COMBINATORIAL MODULUS, THE COMBINATORIAL LOEWNER PROPERTY, AND COXETER GROUPS

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Abstract. We study combinatorial modulus on boundaries of hyperbolic Coxeter groups. We give new examples of hyperbolic groups whose boundary satisfies a combinatorial version of the Loewner property, and prove Cannon's conjecture for Coxeter groups. We also establish some connections with ℓ_p-cohomology.

Contents

1. Introduction

1.1. **Overview.** Every hyperbolic group Γ has a canonical action on its boundary at infinity $\partial\Gamma$; with respect to any visual metric on $\partial\Gamma$, this action is by uniformly quasi-Moebius homeomorphisms. This structure has a central role in the proofs of Mostow's rigidity theorem and numerous other results in the same vein, which are based on the analytic theory of quasiconformal homeomorphisms of the boundary (see the survey papers [GP91], [BP02], [Kle06] and their references). With the aim of extending these rigidity results to a larger class of hyperbolic groups, one may hope to apply the work of J. Heinonen and P. Koskela [HK98] and subsequent authors (e.g. [Che99, Tys98, HKST01, KZ08]), which has generalized much of the classical quasiconformal theory to the setting of Loewner spaces[1], a certain class of metric measure spaces [HK98]. Unfortunately, among the currently known examples of Loewner spaces, the only ones which arise as boundaries of hyperbolic groups are the boundaries of rank one symmetric spaces, and Fuchsian buildings [HK98, BP00]. One of the goals of this paper is to take a step toward improving this situation, by finding infinitely many new examples of hyperbolic groups whose boundaries satisfy the *Combinatorial Loewner Property*, a combinatorial variant of the Loewner property which is conjecturally equivalent to the property of being quasi-Moebius homeomorphic to a Loewner space. In addition to this, using similar techniques, we prove the Cannon conjecture for hyperbolic Coxeter groups, show that the ℓ_p equivalence relation studied in [Gro93, Ele97, Bou04] has a particularly simple form in the case of hyperbolic Coxeter groups, and prove that the standard square Sierpinski carpet in the plane and the standard cubical Menger curve in $\mathbb{R}^3$ satisfy the Combinatorial Loewner Property.

1.2. **Statement of results.** We now present some of the ideas of the paper, illustrating them with non-technical statements. More general results, as well as detailed discussion justifying the statements made here, may be found in the body of the paper.

Combinatorial modulus. Let Z be a compact metric space. For every $k \in \mathbb{N}$, let G_k be the incidence graph of a ball cover $\{B(x_i, 2^{-k})\}_{i \in I}$, where $\{x_i\}_{i \in I} \subset Z$ is a maximal 2^{-k}-separated subset. Given $p \geq 1$ and a curve family $\mathcal{F}$ in Z, we denote by $\mathrm{Mod}_p(\mathcal{F}, G_k)$ the G_k-combinatorial p-modulus of $\mathcal{F}$ (see Subsection 3.1 for the definition); also for any pair of subsets $A, B \subset Z$, we let $\mathrm{Mod}_p(A, B, G_k) = \mathrm{Mod}_p(\mathcal{F}, G_k)$ where $\mathcal{F}$ is the collection of paths joining A and B.

In our study of combinatorial modulus, we will assume that Z is *approximately self-similar* (see Definition 3.11). Examples of approximately self-similar metric spaces include many classical fractals such as the the square Sierpinski carpet or the cubical Menger sponge, boundaries of hyperbolic groups equipped with their visual metrics, metric

[1] We will use the shorthand *Loewner space* for a compact metric space which is Ahlfors p-regular and p-Loewner for some $p > 1$, in the sense of [HK98].

spaces associated with finite subdivision rules, and metric 2-spheres arising from expanding Thurston maps [BM].

One of our principal goals is to find criteria for the *Combinatorial Loewner Property* (CLP). Roughly speaking, a doubling space Z satisfies the CLP if there is a $p \in (1, \infty)$ such that for any pair $A, B \subset Z$ of disjoint nontrivial continua, the p-modulus $\mathrm{Mod}_p(A, B, G_k)$ is controlled by the relative distance

$$\Delta(A, B) = \frac{\mathrm{dist}(A, B)}{\min\{\mathrm{diam}\, A, \mathrm{diam}\, B\}} \ ;$$

see Section 3.2 for the definition. As indicated in the overview, our interest in the CLP stems the fact that a Loewner space satisfies the CLP, and that the converse is conjecturally true for compact approximately self-similar spaces. Thus – modulo the conjecture – this paper would provide new examples of hyperbolic groups to which the recently developed quasiconformal theory would be applicable. Should the conjecture turn out to be false, the CLP would be of independent interest, since it shares many of the features of the Loewner property, *e.g.* quasi-Moebius invariance; see Section 3.2 for more discussion.

Coxeter group boundaries satisfying the CLP. Recall that a group Γ is a *Coxeter group* if it admits a presentation of the form

$$\Gamma = \langle s_i \in S \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \text{ for } i \neq j \rangle,$$

with $|S| < +\infty$, and with $m_{ij} \in \{2, 3, ..., +\infty\}$. A subgroup is *special* if it is generated by a subset of the generating set S. A subgroup is *parabolic* if it is conjugate to a special subgroup. Now suppose in addition that Γ is hyperbolic, and $\partial\Gamma$ is its boundary at infinity. Then a non-empty limit set $\partial P \subset \partial\Gamma$ of a parabolic subgroup $P \leqq \Gamma$ is called a *parabolic limit set*.

In Theorem 6.6 we give a sufficient condition (of combinatorial flavour) for the boundary of a hyperbolic Coxeter group to satisfy the CLP. A special case is the following result, which shows that the CLP holds when the parabolic limit sets form a combinatorially simple collection of subsets.

Theorem 1.1 (Corollary 6.7)**.** *Let Γ be a hyperbolic Coxeter group whose boundary is connected and such that* $\mathrm{Confdim}(\partial\Gamma) > 1$. *Assume that for every proper, connected, parabolic limit set $\partial P \subset \partial\Gamma$, one has*

$$\mathrm{Confdim}(\partial P) < \mathrm{Confdim}(\partial\Gamma).$$

Suppose furthermore that for every pair ∂P, ∂Q of distinct, proper, connected, parabolic limit sets, the subset $\partial P \cap \partial Q$ is at most totally disconnected. Then $\partial\Gamma$ satisfies the CLP.

In the statement $\mathrm{Confdim}(Z)$ denotes the Ahlfors regular conformal dimension of Z *i.e.* the infimum of Hausdorff dimensions of Ahlfors regular metric spaces quasi-Moebius homeomorphic to Z (see [MT] for a survey paper on the conformal dimension).

To illustrate Theorem 1.1 with some simple examples, consider a Coxeter group Γ with a Coxeter presentation

$$\langle s_1, \ldots, s_4 \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \text{ for } i \neq j \rangle \,,$$

where the order m_{ij} is finite for all $i \neq j$, and for every $j \in \{1, ..., 4\}$ one has $\sum_{i \neq j} \frac{1}{m_{ij}} < 1$. For these examples, the proper connected parabolic limit sets are circles, and hence have conformal dimension 1, while $\partial\Gamma$ is homeomorphic to the Sierpinski carpet and therefore has conformal dimension > 1 by a result of J. Mackay [Mac08]. Theorem 1.1 therefore applies, and $\partial\Gamma$ has the CLP.

Applying similar techniques in a simplified setting, we prove:

Theorem 1.2 (Theorem 8.4)**.** *The square Sierpinski carpet and the cubical Menger sponge satisfy the CLP.*

The Cannon conjecture for Coxeter groups. We obtain a proof of Cannon's conjecture in the special case of Coxeter groups:

Theorem 1.3 (Theorem 5.1)**.** *Let Γ be a hyperbolic Coxeter group whose boundary is homeomorphic to the 2-sphere. Then there is a properly discontinuous, cocompact, and isometric action of Γ on $\mathbb{H}^3$.*

This result was essentially known (see the discussion at the end of Section 5). Our view is that the principal value of the proof is that it illustrates the feasibility of the asymptotic approach (using the ideal boundary and modulus estimates), and it may suggest ideas for attacking the general case. It also gives a new proof of the Andreev's theorem on realizability of polyhedra in $\mathbb{H}^3$, in the case when the prescribed dihedral angles are submultiples of π.

ℓ_p-equivalence relations. Let Γ be a hyperbolic group and let $p \geq 1$. The first ℓ_p-cohomology group of Γ induces on $\partial\Gamma$ an equivalence relation – the *ℓ_p-equivalence relation* – which is invariant under quasi-isometries of Γ [Gro93, Ele97, BP03, Bou04]. A natural problem is to determine its cosets. The existence of a non trivial coset, *i.e.* a coset different from a point and from the whole space, was shown to be an obstruction to the Loewner property in [BP03]. Inspired by this, we prove the analogous statement for the CLP in Corollary 9.5. Moreover in consequence of some of our previous results we get:

Corollary 1.4 (Corollaries 9.3 and 9.4)**.** *Assume Γ is a hyperbolic Coxeter group, let $p \geq 1$, and denote by $\sim_p$ the ℓ_p-equivalence relation on $\partial\Gamma$. Then:*

(1) *Each coset of $\sim_p$ is either a point or a connected parabolic limit set.*
(2) *If $\partial\Gamma$ is connected, and $\sim_p$ admits a coset different from a point and the whole $\partial\Gamma$, then $\sim_p$ admits a coset F with* $\mathrm{Confdim}(F) = \mathrm{Confdim}(\partial\Gamma)$.

Beyond the CLP. At present, our understanding of Coxeter groups is still quite limited. We have only been able to show that certain very special groups have boundaries which satisfy the CLP. While this is consistent with our expectation that the CLP should be a highly non-generic property, we only have a few examples which are known not to have the CLP, apart from groups whose boundaries could not have the CLP for topological reasons, see Proposition 3.5 and the remark at the end of Section 9.

It would be desirable to have effective criteria for showing that a group boundary does not have the CLP, as well as new examples of such groups. In addition, when a group boundary does not have the CLP, we expect that alternate structure will be present instead, such as a quasi-Moebius invariant equivalence relation.

1.3. **Discussion of the proofs.** We now give an indication of the ideas that go into some of the proofs.

Combinatorial modulus on approximately self-similar spaces. Let Z be a compact, approximately self-similar space. For $d_0 > 0$, let $\mathcal{F}_0$ be the family of curves $\gamma \subset Z$ with $\operatorname{diam}(\gamma) \geq d_0$. The approximate self-similarity of Z allows one to compare moduli of curve families at different locations and scales with the modulus of $\mathcal{F}_0$, and this observation leads readily to a submultiplicativity relation between combinatorial moduli at different scales:

Proposition 1.5 (Proposition 3.15). *Let Z be an arcwise connected approximately self-similar metric space. Let $p \geq 1$ and set $M_k := \operatorname{Mod}_p(\mathcal{F}_0, G_k)$. Then, for d_0 sufficiently small, there exists a constant $C > 0$ such that for every pair of integers k, ℓ one has :*

$$M_{k+\ell} \leq C \cdot M_k \cdot M_\ell \,. \tag{1.1}$$

In addition when p belongs to a compact subset of $[1, +\infty)$ the constant C may be chosen independent of p.

Iterating (1.1) yields $M_{nk} \leq C^{n-1} M_k^n$, which implies that $\lim_{j\to\infty} M_j = 0$ if $M_k < C^{-1}$ for any k. Therefore if we define the critical exponent to be

$$Q_M = \inf\{p \in [1, \infty) \; ; \; \lim_{k\to\infty} M_k = 0\}$$

then $M_k = \operatorname{Mod}_{Q_M}(\mathcal{F}_0, G_k) \geq C^{-1}$ for all k. In fact, Q_M is the Ahlfors regular conformal dimension of Z, [KK].

When Z is the standard Sierpinski carpet constructed from the unit square, one can exploit the reflectional symmetry to get additional control on the modulus. Using it, one shows that for any nonconstant curve $\eta : [0, 1] \to Z$ and any $\epsilon > 0$, if $\mathcal{U}_\epsilon(\eta)$ denotes the ϵ neighborhood of η in the C^0 topology, then the p-modulus $\operatorname{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k)$ is uniformly comparable to M_k, independent of k. In other words, the modulus of the curves near an arbitrary curve is comparable to the modulus of all curves. From this, and using the planarity of the carpet, one can prove a supermultiplicativity inequality as well:

$$M_{k+\ell} \geq C' \cdot M_k \cdot M_\ell \,, \tag{1.2}$$

where $C' \in (0, \infty)$ may be chosen in terms of an upper bound on p. Reasoning as above, it follows that if $M_k > C'^{-1}$ for any k, then $\lim_{k\to\infty} M_k = \infty$; this implies that at the critical exponent Q_M, the sequence $\{M_k\}$ is bounded away from zero and infinity. From this, one concludes that the statement of the CLP holds for pairs of balls. By imitating an argument from [BK05a], one shows that the CLP holds provided it holds for pairs of balls, and therefore the Sierpinski carpet satisfies the CLP (see Theorem 8.4).

In the case of the Menger curve, instead of (1.2), one obtains an estimate for M_k in terms of the moduli $\{M_j\}_{j<k}$, which is sufficient to verify the CLP.

Dynamics of curves and crossing. Our strategy for understanding the boundary of a hyperbolic Coxeter group Γ is inspired by the analysis of the Sierpinski carpet, although the story is more complicated.

For the purpose of this paper, much of the dynamics of the Γ-action on $\partial\Gamma$ is encoded in the parabolic limit sets. For example we get:

Theorem 1.6 (Corollary 2.6)**.** *Consider a Γ-invariant equivalence relation on $\partial\Gamma$ whose cosets are connected. Then:*

- *The closure of each coset is either a point or a parabolic limit set.*
- *If a nontrivial coset F is path-connected, and P is the parabolic subgroup with $\overline{F} = \partial P$, then for every $\epsilon > 0$ and every path $\eta : [0,1] \to \partial P$, there is a path $\eta' : [0,1] \to F$ such that*
$$d(\eta, \eta') = \max_{t\in[0,1]} d(\eta(t), \eta'(t)) < \epsilon\,.$$

As an illustration, let γ be a nontrivial curve in $\partial\Gamma$, and let $\sim$ be the smallest equivalence relation on $\partial\Gamma$ such that for every $g \in \Gamma$, the curve $g\gamma$ lies in a single coset; in other words, two points $x, y \in \partial\Gamma$ lie in the same coset if there is a finite chain $g_1\gamma, \ldots, g_k\gamma$ joining x to y. Thanks to the previous theorem, the coset closures are either points or parabolic limit sets. In particular, if γ is not contained in any proper parabolic limit set, then any path $\eta : [0,1] \to \partial\Gamma$ is a uniform limit of paths lying in the coset of γ.

A key ingredient in the analysis of the combinatorial modulus on $\partial\Gamma$ is a quantitative version of this phenomenon, which is established in Proposition 2.10.

The proof of the Cannon conjecture for Coxeter groups. By [Sul81], if Γ is a hyperbolic group and $\partial\Gamma$ is quasi-Moebius homeomorphic to the Euclidean 2-sphere, then Γ admits a properly discontinuous, cocompact, isometric action on $\mathbb{H}^3$. Also, as a consequence of the uniformization theorem established in [BK02], we obtain:

Corollary 1.7 (Corollary 3.14)**.** *Suppose Z is an approximately self-similar metric space homeomorphic to the 2-sphere. Assume that for $d_0 > 0$ small enough, there exists a constant $C = C(d_0) \geq 1$ such that for every $k \in \mathbb{N}$ one has*

$$\mathrm{Mod}_2(\mathcal{F}_0, G_k) \leq C. \tag{1.3}$$

Then Z is quasi-Moebius homeomorphic to the Euclidean 2-sphere.

Thus, we are reduced to verifying the hypotheses of the above corollary when Γ is a Coxeter group. We note that an alternate reduction to the same assertion can be deduced using [CFP99].

One of the main results of the paper is the existence of a finite number of "elementary curves families", whose moduli govern the modulus of every (thick enough) curve family in $\partial\Gamma$ (see Theorem 4.1 and Corollary 4.2). Each elementary curve family is associated to a conjugacy class of an infinite parabolic subgroup.

In consequence, to obtain the bound (1.3), it is enough to establish that every connected parabolic limit set ∂P enjoys the following property: there exists a non constant continuous curve $\eta \subset \partial P$, such that letting $\mathcal{U}_\epsilon(\eta)$ be the ϵ-neighborhood of η in the C^0 topology, the modulus $\mathrm{Mod}_2(\mathcal{U}_\epsilon(\eta), G_k)$ is bounded independently of k, for $\epsilon > 0$ small enough.

To do so, two cases are distinguished: either ∂P is a circular limit set *i.e.* it is homeomorphic to the circle, or it is not.

In the second case one can find two crossing curves $\eta_1, \eta_2 \subset \partial P$. Since $\partial\Gamma$ is a planar set, one gets that $\min_{i=1,2} \mathrm{Mod}_2(\mathcal{U}_\epsilon(\eta_i), G_k)$ is bounded independently of k, for ϵ small enough. Note that crossing type arguments in relation with the combinatorial 2-modulus, appear frequently in the papers [Can94], [CS98] and [CFP99] (not to mention the whole body of literature on 2-dimensional quasiconformal geometry).

Let $r > 0$, and denote by $\mathcal{F}_1$ the subfamily of $\mathcal{F}_0$ consisting of the curves $\gamma \in \mathcal{F}_0$ which do not belong to the r-neighborhood $N_r(\partial P)$ of any circular limit set ∂P. At this stage one knows that for r small enough, $\mathrm{Mod}_2(\mathcal{F}_1, G_k)$ is bounded independently of k. To bound the modulus of $\mathcal{F}_0 \setminus \mathcal{F}_1$, we proceed as follows. Consider a curve $\gamma \in \mathcal{F}_0$ contained in $N_r(\partial P)$, where ∂P is a circular parabolic limit set. The idea is to break γ into pieces $\gamma_1, \dots, \gamma_i$, such that for each $j \in \{1, \dots, i\}$, the maximal distance $\max\{d(x, \partial P) \,;\, x \in \gamma_j\}$ is comparable to $\mathrm{diam}(\gamma_j)$. Then for each j, applying a suitable group element $g \in \Gamma$, we can arrange that both $g\gamma_j$ and $g\partial P$ have roughly unit diameter.

