CONFORMAL DIMENSION AND GROMOV HYPERBOLIC GROUPS WITH 2-SPHERE BOUNDARY

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Abstract. Suppose G is a Gromov hyperbolic group, and $\partial_\infty G$ is quasisymmetrically homeomorphic to an Ahlfors Q-regular metric 2-sphere Z with Ahlfors regular conformal dimension Q. Then G acts discretely, cocompactly, and isometrically on $\mathbb{H}^3$.

1. Introduction

According to a well-known conjecture by Cannon, for every Gromov hyperbolic group G whose boundary at infinity $\partial_\infty G$ is homeomorphic to the 2-sphere $\mathbb{S}^2$, there should exist a discrete, cocompact, and isometric action of G on hyperbolic 3-space $\mathbb{H}^3$. In the present paper we establish Cannon's conjecture under the additional assumption that the Ahlfors regular conformal dimension of $\partial_\infty G$ is realized.

Theorem 1.1. *Let G be a Gromov hyperbolic group with boundary $\partial_\infty G$ homeomorphic to $\mathbb{S}^2$. If the Ahlfors regular conformal dimension of $\partial_\infty G$ is attained, then there exists an action of G on $\mathbb{H}^3$ which is discrete, cocompact and isometric.*

By definition, the *Ahlfors regular conformal dimension* of a metric space Z is the infimal Hausdorff dimension of all Ahlfors regular metric spaces (see Section 2 for the precise definition) quasisymmetrically homeomorphic to Z. This notion occurs implicitly in a paper by Bourdon and Pajot [7, Section 0.2] and is a variant of Pansu's *conformal dimension* for metric spaces (the conformal dimension of a metric space Z is the infimal Hausdorff dimension of *all* metric spaces quasisymmetrically homeomorphic to Z).

We recall that the boundary of a Gromov hyperbolic group G carries a canonical family of *visual metrics*; these are Ahlfors regular and pairwise quasisymmetrically homeomorphic by the identity map. In particular, it is meaningful to speak about quasisymmetric homeomorphisms between $\partial_\infty G$ and other metric spaces. The assumption on the Ahlfors regular conformal dimension of $\partial_\infty G$ says more explicitly that there is an Ahlfors Q-regular metric space Z quasisymmetrically homeomorphic

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to $\partial_\infty G$ with smallest possible Q among all such Ahlfors regular spaces. We necessarily have $Q \geq 2$, since the Hausdorff dimension of a space cannot be smaller than its topological dimension. The case $Q = 2$ of Theorem 1.1 can easily be deduced from [2, Theorem 1.1] or [3, Theorem 1.1].

The converse of Theorem 1.1 is well-known: if a group acts discretely, cocompactly and isometrically on hyperbolic 3-space, then its boundary is quasisymmetrically homeomorphic to the standard 2-sphere [17], which is a 2-regular space of conformal dimension 2. So by Theorem 1.1, Cannon's conjecture is equivalent to:

Conjecture 1.2. *If G is a hyperbolic group with 2-sphere boundary, then the Ahlfors regular conformal dimension of $\partial_\infty G$ is attained.*

We derive Theorem 1.1 from [2, Theorem 1.2] and a more general result about hyperbolic groups:

Theorem 1.3. *Let Z be an Ahlfors Q-regular metric space, $Q > 1$, where Q is the Ahlfors regular conformal dimension of Z. If Z admits a uniformly quasi-Möbius action $G \curvearrowright Z$ which is fixed point free and for which the induced action on the space of triples $\mathrm{Tri}(Z)$ is cocompact, then Z is Q-Loewner.*

The terminology will be explained in Section 2. The hypotheses of this theorem will hold, for example, if Z is a Q-regular space of Ahlfors regular conformal dimension Q, where $Q > 1$, and Z is quasisymmetrically homeomorphic to the boundary of a hyperbolic group.

Another way to state the conclusion of Theorem 1.3 is by saying that Z satisfies a $(1, Q)$-Poincaré inequality in the sense of Heinonen and Koskela [13]. They showed that for a proper Q-regular metric space such a Poincaré inequality holds if and only if the space is Q-Loewner; they also extended many classical results about quasiconformal and quasisymmmetric homeomorphisms to the setting of Q-regular Q-Loewner spaces.

By now there is a substantial body of literature about metric spaces satisfying Poincaré inequalities; see for example [13, 11, 19, 18, 16, 15]. These spaces play a central role in Cheeger's theory of differentiability of Lipschitz functions [9], and the Bourdon-Pajot rigidity theorem for quasi-isometries of hyperbolic buildings [6]. Theorem 1.3 suggests that one might obtain more examples of these nice spaces by minimizing the Hausdorff dimension of Ahlfors regular metrics on the boundary of a hyperbolic group.

The full strength of the group action $G \curvearrowright Z$ is actually not needed in the proof of Theorem 1.3. It is sufficient to have a collection $\mathcal{G}$ of uniformly quasi-Möbius homeomorphisms which is large enough to map any triple in Z to a uniformly separated triple, and which does not have a common fixed point. However, the assumption that the action $G \curvearrowright Z$ is fixed point free is essential. Starting with the Ahlfors 3-regular metric on $\mathbb{R}^2$ defined by the formula

$$d((x_1, y_1), (x_2, y_2)) := |x_1 - x_2| + |y_1 - y_2|^{1/2},$$

one can construct an Ahlfors 3-regular metric on $\mathbb{S}^2$ admitting a uniformly quasi-Möbius action which is transitive on the complement of a point, and cocompact on triples. The sphere $\mathbb{S}^2$ equipped with this metric has Ahlfors regular conformal dimension 3, but does not satisfy a $(1, p)$-Poincaré inequality for any $p \geq 1$.

Similar in spirit to Theorem 1.1 is another immediate consequence of Theorem 1.3 for convex cocompact Kleinian groups.

Theorem 1.4. *Suppose $G \curvearrowright \mathbb{H}^{n+1}$ is a convex cocompact isometric action of a discrete group G on hyperbolic n-space $\mathbb{H}^{n+1}$, $n \geq 1$. Let $\Lambda(G) \subseteq \mathbb{S}^n = \partial_\infty \mathbb{H}^{n+1}$ be the limit set of G, and assume that $Q > 1$, where Q is the Hausdorff dimension of $\Lambda(G)$. If the Ahlfors regular conformal dimension of $\Lambda(G)$ is equal to Q, then $Q = k \in \mathbb{N}$ is an integer and Γ stabilizes a totally geodesic subspace of $\mathbb{H}^{n+1}$ isometric to $\mathbb{H}^{k+1}$ on which Γ acts cocompactly.*

Note that if under the assumptions of this theorem $Z = \Lambda(G)$ carries a family of nonconstant curves with positive Q-modulus, then Q is equal to the Ahlfors regular conformal dimension of Z [12, Thm. 15.10]. One can also replace the condition on the Ahlfors regular dimension in the previous theorem by the requirement that Z satisfies a $(1, p)$-Poincaré inequality for some $p > 1$ (see Section 5 for further discussion).

We now sketch the proof of Theorem 1.3. Let Z and $G \curvearrowright Z$ be as in the statement of the theorem. A key ingredient used repeatedly in our proof is a result of Tyson [23] that implies that elements of G preserve Q-modulus to within a controlled factor. Our point of departure is a result of Keith and Laakso [14]:

Theorem 1.5 (Keith-Laakso)**.** *Let X be a proper Ahlfors Q-regular metric space, where $Q > 1$ is the Ahlfors regular conformal dimension of X. Then there exists a weak tangent W of X which carries a family of nonconstant paths with positive Q-modulus.*

This theorem can easily be derived from [14, Cor. 1.0.2]. For the definition of weak tangents and related discussion see [3, Section 4]; see Section 2 or [12] for a discussion of modulus. In our "self-similar" situation we can combine Theorem 1.5 with results from [3] and [23] to obtain the following corollary, which may be of independent interest.

Corollary 1.6. *Let Z be a compact Ahlfors Q-regular metric space, where $Q > 1$ is the Ahlfors regular conformal dimension of Z. If Z admits a uniformly quasi-Möbius action $G \curvearrowright Z$ for which the induced action on the space of triples* $\mathrm{Tri}(Z)$ *is cocompact, then there is a family of nonconstant paths in Z with positive Q-modulus.*

As we already pointed out, every (proper) Ahlfors Q-regular space carrying a family of nontrivial paths with positive Q-modulus has Ahlfors regular conformal dimension Q; the corollary may be viewed as a partial converse of this fact.

The next step in the proof of Theorem 1.3 is to show that Z satisfies a Loewner type condition for pairs of balls: if the Q-modulus for a pair of balls is small, then their relative distance is big, quantitatively. To prove this ball-Loewner condition, we introduce the notion of a *thick path*. Thick paths correspond to points in the support of Q-modulus, viewed as an outer measure on the space of (nonconstant) paths. Using the dynamics of the action $G \curvearrowright Z$, we show that any two open sets can be joined by a thick path, and this quickly leads to the ball-Loewner condition. The remaining step, which is the bulk of our argument, shows that any proper Q-regular space satisfying the ball-Loewner condition is Q-Loewner. By the result of Heinonen-Koskela mentioned above, this implies that Z satisfies a $(1, Q)$-Poincaré inequality.

