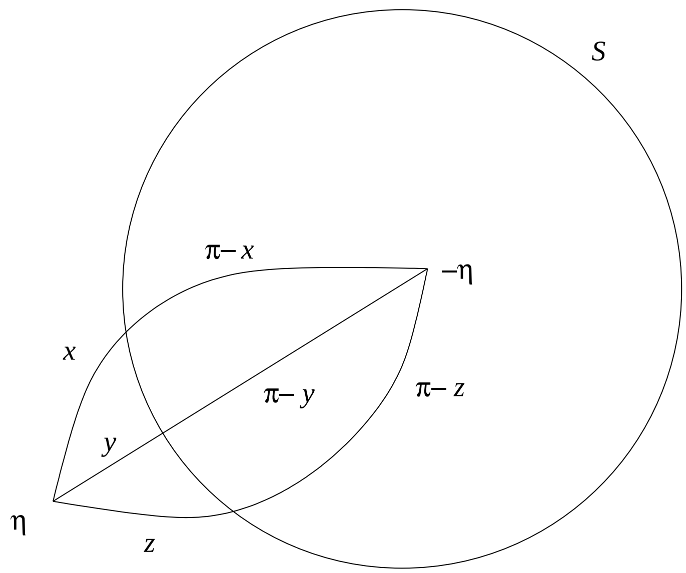
We will say that a parallel set $\mathcal{P}(l)$ is *nontrivial* if $\mathcal{P}(l) = l \times Y$ where $|\partial_\infty Y| \geq 3$.

It follows (from convexity of $S \cap S'$ with respect to the Tits metric) that in the case S, S' cross, the intersection $S \cap S'$ consists of a pair of points which are antipodal with respect to the Tits metric. Note that crossing is a symmetric relation.

Lemma 13. *Suppose that $\mathcal{P}(l)$ is a nontrivial parallel set, and $F \subset X$ is a flat. If $S := \partial_T F$ crosses a circle in $\partial_T \mathcal{P}(l)$, then $F \subset \mathcal{P}(l)$.*

Proof. Suppose S crosses a circle $S' \subset \partial_T \mathcal{P}(l)$. Recall that $\partial_T \mathcal{P}(l)$ is the metric join of $\{\eta, \eta'\} = \partial_T l$ and $\partial_T Y$. As observed before, the intersection $S \cap S'$ is a pair of antipodal points in $\partial_T \mathcal{P}(l)$. If $S \cap S'$ contained one of the points $\{\eta, \eta'\}$, it would therefore contain the other (since the other is the unique antipode in $\partial_T \mathcal{P}(l)$). This would imply that $F \subset \mathcal{P}(l)$, and we are done in this case.

Therefore, we may assume that S does not pass through $\partial_\infty l$ and so the configuration $\{\partial_\infty P, S\}$ has to look like the one on Figure 4, where x, y, z denote the distances



from η to the points of intersection between $\partial_\infty P$ and S. It follows that $x + y = \pi$, $y + z = \pi$, $x + z = \pi$ and thus

$$x = y = z = \pi/2.$$

This implies that the 3 intersection points have pairwise distance π, which is absurd since S is a standard circle. $\square$

Suppose $\mathcal{P}(l)$ is a nontrivial parallel set. We refer to the standard semi-circles in $\partial_T \mathcal{P}(l)$ running between the "poles" $\partial_\infty l = \{\eta, \eta'\}$ as *longitudes*. By our assumptions we have $\partial_\infty X \setminus \partial_\infty \mathcal{P}(l) \neq \emptyset$; let U be a connected component of $\partial_\infty X \setminus \partial_\infty \mathcal{P}(l)$. We claim that the frontier of U is a pair of longitudes. To see this, note that if $\bar{U} \setminus \{\eta, \eta'\}$ intersected at most one longitude L, then we would necessarily have $U = \partial_\infty X \setminus L$, which contradicts our assumption that $\mathcal{P}(l)$ is nontrivial. Hence there are at least two longitudes L_1 and L_2 intersecting $\bar{U} \setminus \{\eta, \eta'\}$. Let U' be the component of $\partial_\infty X \setminus L_1 \cup L_2$ intersecting U. Clearly no other longitute L' can intersect U', and hence we have $U = U'$, and the frontier of U is a standard circle (the union of two longitudes of $\partial_T \mathcal{P}(l)$). We will refer to these circles as *peripheral circles* of $\partial_\infty \mathcal{P}(l)$. A flat in X whose boundary is a peripheral circle will be called a *peripheral flat*.