Since $g\,\gamma_j$ lies close to $g\,\partial P$, but not too close, it cannot lie very close to a circular limit set; it follows that $g\gamma_j$ belongs to a curve family with controlled modulus. We then apply g^{-1} to the corresponding admissible function, and renormalize it suitably; by summing the collection of functions which arise in this fashion from all such configurations, we arrive at an admissible function for all such curves γ. The fact that the conformal dimension of S^1 is < 2 allows us to bound the 2-mass of this admissible function, and this yields the desired bound (1.3).

We note that in the body of the paper, the argument in the preceding paragraph appears in Theorem 4.3, where it is formulated in greater generality. It is also used in the proofs of Theorem 6.6 and Corollary 9.4.

1.4. **Organization of the paper.** Section 2 contains preliminary results about various dynamical aspects of the action of a hyperbolic Coxeter group on its boundary. Combinatorial modulus, combinatorial Loewner property, and approximately self-similar metric spaces are presented in Section 3. Section 4 is the heart of the paper, it focusses on the combinatorial modulus on boundaries of hyperbolic Coxeter groups. Section 5 discusses a proof of the Cannon's conjecture in the Coxeter group case. An application to Coxeter groups with Sierpinski carpet boundary is given. Section 6 establishes a sufficient condition for a Coxeter boundary to satisfy the combinatorial Loewner property. Examples are presented in Section 7. In Section 8 the combinatorial Loewner property is established for the square Sierpinski carpet and the cubical Menger sponge. Section 9 discusses applications to the ℓ_p-equivalence relations.

1.5. **Suggestions to the reader.** Readers who are concerned only with the Coxeter group case of the Cannon's conjecture may read Section 2, Subsection 3.1, Subsection 3.3 until Corollary 3.14, and Sections 4 and 5.

For those who want to quickly understand the combinatorial Loewner property for the square Sierpinski carpet and the cubical Menger sponge cases, we recommand the following abbreviated itinerary : start with Section 3, then go directly to Section 8 and follow the indications there to find the required arguments in the rest of the paper.

Acknowledgements. We would like to thank Mario Bonk and Frederic Haglund for several helpful discussions on the subject of this article. Special thanks to Peter Haïssinsky for his interest and for several comments on a first version of the paper. M. B. was partially supported by ANR grant "Cannon", and B. K. by NSF grant DMS-0701515.

Notation and conventions. Any curve $\gamma : [a, b] \to X$ is assumed to be continuous. Often we do not distinguish between γ and its image $\gamma([a, b])$. Two real valued functions f, g defined on a space X are said to be *comparable*, and then we write $f \asymp g$, if there exists a constant $C > 0$ such that $C^{-1}f \leq g \leq Cf$. A *continuum* is a non-empty compact connected topological space, it is *non-degenerate* if it contains more than one point. For an open ball $B = B(x, r)$ in a metric space and for $\lambda > 0$, we denote by λB the ball $B(x, \lambda r)$. The radius of a ball B is denoted by $r(B)$. The open r-neighborhood of a subset E is denoted by $N_r(E)$. A subset E of a set F is called a *proper subset of* F if $E \subsetneqq F$.

2. Hyperbolic Coxeter groups

This section exhibits some specific dynamical properties of the action of a (word) hyperbolic Coxeter group on its boundary.

2.1. **Invariant subsets and parabolic subgroups.** We start by recalling some standard definitions, see [Dav08] for more details. Let (Γ, S) be a Coxeter system. A *special subgroup* of Γ is a subgroup generated by a non-empty subset I of S, we shall denote it by Γ_I. A *parabolic subgroup* of Γ is a conjugate of a special subgroup. Let G be the Cayley graph of (Γ, S). We define G^0 and G^1 to be the set of vertices and of (non-oriented) open edges respectively. Each edge carries a *type* which is an element of S. The distance between two vertices x, y of G is denoted by $|x - y|$. For $s \in S$ the *wall associated to* s is the subset $M_s \subset G^1$ of s-invariant (open) edges. The graph $G \setminus M_s$ consists of two disjoint convex closed subsets of G, denoted by $H_-(M_s)$ and $H_+(M_s)$, and called the *half-spaces bounded by* M_s. They satisfy the relations

$$H_-(M_s)^0 = \{g \in \Gamma \, ; \, |sg| = |g|+1\}, \;\; H_+(M_s)^0 = \{g \in \Gamma \, ; \, |sg| = |g|-1\},$$

and s permutes $H_-(M_s)$ and $H_+(M_s)$. A *wall* of G is a subset of the form $g(M_s)$, with $g \in \Gamma$ and $s \in S$. The involution gsg^{-1} is called the *reflection along the wall* $g(M_s)$. The set of walls forms a partition of G^1. Each wall M divides G in two disjoint convex closed subsets, called the *half-spaces* bounded by M and denoted by $H_-(M)$ and $H_+(M)$, with the convention that $e \in H_-(M)$, where e denotes the identity of Γ.

Assume now that Γ is a hyperbolic Coxeter group (see *e.g.* [Gro87], [BH99] and their references for hyperbolic groups and related topics). We denote by $\partial\Gamma$ its boundary at infinity equipped with a visual metric. For a subset E of G we denote by ∂E its limit set in $\partial\Gamma$. A non-empty limit set of a parabolic subgroup of Γ will be called a *parabolic limit set*. If in addition it is a topological circle we shall call it a *circular limit set*.

The limit set of a wall M is either of empty interior or equal to $\partial\Gamma$. Indeed this property is well-known for limit sets of subgroups, and the stabilizer of M in Γ acts cocompactly on M. Using the convexity of the half-spaces one easily sees that

$$\partial H_-(M) \cup \partial H_+(M) = \partial\Gamma, \quad \partial H_-(M) \cap \partial H_+(M) = \partial M.$$

In consequence, in $\partial\Gamma$, the fixed point set of a reflection is the limit set of its wall.

The following property asserts in particular that the limit sets of half-spaces form a basis of neighborhoods in $\partial\Gamma$.

Proposition 2.1. *There exists a constant $\lambda \geq 1$ such that for every $z \in \partial\Gamma$ and every $0 < r \leq \operatorname{diam}(\partial\Gamma)$ there exists a half-space H of G with*

$$B(z, \lambda^{-1}r) \subset \partial H \subset B(z, \lambda r).$$

Proof. Let δ_G be the triangle fineness constant of G. Using the convexity of the half-spaces and the visual metric one sees that it is enough to establish the existence of a constant $L > 0$ with the following property. Let x, x_1, y_1, y be arbitrary 4 points of G lying in this order on a geodesic line. If $|x_1 - y_1| \geq L$ then there exists a wall M passing between x_1 and y_1 such that

$$\operatorname{dist}(x, M) \geq |x - x_1| - 2\delta_G \text{ and } \operatorname{dist}(y, M) \geq |y - y_1| - 2\delta_G.$$

In order to prove this property observe that different edges of a geodesic segment give rise to different walls. Thus there are as many walls passing between x_1 and y_1 as many edges in $[x_1, y_1]$. Let (a, b) be such an edge and let M be the corresponding wall. Let $p \in G^0$ be a vertex adjacent to M realizing $\operatorname{dist}(x, M)$. Assume that $|x-p| < |x-x_1|-2\delta_G$.

Then by Gromov hyperbolicity one has $\operatorname{dist}(p, [x, x_1]) \leq 2\delta_G$. Applying Gromov hyperbolicity once again we obtain that

$$\operatorname{dist}([p, a], x_1) \leq 3\delta_G.$$

Observe that the geodesic segment $[p, a]$ lies in a δ_G-neighborhood of M. Indeed the Hausdorff distance between $[p, a]$ and its image by the reflection along M is smaller than δ_G. Therefore the distance between M and x_1 is smaller than $4\delta_G$. A similar argument applies to y and y_1. Let N be the number of walls $M \subset G$ such that $\operatorname{dist}(e, M) \leq 4\delta_G$. The property follows letting $L = 2N + 1$. $\square$

We study now the parabolic subgroups of Γ in connection with the action of Γ on its boundary.

Definition 2.2. Let M be a wall of G and let F be a subset of $\partial\Gamma$. We say that M *cuts* F if F meets both subsets $\partial H_-(M)$ and $\partial H_+(M)$.

Theorem 2.3. *Let F be a subset of $\partial\Gamma$ containing at least two distinct points. Assume that F is invariant under each reflection whose wall cuts F, and let P be the subgroup of Γ generated by these reflections. Then P is a parabolic subgroup of Γ, the closure of F is the limit set of P, and P is the stabilizer of F in Γ.*

To prove the theorem we introduce the following notion which will also be useful in the sequel:

Definition 2.4. Let F be a subset of $\partial\Gamma$. Assume it contains at least two distinct points. The *convex hull* $\mathcal{C}_F$ of F is the intersection of all the half-spaces H in G such that F belongs to the interior of ∂H in $\partial\Gamma$. It is a convex subgraph of G.

Lemma 2.5. *The limit set $\partial\mathcal{C}_F$ is the closure of F. In addition every wall of G which intersects $\mathcal{C}_F$ cuts F.*

Proof. Since $\mathcal{C}_F$ contains every geodesic in G with both endpoints in F, one has $F \subset \partial\mathcal{C}_F$, and thus $\overline{F} \subset \partial\mathcal{C}_F$. To prove the converse inclusion pick a point $z \in \partial\Gamma \setminus \overline{F}$. Since $\overline{F}$ is a closed set, Proposition 2.1 insures the existence of a wall M, with associated half-spaces H_+ and H_-, such that

$$z \in (\partial H_+ \setminus \partial M) \text{ and } \partial H_+ \subset (\partial\Gamma \setminus F).$$

It follows that $z \notin \partial H_-$ and that $F \subset (\partial\Gamma \setminus \partial H_+) = \operatorname{int}(\partial H_-)$, therefore $z \notin \partial\mathcal{C}_F$.

Let M be a wall intersecting $\mathcal{C}_F$ and let (x, y) be an edge in the intersection. Since $\mathcal{C}_F$ is a closed subset of G it contains x and y.

Hence both half-spaces H_+, H_- bounded by M intersect $\mathcal{C}_F$. It follows that F is contained neither in $\mathrm{int}(\partial H_+) = \partial\Gamma \setminus \partial H_-$, nor in $\mathrm{int}(\partial H_-) = \partial\Gamma \setminus \partial H_+$. Thus M cuts F. □

Proof of Theorem 2.3. Consider the convex hull $\mathcal{C}_F$ of F in G. This is a closed non-empty P-invariant convex subset of G. Note also that any wall M which cuts F must intersect $\mathcal{C}_F$, since $\mathcal{C}_F$ is connected and invariant under reflection in M.

Up to conjugacy we can assume that e belongs to $\mathcal{C}_F$. We claim that P is generated by the elements $s \in S$ such that the walls M_s intersect $\mathcal{C}_F$. To this aim let $x \in \mathcal{C}_F^0$ and let $x_0 = e, x_1, ..., x_n = x$ be the successive vertices of a geodesic segment joining e to x in G. The wall M_k between x_k and x_{k+1} intersects $\mathcal{C}_F$, thus according to Lemma 2.5 it cuts F. Therefore the reflection along the wall M_k belongs to P. So we get that P acts freely and transitively on $\mathcal{C}_F^0$. The claim and the theorem now follow easily. □

A first corollary concerns a special class of equivalence relations on $\partial\Gamma$. Examples of such equivalence relations will be given in Section 9.

Corollary 2.6. *Consider a Γ-invariant equivalence relation on $\partial\Gamma$ whose cosets are connected. Then:*

(1) *The closure of each coset is either a point or a parabolic limit set.*
(2) *If a nontrivial coset F is path-connected, and P is the parabolic subgroup with $\bar{F} = \partial P$, then for every $\epsilon > 0$ and every path $\eta : [0,1] \to \partial P$, there is a path $\eta' : [0,1] \to F$ such that*
$$d(\eta, \eta') = \max_{t\in[0,1]} d(\eta(t), \eta'(t)) < \epsilon\,.$$

Proof. (1). If F is a coset, and a wall M cuts F, then the limit set ∂M intersects F, because F is connected. Since the reflection in M is the identity map on ∂M, the coset and its image intersect, so they are equal. Therefore the assertion follows from Theorem 2.3.

(2). Suppose F is path-connected, $\overline{F} = \partial P$, $\eta \subset \partial P$ is a path, and $\epsilon > 0$. By Proposition 2.1, we can find a finite collection of half-spaces $H_1, \ldots, H_k \subset G$, such that the limit sets $\partial H_1, \ldots, \partial H_k \subset \partial\Gamma$ each have diameter $< \epsilon$, and their interiors cover the image of η. We can then choose $0 = t_0 < t_1 < \ldots < t_n = 1$ such that for each i, the pair $\eta(t_{i-1}), \eta(t_i)$ is contained in ∂H_{j_i} for some $j_i \in \{1, \ldots, n\}$, and $\mathrm{diam}(\eta([t_{i-1}, t_i])) < \epsilon/2$. Since F is dense in ∂P, for each $j \in \{1, \ldots, n\}$ we may choose $s_j \in F$ close enough to $\eta(t_j)$, such that for all i, the pair

s_{i-1}, s_i also lies in ∂H_{j_i}. By the path connectedness of F, we may join s_{i-1} to s_i by a path $\bar\gamma_i \subset F$; reflecting the portion of $\bar\gamma_i$ lying outside ∂H_{j_i} into ∂H_{j_i} using the reflection whose wall bounds H_{j_i}, we get a path $\gamma_i \subset \partial H_{j_i}$ joining s_{i-1} to s_i. Concatenating the γ_i's, we obtain the desired path η'. ☐

The same proof gives:

Corollary 2.7. *Let Φ be a quasiconvex subgroup of Γ with connected limit set. Assume that for all $g \in \Gamma$, the intersection $g\partial\Phi \cap \partial\Phi$ is either empty, or equal to $\partial\Phi$. Then Φ is virtually a parabolic subgroup of Γ. In other words there exists a parabolic subgroup P of Γ such that $\Phi \cap P$ is of finite index in both subgroups Φ and P.*

☐

Corollary 2.8. *Each connected component of $\partial\Gamma$ containing more than one point is a parabolic limit set.*

☐

2.2. **Shadowing curves.** In this paragraph we present a sort of a quantitative version of Theorem 2.3.

Let γ be a non constant curve in $\partial\Gamma$. Up to a translation by a group element, we can assume that the convex hull $\mathcal{C}_\gamma$ of γ contains e the identity of Γ (the convex hull of a subset of $\partial\Gamma$ is defined in Definition 2.4).

Definition 2.9. Let γ be a non constant curve in $\partial\Gamma$, let I be a non-empty subset of S and let $L \geq 0$. We say that γ is a (L, I)*-curve* if $e \in \mathcal{C}_\gamma$, and if for every $s \in I$ there exists an edge a_s of type s in $\mathcal{C}_\gamma$ with $\mathrm{dist}(e, a_s) \leq L$.

Proposition 2.10. *Let $\epsilon > 0$ and let L, I be as in the above definition. Let $P \leqq \Gamma$ be a conjugate of Γ_I, and let η be a (parametrized) curve contained in ∂P. There exists a finite subset $E \subset \Gamma$ such that for any (L, I)-curve γ the subset $\bigcup_{g\in E} g\gamma \subset \partial\Gamma$ contains a curve which approximates η to within ϵ with respect to the C^0 distance.*

Proof. Since ∂P is a translate of $\partial\Gamma_I$ we may assume that $P = \Gamma_I$.

First step : we show that for every (L, I)-curve γ the subset

$$\bigcup_{\{g\in\Gamma\ ;\ |g|\leq L\}} g\gamma$$

of $\partial\Gamma$ contains a curve passing through every ∂M_s with $s \in I$. For this purpose pick $s \in I$, let $a_s \subset \mathcal{C}_\gamma$ be an edge of type s with $\mathrm{dist}(e, a_s) \leq L$, and let M be the wall containing a_s. Let $g \in H_-(M)$ such that $a_s = (g, gs)$. The geodesic segment $[e, g]$ belongs to $\mathcal{C}_\gamma$ (by convexity). Denote by

$$g_0 = e,\ g_1 = \sigma_1,\ g_2 = \sigma_1\sigma_2,\ \ldots\ ,\ g_n = g = \sigma_1 \ldots \sigma_n,$$

the successive vertices of the segment $[e, g]$ with $\sigma_i \in S$. According to Lemma 2.5 the wall M_i passing between g_i and g_{i+1} cuts γ. Thus for every $i \in \{0, \ldots, n-1\}$ the curves γ and $g_i\sigma_{i+1}g_i^{-1}\gamma$ intersect. One has $g_i\sigma_{i+1}g_i^{-1} = g_i g_{i+1}^{-1}$. It follows that the subset $\gamma \cup g_1^{-1}\gamma \cup \ldots \cup g_n^{-1}\gamma$ is an arcwise connected set. It intersects the limit set of the wall $g^{-1}M = M_s$. Thus the subset

$$\bigcup_{\{g \in \Gamma\ ;\ |g| \leq L\}} g\gamma$$

is an arcwise connected set which intersects the limit set of every walls M_s with $s \in I$.