In view of Conjecture 1.2 and Theorem 1.3, it is interesting to look for spaces whose Ahlfors regular conformal dimension is (or is not) attained. There are now several examples known where the Ahlfors regular conformal dimension is actually not realized; see Section 6 for more discussion. It is particulary interesting that Bourdon and Pajot [7] have found Gromov hyperbolic groups G for which $\partial_\infty G$ is not quasisymmetrically homeomorphic to an Ahlfors regular Loewner space; so by Theorem 1.3 the Ahlfors regular conformal dimension of $\partial_\infty G$ is not attained.

Additional remarks and open problems related to the discussion in this introduction can be found in the final Section 7 of the paper.

2. Notation and preliminaries

In this section, we will fix notation and review some basic definitions and facts. We will be rather brief, since by now there is a standard reference on these subjects [12] and most of the material has been discussed in greater detail in our previous papers [3, 2].

Notation. If (Z, d) is a metric space, we denote the open and the closed ball of radius $r > 0$ centered at $a \in Z$ by $B_Z(a, r)$ and $\bar{B}_Z(a, r)$, respectively. We will drop the subscript Z if the space Z is understood. If $B = B(a, r)$ is a ball and $\lambda > 0$ we let $\lambda B := B(a, \lambda r)$. We use $\operatorname{diam}(A)$ for the diameter of a set $A \subseteq Z$. If $z \in Z$ and $A, B \subseteq Z$, then $\operatorname{dist}(z, A)$ and $\operatorname{dist}(A, B)$ are the distances of z and A and of A and B, respectively. If $A \subseteq Z$ and $r > 0$, then we let $N_r(A) := \{z \in Z : \operatorname{dist}(z, A) < r\}$. The *Hausdorff distance* between two sets $A, B \subseteq Z$ is defined by

$$\operatorname{dist}_{\mathrm{H}}(A, B) := \max\Big\{ \sup_{a\in A} \operatorname{dist}(a, B),\ \sup_{b\in A} \operatorname{dist}(b, A)\Big\}.$$

If $f\colon X \to Y$ is a map between two spaces X and Y, we let $\operatorname{Im}(f) := \{f(x) : x \in X\}$. If $A \subseteq X$, then $f|A$ denotes the restriction of the map f to A.

Cross-ratios and quasi-Möbius maps. Let (Z,d) be a metric space. The *cross-ratio*, $[z_1,z_2,z_3,z_4]$, of a four-tuple of distinct points (z_1,z_2,z_3,z_4) in Z is the quantity

$$[z_1,z_2,z_3,z_4] := \frac{d(z_1,z_3)d(z_2,z_4)}{d(z_1,z_4)d(z_2,z_3)}.$$

Let $\eta\colon[0,\infty)\to[0,\infty)$ be a homeomorphism, and let $f\colon X\to Y$ be an injective map between metric spaces (X,d_X) and (Y,d_Y). The map f is an *η-quasi-Möbius map* if for every four-tuple (x_1,x_2,x_3,x_4) of distinct points in X, we have

$$[f(x_1),f(x_2),f(x_3),f(x_4)] \le \eta([x_1,x_2,x_3,x_4]).$$

The map f is *η-quasisymmetric* if

$$\frac{d_Y(f(x_1),f(x_2))}{d_Y(f(x_1),f(x_3))} \le \eta\left(\frac{d_X(x_1,x_2)}{d_X(x_1,x_3)}\right)$$

for every triple (x_1,x_2,x_3) of distinct points in X.

We will make repeated use of the following lemma. We refer to [3, Lemma 5.1] for the proof.

Lemma 2.1. *Let (Z,d) be a compact metric space. Suppose that for each $k\in\mathbb{N}$ we are given a ball $B_k=B(p_k,R_k)\subseteq Z$, distinct points $x_k^1,x_k^2,x_k^3\in\bar{B}(p_k,\lambda_kR_k)$ with*

$$d_Z(x_k^i,x_k^j) > \delta_kR_k \quad \text{for} \quad i,j\in\{1,2,3\},\, i\ne j,$$

where $\lambda_k,\delta_k>0$, and an η-quasi-Möbius homeomorphism $g_k\colon Z\to Z$ such that for $y_k^i := g_k(x_k^i)$ we have

$$d_Z(y_k^i,y_k^j) > \delta' \quad \text{for} \quad i,j\in\{1,2,3\},\, i\ne j,$$

where η and $\delta'>0$ are independent of k.

(i) *If $\lim_{k\to\infty}\lambda_k=0$ and the sequence $(R_k)_{k\in\mathbb{N}}$ is bounded, then*

$$\operatorname{diam}(Z\setminus g_k(B_k))\to 0 \quad \text{for} \quad k\to\infty.$$

(ii) *Suppose for $k\in\mathbb{N}$ the set $D_k\subseteq B_k$ is (ϵ_kR_k)-dense in B_k, where $\epsilon_k>0$. If $\lim_{k\to\infty}\lambda_k=0$ and the sequence $(\epsilon_k/\delta_k^2)_{k\in\mathbb{N}}$ is bounded, then*

$$\operatorname{dist}_{\mathrm{H}}(g_k(D_k),Z)\to 0 \quad \text{for} \quad k\to\infty.$$

If a group G acts on a compact metric space (Z,d) by homeomorphisms, we write $G\curvearrowright Z$ and consider the group elements as self-homeomorphisms of Z. We do not require that the action is effective; so it may well happen that a group element different from the unit element is represented by the identity map. We denote by $\operatorname{Tri}(Z)$ the space of distinct triples in Z. An action $G\curvearrowright Z$ induces an action $G\curvearrowright\operatorname{Tri}(Z)$. The action $G\curvearrowright\operatorname{Tri}(Z)$ is cocompact, if and only if every triple in $\operatorname{Tri}(Z)$ can be mapped to a uniformly separated triple by a group element. More precisely, this means that there exists $\delta>0$ such that for every triple $(z_1,z_2,z_3)\in\operatorname{Tri}(Z)$ there exists a group element $g\in G$ such that the image triple $(g(z_1),g(z_2),g(z_3))\in\operatorname{Tri}(Z)$ is δ-separated, i.e., $d(g(z_i),g(z_j))\ge\delta$ for $i\ne j$. We call the action $G\curvearrowright Z$ *fixed point free* if the maps $g \in G$ have no common fixed point, i.e., for each $z \in Z$ there exists $g \in G$ such that $g(z) \neq z$. The action $G \curvearrowright Z$ is called *uniformly quasi-Möbius* if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that every $g \in G$ is an η-quasi-Möbius homeomorphism of Z.

Lemma 2.2. *Suppose (Z,d) is a compact, uniformly perfect metric space, and $G \curvearrowright Z$ is a fixed point free uniformly quasi-Möbius action which is cocompact on* $\mathrm{Tri}(Z)$. *Then the action is minimal, i.e., for all $z, z' \in Z$ and all $\epsilon > 0$, there is a group element $g \in G$ such that $d(g(z'), z) < \epsilon$.*

Proof. Let $z, z' \in Z$ and $\epsilon > 0$ be arbitrary. Since Z is uniformly perfect, we can choose distinct points $x_k^1, x_k^2 \in Z$ for $k \in \mathbb{N}$ such that the distances $d(z, x_k^1)$, $d(z, x_k^2)$, and $d(x_k^1, x_k^2)$ agree to within a factor independent of k, and $\lim_{k\to\infty} d(z, x_k^1) = 0$. Set $r_k := d(z, x_k^1)$. Since $Z \curvearrowright \mathrm{Tri}(Z)$ is cocompact, we can find $g_k \in G$ such that the triples $(g_k(z), g_k(x_k^1), g_k(x_k^2))$ are δ-separated where $\delta > 0$ is independent of k. Choose $R_k > 0$ such that $\lim_{k\to\infty} R_k = 0$ and $\lim_{k\to\infty} R_k/r_k = \infty$. By Lemma 2.1 we then have $\lim_{k\to\infty} \mathrm{diam}(Z \setminus g_k(B_k)) = 0$, where $B_k := B(z, R_k)$. Pick $g \in G$ such that $g(z') \neq z'$. Then for large k, either $z' \in g_k(B_k)$ or $g(z') \in g_k(B_k)$, which means that either $g_k^{-1}(z') \in B_k$ or $g_k^{-1} \circ g(z') \in B_k$. Hence for sufficiently large k one of the points $g_k^{-1}(z')$ or $g_k^{-1} \circ g(z')$ is contained in $B(z,\epsilon)$. □

Modulus in metric measure spaces. Suppose (Z, d, μ) is a metric measure space, i.e., (Z,d) is a metric space and μ a Borel measure on Z. Moreover, we assume that that (Z,d) is locally compact and that μ is locally finite and has dense support.