Lemma 14. *If $\mathcal{P}(l)$ and $\mathcal{P}(l')$ are distinct nontrivial parallel sets, then $\partial_\infty \mathcal{P}(l')$ lies in $\bar{U}$, where U is a component of $\partial_\infty X \setminus \partial_\infty \mathcal{P}(l)$.*

Proof. Suppose $\xi \in \partial_\infty \mathcal{P}(l') \setminus \partial_\infty \mathcal{P}(l)$. Let U be the component of $\partial_\infty X \setminus \partial_\infty \mathcal{P}(l)$ containing ξ, and let S be the standard circle bounding U. If $\partial_\infty \mathcal{P}(l')$ contains a point $\xi' \in \partial_\infty X \setminus \bar{U}$, then there is a standard circle $S' \subset \mathcal{P}(l')$ containing $\{\xi, \xi'\}$, and S' will cross S. But then Lemma 13 implies that $S' \subset \partial_\infty \mathcal{P}(l)$, which is a contradiction.

Thus $\partial_\infty \mathcal{P}(l) \subset \partial_\infty \mathcal{P}(l')$. Since the standard circles in $\partial_T \mathcal{P}(l)$ are unions of pairs of longitudes, this clearly forces $\partial_T l = \partial_T l'$, and hence $\mathcal{P}(l) = \mathcal{P}(l')$. $\square$

As a consequence of the lemma, if $\mathcal{P}(l)$ is a nontrivial parallel set and $S \subset \partial_T \mathcal{P}(l)$ is a peripheral circle, then the circle collection $\{g(S)\}_{g\in G}$ contains no crossing pairs. On the other hand, if two flats cross, their union is contained in a nontrivial parallel set, and so there is necessarily a G-invariant collection $\mathcal{F}$ of flats whose boundaries are pairwise noncrossing circles. For the remainder of the proof, we fix one such collection $\mathcal{F}$, and endow it with the pointed Hausdorff topology. Since taking closure preserves the noncrossing property, we will assume that $\mathcal{F}$ is a closed subset of the space of subsets of X (equipped with the pointed Hausdorff topology). Therefore $\mathcal{F}$ is locally compact.

Suppose that we have three flats $F, F', F'' \in \mathcal{F}$ with pairwise distinct ideal boundaries. We will say that F' *separates* F from F'' if the following holds:

$$\partial_\infty F \subset \overline{D}, \partial_\infty F'' \subset \overline{D}'',$$

where D and D'' are the connected components of $\partial_\infty X \setminus \partial_\infty F'$. We use $\beta(F, F', F'')$ to denote this ternary relation on $\mathcal{F}$.

We leave it to the reader to verify that with this ternary relation the set $\mathcal{P}$ of all peripheral flats in X satisfies the axioms of a pretree.

Proposition 15. *Let F, F'' be flats in X. Then the set $S(F, F'')$ of flats F' separating F from F'' is compact with respect to the Gromov-Hausdorff topology.*

Proof. If $\partial_\infty F = \partial_\infty F''$ then for each flat F' separating F and F'' we have: $\partial_\infty F' = \partial_\infty F$ and thus the flats F, F' are parallel. Since the parallel set $\mathcal{P}(F)$ is isometric to $K \times F$, where K is compact, it follows that $S(F, F'')$ is compact.