Second step : Consider a collection $H_1, \ldots, H_k$ of half-spaces in G all of them intersecting the special subgroup Γ_I *properly i.e.* $\Gamma_I \cap H_i \neq \emptyset$ nor Γ_I. We will show that there exists a finite subset E_0 of Γ such that for any (L, I)-curve γ, one can find in the subset $\bigcup_{g \in E_0} g\gamma \subset \partial\Gamma$ a curve passing through every $\partial H_1, \ldots, \partial H_k$. To do so, pick for each $i \in \{1, \ldots, k\}$ an element $p_i \in \Gamma_I$ adjacent to the wall which bounds H_i. Let c be a path in G which joins successively $p_1, \ldots, p_k$ and whose vertices $c_1, \ldots, c_n$ lie in Γ_I. Define

$$E_0 = \{c_i g \in \Gamma\ ;\ |g| \leq L,\ 1 \leq i \leq n\},$$

and let θ be a curve made of translates of γ passing through every ∂M_s with $s \in I$ (as constructed in step 1). The subset $\bigcup_{1 \leq i \leq n} c_i\theta$ meets the limit set of any wall $c_i M_s$ with $i \in \{1, \ldots, n\}$ and $s \in I$. In particular it intersects $\partial H_1, \ldots, \partial H_k$. In addition this is an arcwise connected set. Indeed write $c_{i+1} = c_i s = \sigma c_i$, with $s \in I$ and $\sigma = c_i s c_i^{-1}$. Then $c_{i+1}\theta = \sigma c_i\theta$. The curve θ intersects ∂M_s, thus $c_i\theta$ intersects $c_i\partial M_s$. The intersection set is pointwise invariant by the reflection σ and thus it belongs to $c_{i+1}\theta$ too.

Last step : We finally prove the proposition. By Proposition 2.4 there exists a collection of half-spaces $H'_1, \ldots, H'_{k+1}$ of G such that the union of their limit sets is a neighborhood of η contained in the $\epsilon/2$-neighborhood of η. Reordering if necessary we may assume that the curve η enters successively $\partial H'_1, \ldots, \partial H'_k$. Pick a collection of half-spaces

$H_1, ..., H_k$ each of them intersecting Γ_I properly, and such that for every $i \in \{1, ..., k\}$ one has

$$\partial H_i \subset \partial H_i' \cap \partial H_{i+1}'.$$

Existence of the H_i's follows from Proposition 2.4. According to step 2 there exists a subset E_0 of Γ such that for every (L, I)-curve γ, one can find in the subset $\bigcup_{g \in E_0} g\gamma$ a curve passing through every $\partial H_1, ..., \partial H_k$. Denote by θ such a curve. The part of θ between ∂H_{i-1} and ∂H_i may exit from $\partial H_i'$. If it happens reflect the outside part of θ along the wall which bounds H_i'. The resulting curve can be parametrized to approximate η to within ϵ with respect to the C^0 distance. Let σ_i be the reflection along the wall of H_i'. The following subset of Γ

$$E = E_0 \cup \bigcup_{i=2}^{k} \sigma_i E_0$$

satisfies the property we were looking for. □

We now establish the abundance of (L, I)-curves. Denote by $N_r(E)$ the open r-neighborhood in $\partial\Gamma$ of a subset $E \subset \partial\Gamma$.

Proposition 2.11. *Let $I \subset S$ and let $P \leqq \Gamma$ be a conjugate of Γ_I. For all $r > 0$ there exist $L \geq 0$ and $\delta > 0$ such that every curve $\gamma \subset \partial\Gamma$ satisfying the following conditions is a (L, I)-curve :*

(i) *its convex hull contains e,*
(ii) $\gamma \subset N_\delta(\partial P)$,
(iii) $\gamma \nsubseteq N_r(\partial Q)$ *for any parabolic $Q \lneqq P$ with connected limit set.*

Proof. Assume by contradiction that for every $L \geq 0$ and $\delta > 0$ there exists a curve γ which satisfies property (i), (ii) and (iii), and which is not a (L, I)-curve. Choose $L = n$, $\delta = 1/n$ and pick γ_n a corresponding curve. We may assume, by extracting a subsequence if necessary, that there exists an element $s \in I$ such that for every $n \geq 1$ no edge of $\mathcal{C}_{\gamma_n} \cap B(e, n)$ is of type s. We also may assume that the sequence of compact subsets $\{\gamma_n\}_{n\geq 1}$ converges for the Hausdorff distance to a non-degenerate continuum $\mathcal{L} \subset \partial P$, and that the sequence $\{\mathcal{C}_{\gamma_n}\}_{n\geq 1}$ converges to a convex subset $\mathcal{C} \subset G$ on every compact subset of G. With item (i) one has $e \in \mathcal{C}$, moreover one sees easily that $\mathcal{L} \subset \partial\mathcal{C}$. The fact that no edge of $\mathcal{C}$ is of type s implies that $\mathcal{L}$ is contained in the limit set of the special subgroup generated by $S \setminus \{s\}$. Intersections of parabolic subgroups are again parabolic subgroups (see [Dav08] Lemma 5.3.6), thus $\mathcal{L}$ is contained in the limit set of a proper parabolic subgroup of P. By Corollary 2.8 the connected component which contains $\mathcal{L}$ is the limit set of a proper parabolic subgroup Q of P. So we get a contradiction with the hypothesis (iii). $\square$

Remarks and questions : 1) Theorem 2.3 admits a partial converse. Indeed let $P = g\Gamma_I g^{-1}$ be a parabolic subgroup of Γ and let M be a wall of G such that ∂P meets both open subsets $\partial H_-(M) \setminus \partial M$ and $\partial H_+(M) \setminus \partial M$. Then a convexity argument shows that M admits an edge whose end-points lie in $g\Gamma_I$. Thus letting σ be the reflection along M, there exist $h, h' \in \Gamma_I$ such that $\sigma g h = g h'$. It follows that $\sigma = g h' h^{-1} g^{-1} \in P$, and so ∂P is σ-invariant.

2) Let Γ be an arbitrary hyperbolic group. Recall that the limit set of the intersection of two quasiconvex subgroups is the intersection of their limit sets (see [Gro93] p. 164). Hence for any quasiconvex subgroup $\Phi \leqq \Gamma$ the following properties are equivalent :

- for every $g \in \Gamma$, $\; g\partial\Phi \cap \partial\Phi = \partial\Phi \;$ or $\; \emptyset$,
- for every $g \in \Gamma$, either $g\Phi g^{-1} \cap \Phi$ is finite or is of finite index in both subgroups $g\Phi g^{-1}$ and Φ.

3) Given a subset $E \subset \partial\Gamma$ there exists a unique smallest parabolic limit set ∂P containing E, moreover if E is connected and non reduced to a point then ∂P is so. Indeed this follows from the fact that parabolic subgroups are stable by intersection (see [Dav08] Lemma 5.3.6), from the property of intersections of limit sets recalled in Remark 2 above, and from Corollary 2.8.

4) Let Γ be a hyperbolic group and consider a closed Γ-invariant equivalence relation on $\partial\Gamma$ whose cosets are continua. What one can say about such equivalence relations ? In particular for which groups Γ do the nontrivial cosets arise as the limit sets of a finite collection of conjugacy classes of quasiconvex subgroups ?

Note that the quotient space of $\partial\Gamma$ by a Γ-invariant closed equivalence relation $\sim$ is a compact metrizable space on which Γ acts as a convergence group, *i.e.* Γ acts properly discontinuously on the set of triples of distinct points of $\partial\Gamma/\sim$ (see [Bow99]).

3. Combinatorial modulus

This section develops the theory of combinatorial modulus in a general setting. The combinatorial Loewner property and related topics are discussed. For the classical notions of geometric function theory used in this paper, we refer to [Hei01].

Versions of combinatorial modulus have been considered by several authors in connection with Cannon's conjecture on groups with 2-sphere boundary (see *e.g.* [Can94, CS98, CFP99, BK02, Haib]), and in a more general context [Pan89b, Tys98].

3.1. **Definitions and first properties.** Let (Z, d) be a compact metric space, let $k \in \mathbb{N}$ and let $\kappa \geq 1$. A finite graph G_k is called a *κ-approximation of Z on scale k*, if it is the incidence graph of a covering of Z such that for every $v \in G_k^0$ there exists $z_v \in Z$ with

$$B(z_v, \kappa^{-1}2^{-k}) \subset v \subset B(z_v, \kappa 2^{-k}),$$

and for $v, w \in G_k^0$ with $v \neq w$:

$$B(z_v, \kappa^{-1}2^{-k}) \cap B(z_w, \kappa^{-1}2^{-k}) = \emptyset.$$

Note that we identify every vertex v of G_k with the corresponding subset in Z. A collection of graphs $\{G_k\}_{k\in\mathbb{N}}$ is called a *κ-approximation* of Z, if for each $k \in \mathbb{N}$ the graph G_k is a κ-approximation of Z on scale k.

Let $\gamma : [a, b] \to Z$ be a curve and let $\rho : G_k^0 \to \mathbb{R}_+$ be any function. The *ρ-length* of γ is

$$L_\rho(\gamma) = \sum_{v\cap\gamma\neq\emptyset} \rho(v).$$

For $p \geq 1$ the *p-mass* of ρ is

$$M_p(\rho) = \sum_{v\in G_k^0} \rho(v)^p.$$

Let $\mathcal{F}$ be a non-void family of curves in Z, we define its *G_k-combinatorial p-modulus* by

$$\mathrm{Mod}_p(\mathcal{F}, G_k) = \inf_\rho M_p(\rho),$$

where the infimum is over all *$\mathcal{F}$-admissible functions i.e.* functions $\rho : G_k^0 \to \mathbb{R}_+$ which satisfy $L_\rho(\gamma) \geq 1$ for every $\gamma \in \mathcal{F}$. If $\mathcal{F} = \emptyset$ we set $\mathrm{Mod}_p(\mathcal{F}, G_k) = 0$. Observe that admissible functions with minimal p-mass are smaller than or equal to 1.

We denote by $\mathcal{F}(A, B)$ the family of curves joining two subsets A and B of Z and by $\mathrm{Mod}_p(A, B, G_k)$ its G_k-combinatorial p-modulus. The following properties are routine.

Proposition 3.1. *1) If* $\mathcal{F}_1 \subset \mathcal{F}_2$ *then* $\mathrm{Mod}_p(\mathcal{F}_1, G_k) \leq \mathrm{Mod}_p(\mathcal{F}_2, G_k)$.

2) Let $\mathcal{F}_1, ..., \mathcal{F}_n$ *be curves families then*

$$\mathrm{Mod}_p(\cup \mathcal{F}_j, G_k) \leq \sum \mathrm{Mod}_p(\mathcal{F}_j, G_k).$$

3) Let $\mathcal{F}_1, \mathcal{F}_2$ *be two curve families. Suppose that each curve in* $\mathcal{F}_1$ *admits a subcurve in* $\mathcal{F}_2$*, then* $\mathrm{Mod}_p(\mathcal{F}_1, G_k) \leq \mathrm{Mod}_p(\mathcal{F}_2, G_k)$.

□

Recall that a metric space Z is called a *doubling metric space* if there is a constant $n \in \mathbb{N}$ such that every ball B can be covered by at most n balls of radius $\frac{r(B)}{2}$. For a doubling metric space the combinatorial modulus does not depend on the choice of the graph approximation up to a multiplicative constant. More precisely we have :

Proposition 3.2. *Assume that* Z *is a doubling metric space. Then for every* $\kappa, \kappa' \geq 1$ *and every* $p \geq 1$ *there exists a constant* $D \geq 1$ *such that for any* $k \in \mathbb{N}$ *and for any graphs* G_k, G'_k *which are respectively* κ *and* κ'*-approximations of* Z *on scale* k*, one has*

$$D^{-1} \mathrm{Mod}_p(\cdot, G_k) \leq \mathrm{Mod}_p(\cdot, G'_k) \leq D \, \mathrm{Mod}_p(\cdot, G_k).$$

Proof. The doubling property allows one to bound the maximal number N of pieces of G_k^0 which overlap a given piece of ${G'_k}^0$, in terms of n, κ, κ'.

Let $\mathcal{F}$ be a family of curves in Z. For every $\mathcal{F}$-admissible function $\rho : G_k^0 \to \mathbb{R}_+$ define $\rho' : {G'_k}^0 \to \mathbb{R}_+$ by

$$\rho'(v') = \max\{\rho(v) \; ; \; v \in G_k^0, v \cap v' \neq \emptyset\}.$$

Then $N\rho'$ is a $\mathcal{F}$-admissible function for the graph G'_k. In addition we have

$$M_p(\rho') \leq N M_p(\rho),$$

so we get

$$\mathrm{Mod}_p(\mathcal{F}, G'_k) \leq N^p \, \mathrm{Mod}_p(\mathcal{F}, G_k).$$

□

The following lemma is sometimes useful to understand the asymptotic behaviour of a minimal admissible function when k tends to $+\infty$.

Lemma 3.3. *Let $\mathcal{F}$ be a curve family in Z, let G_k be a κ-approximation of Z on scale k, and let $\rho : G_k^0 \to \mathbb{R}_+$ be a $\mathcal{F}$-admissible function with minimal p-mass. For $v \in G_k^0$ define*

$$\mathcal{F}_v = \{\gamma \in \mathcal{F} \ ; \ \gamma \cap v \neq \emptyset\}.$$

Then one has $\rho(v) \leq \mathrm{Mod}_p(\mathcal{F}_v, G_k)^{1/p}$.

Proof. Let $\rho_v : G_k^0 \to \mathbb{R}_+$ be a minimal $\mathcal{F}_v$-admissible function, and let $\tilde{\rho}$ be the function on G_k^0 defined by $\tilde{\rho}(w) = \max\{\rho(w), \rho_v(w)\}$ for $w \neq v$, and by $\tilde{\rho}(v) = \rho_v(v)$. Clearly $\tilde{\rho}$ is a $\mathcal{F}$-admissible function, thus:

$$\mathrm{Mod}_p(\mathcal{F}, G_k) \leq M_p(\tilde{\rho}) \leq \sum_{w \neq v} \rho(w)^p + \sum_{w} \rho_v(w)^p,$$

which implies that

$$\mathrm{Mod}_p(\mathcal{F}, G_k) \leq \mathrm{Mod}_p(\mathcal{F}, G_k) - \rho(v)^p + \mathrm{Mod}_p(\mathcal{F}_v, G_k).$$

The statement follows. □

3.2. **The Combinatorial Loewner property (CLP).** In this subsection we define a combinatorial analogue of the Loewner property introduced by J. Heinonen and P. Koskela in [HK98]. This notion appears in [Kle06] Section 7. We will show that the CLP has a number of features in common with the Loewner property. Examples of spaces satisfying the combinatorial Loewner property will be given in Sections 7 and 8.

We assume that Z is a compact arcwise connected doubling metric space, in particular Proposition 3.2 holds. Let $\{G_k\}_{k \in \mathbb{N}}$ be a κ-approximation of Z. Recall that the *relative distance* between two disjoint non-degenerate continua $A, B \subset Z$ is

$$\Delta(A, B) = \frac{\mathrm{dist}(A, B)}{\min\{\mathrm{diam}\, A, \mathrm{diam}\, B\}}.$$

Definition 3.4. Suppose $Q > 1$. Then Z satisfies the *Combinatorial p-Loewner Property* if there exist two positive increasing functions ϕ, ψ of $(0, +\infty)$ with $\lim_{t \to 0} \psi(t) = 0$, such that for all disjoint non-degenerate continua $A, B \subset Z$ and for all k with $2^{-k} \leq \min\{\mathrm{diam}\, A, \mathrm{diam}\, B\}$ one has :

$$\phi(\Delta(A, B)^{-1}) \leq \mathrm{Mod}_p(A, B, G_k) \leq \psi(\Delta(A, B)^{-1}).$$

The following properties have been established by J. Heinonen and P. Koskela for the Loewner spaces (see [HK98] Theorem 3.13 and Remark 3.19). Their proof generalizes verbatim to the spaces which satisfy the combinatorial Loewner property.

Proposition 3.5. *Assume that Z satisfies the CLP, then :*

(1) *It is* linearly connected*, in other words there exists a constant $C \geq 1$ such that any two points $z_1, z_2 \in Z$ can be joined by a path of diameter less or equal to $Cd(z_1, z_2)$,*
(2) *It has* no local cut point*, in other words no connected open subset is disconnected by removing a point.*

We also have:

Theorem 3.6.

(1) *If Z is a compact Ahlfors p-regular, p-Loewner metric space, then Z satisfies the combinatorial p-Loewner property.*
(2) *If Z' is quasi-Moebius homeomorphic to a compact space Z satisfying the CLP, then Z' also satisfies the CLP (with the same exponent).*

The proof of (1) involves tranferring admissible functions on the metric measure space to admissible functions on an associated discrete approximation, and vice-versa. The arguments are straightforward imitations of those appearing in [HK98, BK02, Haib], so we omit them.

The proof of (2) is similar in spirit, except that the admissible functions are transferred between two discrete approximations. It involves some of the techniques that will be used frequently in the sequel.

Proof of (2). We start by some general observations. Let $f : Z \to Z'$ be a quasi-Moebius homeomorphism. Given a κ'-approximation G'_ℓ of Z', the preimages $f^{-1}(v')$ of the pieces $v' \in G'^0_\ell$ form a covering of Z that we denote by $\mathcal{U}$. It enjoys the following properties : there exist constants $\lambda \geq 1$ and $N \in \mathbb{N}$, depending only on κ' and the geometric data of Z, Z', f, such that

(i) For every $u \in \mathcal{U}$, there is a ball $B_u \subset Z$ with $\frac{1}{\lambda} B_u \subset u \subset B_u$.
(ii) For every $z \in Z$, the number of balls B_u containing z is bounded by N.
(iii) If $2B_u$ and $2B_v$ intersect, their radii satisfy $r(B_u) \leq \lambda r(B_v)$.

For disjoint continua $E, F \subset Z$ one defines in an obvious way their *p-modulus relatively to the covering $\mathcal{U}$*, denoted by $\mathrm{Mod}_p(E, F, \mathcal{U})$, so that $\mathrm{Mod}_p(E, F, \mathcal{U}) = \mathrm{Mod}_p(f(E), f(F), G'_\ell)$.