The space (Z,d,μ) is called *(Ahlfors) Q-regular*, $Q > 0$, if the measure μ satisfies

$$C^{-1} R^Q \leq \mu(B(a,R)) \leq C R^Q \tag{2.3}$$

for each open ball $B(a,R)$ of radius $0 < R \leq \mathrm{diam}(Z)$ and for some constant $C \geq 1$ independent of the ball. If the measure is not specified, then it is understood that μ is Q-dimensional Hausdorff measure.

A *density* (on Z) is a Borel function $\rho\colon Z \to [0,\infty]$. A density ρ is called *admissible* for a path family Γ in Z, if

$$\int_\gamma \rho\, ds \geq 1$$

for each rectifiable path $\gamma \in \Gamma$. Here integration is with respect to arclength on γ. If $Q \geq 1$, the *Q-modulus* of a family Γ of paths in Z is the number

$$\mathrm{Mod}_Q(\Gamma) = \inf \int \rho^Q \, d\mu, \tag{2.4}$$

where the infimum is taken over all densities $\rho\colon Z \to [0,\infty]$ that are admissible for Γ. If E and F are subsets of Z with positive diameter, we denote by

$$\Delta(E,F) := \frac{\mathrm{dist}(E,F)}{\min\{\mathrm{diam}(E), \mathrm{diam}(F)\}} \tag{2.5}$$

the *relative distance* of E and F, and by $\Gamma(E,F)$ the family of paths in Z connecting E and F.

Suppose (Z,d,μ) is a connected metric measure space. Then Z is called a *Q-Loewner space*, $Q\geq 1$, if there exists a positive decreasing function $\Psi\colon(0,\infty)\to(0,\infty)$ such that

$$\operatorname{Mod}_Q(\Gamma(E,F))\geq\Psi(\Delta(E,F)) \tag{2.6}$$

whenever E and F are disjoint continua in Z. Note that in [13] it was also required that Z is rectifiably connected. In case that the (locally compact) space (Z,d,μ) is Q-regular and $Q>1$, it is unnecessary to make this additional assumption, because property (2.6) alone implies that (Z,d) is even quasiconvex, i.e., for every pair of points there exists a connecting path whose length is comparable to the distance of the points.

We will need the following result due to Tyson.

Theorem 2.7. *Let X and Y be Ahlfors Q-regular locally compact metric spaces, $Q\geq 1$, and let $f\colon X\to Y$ be an η-quasi-Möbius homeomorphism. Then for every family Γ of paths in X, we have*

$$\frac{1}{C}\operatorname{Mod}_Q(\Gamma)\leq\operatorname{Mod}_Q(f\circ\Gamma)\leq C\operatorname{Mod}_Q(\Gamma),$$

where $f\circ\Gamma:=\{f\circ\gamma:\gamma\in\Gamma\}$ and C is a constant depending only on X, Y and η.

Tyson proved this for quasisymmetric mappings f in [23] and for locally quasisymmetric maps in [24, Theorem 6.4 and Lemma 9.2]. Here a map $f\colon X\to Y$ is called locally η-quasisymmetric if every point $x\in X$ has an open neighborhood U such that $f|U$ is η-quasisymmetric. Since η-quasi-Möbius maps are locally $\tilde\eta$-quasisymmetric with $\tilde\eta$ depending only on η, the above theorem follows.

Lemma 2.8. *Assume $Q>1$ and (Z,d,μ) is Ahlfors Q-regular. Then there exists a constant $C>0$ with the following property. If Γ is a family of paths in Z which start in a ball $B\subseteq Z$ of radius $R>0$ and whose lengths are at least LR, where $L\geq 2$, then*

$$\operatorname{Mod}_Q(\Gamma)<C\,(\log L)^{1-Q}.$$

We omit a detailed proof, since the statement is well-known (cf. [12, Lemma 7.18] and [4, Lemma 3.2] for very similar results). The basic idea is to use a test function of the form

$$\frac{c}{R+\operatorname{dist}(x,B)}$$

supported in LB and use the upper mass bound $\mu(B(x,r))\lesssim r^Q$.

Thick paths. We now assume that (Z, d, μ) be a separable locally compact metric measure space, and $Q \geq 1$. Let $I := [0, 1]$, and denote by $\mathcal{P} := C(I, Z)$ the set of (continuous) paths in Z, metrized by the supremum metric. Then $\mathcal{P}$ is a separable complete metric space. Since Q-modulus is monotonic and countably subadditive for path families (cf. [12, p. 51]), we can consider Mod_Q as an outer measure on $\mathcal{P}$.

Definition 2.9. A path $\gamma \in \mathcal{P}$ is *thick* if for all $\epsilon > 0$, the family of nonconstant paths in the ball $B(\gamma, \epsilon) \subseteq \mathcal{P}$ has positive Q-modulus.

In other words, a path $\gamma \in \mathcal{P}$ is thick if $\operatorname{Mod}_Q(B(\gamma, \epsilon) \setminus \mathcal{C}) > 0$ for all $\epsilon > 0$, where $\mathcal{C}$ is the family of constants paths in Z. We have to exclude the constant paths here, because $\operatorname{Mod}_Q(\Gamma) = \infty$ whenever Γ contains a constant path. Constant paths lead to some technicalities later on, which could be avoided if we had defined $\mathcal{P}$ as the space of nonconstant paths in Z. This also has disadvantages, since certain completeness and compactness properties of $\mathcal{P}$ would be lost with this definition.

We denote by $\mathcal{P}_T$ the set of thick paths in $\mathcal{P}$. Ignoring constant paths, the thick paths form the support of the outer measure Mod_Q.

Lemma 2.10. (Properties of thick paths)

(i) (Stability under limits) *The set $\mathcal{P}_T$ is closed in $\mathcal{P}$: if $\gamma_k \in \mathcal{P}$ is thick for $k \in \mathbb{N}$ and $\gamma = \lim_{k\to\infty} \gamma_k$, then γ is thick.*

(ii) (Thickness of subpaths) *The composition of any embedding $I \to I$ with a thick path is a thick path.*

(iii) (Quasi-Möbius invariance) *If (Z, d, μ) is locally compact and Ahlfors Q-regular, $Q \geq 1$, then the image of a thick path under a quasi-Möbius homeomorphism $Z \to Z$ is thick.*

Proof. Property (i) follows immediately from the definitions. Property (iii) is a consequence of Tyson's Theorem 2.7.

To prove property (ii) first note that if γ is any path and $\alpha \colon I \to I$ is an embedding, then the definition of modulus implies that

$$\operatorname{Mod_Q}(B(\gamma, \epsilon)) \leq \operatorname{Mod_Q}(B(\gamma \circ \alpha, \epsilon)). \tag{2.11}$$

If γ is a thick path and $\gamma \circ \alpha$ is nonconstant, then for sufficiently small $\epsilon > 0$ there will be no constant paths in either $B(\gamma, \epsilon)$ or $B(\gamma \circ \alpha, \epsilon)$, and (2.11) implies that $\gamma \circ \alpha$ is thick. If γ is thick and $\gamma \circ \alpha$ is constant, we can assume without loss of generality that $\operatorname{Im}(\alpha)$ is not contained in a larger interval on which γ is constant. If $\operatorname{Im}(\alpha) = I$ then $\gamma \circ \alpha$ is just a reparametrization of a thick path and is therefore thick. Otherwise, we can enlarge $\operatorname{Im}(\alpha)$ slightly and approximate $\gamma \circ \alpha$ by nonconstant subpaths of γ, which are thick by the first part of the argument. Property (i) now implies that $\gamma \circ \alpha$ is also thick. $\square$

Lemma 2.12. *The family $\mathcal{P}_t$ of nonconstant paths in $\mathcal{P}$ which are not thick has zero Q-modulus. In particular, given any family $\Gamma \subseteq \mathcal{P}$ of nonconstant paths, we have* $\mathrm{Mod_Q}(\Gamma \cap \mathcal{P}_T) = \mathrm{Mod_Q}(\Gamma)$.

Proof. For each $\gamma \in \mathcal{P}_t$, we can find an open set U_γ containing γ which consists of nonconstant paths and has zero Q-modulus. The space $\mathcal{P}$ is separable, so we can find a countable subcollection of the sets U_γ which covers $\mathcal{P}_t$. Countable subadditivity of Q-modulus implies that $\mathrm{Mod_Q}(\mathcal{P}_t) = 0$.

The second part of the lemma follows from monotonicity and subadditivity of Q-modulus. $\square$

The previous lemma implies the existence of nonconstant thick paths whenever Z carries a family of nonconstant paths of positive Q-modulus. Moreover, suppose Γ_0 is a family of paths with $\mathrm{Mod_Q}(\Gamma_0) = 0$. Then if γ is thick and $\epsilon > 0$ is arbitrary, we can find a thick path $\alpha \in B(\gamma, \epsilon) \setminus \Gamma_0$. In other words, by a small perturbation of a thick path, we can obtain a thick path avoiding any given family of zero Q-modulus.