Therefore we will assume that $\partial_\infty F' \neq \partial_\infty F''$. Suppose that F_i is a sequence of 2-flats in X which diverge to infinity, i.e.

$$\lim_i d(o, F_i) = \infty$$

where $o \in X$ is a base-point. Consider the functions $f_i := d(x, F_i) - d(x, o)$. Then, according to Lemma 2.3 in [14], the functions to f_i subconverge to a Busemann function b_ξ in X.

Let U denote the horoball $\{x : b_\xi(x) \le 0\}$. We claim that $\partial_\infty F \setminus \partial_\infty U$ is nonempty. Otherwise, the Busemann function b_ξ would restrict to a bounded convex function on F, i.e. a constant function. It follows readily that U would contain an isometrically embedded half space $F \times \mathbb{R}_+$; this contradicts our assumption that all parallel sets have Gromov hyperbolic factors. Similarly, $\partial_\infty F'' \setminus \partial_\infty U$ is nonempty.

We leave the proof of the following lemma to the reader:

Lemma 16. *1. For each i, $\partial_\infty F \cap \partial_\infty F'' \subset \partial_\infty F_i$. 2. $\partial_\infty F_i$ subconverge into $\partial_\infty U$, in particular, $\partial_\infty F \cap \partial_\infty F'' \subset \partial_\infty U$.*

Thus we pick points $\eta \in \partial_\infty F \setminus \partial_\infty U$, $\eta'' \in \partial_\infty F'' \setminus \partial_\infty U$. Lemma 7 implies that $\eta, \eta'' \notin \partial_\infty F \cap \partial_\infty F''$ and that (since $\partial_\infty U$ does not separate $\partial_\infty X$) for large i the points η, η'' belong to the same connected component of $\partial_\infty X \setminus \partial_\infty F_i$. This contradicts the assumption that F_i is between F, F'' for all i. □

Let $\tilde{\mathcal{L}}$ be the collection of pointed flats (F, p) where $F \in \mathcal{F}$, equipped with the pointed Hausdorff topology. Since $\mathcal{F}$ is locally compact, the space $\tilde{\mathcal{L}}$ is a locally compact surface lamination, and the induced action $G \curvearrowright \tilde{\mathcal{L}}$ is properly discontinuous and cocompact. The lamination $\tilde{\mathcal{L}}$ has a G-invariant leafwise flat metric. After passing to a smaller G-invariant subset of $\mathcal{F}$ we may further assume that the action $G \curvearrowright \tilde{\mathcal{L}}$ is minimal (i.e. it contains no proper closed, G-invariant sublaminations). Therefore (see [16]) there exists a transversal G-invariant measure μ on $\tilde{\mathcal{L}}$; minimality of $G \curvearrowright \mathcal{F}$ implies that that this measure has full support. Since $\tilde{\mathcal{L}}$ fibers over $\mathcal{F}$, we obtain a measure on $\mathcal{F}$. If $\tilde{\mathcal{L}}$ contains an isolated leaf/flat F, then it will define an atom in the transverse measure, and hence the collection $\{gF\}_{g\in G}$ of translates of F must be locally finite in X. This means that $\{gF\}_{g\in G}$ defines a closed G-invariant sublamination of $\tilde{\mathcal{L}}$, forcing it to be all of $\tilde{\mathcal{L}}$, and the stabilizer of F in G acts cocompactly on F, and so we are done. Hence we may assume that there are no isolated leaves, and that each flat has at most virtually infinite cyclic stabilizer. In fact, similar reasoning implies that for each flat $F \in \mathcal{F}$, there is a sequence $F_i \in \mathcal{F}$ of flats not parallel to F which converge to F.

NEED TO CLARIFY THE RELATION BETWEEN THE LAMINATION $\mathcal{L}$ AND THE PICTURE AT INFINITY WHERE ONE COLLAPSES PARALLEL FLATS.