Assuming that Z satisfies the combinatorial p-Loewner property, we will compare $\mathrm{Mod}_p(E, F, \mathcal{U})$ and $\mathrm{Mod}_p(E, F, G_k)$ for k large. The statement (2) will follow since $\Delta(E, F)$ and $\Delta(f(E), f(F))$ are quantitatively related (see [BK02] Lemma 3.2). We begin by the :

Left hand side CLP inequalities. To establish the left hand side CLP inequalities for Z' and G'_ℓ, we may assume that $2^{-\ell}$ is small in comparison with $\mathrm{dist}(f(E), f(F))$. Indeed, in the contrary, there is a family of curves joining $f(E)$ and $f(F)$, which lies in the union of a controlled amount of pieces of G'_ℓ ; its G'_ℓ-modulus is bounded from below in terms of the number of these pieces. Thus, via the correspondence induced by f, we may assume in addition that

(iv) None of the $2B_u$ $(u \in \mathcal{U})$ intersects both continua E, F.

Let $\varphi : \mathcal{U} \to \mathbb{R}_+$ be a $\mathcal{F}(E, F)$-admissible function. For $k \gg \ell$ we wish to define a $\mathcal{F}(E, F)$-admissible function of G_k^0 whose p-mass is controlled by above by $M_p(\varphi)$. To this aim, for every $u \in \mathcal{U}$, consider a minimal $\mathcal{F}(\overline{B_u}, Z \setminus \frac{3}{2}B_u)$-admissible function $\rho_u : G_k^0 \to \mathbb{R}_+$, and let $\rho_k : G_k^0 \to \mathbb{R}_+$ be defined by :

$$\forall v \in G_k^0,\ \rho_k(v) = \sum_{u \in \mathcal{U}} \varphi(u)\rho_u(v).$$

For every $\gamma \in \mathcal{F}(E, F)$, one has with (iv) and the definition of ρ_u

$$1 \leq \sum_{u \cap \gamma \neq \emptyset} \varphi(u) \leq \sum_{u \cap \gamma \neq \emptyset} \varphi(u) \Big(\sum_{v \cap \gamma \cap 2B_u \neq \emptyset} \rho_u(v) \Big).$$

Using property (ii) we obtain that $N\rho_k$ is $\mathcal{F}(E, F)$-admissible.

To estimate p-masses, observe that ρ_u being minimal, it is supported on the set of $v \in G_k^0$ such that $v \subset 2B_u$. Moreover its p-mass is smaller than $\psi(1/2)$ since Z satisfies the CLP. In combination with (ii), these properties show that the p-mass of ρ_k is

$$\sum_{v \in G_k^0} \Big(\sum_{u \in \mathcal{U}, v \subset 2B_u} \varphi(u)\rho_u(v) \Big)^p \leq N^{p-1} \sum_{v \in G_k^0} \sum_{u \in \mathcal{U}} \varphi(u)^p \rho_u(v)^p,$$

which is less than $N^{p-1}\psi(1/2)M_p(\varphi)$. Therefore the left hand side CLP inequalities hold for Z'. It remains to establish the :

Right hand side CLP inequalities. They require the following general observation, that will be used frequently in the sequel too :

Lemma 3.7. *let $\mathcal{F}$ be a family of curves in a general metric space Z and let $M > 0$. Then $\mathrm{Mod}_p(\mathcal{F}, G_k) \geq M$ if and only if for every function $\rho : G_k^0 \to \mathbb{R}_+$ there exists a curve $\gamma \in \mathcal{F}$ with*

$$L_\rho(\gamma) \leq \Big(\frac{M_p(\rho)}{M}\Big)^{1/p}.$$

Proof. Just remark that the modulus can be written as

$$\mathrm{Mod}_p(\mathcal{F}, G_k) = \inf_\rho \frac{M_p(\rho)}{L_\rho(\mathcal{F})^p} \text{ with } L_\rho(\mathcal{F}) = \inf_{\gamma\in\mathcal{F}} L_\rho(\gamma),$$

where ρ is any positive function on G_k^0. □

Let $k \gg \ell$ and let $\rho_k : G_k^0 \to \mathbb{R}_+$ be any function. According to Lemma 3.7, to establish the right hand side CLP inequalities for Z', it is enough to construct a curve $\gamma \in \mathcal{F}(E, F)$ whose ρ_k-length is controlled by above by

$$\Big(\frac{M_p(\rho_k)}{\mathrm{Mod}_p(E, F, \mathcal{U})}\Big)^{1/p}.$$

We assume that $2^{-\ell} \leq \min\{\mathrm{diam}\, f(E), \mathrm{diam}\, f(F)\}$. Increasing λ if necessary, it yields :

(v) Whenever B_u intersects E, the diameter of $2B_u \cap E$ is larger than $\frac{1}{\lambda} r(B_u)$ (and the same holds for F too).

Let $\Lambda \geq 1$ be a constant (that will be specified later on), and let $\varphi : \mathcal{U} \to \mathbb{R}_+$ be the function :

$$\forall u \in \mathcal{U},\ \varphi(u) = \Big(\sum_{v\cap 2\Lambda B_u \neq \emptyset} \rho_k(v)^p\Big)^{1/p}.$$

The obvious generalization of property (ii) implies that

$$M_p(\rho_k) \asymp M_p(\varphi)\,. \tag{3.1}$$

Lemma 3.7 shows that there is a curve $\delta \in \mathcal{F}(E, F)$ such that

$$L_\varphi(\delta) \leq \Big(\frac{M_p(\varphi)}{\mathrm{Mod}_p(E, F, \mathcal{U})}\Big)^{1/p}. \tag{3.2}$$

Let $u_i \in \mathcal{U}$ so that δ enters successively $u_1, ..., u_n$, and set $B_i := B_{u_i}$ for simplicity. We will now use the following lemma whose proof is similar to the one of Lemma 3.17 in [HK98].

Lemma 3.8. *Let Z be a compact metric space satisfying the CLP. Then for every $\alpha \in (0,1)$ there exist constants $\Lambda \geq 1$ and $m > 0$, such that for every ball $B \subset Z$ and every disjoint continua $E_1, E_2 \subset B$ with* $\operatorname{diam} E_i \geq \alpha r(B)$*, the G_k-combinatorial p-modulus of the family*

$$\{\eta \in \mathcal{F}(E_1, E_2) \ ; \ \eta \subset \Lambda B\}$$

is greater than m, for every k with $2^{-k} \leq \min\{\operatorname{diam} E_1, \operatorname{diam} E_2\}$.

This lemma in combination with Lemma 3.7 and properties (iii), (v), allows one to construct by induction on $s \in \{1, ..., n-1\}$ a curve $\gamma_s \subset \cup_{i=1}^{s} 2\Lambda B_i$, joining E to B_{s+1}, whose ρ_k-length is bounded linearly by above by $\sum_{i=1}^{s} \varphi(u_i)$. Indeed this follows from letting $\alpha = \frac{1}{2\lambda}$, $\Lambda = \Lambda(\alpha)$ and $B = 2B_s$ in the statement of the above lemma. A step futher gives a curve $\gamma \in \mathcal{F}(E, F)$ whose ρ_k-length is bounded linearly by above by $L_\varphi(\delta)$. Thanks to the estimates (3.1) and (3.2), the curve γ enjoys the expected properties. The theorem follows. $\square$

Our next result is a combinatorial version of Proposition 3.1 in [BK05a]. It asserts that a space which satisfies a combinatorial Loewner type condition for pairs of balls satisfies the combinatorial Loewner condition for all pairs of continua. It is a main tool to exhibit examples of spaces satisfying the combinatorial Loewner property.

Proposition 3.9. *Let $p \geq 1$. Assume that for every $A > 0$ there exist constants $m = m(A) > 0$ and $L = L(A) > 0$ such that if $r > 0$ and $B_1, B_2 \subset Z$ are r-balls with* $\operatorname{dist}(B_1, B_2) \leq Ar$*, then for every $k \geq 0$ with $2^{-k} \leq r$ the G_k-combinatorial p-modulus of the family*

$$\{\gamma \in \mathcal{F}(B_1, B_2) \ ; \ \operatorname{diam} \gamma \leq Lr\}$$

is greater than m. Then there exists a positive increasing function ϕ of $(0, +\infty)$ such that for every disjoint non-degenerate continua $E_1, E_2 \subset Z$ and for every k with $2^{-k} \leq \min\{\operatorname{diam} E_1, \operatorname{diam} E_2\}$, one has :

$$\phi(\Delta(E_1, E_2)^{-1}) \leq \operatorname{Mod}_p(E_1, E_2, G_k).$$

Its proof is a rather straighforward discretization of the "proof of Proposition 3.1" in [BK05a]. For the sake of completeness and because similar ideas will be used in Section 6, we will give the details of the proof. Since it is the most technical part of paper, readers may skip it at the first reading. The proof requires the following lemma which is the analogue of Lemma 3.7 in [BK05a].

Lemma 3.10. *Let Z as in Proposition 3.9 and suppose $0 < \lambda < 1/8$. There exist constants $\Lambda = \Lambda(\lambda)$ and $C = C(\lambda)$ with the following* *property. Let $\rho : G_k^0 \to \mathbb{R}_+$ be any function, let $B = B(z, r)$ be a ball of radius $0 < r < \operatorname{diam} Z$, and let $F_1, F_2 \subset Z$ be two continua with $F_i \cap \frac{1}{4}B \neq \emptyset$ and $F_i \setminus B \neq \emptyset$ for $i = 1, 2$. Then for $2^{-k+2} \leq \lambda r$ there exist disjoint balls B_i, $i = 1, 2$ and a path $\sigma \subset Z$ such that :*

(i) *B_1 and B_2 are disjoint, they are centered on F_i and of radius λr,*
(ii) *$B_i \subset \frac{7}{8}B$ and*
$$\sum_{v \subset B_i} \rho(v)^p \leq 8\lambda \sum_{v \subset B} \rho(v)^p,$$
(iii) *the path σ joins $\frac{1}{4}B_1$ to $\frac{1}{4}B_2$, it is contained in ΛB and it has ρ-length at most*
$$C \cdot \Big(\sum_{v \subset \Lambda B} \rho(v)^p \Big)^{1/p}.$$

Proof. We can find a subcontinuum $E_1 \subset F_1$ which is contained in $\overline{B}(z, \frac{3r}{8}) \setminus B(z, \frac{r}{4})$ and which joins $Z \setminus B(z, \frac{3r}{8})$ to $\overline{B}(z, \frac{r}{4})$. Similarly we can find a subcontinuum $E_2 \subset F_2$ which is contained in $\overline{B}(z, \frac{3r}{4}) \setminus B(z, \frac{5r}{8})$ and which joins $Z \setminus B(z, \frac{3r}{4})$ to $\overline{B}(z, \frac{5r}{8})$. Then we have $\operatorname{diam} E_i \geq r/8$ for $i = 1, 2$ and $\operatorname{dist}(E_1, E_2) \geq r/4$.

Since $\operatorname{diam} E_i \geq r/8$, for every λ with $0 < \lambda < 1/8$ there exist at least $\frac{1}{8\lambda}$ pairwise disjoint balls centered on E_i and of radius λr. So at least one of them - called B_i - satisfies item (ii) of the statement. The condition (i) is clearly satisfied by the pair of balls B_1, B_2.

Item (iii) follows from the hypothesis on the modulus of curves joining the pair of balls $\frac{1}{4}B_1, \frac{1}{4}B_2$ and from Lemma 3.7. Note that the radius of $\frac{1}{4}B_i$ is equal to $\frac{\lambda r}{4}$ therefore our hypotheses requires that $2^{-k} \leq \frac{\lambda r}{4}$ *i.e.* $2^{-k+2} \leq \lambda r$. □

Proof of Proposition 3.9. Let $\lambda \in \mathbb{R}$ subject to the conditions $0 < \lambda < 1/8$ and $2 \cdot (8\lambda)^{1/p} < 1$.

Suppose $E_1, E_2 \subset Z$ are disjoint non-degenerate continua and let $\rho : G_k^0 \to \mathbb{R}_+$. According to Lemma 3.7 we are looking for a curve γ joining E_1 to E_2 whose ρ-length is at most
$$\Big(\frac{M_p(\rho)}{M} \Big)^{1/p},$$
where $M > 0$ depends only on the relative distance between E_1 and E_2.

Pick $p_i \in E_i$ such that $d(p_1, p_2) = \operatorname{dist}(E_1, E_2)$. Set

$$r_0 := \frac{1}{2} \min\{d(p_1, p_2), \operatorname{diam} E_1, \operatorname{diam} E_2\} > 0.$$

Let $B_i = B(p_i, r_0)$ for $i = 1, 2$. Then $B_1 \cap B_2 = \emptyset$ and $E_i \setminus B_i \neq \emptyset$ for $i = 1, 2$. In addition :

$$\operatorname{dist}(\frac{1}{4}B_1, \frac{1}{4}B_2) \leq \Big(\frac{4d(p_1, p_2)}{r_0}\Big)\frac{r_0}{4} \leq t\frac{r_0}{4},$$

where $t := 8\max\{1, \Delta(E_1, E_2)\}$. By our hypotheses one can find a path σ joining $\frac{1}{4}B_1$ to $\frac{1}{4}B_2$ and whose ρ-length is at most $(\frac{M_p(\rho)}{m})^{1/p}$ where $m = m(t)$ is the constant appearing in the statement of Proposition 3.9.

By using Lemma 3.10 inductively, we will construct for successive integers n a family of balls $\mathcal{B}_n$, a collection of continua Ω_n, and a collection of paths Σ_n, such that :

(1) $\mathcal{B}_0 = \{B_1, B_2\}$, $\Omega_0 = \{E_1, E_2\}$ and $\Sigma_0 = \{\sigma\}$ are defined previously .
(2) For $n \geq 1$ the family $\mathcal{B}_n$ consists of 2^{n+1} disjoint balls of radius $\lambda^n r_0$. Each ball $B \in \mathcal{B}_n$ is centered on a continuum $\omega \in \Omega_n$ with $\omega \setminus B \neq \emptyset$. The collection Σ_n consists of 2^n paths, for each element $\sigma \in \Sigma_n$ there are exactly two elements $B_1, B_2 \in \mathcal{B}_n$ such that σ joins $\frac{1}{4}B_1$ to $\frac{1}{4}B_2$.

The induction proceeds as follows. According to item (2) for every ball $B \in \mathcal{B}_n$ there exist $\omega \in \Omega_n$ and $\sigma \in \Sigma_n$ such that the pair $\{F_1, F_2\} := \{\omega, \sigma\}$ satisfies $F_i \cap \frac{1}{4}B \neq \emptyset$ and $F_i \setminus B \neq \emptyset$ for $i = 1, 2$. Thus applying Lemma 3.10 we get two disjoint balls B_1, B_2 and a path σ joining $\frac{1}{4}B_1$ to $\frac{1}{4}B_2$. The balls B_i are of radius $\lambda^{n+1}r_0$, they are centered respectively on ω and σ. The definitions of $\mathcal{B}_{n+1}$ and Σ_{n+1} are therefore clear. We define Ω_{n+1} to be $\Omega_n \cup \Sigma_n$.

With Lemma 3.10 and by construction the following additional properties are satisfied :

(3) For every ball $B \in \mathcal{B}_n$ there exists a ball $B' \in \mathcal{B}_{n-1}$ such that $B \subset \frac{7}{8}B'$ and

$$\sum_{v \subset B} \rho(v)^p \leq 8\lambda \sum_{v \subset B'} \rho(v)^p.$$

(4) For every path $\sigma \in \Sigma_n$ there exists a ball $B' \in \mathcal{B}_{n-1}$ such that σ lies in $\Lambda B'$ and has ρ-length at most
$$C \cdot \Big(\sum_{v \subset \Lambda B'} \rho(v)^p \Big)^{1/p},$$
where $\Lambda = \Lambda(\lambda)$ and $C = C(\lambda)$.

(5) At each stage n one can index the elements of $\bigcup_{\ell \leq n} \Sigma_\ell$ by $\sigma_1, ..., \sigma_m$ and the elements of $\mathcal{B}_n$ by $B_1, ..., B_{m+1}$, in order that B_1 meets E_1 and σ_1, B_i meets σ_{i-1} and σ_i for $2 \leq i \leq m$, and B_{m+1} meets σ_m and E_2.

We iterate this procedure as long as $n \in \mathbb{N}$ satisfies $2^{-k+2} \leq \lambda^n r_0$. Let N be the largest integer satisfying this condition.

It remains to connect the paths σ_i described in item (5) to obtain a curve joining E_1 to E_2. For this purpose observe that the hypotheses of Proposition 3.9 assert that for every pair z_1, z_2 of points in Z there exists a path joining $B(z_1, \frac{1}{4}d(z_1, z_2))$ to $B(z_2, \frac{1}{4}d(z_1, z_2))$ whose diameter is comparable to $d(z_1, z_2)$. This property implies that every pair of points z_1, z_2 are connected by a path whose diameter is comparable to $d(z_1, z_2)$, (see Lemma 3.4 in [BK05a] for more details). Therefore increasing Λ if necessary and using item (5) with $n = N$ we exhibit a family Θ consisting of 2^{N+1} curves such that :

(6) For every $\theta \in \Theta$ the union of the subsets $v \in G_k^0$ with $v \cap \theta \neq \emptyset$ is contained in a ball of the form ΛB with $B \in \mathcal{B}_N$.

(7) The following subset contains a curve γ joining E_1 to E_2 :
$$\bigcup_{\theta \in \Theta} \theta \cup \bigcup_{n \leq N} \bigcup_{\sigma \in \Sigma_n} \sigma.$$

Observe that for $B \in \mathcal{B}_N$ the number of $v \in G_k^0$ with $v \subset \Lambda B$ is bounded in terms of Λ, κ and the doubling constant of Z. Hence increasing C if necessary we obtain with property (6) that

(8) Every $\theta \in \Theta$ has ρ-length at most
$$C \cdot \Big(\sum_{v \subset \Lambda B} \rho(v)^p \Big)^{1/p},$$
where $B \in \mathcal{B}_N$ is the ball attached to θ in item (6).