3. The Loewner condition for balls

In this section we prove the following proposition, which asserts that a space which satisfies a Loewner type condition for pairs of balls, satisfies the Loewner condition for all pairs of continua.

Proposition 3.1. *Let (Z, d, μ) be a proper metric measure space. Assume that for all $C > 0$, there are constants $m = m(C) > 0$ and $L = L(C) > 0$ such that if $R > 0$ and $B, B' \subseteq Z$ are R-balls with* $\mathrm{dist}(B, B') \leq CR$, *then the Q-modulus of the family*

$$\{\gamma \in \Gamma(B, B') : \mathrm{length}(\gamma) \leq LR\}$$

is a least m. Then (Z, d, μ) is Q-Loewner.

Rather than using the hypothesis directly, the proof of the proposition will use the following consequence: if $\rho\colon Z \to [0, \infty]$ is a Borel function and the balls are as in the statement of the proposition, then there is a path $\sigma \in \Gamma(B, B')$ whose length is at most LR and whose ρ-length is at most

$$\frac{1}{m^{1/Q}} \left(\int_{(L+1)B} \rho^Q \, d\mu \right)^{1/Q}. \tag{3.2}$$

Here and in the following we call the integral $\int_\sigma \rho \, ds$ the *ρ-length* of a rectifiable path σ.

We point out the following corollary of Proposition 3.1 which is of independent interest.

Corollary 3.3. *Suppose (Z, d, μ) is a proper Ahlfors Q-regular metric measure space, $Q > 1$. Suppose that there exists a positive decreasing function $\Psi \colon (0, \infty) \to (0, \infty)$ such that*

$$\mathrm{Mod}_{\mathrm{Q}}(\Gamma(B, B')) \geq \Psi(\Delta(B, B'))$$

whenever B and B' are disjoint balls in Z. Then (Z, d, μ) is Q-Loewner.

Proof. This immediately follows from Proposition 3.1, Lemma 2.8 and the subadditivity of modulus (see the end of the proof of Lemma 4.3 for additional details). □

Before we start with the proof of Proposition 3.1, we first indicate a lemma whose proof uses a similar construction in a simpler setting.

Lemma 3.4. *Let X be a complete metric space. Suppose there exist $0 \leq \lambda < 1/2$ and $L < \infty$ such that if $u, v \in X$, then there is a path of length at most $Ld(u, v)$ connecting $B(u, \lambda d(u, v))$ and $B(v, \lambda d(u, v))$. Then X is quasi-convex.*

This lemma can be used to give another proof that a Q-regular space satisfying a $(1, Q)$-Poincaré inequality is quasi-convex.

Outline of proof. Suppose $x, y \in X$ and let $R := d(x, y)$. By assumption we can find a path σ_1 of length $\leq LR$ joining $B(x, \lambda R)$ to $B(y, \lambda R)$. Set $\Sigma_0 := \{\sigma_1\}$. Then we can find paths σ_2, σ_3 of length $\leq L\lambda^2 R$ such that σ_2 joins $B(x, \lambda^2 R)$ to $B(\sigma_1(0), \lambda^2 R)$ and σ_3 joins $B(\sigma_1(1), \lambda^2 R)$ to $B(y, \lambda^2 R)$. Set $\Sigma_2 := \{\sigma_2, \sigma_3\}$. Continuing inductively, we can find path collections Σ_k for all $k \geq 0$. At each step of the induction the "total gap" Δ_k gets multiplied by $2\lambda < 1$, and the total length of the curves generated is $\leq L\Delta_k$. One then concludes that if

$$Y := \bigcup_{k=0}^{\infty} \bigcup_{\sigma \in \Sigma_k} \mathrm{Im}(\sigma),$$

then $\bar{Y}$ is a compact connected set containing $\{x, y\}$, and has 1-dimensional Hausdorff measure at most $\frac{LR}{1-2\lambda}$. Therefore there is an arc of length

$$\leq \left(\frac{L}{1 - 2\lambda}\right) d(x, y)$$

joining x to y. □

The proof of Proposition 3.1 will require two lemmas.

Lemma 3.5. *Let X be a metric space, and ν be a finite Borel measure on X. If $Y \subseteq X$ is a nondegenerate continuum, then we can find a point $y \in Y$ such that for all $r > 0$ we have*

$$\nu(B(y, r)) \leq \frac{10r}{\mathrm{diam}(Y)} \nu(X). \tag{3.6}$$

Proof. We assume that $\nu(X) > 0$, for otherwise the assertion obviously holds.

If the statement were false, then for each $y \in Y$ we could find $r_y > 0$ such that $\nu(B(y, r_y)) > M\nu(X)r_y$, where $M = 10/\operatorname{diam}(Y)$. Then the radii r_y, $y \in Y$, are uniformly bounded from above, and so we can find a disjoint subcollection $\{B_i = B(y_i, r_i)\}_{i\in I}$ of the cover $\{B(y, r_y) : y \in Y\}$ of Y such that the collection $\{5B_i\}_{i\in I}$ is also a cover of Y [12, Theorem 1.2]. Define an equivalence relation on I by declaring that $i \sim i'$ if there are elements $i = i_1, \dots, i_k = i'$ such that $B_{i_j} \cap B_{i_{j+1}} \neq \emptyset$ for $1 \le j < k$. If $I = \sqcup_{j\in J} I_j$ is the decomposition of I into equivalence classes, then the collection $\{\cup_{i\in I_j} 5B_i\}_{j\in J}$ is a cover of Y by disjoint open sets; since Y is connected this implies that $\#J = 1$. It follows that

$$\sum_i r_i \ge \frac{1}{10}\operatorname{diam}(Y) = \frac{1}{M},$$

and so

$$\nu(X) \ge \sum_i \nu(B_i) > \nu(X)M\sum_i r_i \ge \nu(X).$$

This is a contradiction. □

Lemma 3.7. *Let Z be as in Proposition 3.1, and suppose $0 < \lambda < 1/8$. Then there are constants $\Lambda = \Lambda(\lambda) > 0$, $K = K(\lambda) > 0$ with the following property. If $\rho\colon Z \to [0, \infty]$ is a Borel function, $B = B(p, r) \subseteq Z$ is a ball, and F_1, $F_2 \subseteq Z$ are continua such that $F_i \cap \frac{1}{4}B \neq \emptyset$ and $F_i \setminus B \neq \emptyset$ for $i = 1, 2$, then there are disjoint balls $B_i := B(q_i, \lambda r)$ for $i = 1, 2$, and a path $\sigma\colon [0, 1] \to Z$ such that*

(i) *$q_i \in F_i$ for $i = 1, 2$,*

(ii) *$B_i \subseteq \frac{7}{8}B$ and*

$$\int_{B_i} \rho^Q \, d\mu \le 80\lambda \int_B \rho^Q \, d\mu$$

for $i = 1, 2$,

(iii) *the path σ joins $\frac{1}{4}B_1$ and $\frac{1}{4}B_2$, has image contained in ΛB, length at most Λr, and ρ-length at most*

$$K\left(\int_{\Lambda B} \rho^Q \, d\mu\right)^{1/Q}.$$

Proof. We can find a subcontinuum $E_1 \subseteq F_1$ which is contained in $\bar{B}(p, \frac{3r}{8}) \setminus B(p, \frac{r}{4})$ and joins the sets $\partial B(p, \frac{3r}{8})$ and $\partial B(p, \frac{r}{4})$. Similarly, we can find a subcontinuum $E_2 \subseteq F_2$ which is contained in $\bar{B}(p, \frac{3r}{4}) \setminus B(p, \frac{5r}{8})$ and joins the sets $\partial B(p, \frac{3r}{4})$ and $\partial B(p, \frac{5r}{8})$. Then $\operatorname{diam}(E_i) \ge r/8$ for $i = 1, 2$, and $\operatorname{dist}(E_1, E_2) \ge r/4$.

Applying Lemma 3.5 with $X = B$, the measure ν defined by $\nu(N) := \int_N \rho^Q \, d\mu$ for a Borel set $N \subseteq B$, and $Y = E_i$ for $i = 1, 2$, we find $q_i \in E_i$ such that

$$\int_{B(q_i,s)} \rho^Q \, d\mu \leq \frac{10s}{\operatorname{diam}(E_i)} \int_B \rho^Q \, d\mu \leq \frac{80s}{r} \int_B \rho^Q \, d\mu, \tag{3.8}$$

for $0 < s \leq r/4$. Set $B_i := B(q_i, \lambda r)$. By our assumption on Z, we can find a path σ from $\frac{1}{4}B_1$ to $\frac{1}{4}B_2$ with $\operatorname{Im}(\sigma) \subseteq \Lambda B$, length at most Λr, and ρ-length at most

$$K \left(\int_{\Lambda B} \rho^Q \, d\mu \right)^{1/Q},$$

where $\Lambda = \Lambda(\lambda) > 0$ and $K = K(\lambda) > 0$.