Lemma 17. *Suppose that $F \in \mathcal{F}$, $g_n \in G$ is a sequence such that $\lim_{n\to\infty} g_n F = F_\infty \in \mathcal{F}$. Then there exist $x_-, x_+ \in \mathcal{F}$ such that for all sufficiently large n, $g_n F \in [x_-, x_+]$ and $F_\infty \in [x_-, x_+]$.*

Proof. Since $\lim_{n\to\infty} g_n F = F_\infty$, the circles $\partial_T(g_n F)$ converge to the circle $\partial_T F_\infty$ in the Chabauty topology (we again are using here the visual topology on Z). The circles in the collection

$$\{\partial_T(g_n F), \partial_T F_\infty, n \in \mathbb{N}\}$$

are all peripheral and hence do not cross each other (by Lemma 13). This implies that for all large n, m either $\partial_T(g_n F)$ separates $\partial_T(g_m F)$ from $\partial_T F_\infty$ or $\partial_T F_\infty$ separates $\partial_T(g_n F)$ from $\partial_T(g_m F)$. □

The above lemma implies that the natural projection $p : \tilde{\mathcal{L}} \to \mathcal{F}$ is continuous, where we give $\mathcal{F}$ the order topology, whose basis consists of the open intervals (a, b). It is also clear that p is a proper map in the sense that for each interval $[a, b]$ the inverse image $p^{-1}([a, b])$ consists of leaves of $\tilde{\mathcal{L}}$ which intersect a certain compact subset in X: If a sequence of flats F_j leaves every compact subset in X then this sequence subconverges to a point in $\partial_\infty X$, but a point cannot separate one circle in $\partial_T X$ from another.

The measure μ on the pretree $\mathcal{F}$ has the no atoms and (since the measure μ transversal to $\tilde{\mathcal{L}}$ has full support) for each pair of distinct points $x, x' \in \mathcal{F}$, $\mu([x,x']) = 0$ iff the corresponding flats F, F' in X are not separated by any flat in $\mathcal{F}$. We let T be the quotient of $\mathcal{F}$ by the equivalence relation: Points $x, x' \in \mathcal{F}$ are equivalent iff $\mu([x,x']) = 0$. The G-action, the pretree structure and the measure μ project to T (we retain the notation μ for the projection of the measure). As it was explained in section 2, the measure μ yields a metric d on T. Local compactness of $\tilde{\mathcal{L}}$ implies that the restriction of d to each interval in T is a complete metric. It is clear that the group G acts isometrically on T.

Remark 18. The map $\mathcal{F} \to T$ has at most countable multiplicity. Moreover, all but countably many points in T have a unique preimage in $\mathcal{F}$.

Lemma 19. *1. T is an uncountable metric tree. 2. Stabilizers of nondegenrate arcs in T are at most cyclic. 3. G does not have a global fixed point on T.*

Proof. 1. Follows from Lemma 4.

2. By our hypothesis, for each point $F \in \mathcal{F}$ its G-stabilizer is at most cyclic. Since $\mathcal{F}$ is prefect, it is uncountable; hence, by Remark 18, uncountably many points in each nondegenerate arc $[x,y] \subset T$ have at most cyclic stabilizer.

3. The action $G \curvearrowright \mathcal{F}$ is minimal, hence the action $G \curvearrowright T$ is minimal as well. Since T is not a point it follows that G cannot fix a point in T. ☐

We now can applly [3] to conclude that the group G splits as an amalgam over a virtually abelian subgroup A. Since G is a 3-dimensional Poincare duality group over $\mathcal{R}$, it follows from the Meyer-Vietoris sequence that A is a 2-dimensional Poincare duality group over $\mathcal{R}$. Thus implies that A is virtually a surface group, see for instance [13]. Since A is virtually abelian, it follows that A contains $\mathbb{Z}^2$ as a subgroup of finite index. This proves the main theorem. ☐

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