We now compute the ρ-length of the curve γ defined in (7). At first with properties (2) and (3) one obtains that for $0 \leq s \leq n \leq N$ and for every ball $B \in \mathcal{B}_n$ there exists a ball $B' \in \mathcal{B}_{n-s}$ such that $(\frac{1}{8\lambda})^s B \subset B'$. Let $s = s(\lambda)$ be the smallest integer such that $(\frac{1}{8\lambda})^s \geq \Lambda$.

With properties (4) and (3) we get that for $s < n \leq N$ the ρ-length of every curve $\sigma \in \Sigma_n$ is less than

$$C \cdot \Big((8\lambda)^{n-s-1} M_p(\rho) \Big)^{1/p}.$$

Similarly with properties (8) and (3) we get that the ρ-length of every curve $\theta \in \Theta$ is less than

$$C \cdot \Big((8\lambda)^{N-s} M_p(\rho) \Big)^{1/p}.$$

For $0 < n \leq s$ each $\sigma \in \Sigma_n$ has ρ-length at most $C(M_p(\rho))^{1/p}$. Recall that $\sigma \in \Sigma_0$ has ρ-length at most $(\frac{M_p(\rho)}{m})^{1/p}$. Finally :

$$L_\rho(\gamma) \leq M_p(\rho)^{1/p} \Big(\frac{1}{m^{1/p}} + C \sum_{n=1}^{s} 2^n + C \sum_{n=s+1}^{N+1} 2^n (8\lambda)^{(n-s-1)/p} \Big).$$

Therefore letting $D := C \cdot (8\lambda)^{(-s-1)/p}$ and $a := 2 \cdot (8\lambda)^{1/p}$ we get

$$L_\rho(\gamma) \leq M_p(\rho)^{1/p} \Big(\frac{1}{m^{1/p}} + D \sum_{n=1}^{N+1} a^n \Big),$$

and

$$M := \Big(\frac{1}{m^{1/p}} + D \sum_{n=1}^{N+1} a^n \Big)^{-p}$$

satisfies the desired properties since $a < 1$ by assumption. □

3.3. **Self-similarity.** This subsection derives from self-similarity several general principles that will be useful in the sequel. The following definition appears in [Kle06] Section 3.

Definition 3.11. A compact metric space (Z, d) is called *approximately self-similar* if there is a constant $L_0 \geq 1$ such that if $B(z, r) \subset Z$ is a ball of radius $0 < r \leq \operatorname{diam}(Z)$, then there is an open subset $U \subset Z$ which is L_0-bi-Lipschitz homeomorphic to the rescaled ball $(B(z, r), \frac{1}{r} d)$.

Observe that approximately self-similar metric spaces are doubling metric spaces. Examples include some classical fractal spaces like the square Sierpinski carpet and the cubical Menger sponge (their respective definitions are recalled in Sections 5 and 8). Other examples are provided by the following situation.

Definition 3.12. Let Γ be a hyperbolic group. A metric d on $\partial\Gamma$ is called a *self-similar metric*, if there exists a Gromov hyperbolic geodesic metric space X, on which Γ acts by isometries, properly discontinuously and cocompactly, such that d is the preimage of a visual metric on ∂X by the canonical homeomorphism $\partial\Gamma \to \partial X$.

By standard arguments $\partial\Gamma$ equipped with a self-similar metric d is approximately self-similar, the partial bi-Lipschitz maps being restrictions of group elements. Moreover Γ acts on $(\partial\Gamma, d)$ by bi-Lipschitz homeomorphisms, and d is quasi-Moebius equivalent to a visual metric on $\partial\Gamma$. In addition we remark that $(\partial\Gamma, d)$ is linearly connected as soon as it is connected [BK05b].

A further source of examples comes from expanding Thurston maps, [BM]. It follows readily from [BM, Theorem 1.2] that with respect to any visual metric as in [BM], the 2-sphere is approximately self-similar.

For the rest of the subsection Z denotes an arcwise connected, approximately self-similar metric space. Let $\{G_k\}_{k\in\mathbb{N}}$ be a κ-approximation of Z. We fix a positive constant d_0 which is small compared to the diameter of Z and to the constant L_0 of the Definition 3.11. Denote by $\mathcal{F}_0$ the family of curves $\gamma \subset Z$ with $\mathrm{diam}(\gamma) \geq d_0$.

Proposition 3.13. *Let $p > 1$ and suppose there exists a constant $C \geq 0$ such that for every $k \in \mathbb{N}$ one has $\mathrm{Mod}_p(\mathcal{F}_0, G_k) \leq C$. Then there exists a positive increasing function ψ of $(0, +\infty)$ with $\lim_{t\to 0} \psi(t) = 0$, such that for every disjoint non-degenerate continua $A, B \subset Z$ and every integer k satisfying $2^{-k} \leq \min\{\mathrm{diam}(A), \mathrm{diam}(B)\}$, one has*

$$\mathrm{Mod}_p(A, B, G_k) \leq \psi(\Delta(A, B)^{-1}).$$

Proof. Define for $t > 0$:

$$\psi(t) = \sup \mathrm{Mod}_p(A, B, G_k),$$

where the supremum is over all disjoint continua A, B with $\Delta(A, B) \geq 1/t$ and over all integers k such that $2^{-k} \leq \min\{\mathrm{diam}(A), \mathrm{diam}(B)\}$. From the monotonicity of the modulus (Proposition 3.1.1) and from our hypotheses, the function $t \mapsto \psi(t)$ is non-decreasing with non-negative real values. It is enough to prove that $\psi(t)$ tends to 0 when t tends to 0.

Let A and B be disjoint non-degenerate continua, assume that $d := \mathrm{diam}(A)$ is smaller than $\mathrm{diam}(B)$ and let n be the largest integer with

$2^n \cdot d \leq \mathrm{dist}(A, B)$. Pick $z_0 \in A$, we get :

$$A \subset B(z_0, d) \text{ and } B \subset Z \setminus B(z_0, 2^n \cdot d).$$

For $i \in \{1, ..., n-1\}$ define $B_i = B(z_0, 2^i \cdot d)$. There exists a constant C_1 depending only on κ and on the geometry of Z such that for every $k \in \mathbb{N}$ with $2^{-k} \leq d$ and every $i \in \{1, ..., n-1\}$ one has

$$\mathrm{Mod}_p(B_i, Z \setminus B_{i+1}, G_k) \leq C_1.$$

Indeed applying the self-similarity property (Definition 3.11) to B_{i+1} we may inflate $\mathcal{F}(B_i, Z \setminus B_{i+1})$ to a family of curves of essentially unit diameters. Hence the above inequalities follow from the monotonicity of the modulus, from our hypotheses, and from Proposition 3.2.

Choose for every $i \in \{1, ..., n-1\}$ a minimal $\mathcal{F}(B_i, Z \setminus B_{i+1})$-admissible function $\rho_i : G_k^0 \to \mathbb{R}_+$, and let $\rho = \frac{1}{n-1} \sum_{i=1}^{n-1} \rho_i$. This is a $\mathcal{F}(A, B)$-admissible function since every curve joining A and B joins B_i and $Z \setminus B_{i+1}$ too. For $2^{-k} \leq d$ the minimality of the ρ_i's shows that their supports are essentially disjoint, thus

$$M_p(\rho) \leq \frac{C_2}{(n-1)^p} \sum_{i=1}^{n-1} M_p(\rho_i) \leq \frac{C_1 C_2}{(n-1)^{p-1}},$$

where C_2 depends only on κ. Since $p > 1$ and $\Delta(A, B) \leq 2^{n+1}$ we get that for t small enough $\psi(t) \leq C_3 (\log 1/t)^{-1}$ where C_3 depends only on C_1, C_2, p. □

The following uniformization criterion is a consequence of an uniformization theorem established in [BK02] (see also [CFP99] for related results).

Corollary 3.14. *Suppose in addition that Z is homeomorphic to the 2-sphere. Assume that there exists a constant $C \geq 1$ such that for every $k \in \mathbb{N}$ one has*

$$\mathrm{Mod}_2(\mathcal{F}_0, G_k) \leq C .$$

Then Z is quasi-Moebius homeomorphic to the Euclidean 2-sphere.

Proof. The statement is a consequence of the previous proposition in combination with [BK02] Th. 10.4. This theorem supposes that Z is doubling and linearly locally contractible.

The doubling property follows from the fact that Z is approximately self-similar.

Recall that a metric space Z is *linearly locally contractible* if there exists a constant $\lambda \geq 1$, such that every ball $B \subset Z$ with $0 < r(B) <$

$\frac{\operatorname{diam} Z}{\lambda}$, is contractible in λB. Approximately self-similar manifolds enjoy this property. Indeed, if Z is a compact manifold then there is a $R_0 > 0$ and a positive function Φ of $(0, R_0)$ with $\lim_{t\to 0}\Phi(t) = 0$, such that every ball $B \subset Z$ of radius r with $0 < r < R_0$, is contractible in $\Phi(r)B$. Let $\lambda \geq 1$ and let $B(z,r) \subset Z$ be a ball with $\lambda r < \operatorname{diam} Z$. Applying self-similarity to the ball $B(z, \lambda r)$, one obtains a open subset $U \subset Z$ and a bilipschitz map $f : B(z, \lambda r) \to U$ such that for every pair $x, y \in B(z, \lambda r)$:

$$L_0^{-1}\frac{1}{\lambda r}d(x,y) \leq d(f(x), f(y)) \leq L_0\frac{1}{\lambda r}d(x,y).$$

Thus if λ satisfies $\frac{L_0}{\lambda} \leq L_0^{-1}$, one has

$$f(B(z,r)) \subset B(f(z), \frac{L_0}{\lambda}) \subset B(f(z), L_0^{-1}) \subset f(B(z, \lambda r)).$$

Choosing λ subject to the conditions $\Phi(\frac{L_0}{\lambda}) < L_0^{-1}$ and $\frac{L_0}{\lambda} \leq L_0^{-1}$, we get that $f(B(z,r))$ is contractible in $f(B(z, \lambda r))$. Therefore so is $B(z,r)$ in $B(z, \lambda r)$.

Assumptions of Theorem 10.4 of [BK02] also require the graphs G_k to be (essentially) homeomorphic to 1-skeletons of triangulations of S^2. Corollary 6.8 of [BK02] ensures existence of such graphs.

We remark finally that when Z is the boundary of a hyperbolic group, the corollary can also be deduced from [CFP99] Th. 1.5 and 8.2. □

We now study the behaviour of $\operatorname{Mod}_p(\mathcal{F}_0, G_k)$ when k tends to $+\infty$, depending on $p \geq 1$. For this purpose we establish a submultiplicative inequality :

Proposition 3.15. *Let $p \geq 1$ and write $M_k := \operatorname{Mod}_p(\mathcal{F}_0, G_k)$ for simplicity. There exists a constant $C > 0$ such that for every pair of integers k, ℓ one has : $M_{k+\ell} \leq C \cdot M_k \cdot M_\ell$. In addition when p belongs to a compact subset of $[1, +\infty)$ the constant C may be choosen independent of p.*

Proof. Let $\rho_k : G_k^0 \to \mathbb{R}_+$ be a minimal $\mathcal{F}_0$-admissible function on scale k. By definition each $v \in G_k^0$ is roughly a ball of Z of radius 2^{-k}. For every $v \in G_k^0$, let B_v be a ball containing v and whose radius is approximately 2^{-k}.

We inflate every ball $2B_v$ to essentially unit diameter like in Definition 3.11. Let g_v be the corresponding partial bi-Lipschitz map. Define $G_{k+\ell} \cap 2B_v$ to be the incidence graph of the covering of $2B_v$ by the subsets $w \in G_{k+\ell}^0$ with $w \cap 2B_v \neq \emptyset$. Using the map g_v, one can consider the graph $G_{k+\ell} \cap 2B_v$ as a $(2\kappa L_0)$-approximation of $g_v(2B_v)$ on scale ℓ. We pull back on $G_{k+\ell} \cap 2B_v$ a normalized minimal $\mathcal{F}_0$-admissible function on scale ℓ, in order to get for every $v \in G_k^0$ a function $\rho_v : (G_{k+\ell} \cap 2B_v)^0 \to \mathbb{R}_+$ with the following properties

(i) its p-mass is bounded by above by $D \cdot M_\ell \cdot \rho_k(v)^p$, where D is independent of k and v,
(ii) every curve $\gamma \subset 2B_v$ whose diameter is larger than $r(B_v)$ picks up a (ρ_v)-length larger than $\rho_k(v)$, where $r(B_v)$ is the radius of B_v.

Define a function $\rho_{k+\ell}$ on $G^0_{k+\ell}$ by

$$\rho_{k+\ell}(w) = \max \rho_v(w),$$

where the maximun is over all $v \in G_k^0$ with $w \cap 2B_v \neq \emptyset$. Its p-mass is linearly bounded by above by

$$\sum_{v \in G_k^0} M_p(\rho_v),$$

which in turn is linearly bounded by $M_p(\rho_k) \cdot M_\ell = M_k \cdot M_\ell$ (see item (i)).

It remains to prove that $\rho_{k+\ell}$ is a $\mathcal{F}_0$-admissible function – up to a multiplicative constant independent of the scale. For $\gamma \in \mathcal{F}_0$ we have

$$1 \leq \sum_{v \cap \gamma \neq \emptyset} \rho_k(v) \leq \sum_{v \cap \gamma \neq \emptyset} \sum_{\gamma \cap 2B_v \cap w \neq \emptyset} \rho_v(w).$$

Indeed the last inequality follows from item (ii) since the relation $v \cap \gamma \neq \emptyset$ implies that there exists a subcurve of γ of diameter greater than $r(B_v)$ contained in $2B_v$.

Thus the $\rho_{k+\ell}$-length of γ is larger than $1/N$ where N is maximal number of elements $v \in G_k^0$ such that $2B_v$ intersects a given piece $w \in G^0_{k+\ell}$. Therefore $N \cdot \rho_{k+\ell}$ is $\mathcal{F}_0$-admissible. □

For every $k \in \mathbb{N}$, observe that $p \mapsto \mathrm{Mod}_p(\mathcal{F}_0, G_k)$ is a non-increasing continuous function on $[1, +\infty)$ (monotonicity comes from the fact that minimal admissible functions are smaller than or equal to 1). We define a critical exponent associated to the curve family $\mathcal{F}_0$ by

$$Q_M := \inf\{p \in [1, +\infty) \ ; \lim_{k \to +\infty} \mathrm{Mod}_p(\mathcal{F}_0, G_k) = 0\}.$$

With Propositions 3.13, 3.15 and Lemma 3.3 one gets

Corollary 3.16. *(i) For $p > Q_M$ one has* $\lim_{k\to+\infty} \mathrm{Mod}_p(\mathcal{F}_0, G_k) = 0$*,*

(ii) for $1 \le p \le Q_M$ the sequence $\{\mathrm{Mod}_p(\mathcal{F}_0, G_k)\}_{k\ge0}$ admits a positive lower bound,

(iii) if in addition Z is linearly connected, then for $1 \le p < Q_M$ the sequence $\{\mathrm{Mod}_p(\mathcal{F}_0, G_k)\}_{k\ge0}$ is unbounded.

In particular when Z satisfies the combinatorial p-Loewner property one has $p = Q_M$.

Proof. Part (i) is a consequense of the definition. Part (ii) comes from Proposition 3.15 and from the fact that if a positive sequence $\{M_k\}_{k\in\mathbb{N}}$ satisfies $M_{k+\ell} \le C \cdot M_k \cdot M_\ell$, then one has :

$$\lim_{k\to+\infty} M_k = 0 \iff \exists \ell \in \mathbb{N} \text{ with } M_\ell < C^{-1}.$$

To establish part (iii) suppose by contradiction that the sequence is bounded for some Q with $1 < Q < Q_M$. Then the conclusion of Proposition 3.13 holds for the exponent Q. We will prove that for $p > Q$ the sequence $\{\mathrm{Mod}_p(\mathcal{F}_0, G_k)\}_{k\ge0}$ tends to 0, contradicting the definition of Q_M.

By the monotonicity of the combinatorial modulus it is enough to show that for every disjoint non-degenerate continua A and B of Z the sequence $\{\mathrm{Mod}_p(A, B, G_k)\}_{k\ge0}$ tends to 0. For this purpose we will establish that the $\mathcal{F}(A, B)$-admissible functions $\rho_k : G_k^0 \to \mathbb{R}_+$ with minimal Q-mass satisfy

$$\lim_{k\to+\infty} \|\rho_k\|_\infty = 0, \text{ where } \|\rho_k\|_\infty := \sup_{v\in G_k^0} \rho_k(v).$$

For each $v \in G_k^0$ pick a continuum E_v containing v and whose diameter d_v is comparable to $\mathrm{diam}(v) \asymp 2^{-k}$. The existence of E_v follows from the assumption that Z is linear connected. Any curve in $\mathcal{F}(A, B)$ passing through v possesses a subcurve in each family $\mathcal{F}(A, E_v)$ and $\mathcal{F}(B, E_v)$. Hence with Lemma 3.3 and Proposition 3.1 we get

$$\rho_k(v) \le \min\{\mathrm{Mod}_Q(A, E_v, G_k), \mathrm{Mod}_Q(B, E_v, G_k)\}^{1/Q}.$$

For 2^{-k} small enough compared with $\mathrm{dist}(A, B)$ one has

$$\max\{\Delta(A, E_v), \Delta(B, E_v)\} \ge \mathrm{dist}(A, B)/2d_v.$$

Therefore Proposition 3.13 applied with exponent Q shows that

$$\rho_k(v) \le \psi(2d_v/\,\mathrm{dist}(A, B))^{1/Q},$$

and so $\|\rho_k\|_\infty$ tends to 0 when k tends to $+\infty$.