The balls B_1 and B_2 are disjoint since $\lambda < 1/8$ and

$$d(q_1, q_2) \geq \operatorname{dist}(E_1, E_2) \geq r/4.$$

Conditions (i) and (iii) are clearly satisfied. Condition (ii) follows from the facts that $\lambda < 1/8$ and $q_i \in \bar{B}(p, \frac{3r}{4})$ and from (3.8). $\square$

Proof of Proposition 3.1. Fix $0 < \lambda < 1/8$ subject to the condition $2 \cdot (80\lambda)^{1/Q} < 1$.

Suppose E_1, $E_2 \subseteq Z$ are nondegenerate continua, and let $\rho\colon Z \to [0, \infty]$ be a Borel function. We will show that there is a rectifiable path γ joining E_1 to E_2 whose ρ-length is at most

$$M \left(\int_Z \rho^Q \, d\mu \right)^{1/Q}$$

where $M > 0$ only depends on the relative distance $\Delta(E_1, E_2)$ of E_1 and E_2. In fact, the path produced will have length $\lesssim \operatorname{dist}(E_1, E_2)$, though this will not be used anywhere. We will henceforth assume that E_1 and E_2 are disjoint, for otherwise we may use a constant path mapping into $E_1 \cap E_2$.

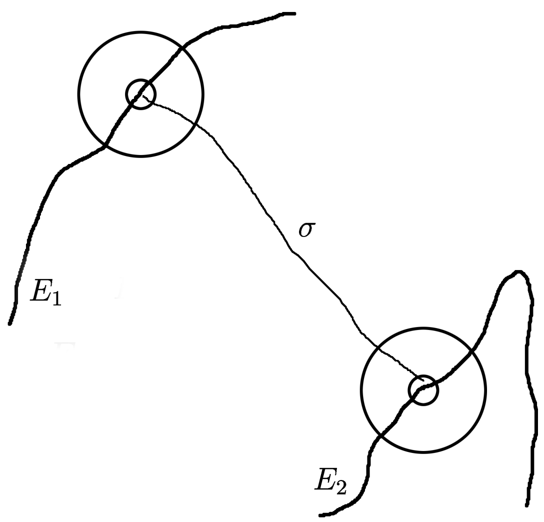
Pick $p_i \in E_i$ such that $d(p_1, p_2) = \operatorname{dist}(E_1, E_2)$. Set

$$r_0 := \frac{1}{2} \min(d(p_1, p_2), \operatorname{diam}(E_1), \operatorname{diam}(E_2)) > 0.$$

Let $B_i = B(p_i, r_0)$ for $i = 1, 2$. Then $B_1 \cap B_2 = \emptyset$ and $E_i \setminus B_i \neq \emptyset$ for $i = 1, 2$. Also, $\operatorname{dist}(B_1, B_2) \leq \left(\frac{d(p_1,p_2)}{r_0} \right) r_0 \leq t r_0$ where $t := 2 \max(1, \Delta(E_1, E_2))$. By our hypotheses we can find a path σ joining $\frac{1}{4}B_1$ to $\frac{1}{4}B_2$ of length at most $L r_0$ and ρ-length at most

$$\frac{1}{m^{1/Q}} \left(\int_Z \rho^Q \, d\mu \right)^{1/Q}, \tag{3.9}$$

where $L = L(t)$, $m = m(t)$ are the constants appearing in the statement of Proposition 3.1. Let $\Sigma_0 := \{\sigma\}$, $\mathcal{B}_0 := \{B_1, B_2\}$, and Ω_0 be the set $\{E_1, \operatorname{Im}(\sigma), E_2\}$ endowed with the linear ordering $E_1 < \operatorname{Im}(\sigma) < E_2$. In addition, we associate the ball B_1 with the pair $E < \operatorname{Im}(\sigma)$, and the ball B_2 with the pair $\operatorname{Im}(\sigma) < E_2$.



Inductively, assume that for $j = 0, \ldots, k$ we have a path collection Σ_j, a ball collection $\mathcal{B}_j$, and a collection of continua Ω_j subject to the following conditions:

1. For $0 \leq j \leq k$, the set Ω_j is linearly ordered.

2. For each pair $\tau_1 < \tau_2$ of successive elements of Ω_j, there is an associated ball $B_{\tau_1,\tau_2} \in \mathcal{B}_j$ such that $\tau_i \setminus B_{\tau_1,\tau_2} \neq \emptyset$ and $\tau_i \cap \frac{1}{4}B_{\tau_1,\tau_2} \neq \emptyset$, for $i = 1, 2$.

3. For $j \geq 1$, the collections Σ_j, $\mathcal{B}_j$, and Ω_j are obtained from Ω_{j-1} and $\mathcal{B}_{j-1}$ using the following procedure. For each pair of successive elements $\tau_1, \tau_2 \in \Omega_{j-1}$ with associated ball $B_{\tau_1,\tau_2} \in \mathcal{B}_{j-1}$, one applies Lemma 3.7 with $B = B_{\tau_1,\tau_2}$ and $\{F_1, F_2\} = \{\tau_1, \tau_2\}$, to obtain a path $\sigma = \sigma(\tau_1, \tau_2)$ and a pair of disjoint balls $B_{\tau_1,\sigma}, B_{\sigma,\tau_2}$. Here $B_{\tau_1,\sigma}$ is centered at a point in τ_1, and B_{σ,τ_2} is centered at a point in τ_2. Then Σ_j is the collection of paths σ and $\mathcal{B}_j$ is the collection of balls $B_{\tau_1,\sigma}, B_{\sigma,\tau_2}$ where $\tau_1 < \tau_2$ ranges over all successive pairs in $\mathcal{B}_{j-1}$. The continuum collection Ω_j is the disjoint union $\Omega_{j-1} \sqcup \{\mathrm{Im}(\sigma) : \sigma \in \Sigma_j\}$. One linearly orders Ω_j by extending the order on Ω_{j-1} subject to $\tau_1 < \mathrm{Im}(\sigma) < \tau_2$; moreover, one associates the ball $B_{\tau_1,\sigma}$ with the pair $\tau_1 < \mathrm{Im}(\sigma)$, and the ball $B_{\tau_2,\sigma}$ with the pair $\mathrm{Im}(\sigma) < \tau_2$.

By our second induction assumption, the hypotheses of Lemma 3.7 hold for each successive pair $\tau_1, \tau_2 \in \Omega_k$ and associated ball $B_{\tau_1,\tau_2} \in \mathcal{B}_k$. Hence we may use the procedure in the third induction assumption (with j replaced by $k+1$) to generate Σ_{k+1}, $\mathcal{B}_{k+1}$, Ω_{k+1}, the linear order on Ω_{k+1}, and an association of balls in $\mathcal{B}_{k+1}$ with successive pairs in Ω_{k+1}. The conditions in Lemma 3.7 guarantee that the induction hypotheses are fulfilled. Therefore by induction there are collections Σ_k, $\mathcal{B}_k$, and Ω_k for all $k \geq 0$ which satisfy the conditions 1–3 for all $k \geq 0$.

By induction, one proves the following:

(a) For each $k \geq 0$, we have $\#\mathcal{B}_k = 2^{k+1}$, and each ball in $\mathcal{B}_k$ has radius $\lambda^{k+1} r_0$ (see Lemma 3.7).

(b) For each $k \geq 0$, $j \leq k$, and $B \in \mathcal{B}_k$, there is a ball $B' \in \mathcal{B}_{k-j}$ such that $B' \supset (\frac{8}{7})^{k-j} B$, (see condition (ii) of Lemma 3.7).

(c) For $k \geq 0$, the ρ-mass of each ball $B \in \mathcal{B}_k$ satisfies

$$\int_B \rho^Q \, d\mu \leq (80\lambda)^k \int_Z \rho^Q \, d\mu$$

(see Lemma 3.7, condition (ii)).

(d) For each $k > 0$, we have $\#\Sigma_k = 2^k$. Each $\sigma \in \Sigma_k$ has length at most $\Lambda \lambda^k r_0$, for a suitable ball $B \in \mathcal{B}_{k-1}$ we have $\mathrm{Im}(\sigma) \subseteq \Lambda B$, and the ρ-length of σ is at most

$$K \left(\int_{\Lambda B} \rho^Q \, d\mu \right)^{1/Q}.$$

(e) For $k \geq 0$ set

$$Y_k := \bigcup_{j=0}^{k} \bigcup_{\sigma \in \Sigma_j} \mathrm{Im}(\sigma).$$

Then $\mathrm{dist}(Y_k, E_1), \mathrm{dist}(Y_k, E_2) \leq \lambda^k r_0/4$, and Y_k is $(\lambda^k r_0/4)$-connected: given y, $y' \in Y_k$ there are points $y = y_1, \ldots, y_l = y'$ such that $d(y_j, y_{j+1}) \leq \lambda^k r_0/4$ for $1 \leq j \leq l$.

(f) For $k > 0$, the union

$$Z_k := \bigcup_{j>k} \left(\bigcup_{\sigma \in \Sigma_j} \mathrm{Im}(\sigma) \right)$$

is covered by the collection $\{(\Lambda + 1)B\}_{B \in \mathcal{B}_k}$.

Set

$$Y := \bigcup_{j=0}^{\infty} \bigcup_{\sigma \in \Sigma_j} \mathrm{Im}(\sigma).$$

By (e) the closure $\bar{Y}$ of Y is compact, connected, and intersects both E_1 and E_2. Then (f) and (a) imply that $\bar{Y} \setminus Y$ has 1-dimensional Hausdorff measure zero. Combining this with (d) and the fact that $\lambda < 1/8$, we get that the 1-dimensional Hausdorff measure of $\bar{Y}$ is at most

$$\sum_{k=0}^{\infty} \sum_{\sigma \in \Sigma_k} \mathrm{length}(\sigma) \leq \sum_{k=0}^{\infty} (2^k)(\Lambda \lambda^k r_0) = \frac{\Lambda r_0}{1 - 2\lambda} < \infty.$$

Hence there is a rectifiable path $\gamma \colon [0,1] \to Z$ contained in $\bar{Y}$ joining E_1 to E_2 with

$$\mathrm{length}(\gamma) \leq \frac{\Lambda r_0}{1 - 2\lambda}.$$

Moreover, we may assume that $\mathrm{Im}(\gamma)$ is an arc and γ is an injective map.

Pick an integer s such that $(\frac{8}{7})^{s-1} > \Lambda$. Then by (d), (b) and (c), for every $k \geq s$ and $\sigma \in \Sigma_k$ there is a ball $B' \in \mathcal{B}_{k-s}$ such that the ρ-length of σ is at most

$$K \left(\int_{B'} \rho^Q \, d\mu \right)^{1/Q} \leq K \left((80\lambda)^{k-s} \int_Z \rho^Q \, d\mu \right)^{1/Q}.$$

For $0 < k < s$, each $\sigma \in \Sigma_k$ has ρ-length at most

$$K \left(\int_Z \rho^Q \, d\mu \right)^{1/Q}.$$

Recall that $\sigma \in \Sigma_0$ has ρ-length

$$\frac{1}{m^{1/Q}} \left(\int_Z \rho^Q \, d\mu \right)^{1/Q}.$$

Using these bounds for ρ-length and the fact that γ parametrizes an arc, we get

$$\begin{aligned}
\int_\gamma \rho \, ds &\leq \int_{\bar{Y}} \rho \, d\mathcal{H}^1 \leq \sum_{k=0}^{\infty} \sum_{\sigma \in \Sigma_k} \int_{\mathrm{Im}(\sigma)} \rho \, d\mathcal{H}^1 \\
&\leq \left(\int_Z \rho^Q \, d\mu \right)^{1/Q} \left(\frac{1}{m^{1/Q}} + K \sum_{k=1}^{s-1} 2^k + K \sum_{k=s}^{\infty} 2^k (80\lambda)^{k-s} \right) \\
&=: M \left(\int_Z \rho^Q \, d\mu \right)^{1/Q}.
\end{aligned}$$

Note that M is finite since $2 \cdot (80\lambda)^{1/Q} < 1$ by our initial choice of λ. Moreover, M depends only on $\Delta(E_1, E_2)$. This shows that the path γ has the desired properties. $\square$

4. Rescaling and abundance of thick paths

We now let (Z, d, μ) be a Ahlfors Q-regular compact metric space, $Q > 1$, which carries a family of nonconstant paths with positive Q-modulus, and we let $G \curvearrowright Z$ be a uniformly quasi-Möbius action which is fixed point free, and acts cocompactly on triples in Z. As we have seen, Lemma 2.12 implies that there exist nonconstant thick paths in Z.

Lemma 4.1. *There exist disjoint open balls B and B' in Z such that the set of endpoints of thick paths connecting B and B' has a point of density in B.*

Proof. Let $\gamma \colon I \to Z$ be a nonconstant thick path. Pick $t \in (0,1)$ so that $x := \gamma(t)$ is distinct from the endpoints $x_0 := \gamma(0)$ and $x_1 := \gamma(1)$ of γ. Define $R := \frac{1}{10} \operatorname{dist}(x, \{x_0, x_1\})$, and let $\epsilon := R/10$. Set $B := B(x, R)$ and $B' := B(x_1, R)$.

Every path $\alpha \in B(\gamma, \epsilon)$ has an image intersecting the open ball B and picks up length in B which is comparable to R. In particular, each path in $B(\gamma, \epsilon)$ is nonconstant. Let

$$S := B \cap \{\mathrm{Im}(\alpha) : \alpha \in B(\gamma, \epsilon) \cap \mathcal{P}_T\}.$$

By Lemma 2.10, every point in S is the initial point of a thick path ending in B'. Hence it is enough to show that S has positive Q-dimensional Hausdorff measure. If this is not the case, we can find a Borel set $S' \supset S$ of vanishing Hausdorff Q-measure. Then the function $\rho \colon Z \to [0, \infty]$ defined to be infinity on S' and 0 elsewhere is Borel and an admissible test function for the path family $B(\gamma, \epsilon) \cap \mathcal{P}_T$. Since the total Q-mass of ρ is zero, Lemma 2.12 implies

$$\mathrm{Mod_Q}(B(\gamma, \epsilon) \setminus \mathcal{C}) = \mathrm{Mod_Q}(B(\gamma, \epsilon)) = \mathrm{Mod_Q}(B(\gamma, \epsilon) \cap \mathcal{P}_T) = 0,$$

which contradicts the thickness of γ. □

Lemma 4.2. *Let $M \subseteq Z \times Z$ be the set of pairs of points that can be joined by a thick path. Then M is dense in $Z \times Z$.*

Note that this implies in particular that Z is connected.

Proof. By Lemma 4.1, we can find a pair of disjoint open balls B and B' so that there exists a density point $x \in B$ of the set of initial points of the family Γ of thick paths starting in B and ending in B'. For $k \in \mathbb{N}$ pick $R_k > 0$ with $\lim_{k \to \infty} R_k = 0$, and let D_k be the set of initial points of paths in Γ which start in $B_k := B(x, R_k)$. Then

$$\epsilon_k := \frac{\mathrm{dist_H}(D_k, B_k)}{R_k} \to 0.$$

We let $\delta_k := \sqrt{\epsilon_k}$ and $\lambda_k := 2\sqrt{\epsilon_k}$. Set $x_k^1 := x$. Since Z is connected, for large k we can choose points $x_k^2 \in \partial B(x, \delta_k R_k)$ and $x_k^3 \in \partial B(x, 2\delta_k R_k)$. Using the cocompactness of the action $G \curvearrowright \mathrm{Tri}(Z)$, we can find $g_k \in G$ such that the image of the triple (x_k^1, x_k^2, x_k^3) under g_k is δ'-separated where $\delta' > 0$ is independent of k. Applying Lemma 2.1, we conclude that $\mathrm{dist_H}(g_k(D_k), Z) \to 0$ and $\mathrm{diam}(Z \setminus g_k(B_k)) \to 0$ as $k \to \infty$. After passing to a subsequence if necessary, we may assume that the sets $Z \setminus g_k(B_k)$ Hausdorff converge to $\{z\}$ for some $z \in Z$. Since B' and B_k are disjoint for large k, we then also have $\mathrm{dist_H}(g_k(B'), \{z\}) \to 0$ as $k \to \infty$.

Now let $z_1, z_2 \in Z$ and $\epsilon > 0$ be arbitrary. By Lemma 2.2, we can find $g \in G$ such that $g(z) \in B(z_1, \epsilon)$; then $g \circ g_k(B') \subseteq B(z_1, \epsilon)$ for large k. In addition, for large k we will also have $g \circ g_k(D_k) \cap B(z_2, \epsilon) \neq \emptyset$. Using this and the invariance of thickness under quasi-Möbius homeomorphisms we see that there is a thick path starting in $B(z_1, \epsilon)$ and ending in $B(z_2, \epsilon)$. □

Lemma 4.3. *For each $C > 0$ there are constants $m > 0$ and $L > 0$ such that if B, $B' \subseteq Z$ are R-balls with $\mathrm{dist}(B, B') \leq CR$, then the modulus of the family of paths of length at most LR joining B to B' has Q-modulus at least m.*