Finally the CLP assertion follows from Proposition 3.5.1. □

Remarks :

1) P. Haïssinsky [Haia] has proved that if Z satisfies the combinatorial Q-Loewner property then for every disjoint non-degenerate continua $A, B \subset Z$ one has for $1 \leq p < Q$:

$$\lim_{k \to +\infty} \mathrm{Mod}_p(A, B, G_k) = +\infty.$$

2) In [KK] S. Keith and the second (named) author establish that Q_M is equal to the *Ahlfors regular conformal dimension* of Z *i.e.* the infimum of Hausdorff dimensions of Ahlfors regular metric spaces quasi-Moebius homeomorphic to Z. In particular Q_M is a quasi-Moebius invariant of Z.

3) M. Barlow and R. Bass have established submultiplicative and supermultiplicative inequalities for the combinatorial 2-modulus on some self-similar space like the square Sierpinski carpet. Their method relies on the analysis of random walks on graph approximations (see [Bar03] Lemma 3.2).

4) One may formulate a variant of Proposition 3.15 for doubling spaces in general, by modifying the definition of M_k. For constants C_1, C_2, and each k, let M_k be the supremum, as x ranges over Z, of the p-modulus of the family of curves of diameter at least $C_1 2^{-k}$ lying in the ball $B(x, C_2 2^{-k})$. Then for suitably chosen C_1, C_2, one obtains a submultiplicative inequality $M_{k+l} \leq C \cdot M_k M_l$.

5) We could have chosen to define the CLP using the more general notion of κ-approximations from [BK02], so as to make the quasi-Moebius invariance automatic from the definition. However, this would simply make it harder to verify in examples, forcing one to prove the equivalence of the two definitions anyway.

4. Combinatorial modulus and Coxeter groups

In this section we study the combinatorial modulus in the case of hyperbolic Coxeter groups. Consider a hyperbolic Coxeter group Γ with connected boundary, and let Z be the metric space $\partial\Gamma$ equipped with a self-similar metric (see Definition 3.12). We fix in the sequel some κ-approximation $\{G_k\}_{k \in \mathbb{N}}$ of Z.

We wish to establish a kind of a filtration for the combinatorial modulus of curve families in Z. Let d_0 be a fixed (small) positive constant. We are mainly concerned with the following families of curves. Let $\partial P \subset Z$ be a parabolic limit set and let $\delta, r > 0$. Denote by $\mathcal{F}_{\delta,r}(\partial P)$ the family of curves γ in Z satisfying the following conditions :

- $\gamma \subset N_\delta(\partial P)$ and $\operatorname{diam} \gamma \geq d_0$,
- $\gamma \not\subseteq N_r(\partial Q)$ for any connected parabolic limit set $\partial Q \subsetneqq \partial P$.

For $\epsilon > 0$ and for any (parametrized) curve η in Z, let $\mathcal{U}_\epsilon(\eta)$ be the set of curves whose C^0 distance to η is smaller than ϵ.

Theorem 4.1. *Let $p \geq 1$, ϵ, $r > 0$, let $\partial P \subset Z$ be a parabolic limit set and let η be any curve in ∂P. There exist $\delta = \delta(r) > 0$ and $C = C(p, \epsilon, \eta, r) > 0$ such that one has for every $k \in \mathbb{N}$*

$$\operatorname{Mod}_p(\mathcal{F}_{\delta,r}(\partial P), G_k) \leq C \operatorname{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

In addition when p belongs to a compact subset of $[1, +\infty)$ the constant C may be choosen independent of p.

In combination with Proposition 3.1.1, this leads to:

Corollary 4.2. *Let $\eta \subset Z$ be a curve whose diameter is larger than d_0, and let ∂P be the smallest parabolic limit set containing η. Let $r > 0$ be small enough in order that $\eta \not\subseteq \overline{N}_r(\partial Q)$ for any parabolic limit set $\partial Q \subsetneqq \partial P$. Then for $\epsilon > 0$ small enough we have for every $k \in \mathbb{N}$:*

$$\operatorname{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k) \leq \operatorname{Mod}_p(\mathcal{F}_{\delta,r}(\partial P), G_k) \leq C \operatorname{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k),$$

where δ and C are the constants defined in the previous theorem.

□

Proof of Theorem 4.1. Instead of considering all curves with diameter larger than d_0 – which might mean translating curves by group elements with word-length bounded by above in terms of d_0 – we may restrict ourselves to those whose convex hulls contain e. In the sequel of the proof we make this restriction and we keep the same notation for the restriction of $\mathcal{F}_{\delta,r}(\partial P)$. We shall also use – without further mention – the metric equivalence between Z and $\partial\Gamma$.

Let $I \subset S$ such that P is a conjugate of Γ_I. Thanks to Proposition 2.11 there exists L and $\delta > 0$ depending only on r, such that every element of $\mathcal{F}_{\delta,r}(\partial P)$ is a (L, I)-curve. By Proposition 2.10 there exists a finite subset $E \subset \Gamma$ such that for every $\gamma \in \mathcal{F}_{\delta,r}(\partial P)$ the subset $\bigcup_{g \in E} g\gamma \subset Z$ contains a curve of $\mathcal{U}_\epsilon(\eta)$. Let G'_k be the incidence graph of the covering of Z by the elements of $\bigcup_{g\in E} g^{-1}G_k^0$. According to Proposition 3.2 the moduli $\mathrm{Mod}_p(\cdot, G'_k)$ and $\mathrm{Mod}_p(\cdot, G_k)$ are comparable (since Γ acts on Z by bi-Lipschitz homeomorphisms). For any $\mathcal{U}_\epsilon(\eta)$-admissible function $\rho : G_k^0 \to \mathbb{R}_+$ define a function $\rho' : {G'_k}^0 \to \mathbb{R}_+$ by

$$\rho'(g^{-1}v) = \rho(v),$$

where $v \in G_k^0$ and $g \in E$. Using the above property of E one gets immediately that ρ' is a $\mathcal{F}_{\delta,r}(\partial P)$-admissible function. Thus

$$\mathrm{Mod}_p(\mathcal{F}_{\delta,r}(\partial P), G'_k) \leq |E| \sum_{v\in G_k^0} \rho(v)^p,$$

and so

$$\mathrm{Mod}_p(\mathcal{F}_{\delta,r}(\partial P), G'_k) \leq |E| \,\mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

□

We now present a companion result to Theorem 4.1. Its statement requires some notation. As in paragraph 3.3, $\mathcal{F}_0$ denotes the family of curves $\gamma \subset Z$ with $\operatorname{diam}\gamma \geq d_0$. For a parabolic subgroup $P \lneqq \Gamma$ let $\mathcal{N}_r(\partial P)$ be the family of curves $\gamma \subset N_r(\partial P)$ with $\operatorname{diam}\gamma \geq d_0$. Let $\mathcal{L}$ be a collection of parabolic limit sets. For $r > 0$ set

$$\mathcal{F}_r(\mathcal{L}) := \mathcal{F}_0 \setminus \bigcup_{\partial Q\in\mathcal{L}} \mathcal{N}_r(\partial Q).$$

We denote by $\mathrm{Confdim}(\partial P)$ the Ahlfors regular conformal dimension of ∂P (its definition is recalled in Remark 2 at the end of Section 3), and we set

$$\mathrm{Confdim}(\mathcal{L}) := \max_{\partial Q\in\mathcal{L}} \mathrm{Confdim}(\partial Q).$$

The following property "controls" the modulus of neighborhoods of some limit sets by the modulus of the complementary subsets.

Theorem 4.3. *Let $\mathcal{L}$ be a Γ-invariant collection of connected proper parabolic limit sets. Assume that there exists $\partial P \in \mathcal{L}$ such that for every $\partial Q \in \mathcal{L}$ with $\partial Q \neq \partial P$, the set $\partial P \cap \partial Q$ is at most totally disconnected. Let $p > \mathrm{Confdim}(\partial P)$. There exist constants $C > 0$ and $a \in (0, 1)$ such that for $r > 0$ small enough and for every $k \in \mathbb{N}$ one has :*

$$\mathrm{Mod}_p(\mathcal{N}_r(\partial P), G_k) \leq C \cdot \sum_{\ell=0}^{k} \mathrm{Mod}_p(\mathcal{F}_r(\mathcal{L}), G_{k-\ell})\; a^\ell.$$

With the basic properties of the combinatorial modulus one obtains:

Corollary 4.4. *Let $\mathcal{L}$ be a Γ-invariant collection of connected proper parabolic limit sets. Assume that for every $\partial P, \partial Q \in \mathcal{L}$ with $\partial Q \neq \partial P$, the set $\partial P \cap \partial Q$ is at most totally disconnected. Let $p > \mathrm{Confdim}(\mathcal{L})$. There exist constants $C > 0$ and $a \in (0,1)$ such that for $r > 0$ small enough and for every $k \in \mathbb{N}$:*

$$\mathrm{Mod}_p(\mathcal{F}_0, G_k) \leq C \cdot \sum_{\ell=0}^{k} \mathrm{Mod}_p(\mathcal{F}_r(\mathcal{L}), G_{k-\ell})\ a^{\ell}.$$

□

As an example the collection $\mathcal{L}$ of all circular limit sets satisfies the above hypotheses with $\mathrm{Confdim}(\mathcal{L}) = 1$.

The proof of the theorem relies on some lemmata. The first one shows that two connected parabolic limit sets whose intersection is at most totally disconnected, have to "move away one from another linearly".

Lemma 4.5. *There is a constant $C_0 \geq 1$ with the following property. Let $\partial P_1, \partial P_2$ be two connected parabolic limit sets such that either $\partial P_1 = \partial P_2$ or $\partial P_1 \cap \partial P_2$ is at most totally disconnected. Assume that there exist $r > 0$ and a curve $\gamma \subset N_r(\partial P_1) \cap N_r(\partial P_2)$ with $\mathrm{diam}(\gamma) \geq C_0 r$. Then $\partial P_1 = \partial P_2$.*

Proof. Applying self-similarity property 3.11 we may assume that γ has unit diameter. Then ∂P_1 and ∂P_2 have diameter bounded away from zero, and hence P_1 and P_2 belong to a finite collection of parabolic subgroups with connected limit sets. But if $P_1 \neq P_2$, then $\partial P_1 \cap \partial P_2$ cannot contain an arc, because the intersection is totally disconnected. Hence taking C_0 large enough, the lemma follows. □

From now on we consider a connected parabolic limit set $\partial P \subset Z$.

Lemma 4.6. *Let $r > 0$, $C \geq 1$ and let $\gamma \subset N_r(\partial P)$ be a curve with $\mathrm{diam}(\gamma) > Cr$. For every $z \in (\gamma \setminus \partial P)$ there exists a subcurve γ' of γ such that letting $d := \mathrm{diam}(\gamma')$, one has :*

$$\gamma' \subset B(z, 4d) \cap N_{d/C}(\partial P) \text{ and } \gamma' \nsubseteq N_{d/(8C)}(\partial P).$$

Proof. Either there exists a point z_1 on γ with $d(z, z_1) < 2C\,\mathrm{dist}(z, \partial P)$ and $\mathrm{dist}(z_1, \partial P) = 2\,\mathrm{dist}(z, \partial P)$, or there does not.

If not then the part of γ contained in the ball centered on z of radius $2C\operatorname{dist}(z,\partial P)$ lies in the $(2\operatorname{dist}(z,\partial P))$-neighborhood of ∂P. The assumption on $\operatorname{diam}(\gamma)$ implies that γ exits the ball $B(z, 2C\operatorname{dist}(z,\partial P))$. Thus γ contains a desired subcurve which moreover contains z.

If z_1 exists then we repeat the process with z_1 instead of z. Eventually we get a sequence of points $z_0, z_1, ..., z_n$ on γ with $z_0 = z$ such that for all indices i :

$$\operatorname{dist}(z_i,\partial P) = 2^i \operatorname{dist}(z,\partial P) \text{ and } d(z_i, z_{i+1}) < 2C\operatorname{dist}(z_i,\partial P).$$

Since $\gamma \subset N_r(\partial P)$ and $\operatorname{dist}(z,\partial P) > 0$ the process has to stop, let z_n be the last point. From the first case argument we get a subcurve γ' of γ such that letting $d = \operatorname{diam}(\gamma')$ we have :

$$z_n \in \gamma' \subset B(z_n, 2C\operatorname{dist}(z_n,\partial P)) \ , \ d \geq 2C\operatorname{dist}(z_n,\partial P),$$

and

$$\gamma' \subset N_{d/C}(\partial P) \ , \ \gamma' \nsubseteq N_{d/(8C)}(\partial P).$$

Finally we compute

$$d(z, z_n) \leq \sum_{i=0}^{n-1} d(z_i, z_{i+1}) \leq 2^{n+1} C\operatorname{dist}(z,\partial P) = 2C\operatorname{dist}(z_n,\partial P) \leq d.$$

Therefore $\gamma' \subset B(z, 4d)$. $\square$

Pick for every $\ell \in \mathbb{N}$ a collection $\mathcal{B}_\ell$ of balls in Z centered on ∂P and of radius $2^{-\ell}$ such that the set $\{\frac{1}{2}B \ ; \ B \in \mathcal{B}_\ell\}$ is a minimal covering of ∂P. Let $\mathcal{B} = \bigcup_{\ell\in\mathbb{N}} \mathcal{B}_\ell$. Recall that the radius of a ball B is denoted by $r(B)$. From the previous lemma we get :

Lemma 4.7. *Let $r > 0, C \geq 1$ and let $\gamma \subset N_r(\partial P)$ be a curve with $\operatorname{diam}(\gamma) > Cr$. For every $z \in (\gamma \setminus \partial P)$ there exists a ball $B \in \mathcal{B}$ and a subcurve γ' of γ such that letting $d = \operatorname{diam}(\gamma')$ we have :*

$$\gamma' \cup \{z\} \subset B \ , \ r(B) \leq 36d \ , \ \gamma' \subset N_{d/C}(\partial P) \text{ and } \gamma' \nsubseteq N_{d/(8C)}(\partial P).$$

Proof. Consider the subcurve γ' obtained in the previous lemma. We compute :

$$\operatorname{dist}(z,\partial P) \leq 4d + \operatorname{dist}(\gamma',\partial P) \leq 5d.$$

Let w be a point in ∂P which realises $\operatorname{dist}(z,\partial P)$, we have $\gamma' \cup \{z\} \subset B(w, 9d)$. Pick a ball $B \in \mathcal{B}$ such that $w \in \frac{1}{2}B$ and $r(B)/4 \leq 9d \leq r(B)/2$. Then B contains $B(w, 9d)$ and hence it contains $\gamma' \cup \{z\}$. In addition we have $r(B) \leq 36d$. $\square$

Proof of Theorem 4.3. Let $\partial P \in \mathcal{L}$ be the connected proper limit set of the statement, and let $p > \mathrm{Confdim}(\partial P)$. For appropriate $r > 0$ we wish to construct on every scale a $\mathcal{N}_r(\partial P)$-admissible function $\rho_k : G_k^0 \to \mathbb{R}_+$ with controlled p-mass. For this purpose we equip ∂P with an Ahlfors regular metric $\delta_{\partial P}$, which is quasi-Moebius to the restriction of the Z-metric to ∂P, and whose Hausdorff dimension Q satisfies $Q < p$. Let μ be the Q-Hausdorff measure of $(\partial P, \delta_{\partial P})$. Given any ball $B \subset Z$ centered on ∂P, its trace on ∂P is comparable to a ball of $(\partial P, \delta_{\partial P})$. We denote by $\tau(B)$ the radius of a minimal ball of $(\partial P, \delta_{\partial P})$ containing $B \cap \partial P$. One has

$$\mu(B \cap \partial P) \asymp \tau(B)^Q.$$

Let $\mathcal{B}$ be a ball family as considered in the statement of the previous lemma. We inflate every element $B \in \bigcup_{\ell=0}^k \mathcal{B}_\ell$ to essentially unit diameter as in Definition 3.11. Let g_B be the corresponding group element. Define $G_k \cap B$ to be the incidence graph of the covering of B by the subsets $v \in G_k^0$ with $v \cap B \neq \emptyset$. Note that for $B \in \mathcal{B}_\ell$, the graph $G_k \cap B$ may be considered as a $(2\kappa L_0)$-approximation of $g_B(B)$ on scale $k - \ell$ (via the group element g_B).

For $k \in \mathbb{N}$ and $r > 0$ write $m_k := \mathrm{Mod}_p(\mathcal{F}_r(\mathcal{L}), G_k)$ for simplicity. Suppose that r is small compared to L_0, C_0, d_0 (L_0 and C_0 are defined respectively in statements 3.11 and 4.5) . Then using the group element g_B and the Γ-invariance of $\mathcal{L}$, we may pull back on $G_k \cap B$ a normalized minimal $\mathcal{F}_r(\mathcal{L})$-admissible function on scale $k - \ell$, in order to get for every ball $B \in \bigcup_{\ell=0}^k \mathcal{B}_\ell$ a function $\rho_B : (G_k \cap B)^0 \to \mathbb{R}_+$, with the following properties

(i) its p-mass satisfies, with $D \geq 1$ independent of k and of B :

$$M_p(\rho_B) \leq D \cdot m_{k-\ell} \cdot \tau(B)^p,$$

(ii) every curve $\gamma \subset B$ whose diameter is larger than $r(B)/36$ picks up a (ρ_B)-length larger than $\tau(B)$, unless it lies in the $(10^{-3} C_0^{-1} r(B))$-neighborhood of a limit set of $\mathcal{L}$.

Define ρ_k by

$$\forall v \in G_k^0,\ \rho_k(v) = \max \rho_B(v),$$

where the maximum is over all $B \in \bigcup_{\ell=0}^k \mathcal{B}_\ell$ with $v \cap B \neq \emptyset$. Its p-mass is linearly bounded by above by

$$\sum_{\ell=0}^{k} \sum_{B \in \mathcal{B}_\ell} M_p(\rho_B).$$

Since the metric $\delta_{\partial P}$ and the rectriction of the Z-metric are quasi-Moebius, they are Hölder equivalent (see [Hei01]). In particular there exists $\alpha > 0$ such that for every $B \in \mathcal{B}_\ell$ one has $\tau(B) \leq 2^{-\alpha\ell}$ (up to a multiplicative constant). With item (i) one obtains that the p-mass of ρ_k is linearly bounded by above by :

$$\sum_{\ell=0}^{k} m_{k-\ell} \sum_{B\in\mathcal{B}_\ell} \tau(B)^p \leq \sum_{\ell=0}^{k} m_{k-\ell} \sum_{B\in\mathcal{B}_\ell} \tau(B)^Q a^\ell \asymp \mu(\partial P) \sum_{\ell=0}^{k} m_{k-\ell}\ a^\ell,$$

with $a = 2^{-\alpha(p-Q)}$. Because $p > Q$ one has $0 < a < 1$.

It remains to prove that ρ_k is a $\mathcal{N}_r(\partial P)$-admissible function up to a multiplicative constant independent of k. Let $\gamma \in \mathcal{N}_r(\partial P)$.

Claim : For every $z \in \gamma$ there exists a ball $B_z \in \bigcup_{\ell=0}^k \mathcal{B}_\ell$ with $z \in B_z$ and such that the ρ_k-length of $\gamma \cap B_z$ is larger than $\tau(B_z)$.

Indeed, assume at first that $z \in \gamma \cap \partial P$. Pick $B_z \in \mathcal{B}_k$ with $z \in B_z$. Since for $B \in \mathcal{B}_k$ we may assume that the function ρ_B is larger than $\tau(B)$, the ball B_z admits the desired property.

Now assume that $z \in \gamma \setminus \partial P$. Because r is small compared with d_0 the hypotheses of Lemma 4.7 are satisfied with $C = C_0$. Let $B \in \mathcal{B}$ as in this lemma. If B belongs to $\bigcup_{\ell\geq k} \mathcal{B}_\ell$, then choosing $B_z \in \mathcal{B}_k$ with $B \subset B_z$ we are back to the previous case.

If $B \in \bigcup_{\ell=0}^{k-1} \mathcal{B}_\ell$, let $B_z = B$ and consider the subcurve γ' given by Lemma 4.7 applied with $C = C_0$. It lies in the (d/C_0)-neighborhood of ∂P, hence our hypotheses and Lemma 4.5 assert that there is no other limit set in $\mathcal{L}$ whose (d/C_0)-neighborhood contains γ'. In addition we have

$$10^{-3}C_0^{-1} r(B_z) \leq 10^{-3}C_0^{-1} 36d \leq d/(8C_0),$$

thus γ' does not lie in any $(10^{-3}C_0^{-1}r(B_z))$-neighborhood of a limit set of $\mathcal{L}$. Therefore the claim follows from item (ii).

According to $5r$-covering theorem (see [Mat95]) we may extract from the collection $\{10B_z\ ;\ z \in \gamma\}$ a finite cover $\{10B_1, ..., 10B_n\}$ of γ such that the balls $2B_1, ..., 2B_n$ are pairwise disjoint. This last property ensures that for every $v \in G_k^0$ the number of elements of $\{B_1, ..., B_m\}$ which meet v is bounded by above by a constant C_1 which depends only on κ and on the geometry of Z. With the claim we obtain

$$C_1 L_{\rho_k}(\gamma) \geq \sum_{i=1}^{n} L_{\rho_k}(\gamma \cap B_i) \geq \sum_{i=1}^{n} \tau(B_i).$$

Reordering if necessary we may assume that γ enters successively the $10B_i$'s. Let $w_i \in \partial P$ be the center of B_i. For every $i \in \{1, ..., n-1\}$ the balls $10B_i$ and $10B_{i+1}$ intersect, so one of two balls $20B_i$ and $20B_{i+1}$ contains the pair $\{w_i, w_{i+1}\}$. Thus $20B_i \cap \partial P$ and $20B_{i+1} \cap \partial P$ intersect. Since the metric $\delta_{\partial P}$ and the rectriction of the Z-metric are quasi-Moebius, it follows from the triangle inequality that the last sum is linearly bounded from below by the $\delta_{\partial P}$-diameter of $\bigcup_{i=1}^{n} 20B_i \cap \partial P$. Therefore $L_{\rho_k}(\gamma)$ is bounded from below in terms of d_0. □

5. Cannon Coxeter groups

Using the techniques developed in the previous section we prove a special case of the Cannon's conjecture (Theorem 5.1). This result was already known, we review a proof due to M. Davis at the end of the section. However our methods are new and of different flavour. An application to the hyperbolic Coxeter groups whose boundary is homeomorphic to the Sierpinski carpet is presented in Corollary 5.5.

Theorem 5.1. *Let Γ be a hyperbolic Coxeter group whose boundary is homeomorphic to the 2-sphere. Then there is a properly discontinuous, cocompact, and isometric action of Γ on $\mathbb{H}^3$, the real hyperbolic space.*

Let $\{G_k\}_{k\in\mathbb{N}}$ be a κ-approximation of $\partial\Gamma$, and let d_0 be a fixed small positive constant. As before we denote by $\mathcal{F}_0$ the family of curves $\gamma \subset \partial\Gamma$ with $\operatorname{diam}\gamma \geq d_0$. The following proposition is the main step of the proof.

Proposition 5.2. *For $p = 2$ the modulus $\operatorname{Mod}_p(\mathcal{F}_0, G_k)$ is bounded independently of the scale k.*

Its proof will use the following general dichotomy :

Lemma 5.3. *Let Φ be a hyperbolic group whose boundary is connected. Then either $\partial\Phi$ is a topological circle, or it contains a subset homeomorphic to the capital letter* I.

Proof. Since $\partial\Phi$ is locally connected and nontrivial, it contains an arc X (see [BK05b]). If $x \in X$ is not an endpoint and if some neighborhood of x in $\partial\Phi$ is contained in X, then clearly $\partial\Phi$ is a 1-manifold and so is homeomorphic to the circle. Otherwise, if $x_1, x_2 \in X$ are distinct points which are not endpoints of X, then we may find $y_1, y_2 \in \partial\Phi \setminus X$ and disjoint arcs $J_i \subset \partial\Phi$, $(i = 1, 2)$, joining y_i to x_i. Trimming the J_i's and taking the union $X \cup J_1 \cup J_2$, we get a subset homeomorphic to the capital letter I, provided y_i is sufficiently close to x_i. □

Proof of Proposition 5.2. With the notation of Section 4, let $\mathcal{L}$ be the collection of circular limit sets. We will prove that for every $r > 0$ there exists $\delta = \delta(r) > 0$ such that for every parabolic subgroup P of Γ with connected non-circular limit set, $\mathrm{Mod}_2(\mathcal{F}_{\delta,r}(\partial P), G_k)$ is bounded independently of the scale k. This together with Proposition 3.1.2 imply that for every $r > 0$ the modulus $\mathrm{Mod}_2(\mathcal{F}_r(\mathcal{L}), G_k)$ is so. Therefore with Corollary 4.4 the statement of Proposition 5.2 will follow.

Let ∂P be a non-circular connected parabolic limit set. According to the above lemma, ∂P contains a graph homeomorphic to the capital letter I. Express the capital letter I as the union of two arcs α, α' where α joins the upper left endpoint to the lower right endpoint, and α' joins the upper right endpoint to the lower left endpoint. Bending the two horizontal segments of α' we obtain a vertical segment denoted by β. Consider a thin vertical rectangle $\mathcal{R}$ which is a small planar neighborhood of the vertical segment $\alpha \cap \beta$. Then every curve which lies within sufficiently small C^0 distance from α cross-connects in $\mathcal{R}$ its vertical sides, and every curve which lies within sufficiently small C^0 distance from β cross-connects in $\mathcal{R}$ its horizontal sides.

By assumption a similar picture appears in $\partial\Gamma$ and ∂P too : there exist two arcs η_1, η_2 in ∂P and there exists a topological rectangle $\mathcal{R} \subset \partial\Gamma$ such that every curve in $\partial\Gamma$ which lies within sufficiently small C^0 distance from η_1 cross-connects in $\mathcal{R}$ its vertical sides, and every curve which lies within sufficiently small C^0 distance from η_2 cross-connects in $\mathcal{R}$ its horizontal sides. Let $\mathcal{F}_h(\mathcal{R})$ (resp. $\mathcal{F}_v(\mathcal{R})$) be the family of curves contained in $\mathcal{R}$ and joining its horizontal (resp. vertical) sides. With Proposition 3.1 we get that for $\epsilon > 0$ small enough:

$$\mathrm{Mod}_2(\mathcal{U}_\epsilon(\eta_1), G_k) \leq \mathrm{Mod}_2(\mathcal{F}_v(\mathcal{R}), G_k),$$

$$\text{and } \mathrm{Mod}_2(\mathcal{U}_\epsilon(\eta_2), G_k) \leq \mathrm{Mod}_2(\mathcal{F}_h(\mathcal{R}), G_k).$$

The following lemma shows that $\min_{i=1,2} \mathrm{Mod}_2(\mathcal{U}_\epsilon(\eta_i), G_k)$ is bounded independently of k. Therefore Theorem 4.1 applied to η_1 or η_2 completes the proof of the Proposition 5.2. $\square$

Lemma 5.4. *There exists a constant $C \geq 1$ such that for any topological rectangle $\mathcal{R} \subset \partial\Gamma$ one has for every $k \in \mathbb{N}$ large enough :*

$$\mathrm{Mod}_2(\mathcal{F}_v(\mathcal{R}), G_k) \cdot \mathrm{Mod}_2(\mathcal{F}_h(\mathcal{R}), G_k) \leq C.$$

Proof. Let $\mathrm{mod}(\cdot)$ denotes the classical analytic modulus on the Euclidean sphere S^2. A well-known result asserts that for $\mathcal{R} \subset S^2$:

$$\mathrm{mod}(\mathcal{F}_v(\mathcal{R})) \cdot \mathrm{mod}(\mathcal{F}_h(\mathcal{R})) = 1.$$

The lemma follows from this fact and from some process to relate the combinatorial 2-modulus on $\partial\Gamma$ with the analytic modulus on S^2 (see [CFP99] Th. 1.5 or [BK02] Cor. 8.8 for more details). See also [Haib] lemme 2.14 for an alternative and more direct proof of this lemma. □

Proof of Theorem 5.1. According to Proposition 5.2 and Corollary 3.14 one obtains that $\partial\Gamma$ is quasi-Moebius homeomorphic to the Euclidean 2-sphere. As explained in the sketch of proof in Subsection 1.3, a theorem of Sullivan completes the proof ([Sul81] p. 468, see also [Tuk86]). □

Recall that the *Sierpinski carpet* is the compact topological space constructed as follows : start with the unit square in the plane, subdivide it into nine equal subsquares, remove the middle open square, and then repeat this procedure inductively on the remaining squares.

Corollary 5.5. *Let Γ be a hyperbolic Coxeter group whose boundary is homeomorphic to the Sierpinski carpet, then Γ acts properly discontinuously by isometries on $\mathbb{H}^3$, and cocompactly on the convex-hull of its limit set.*

Proof. Recall that a *peripheral circle* in $\partial\Gamma$ is a nonseparating topological circle. A *peripheral subgroup* of Γ is the stabilizer of a peripheral circle. In [KK00] Th. 5, the following results are proved :

(i) there is only a finite number of conjugacy classes of peripheral subgroups ;
(ii) let $H_1, ..., H_k$ be a set of representatives of these classes, then the double group $\Gamma \star_{H_i} \Gamma$ [2] is a hyperbolic group with 2-sphere boundary.

Observe that Theorem 2.3 shows that every peripheral subgroup is a parabolic subgroup. Choose subsets $I_1, ..., I_k$ of S such that the subgroups P_i generated by I_i form a set of representatives of conjugacy classes of peripheral subgroups. Then $\hat{\Gamma} := \Gamma \star_{P_i} \Gamma$ is obviously an index 2 subgroup of a Coxeter group. Therefore the corollary follows from item (ii) and Theorem 5.1. □

[2] The notation $\Gamma \star_{H_i} \Gamma$ is a bit misleading as it suggests an amalgamated product of some kind, while in reality this is the fundamental group of a graph of groups with two vertices and k edges between them, so it is never an amalgamated product unless k=1.

Remark : The following proof of Theorem 5.1 has been communicated to us by M. Davis. A theorem of Bestvina-Mess [BM91] and the boundary hypothesis show that Γ is a virtual 3-dimensional Poincaré duality group. Then Theorem 10.9.2 of [Dav08] implies that Γ decomposes as $\Gamma = \Gamma_0 \times \Gamma_1$, where Γ_0 is a finite Coxeter group and where Γ_1 is a Coxeter group whose nerve is a 2-sphere. Applying Andreev's theorem to the dual polyhedron to the nerve, one obtains that Γ_1 acts on $\mathbb{H}^3$ as a cocompact reflection group.

6. The Combinatorial Loewner Property for Coxeter groups

This section gives a sufficient condition for the boundary of a hyperbolic Coxeter group to satisfy the combinatorial Loewner property (Theorem 6.6). Some examples of groups for which the condition applies are presented in the next section.

6.1. **Generic curves.** Let Γ be a hyperbolic Coxeter group whose boundary is connected. Let Z be the metric space $\partial\Gamma$ equipped with a self-similar metric (see Definition 3.12). We fix in the sequel some κ-approximation $\{G_k\}_{k\in\mathbb{N}}$ of Z.

As before d_0 is a fixed small positive constant. For $r > 0$ consider the family of curves $\gamma \subset Z$ with $\mathrm{diam}(\gamma) \geq d_0$, and such that $\gamma \not\subseteq N_r(\partial Q)$ for any connected proper parabolic limit set $\partial Q \subset Z$. It is a non-empty curve family provided r is small enough. According to Corollary 4.2 its combinatorial modulus is comparable to the one of any of its subsets of the form $\mathcal{U}_\epsilon(\eta)$. In particular it does not depend on r up to a multiplicative constant independent of the scale. We shall denote such a family of curves by $\mathcal{F}^g$ and we shall call it *a family of generic curves*. Similarly we will call its combinatorial modulus the *combinatorial modulus of generic curves*.

When $p \geq 1$ is understood we denote $\mathrm{Mod}_p(\mathcal{F}^g, G_k)$ by m_k for simplicity. It will be also convenient to set $L_k := m_k^{-1/p}$. It satisfies :

$$L_k = \sup_\rho \frac{L_\rho(\mathcal{F}^g)}{M_p(\rho)^{1/p}} \text{ with } L_\rho(\mathcal{F}^g) := \inf_{\gamma\in\mathcal{F}^g} L_\rho(\gamma),$$

where the supremum is over all positive functions of G_k^0.

We now study the asymptotic behaviour of the sequence $\{L_k\}_{k\in\mathbb{N}}$ depending on $p \geq 1$. Our main result establishes a weak type submultiplicative inequality for the sequence $\{L_k\}_{k\in\mathbb{N}}$ (Proposition 6.3). The results of this paragraph must be compared with those of paragraph 3.3 concerning the family $\mathcal{F}_0$.

For the rest of the paragraph p is an arbitrary number in $[1, +\infty)$. In the statement of the following two lemmata, A_0 denotes a fixed (large) positive number.

Lemma 6.1. *There exist a constant $\Lambda \geq 1$ and a positive function Φ of $(0, +\infty)$ with the following properties. Let B be a ball in Z and let $k \in \mathbb{N}$ such that the radius $r(B)$ satisfies*

$$A_0^{-1} \leq \frac{r(B)}{2^{-k}} \leq A_0.$$

Consider two balls of same radius $B_1, B_2 \subset B$, and let $t := \frac{r(B_i)}{r(B)}$. Then for every $\ell \in \mathbb{N}$ the $G_{k+\ell}$-combinatorial p-modulus of the family

$$\{\gamma \in \mathcal{F}(B_1, B_2) \ ; \ \gamma \subset \Lambda B\}$$

is greater than $m_\ell \cdot \Phi(t)$.

Proof. Using self-similarity property 3.11 we can restrict ourself to the case $k = 0$. For every $0 < r \leq \operatorname{diam} Z$ pick a maximal $\frac{r}{2}$-separated subset $E_r \subset Z$. Since Z is linearly connected there exists a constant $\lambda > 0$ such that every pair of points $x, y \in Z$ can be joined by a path of diameter less than $\lambda d(x, y)$. For every pair of points $x, y \in E_r$ we choose such a path and we call it $\eta_{x,y}$.

Let Λ be a large number compared with λ. Consider any two r-balls B_1, B_2 contained in B. Pick two points $z_1, z_2 \in E_r$ such that $B(z_i, r/2) \subset B_i$ for $i = 1, 2$. Then with the notation of Section 4 we have

$$\mathcal{U}_{r/2}(\eta_{z_1,z_2}) \subset \{\gamma \in \mathcal{F}(B_1, B_2) \ ; \ \gamma \subset \Lambda B\}.$$

Given $r > 0$ there is only a finite number of left handside terms. Therefore the desired inequality follows from Theorem 4.1 and from the fact that $r \asymp t$ by the rescaling assumption. □

Lemma 6.2. *There exist constants $\Lambda, D \geq 1$ and $b \in (0, 1)$ with the following properties. Let B be a ball in Z and let $k \in \mathbb{N}$ such that the radius $r(B)$ satisfies*

$$A_0^{-1} \leq \frac{r(B)}{2^{-k}} \leq A_0.$$

Consider two continua $F_1, F_2 \subset Z$ with $F_i \cap \frac{1}{4}B \neq \emptyset$ and $F_i \setminus B \neq \emptyset$. Then for every $\ell \in \mathbb{N}$ and every positive function ρ on $G^0_{k+\ell}$ there exists *a path in ΛB joining F_1 to F_2 whose ρ-length is smaller than*

$$D \cdot M_p(\rho)^{1/p} \cdot \sum_{n=0}^{\ell} L_{\ell-n}\ b^n.$$

Proof. The arguments are basically the same as those used in the proof of Proposition 3.9. Indeed pick $q \in \mathbb{N}$ such that $\lambda := 2^{-q}$ and $a := 2 \cdot (8\lambda)^{1/p}$ satisfy $\lambda < 1/8$ and $a < 1$. Let Λ be as in Lemma 6.1. Using Lemma 6.1 we can construct – like in the proof of Proposition 3.9 – a path joining F_1 to F_2 in ΛB whose ρ-length is smaller than

$$D \cdot M_p(\rho)^{1/p} \cdot \sum_{n=0}^{[\ell/q]} L_{\ell-nq}\ a^n,$$

where $D > 0$ depends only on Λ, q, p, κ and the geometry of Z, and where $[\cdot]$ denotes the integer part. Letting $b := a^{1/q}$ the lemma follows. □

With the above two lemmata we obtain :

Proposition 6.3. *There exist constants $b \in (0,1)$ and $C \geq 1$ such that for every pair of integers k, ℓ one has :*

$$L_{k+\ell} \leq C \cdot L_k \cdot \sum_{n=0}^{\ell} L_{\ell-n}\ b^n.$$

Morever when p belongs to a compact subset of $[1, +\infty)$ the constants C and b may be choosen independent of p.