Proof. Suppose $C > 0$ is arbitrary. We first claim that there is a number $m_0 > 0$ such that if B, $B' \subseteq Z$ are R-balls with $\operatorname{dist}(B, B') \leq CR$, then $\mathrm{Mod_Q}(\Gamma(B, B')) > m_0$. If this assertion were false, there would be balls $B_k = B(z_k, R_k)$ and $B'_k = B(z'_k, R_k)$ for $k \in \mathbb{N}$ such that $\operatorname{dist}(B_k, B'_k) \leq CR_k$ for all k, but

$$\lim_{k\to\infty} \mathrm{Mod_Q}(\Gamma(B_k, B'_k)) = 0. \tag{4.4}$$

Passing to a subsequence, we may assume that the sequences (z_k) and (z'_k) converge to points $z \in Z$ and $z' \in Z$ respectively. Since Z is compact, we deduce from Lemma 4.2 that $\lim_{k\to\infty} R_k = 0$. So after passing to another subsequence we can choose $x_k^1 \in \partial B(z_k, R_k)$, and $x_k^2 \in \partial B(z_k, 2R_k)$ for all k. Choose $g_k \in G$ such that the triples $(g_k(z_k), g_k(x_k^1), g_k(x_k^2))$ are δ-separated where $\delta > 0$ is independent of k. Since the homeomorphisms g_k are uniformly quasi-Möbius it is easy to see that there is $\epsilon > 0$ such that $B(g_k(z_k), 2\epsilon) \subseteq g_k(B_k)$ and $B(g_k(z'_k), 2\epsilon) \subseteq g_k(B'_k)$ for all k. Hence for large k we have $\tilde{B} := B(z, \epsilon) \subseteq g_k(B_k)$, $\tilde{B}' := B(z', \epsilon) \subseteq g_k(B'_k)$. Tyson's theorem (Theorem 2.7) gives

$$\mathrm{Mod_Q}(\Gamma(B_k, B'_k)) \geq c\,\mathrm{Mod_Q}\left(\Gamma(g_k(B_k), g_k(B'_k))\right) \geq c\,\mathrm{Mod_Q}(\Gamma(\tilde{B}, \tilde{B}')),$$

where $c > 0$ is a constant independent of k. Since $\mathrm{Mod_Q}(\Gamma(\tilde{B}, \tilde{B}')) > 0$ by Lemma 4.2, this contradicts (4.4), and hence the claim is true.

According to Lemma 2.8 we can choose $L \geq 2$ such that every family of paths in Z which start in a given ball of radius R and have length at least LR has modulus at most $m_0/2$. Now B and B' are arbitrary balls of radius $R > 0$ in Z and let Γ_1 and Γ_2 be the families of paths in Z a which connect B and B' and have length at most LR and length at least LR, respectively. Then by the choice of L and by subadditivity of modulus we have

$$m_0 \leq \mathrm{Mod_Q}(\Gamma(B, B')) \leq \mathrm{Mod_Q}(\Gamma_1) + \mathrm{Mod_Q}(\Gamma_2) \leq \mathrm{Mod_Q}(\Gamma_1) + m_0/2.$$

It follows that $\mathrm{Mod_Q}(\Gamma_1) \geq m := m_0/2 > 0$. □

5. The proofs of the theorems

Proof of Theorem 1.3. By Lemma 4.3, (Z, d, μ) satisfies the hypotheses of Proposition 3.1, and is therefore a Q-Loewner space. □

Proof of Corollary 1.6. Under the hypotheses of the corollary, for every weak tangent W of Z there exist a point $z \in Z$ and a quasi-Möbius homeomorphism between W and $Z \setminus \{z\}$ (cf. [3, Lemma 5.2]). According to Theorem 1.5 there exists a weak tangent W of Z which carries a family of nonconstant paths with positive Q-modulus. Hence by Tyson's Theorem 2.7, the space Z also carries a family of nonconstant paths with positive Q-modulus. □

Proof of Theorem 1.1. Let G be as in the statement of Theorem 1.1, suppose Z is an Ahlfors Q-regular metric space where $Q \geq 2$ is the Ahlfors regular conformal dimension of $\partial_\infty G$, and $\phi\colon \partial_\infty G \to Z$ is a quasisymmetric homeomorphism. Conjugating the canonical action $G \curvearrowright \partial_\infty G$ by ϕ, we obtain a uniformly quasi-Möbius action $G \curvearrowright Z$ which is fixed point free and for which the induced action on triples is both properly discontinuous and cocompact. Now by Corollary 1.6 the space Z carries a family of nonconstant paths with positive Q-modulus, and so it is Q-Loewner by Theorem 1.3.

According to Theorem 1.2 of [2] every Ahlfors Q-regular and Q-Loewner 2-sphere is quasisymmetrically homeomorphic to the standard 2-sphere $\mathbb{S}^2$. This applies to Z and so there exists a quasisymmetric homeomorphism $\psi\colon Z \to \mathbb{S}^2$. Conjugating our action $G \curvearrowright Z$ by ψ, we get a uniformly quasiconformal action $G \overset{1}{\curvearrowright} \mathbb{S}^2$ (we use the superscript "1" to distinguish this action from another action discussed below). By a theorem of Sullivan and Tukia (cf. [21, p. 468] and [22, Theorem F and Remark F2]) every uniformly quasiconformal action on $\mathbb{S}^2$ is quasiconformally conjugate to an action by Möbius transformations. Hence $G \overset{1}{\curvearrowright} \mathbb{S}^2$ is quasiconformally conjugate to an action $G \overset{2}{\curvearrowright} \mathbb{S}^2$ by Möbius transformations. If we represent $\mathbb{H}^3$ by the unit ball model so that $\partial_\infty \mathbb{H}^3 = \mathbb{S}^2$, the action $G \overset{2}{\curvearrowright} \mathbb{S}^2$ naturally extends to an isometric action $G \curvearrowright \mathbb{H}^3$. Being topologically conjugate to $G \overset{1}{\curvearrowright} \mathbb{S}^2$ and hence to $G \curvearrowright Z$, the action $G \overset{2}{\curvearrowright} \mathbb{S}^2$ is also properly discontinuous and cocompact on triples. Therefore, the corresponding isometric action $G \curvearrowright \mathbb{H}^3$ is discrete and cocompact. $\square$

Proof of Theorem 1.4. Let G be group as in the theorem, and assume that $Q > 1$ is equal to the Ahlfors regular conformal dimension of $Z = \Lambda(G)$. Note that $Z \subseteq \mathbb{S}^n$ equipped with the ambient Euclidean metric on $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is Ahlfors Q-regular [20, Thm. 7], and that the induced action $G \curvearrowright Z$ satisfies the hypotheses of Theorem 1.3. It follows that Z is a Q-Loewner space. Hence Z satisfies a $(1, Q)$-Poincaré inequality.

Since the metric space $Z \subseteq \mathbb{R}^{n+1}$ is Ahlfors regular and satisfies a $(1, Q)$-Poincaré inequality, a theorem by Cheeger [9, Thm. 14.3] implies that Z has a weak tangent which is bi-Lipschitz equivalent to some Euclidean space $\mathbb{R}^k$, $k \geq 1$. Since each weak tangent of an Ahlfors Q-regular space is also Ahlfors Q-regular, we conclude that $Q = k \in \mathbb{N}$; in particular, the topological dimension of Z is equal to its Hausdorff dimension. Moreover, since $Q > 1$, we have $k \geq 2$. The desired conclusion now follows from [3, Thm. 1.2]. $\square$

The proof shows that if $G \curvearrowright X$ is a properly discontinuous, quasi-convex cocompact, and isometric action on a CAT(-1)-space X, if the limit set $\Lambda(G)$ has Hausdorff dimension equal to its Ahlfors regular conformal dimension, and if $\Lambda(G)$ embeds in some Euclidean space by a bi-Lipschitz map, then there is a convex G-invariant copy of a hyperbolic space $Y \subseteq X$ on which G acts cocompactly.

The conclusion of the quoted result by Cheeger already holds if Z satisfies a $(1,p)$-Poincaré inequality for some $p>1$. So it would be enough to stipulate this condition in Theorem 1.4 instead of requiring that the Ahlfors regular conformal dimension of $Z=\Lambda(G)$ is equal to Q.

The converse of Theorem 1.4 and its indicated modifications lead to a statement that is worth recording: if the limit set $\Lambda(G)$ of a group G as in the theorem is *not* a "round" subsphere of $\mathbb{S}^n$, then the Ahlfors regular dimension of $Z=\Lambda(G)$ is strictly less than Q, Z does not carry a family of nonconstant curves with positive Q-modulus, and Z does not satisfy a $(1,p)$-Poincaré inequality for any $p>1$. In particular, limits sets of such groups G cannot lead to new examples of Loewner spaces.

6. Spaces whose Ahlfors regular conformal dimension is not realized

In our discussion below, we will refer to the Ahlfors regular conformal dimension simply as the conformal dimension.

The most basic example of a space whose conformal dimension is not realized is the standard Cantor set C. This dimension is equal to 0 for C, but it is not attained, since any quasisymmetric homeomorphism between C and a metric space Z is bi-Hölder [5], and so the Hausdorff dimension of Z is strictly positive.

To our knowledge, the first connected and locally connected example of this type is due to Pansu, which we learned of through M. Bourdon. Essentially the same example was considered also in [8]: if one glues together two closed hyperbolic surfaces N_1 and N_2 isometrically along embedded geodesics $\gamma_i \subseteq N_i$ of equal length, then one obtains a 2-complex M with curvature bounded from above by -1 and the boundary at infinity $\partial_\infty \widetilde{M}$ of the universal cover $\widetilde{M}$ has conformal dimension 1. To see this one pinches the hyperbolic structures along the closed geodesics γ_i, and observes that the volume entropy of the resulting universal covers tend to 1 ("branching becomes less and less frequent"). The space $\partial_\infty \widetilde{M}$ is not quasisymmetrically homeomorphic to an Ahlfors 1-regular space, because in this case it would have to be a topological circle by [3, Theorem 1.1]; in fact it is not difficult to see directly that $\partial_\infty \widetilde{M}$ is not quasisymmetrically homeomorphic to a space with finite 1-dimensional Hausdorff measure.

Bishop and Tyson [1] have shown that "antenna sets"—certain self-similar dendrites in the plane—have conformal dimension 1, but are not quasisymmetrically homeomorphic to any space of Hausdorff dimension 1.

Another example of a similar flavor is due to Laakso. He has shown that the standard Sierpinski gasket has conformal dimension 1, but again, this dimension cannot be realized. By considering pairs of points whose removal disconnects the set, it is not hard to show that the homeomorphism group of the gasket is the same as its isometry group for the usual embedding in $\mathbb{R}^2$. It follows that this example is not homeomorphic to Pansu's example.

There are translation invariant Ahlfors regular metrics on $\mathbb{R}^2$ for which the 1-parameter group of linear transformations e^{tA}, where

$$A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

is a family of homotheties. Their conformal dimension is 2, but it cannot be realized. The second author would like to thank L. Mosher for bringing these examples to his attention. One can also describe them as follows. Let G be the semi-direct product of $\mathbb{R}$ with $\mathbb{R}^2$, where $\mathbb{R}$ acts on $\mathbb{R}^2$ by the 1-parameter group above. Then the solvable Lie group G admits left invariant Riemannian metrics with curvature pinched arbitrarily close to -1; if one removes the unique fixed point from $\partial_\infty G$, one gets the "twisted plane" example above.

The examples discussed so far are all either disconnected, have local cut points (i.e. by removing a single point one can disconnect a connected neighborhood), or cannot be the boundary of a hyperbolic group.

Suppose an Ahlfors Q-regular space Z is quasisymmetric to the boundary of a hyperbolic group G, where Q is the conformal dimension of Z. If $Q < 1$, then the topological dimension of Z is zero; thus there is a free subgroup F_k sitting in G with finite index. But then $k = 1$, $|Z| = 2$, and $Q = 0$, for otherwise $k > 1$ which implies that Z is quasisymmetric to the standard Cantor set, whose conformal dimension is not realized. If $Q = 1$, then [3, Theorem 1.1] implies that Z is quasisymmetric to the standard circle. The case $Q > 1$ is covered by Theorem 1.3. Disconnected spaces, or spaces with local cut points cannot satisfy a Poincaré inequality, so Theorem 1.3 implies that if $Q > 1$, then Z is connected and has no local cut points. The examples of Bourdon and Pajot [7] give boundaries of hyperbolic groups which possess these two topological properties, but which are not quasisymmetrically homeomorphic to a Q-regular space satisfying a $(1, Q)$-Poincaré inequality. Thus by Theorem 1.3 even under these topological conditions the conformal dimension is not necessarily realized.

Based on the examples mentioned above, one may speculate that if the conformal dimension of a self-similar space fails to be attained, then this is due to degeneration which leads to a limiting structure resembling a foliation or lamination.

We conclude this section with two questions related to the realization problem.

Problem 6.1. *Can one algebraically characterize the hyperbolic groups whose boundary has (Ahlfors regular) conformal dimension equal to* 1*? In particular, if the boundary of a hyperbolic group is homeomorphic to a Sierpinski carpet or a Menger curve, is the Ahlfors regular conformal dimension strictly greater than* 1*?*

Problem 6.2. *Is the (Ahlfors regular) conformal dimension of the standard square Sierpinski carpet S attained?*

If it is, it seems to be the case that S equipped with any Ahlfors regular metric realizing its conformal dimension is a Loewner space. We remark that it follows from [14] that the conformal dimension of S is less than its Hausdorff dimension. A calculation by the second author had earlier given an explicit upper bound for the conformal dimension of S.

7. Remarks and open problems

The themes explored in this paper lead to various general questions. To further exploit the relation between the algebraic structure of a Gromov hyperbolic group and the analysis of its boundary one needs analytic tools from the general theory of analysis on metric spaces, perhaps tailored to the setting of self-similar spaces or spaces admitting group actions as considered in this paper. In particular, it would be interesting to find classes of function spaces that are invariant under quasisymmetric homeomorphisms. They could be used to define quasisymmetric invariants and answer structure and rigidity questions for quasisymmetric homeomorphisms.

The setting of Loewner spaces is relatively well-understood, but it is not clear how natural this framework really is. At present there is a somewhat limited supply of these spaces, and one would like to have more examples. As Theorem 1.4 indicates, the Loewner condition seems to lead to strong conclusions in the presence of group actions and probably also in the presence of self-similarity. In view of this theorem the following problem suggests itself.

Problem 7.1. *Can one classify all quasi-convex cocompact isometric actions $G \curvearrowright X$, where X is a* $\mathrm{CAT}(-1)$*-space and the Ahlfors regular conformal dimension of the limit set $\Lambda(G)$ is realized and strictly greater than* 1*?*

Note that in this situation $Z = \Lambda(G)$ is a Loewner space, so the problem asks for a classification of all Loewner spaces that arise as limit sets of quasi-convex cocompact isometric group actions on $\mathrm{CAT}(-1)$-spaces.

Conversely, one could start with an Ahlfors Q-regular Q-Loewner Z space quasisymmetrically homeomorphic to the boundary of a Gromov hyperbolic group G and ask whether Z appears as the limit set $\Lambda(G)$ of some isometric action $G \curvearrowright X$, where X is a negatively curved metric space. It is natural to require that X is Gromov hyperbolic. One can interpret the relation between Z and $\Lambda(G)$ in a measure theoretic sense. The obvious measure on Z is Hausdorff Q-measure, and the measure on $\Lambda(G)$ related to the dynamics $G \curvearrowright \Lambda(G)$ is the so-called Patterson-Sullivan measure (cf. [10]). We arrive at the following question:

Problem 7.2. *Suppose $\phi\colon Z \to \partial_\infty G$ is a quasisymmetric homeomorphism from a compact Ahlfors regular Loewner space Z to the boundary $\partial_\infty G$ of Gromov hyperbolic group G. Is there a discrete, cocompact, isometric action $G \curvearrowright X$ of G on a Gromov* *hyperbolic space X whose Patterson-Sullivan measure lies the same measure class as push-forward of Hausdorff measure under ϕ?*

More generally, one may ask when the measure class of a given measure on the boundary of a Gromov hyperbolic group is represented by the Patterson-Sullivan measure for some Gromov hyperbolic "filling" $G \curvearrowright X$ of the boundary action $G \curvearrowright \partial_\infty G$.

The general problem behind Cannon's conjecture is the desire to find canonical metric spaces on which a given Gromov hyperbolic group G acts. Since the dynamics of an isometric action G on a Gromov hyperbolic space X is encoded in the Patterson-Sullivan measure on $\Lambda(G)$, a first step in this direction is to find a natural measure, or at least a natural measure class on $\partial_\infty G$.

Problem 7.3. *Given a Gromov hyperbolic group G, when is there a natural measure class on $\partial_\infty G$?*

Here "natural" can be interpreted in various ways. One could require the measure class to be invariant under all (local) quasisymmetric homeomorphisms. For instance, if G acts discretely cocompactly on a rank 1 symmetric space X space other than $\mathbb{H}^2$, then the measure class of the Lebesgue measure associated with the standard smooth structure on $\partial_\infty X$ is invariant under the full group of quasisymmetric self-homeomorphisms of $\partial_\infty X \simeq \partial_\infty G$. When $X = \mathbb{H}^2$ this fails, since the "Mostow map" $\partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^2$ induced between two non-conjugate discrete, cocompact and isometric actions $G \overset{1}{\curvearrowright} \mathbb{H}^2$ and $G \overset{2}{\curvearrowright} \mathbb{H}^2$ will not be absolutely continuous with respect to Lebesgue measure. One expects a similar phenomenon whenever G virtually splits over a virtually cyclic group, or equivalently, when $\partial_\infty G$ has local cut points. Due to this, one can hope for an affirmative answer to Problem 7.3 only under the assumption that G does not have this property.

In many cases one expects that G is a subgroup of finite index in the group $\mathrm{QS}(\partial_\infty G)$ of all quasisymmetric self-homeomorphisms of $\partial_\infty G$. Then the requirement that the measure class be invariant under $\mathrm{QS}(\partial_\infty G)$ is rather weak. This suggests another (stronger) interpretation of Problem 7.3: the measure class should be constructed in a quasisymmetrically invariant way.

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