Proof. Given a positive function $\rho_{k+\ell}$ of $G^0_{k+\ell}$ we wish to construct a curve $\gamma \in \mathcal{F}^g$ with controlled $\rho_{k+\ell}$-length. For this purpose we pick two disjoint balls $E_1, E_2 \subset Z$ such that $\mathcal{F}(E_1, E_2) \subset \mathcal{F}^g$, and we will look for γ in $\mathcal{F}(E_1, E_2)$.

For any $v \in G^0_k$, let B_v be a ball containing v and whose radius is approximately 2^{-k}. Let $\rho_k : G^0_k \to \mathbb{R}_+$ be defined by

$$\rho_k(v)^p = \sum_{w \cap 4\Lambda B_v \neq \emptyset} \rho_{k+\ell}(w)^p,$$

where Λ is the constant appearing in Lemma 6.2. Since Z is a doubling metric space one has

$$M_p(\rho_{k+\ell}) \asymp M_p(\rho_k)\,. \tag{6.1}$$

From Theorem 4.1 there exists a curve $\delta \in \mathcal{F}(E_1, E_2)$ whose ρ_k-length is linearly bounded by above by

$$M_p(\rho_k)^{1/p} \cdot L_k \,. \tag{6.2}$$

Let $v_i \in G_k^0$ so that δ enters successively $v_1, ..., v_n$, and set $B_i := B_{v_i}$ for simplicity. Then, providing 2^{-k} is small enough compared to $\operatorname{diam} E_i$, Lemma 6.2 allows one to construct by induction on $s \in \{1, ..., n\}$ a curve $\gamma_s \subset \cup_{i=1}^{s} 4\Lambda B_i$, joining E_1 to $\cup_{i=s+1}^{n} B_i \cup E_2$, whose $\rho_{k+\ell}$-length is bounded linearly by above by

$$\sum_{i=1}^{s} \rho_k(v_i) \cdot \sum_{n=0}^{\ell} L_{\ell-n}\ b^n.$$

Therefore $\gamma := \gamma_n$ belongs to $\mathcal{F}(E_1, E_2)$ and its $\rho_{k+\ell}$-length is bounded linearly by above by $L_{\rho_k}(\delta) \cdot \sum_{n=0}^{\ell} L_{\ell-n}\ b^n$. Thanks to estimates (6.1) and (6.2), the proposition follows. □

Associated to generic curves, we introduce the following critical exponent

$$Q_m := \sup\{p \in [1, +\infty) \ ; \ \lim_{k\to+\infty} \operatorname{Mod}_p(\mathcal{F}^g, G_k) = +\infty\}.$$

The previous theorem combined with the monotonicity and the continuity of the functions $p \mapsto \operatorname{Mod}_p(\mathcal{F}^g, G_k)$ gives

Corollary 6.4. *The set of $p \geq 1$ such that $\lim_{k\to+\infty} \operatorname{Mod}_p(\mathcal{F}^g, G_k) = +\infty$ is equal to the interval $[1, Q_m)$, in particular it is void if $Q_m = 1$.*

Proof. As a consequence of the weak submultiplicative inequality stated in Proposition 6.3 one has :

$$\lim_{k\to+\infty} L_k = 0 \iff \exists \ell \in \mathbb{N} \text{ with } \sum_{n=0}^{\ell} L_{\ell-n}\ b^n < C^{-1},$$

where C is the constant appearing in the weak submultiplicative inequality. Hence the set of $p \geq 1$ such that $\lim_{k\to+\infty} \operatorname{Mod}_p(\mathcal{F}^g, G_k) = +\infty$ is open in $[1, +\infty)$. □

Proposition 6.5. *One has $Q_m > 1$ if and only if Z has no local cut point.*

Proof. The above corollary shows that $Q_m > 1$ is equivalent to the fact that for $p = 1$ one has $\lim_{k\to+\infty} \operatorname{Mod}_p(\mathcal{F}^g, G_k) = +\infty$.

Assume z_0 is a local cut point of Z, and let $\eta \subset Z$ be a curve containing z_0 in its interior. Enlarging η if necessary, we may assume that for $\epsilon > 0$ small enough, the curve family $\mathcal{U}_\epsilon(\eta)$ is contained in a family $\mathcal{F}^g$ of generic curves. Therefore for any $p \in [1, +\infty)$ the moduli $\mathrm{Mod}_p(\mathcal{F}^g, G_k)$ and $\mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k)$ are comparable. Since z_0 is a local cut point, every curve which belongs to $\mathcal{U}_\epsilon(\eta)$ passes through z_0, provided ϵ is small enough. Choose for every $k \in \mathbb{N}$ an element $v_k \in G_k^0$ containing z_0. The function of G_k^0 whose value is 1 on v_k and 0 otherwise, is a $\mathcal{U}_\epsilon(\eta)$-admissible function of p-mass 1. Hence the sequence $\{\mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k)\}_{k\in\mathbb{N}}$ is bounded.

Conversely assume that Z has no local cut point. Let $\mathcal{F}^g$ be a family of generic curves and let $\eta \subset Z$, $\epsilon > 0$ such that $\mathcal{U}_\epsilon(\eta) \subset \mathcal{F}^g$. Since Z has no local cut point, a construction of J. Mackay [Mac08] shows that the family $\mathcal{U}_\epsilon(\eta)$ contains an infinite collection of pairwise disjoint curves. It implies obviously that $\lim_{k\to+\infty} \mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k) = +\infty$ for $p = 1$. □

6.2. **A sufficient condition for the CLP.** As in the previous subsection Γ is a hyperbolic Coxeter group with connected boundary, and $\partial\Gamma$ is equipped with a self-similar metric.

Among the curve families already considered, the combinatorial modulus of generic curves is the lowest one, while the combinatorial modulus of $\mathcal{F}_0$ is the largest one. Intermediate curve families are the $\mathcal{F}_r(\mathcal{L})$'s introduced in Section 4.

In the following statement we allow $\mathcal{L}$ to be the empty collection $\emptyset$. In this case we set $\mathcal{F}_r(\emptyset) := \mathcal{F}_0$ and $\mathrm{Confdim}(\emptyset) := 1$. The critical exponents Q_M and Q_m are respectively defined in paragraphs 3.3 and 6.1.

Theorem 6.6. *Let $\mathcal{F}^g$ be a family of generic curves and let $\mathcal{L}$ be a Γ-invariant collection of connected proper parabolic limit sets. Assume that for every pair of distinct elements $\partial P, \partial Q \in \mathcal{L}$, the set $\partial P \cap \partial Q$ is at most totally disconnected. Suppose that there exists $p \in [1, +\infty)$ satisfying*

$$p > \mathrm{Confdim}(\mathcal{L}) \text{ and } Q_m \leq p \leq Q_M,$$

such that for $r > 0$ small enough and for every $k \in \mathbb{N}$:

$$\mathrm{Mod}_p(\mathcal{F}_r(\mathcal{L}), G_k) \leq D\, \mathrm{Mod}_p(\mathcal{F}^g, G_k),$$

with $D = D(r) \geq 1$ independent of k. Then $p = Q_M$ and $\partial\Gamma$ satisfies the combinatorial Loewner property.

This leads to

Corollary 6.7. *Assume that* $\mathrm{Confdim}(\partial\Gamma) > 1$*, and suppose that for every proper, connected, parabolic limit set* $\partial P \subset \partial\Gamma$*, one has*

$$\mathrm{Confdim}(\partial P) < \mathrm{Confdim}(\partial\Gamma)\,.$$

Suppose furthermore that for every pair $\partial P, \partial Q$ *of distinct, proper, connected, parabolic limit sets, the subset* $\partial P \cap \partial Q$ *is at most totally disconnected. Then* $\partial\Gamma$ *satisfies the CLP.*

Proof. One knows that $Q_M = \mathrm{Confdim}(\partial\Gamma)$ from Remark 2 at the end of Section 3. Therefore the statement follows from Theorem 6.6 by considering the collection $\mathcal{L}$ of all proper connected parabolic limit sets. □

The rest of the paragraph is devoted to the proof of the theorem. We will slightly abuse notation writing L_k instead of $C \cdot L_k$ where C be the constant appearing in the statement of Proposition 6.3. With this convention the weak submultiplicative inequality in Proposition 6.3 simply writes

$$L_{k+\ell} \leq L_k \cdot \sum_{n=0}^{\ell} L_{\ell-n}\; b^n.$$

Letting $u_\ell := \sum_{n=0}^{\ell} L_{\ell-n}\; b^n$, it becomes $L_{k+\ell} \leq L_k \cdot u_\ell$.

Lemma 6.8. *Let* $p \in [1, +\infty)$ *and* $a \in (0, 1)$ *such that* $\sum_{\ell=0}^{+\infty} u_\ell^p\; a^\ell = +\infty$*. Then the sequence* $\{L_k\}_{k\in\mathbb{N}}$ *tends to* $+\infty$ *exponentially fast.*

Proof. Observe that $\{u_k\}_{k\in\mathbb{N}}$ is submultiplicative. Indeed from its definition and with the weak submultiplicative inequality one has

$$u_{k+\ell} = \Big(\sum_{n=0}^{\ell-1} L_{k+\ell-n}\; b^n\Big) + b^\ell u_k \;\leq\; \Big(\sum_{n=0}^{\ell-1} L_{\ell-n}\; b^n\Big) u_k + b^\ell u_k.$$

The weak submultiplicative inequality applied with $k = \ell = 0$ shows that $L_0 \geq 1$. With the above inequalities we get that $u_{k+\ell} \leq u_k \cdot u_\ell$.

Therefore there exists $\alpha \in \mathbb{R}_+$ such that $\lim_{k\to+\infty} \frac{\log u_k}{k} = \log \alpha$. With our hypotheses we obtain that $\alpha > 1$. It follows that there exists a constant $C \geq 1$ such that for every $k \in \mathbb{N}$

$$C^{-1} \alpha^{\frac{3}{4}k} \leq u_k \leq C \alpha^{\frac{5}{4}k}.$$

In another hand one has for every $k, \ell \in \mathbb{N}$:

$$u_{k+\ell} = \Big(\sum_{n=0}^{\ell-1} L_{k+\ell-n}\; b^n\Big) + b^\ell u_k \;\leq\; L_k \Big(\sum_{n=0}^{\ell-1} u_{\ell-n}\; b^n\Big) + b^\ell u_k.$$

Because $b \in (0,1)$ the above inequalities and a simple computation show the existence of constants $C_1, C_2 \geq 1$ with

$$\alpha^{\frac{3}{4}(k+\ell)} \leq C_1 L_k \alpha^{\frac{5}{4}\ell} + C_2 \alpha^{\frac{5}{4}k}.$$

Letting $k = \ell$ yields the desired conclusion. □

Proof of Theorem 6.6. Let p satisfying the hypotheses of the statement. At first we prove that the sequence $\{L_k\}_{k\in\mathbb{N}}$ is bounded. Since $p \leq Q_M$ part (ii) of Corollary 3.16 shows that the sequence $\{\mathrm{Mod}_p(\mathcal{F}_0, G_k)\}_{k\in\mathbb{N}}$ admits a positive lower bound. With our assumptions and with Corollary 4.4 we obtain the existence of constants $C > 0$ and $a \in (0,1)$ such that for every $k \in \mathbb{N}$

$$C \leq \sum_{\ell=0}^{k} m_{k-\ell}\ a^\ell = \sum_{\ell=0}^{k} L_{k-\ell}^{-p}\ a^\ell.$$

In particular $\{L_k\}_{k\in\mathbb{N}}$ does not tend to $+\infty$, and so according to the previous lemma the sum $\sum_{\ell=0}^{+\infty} u_\ell^p\ a^\ell$ is finite. The weak submultiplicativity inequality writes $L_{k-\ell} \geq L_k \cdot u_\ell^{-1}$, and hence

$$C \leq L_k^{-p} \cdot \sum_{\ell=0}^{k} u_\ell^p\ a^\ell.$$

Therefore $\{L_k\}_{k\in\mathbb{N}}$ is bounded.

We now claim that $\{L_k\}_{k\in\mathbb{N}}$ admits a positive lower bound. For this purpose observe that from its definition and with the weak submultiplicative inequality one has

$$u_{k+\ell} = \Big(\sum_{n=0}^{\ell-1} L_{k+\ell-n}\ b^n\Big) + b^\ell u_k\ \leq\ L_k\Big(\sum_{n=0}^{\ell-1} u_{\ell-n}\ b^n\Big) + b^\ell u_k.$$

Since $\{L_k\}_{k\in\mathbb{N}}$ is bounded so is $\{u_k\}_{k\in\mathbb{N}}$. Let M be an upper bound for $\{u_k\}_{k\in\mathbb{N}}$, we obtain

$$u_{k+\ell} \leq L_k S_\ell + b^\ell M \text{ with } S_\ell := \sum_{n=0}^{\ell-1} u_{\ell-n}\ b^n.$$

Assume by contradiction that $\liminf_{k\to+\infty} L_k = 0$. At first choose ℓ such that $b^\ell M < 1/2$, then pick k such that $L_k S_\ell < 1/2$; we get that $u_{k+\ell} < 1$. As seen already in the proof of Corollary 6.4, this implies that $\{L_k\}_{k\in\mathbb{N}}$ tends to 0. Since $p \geq Q_m$ it contradicts Corollary 6.4, the claim follows.

The claim combined with our assumption imply that $\mathrm{Mod}_p(\mathcal{F}_r(\mathcal{L}), G_k)$ is bounded independently of k. With Corollary 4.4 we get that the same holds for $\mathrm{Mod}_p(\mathcal{F}_0, G_k)$. Summarizing there exits a constant $C > 0$ such that for every $k \in \mathbb{N}$:

$$C^{-1} \leq \mathrm{Mod}_p(\mathcal{F}^g, G_k) \leq \mathrm{Mod}_p(\mathcal{F}_0, G_k) \leq C.$$

The p-combinatorial Loewner property comes now from Lemma 6.1 and from Propositions 3.9 and 3.13. Note that Proposition 3.13 requires that $p > 1$. This inequality is satisfied because by assumption $p > \mathrm{Confdim}(\mathcal{L}) \geq 1$. Finally one has $p = Q_M$ according to Corollary 3.16. □

Remarks and questions : 1) The equality $Q_M = Q_m$ is a necessary condition for the CLP. This follows from Remark 1 at the end of Section 3, by choosing the continua A, B in order that $\mathcal{F}(A, B)$ is contained in a family of generic curves.

2) Recall that Q_M is a quasi-Moebius invariant of Z (see Remark 2 at the end of Section 3). Is it true for Q_m too ?

7. Examples

We give various examples of boundaries of hyperbolic Coxeter groups satisfying the combinatorial Loewner property.

7.1. **Simplex groups.** Let (Γ, S) be an infinite hyperbolic Coxeter group such that every proper parabolic subgroup is finite. This last condition is equivalent to the fact that the Davis chamber of (Γ, S) is a simplex of dimension $|S| - 1$ (see [Dav08]). Using Proposition 6.5 one sees easily that $Q_m > 1$ if and only if $|S| \geq 4$. Since the only infinite parabolic subgroup is Γ itself, the assumptions of Corollary 6.7 are clearly satisfied for $|S| \geq 4$. Therefore when $|S| \geq 4$, the boundary of Γ admits the CLP.

This result was already known by other methods. Indeed it is a well-known theorem due to Lannér that such a group acts by isometries, properly discontinuously and cocompactly on the real hyperbolic space $\mathbb{H}^n$ of dimension $n = |S| - 1$. Moreover the boundary of $\mathbb{H}^n$ is the Euclidean $(n-1)$-sphere which is a Loewner space for $n - 1 \geq 2$ (see [HK98]).

7.2. **Prism groups.** Let $n \geq 3$, consider a n-simplex and truncate an open neighborhood of one of its vertices. The resulting polytope is a n-prism (*i.e.* isomorphic to the product of a segment with a $(n-1)$-simplex). Let $\Sigma \subset \mathbb{H}^n$ be a geodesic n-prism whose dihedral angles are submultiples of π, and those of the truncated face are equal to $\frac{\pi}{2}$. Such polytopes have been classified by Kaplinskii ([Kap74], see also [Vin85] table 4), they exist only when $n \leq 5$. Let Γ be the discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ generated by reflections along the codimension 1 faces of Σ except the truncated one. The subset $\Gamma \cdot \Sigma$ is equal to $\mathbb{H}^n$ minus a countable union of disjoint totally geodesic half-spaces. Therefore Γ is a (word) hyperbolic Coxeter group and its boundary identifies with S^{n-1} minus a countable union of disjoint Euclidean $(n-1)$-balls.

Up to conjugacy the only infinite proper parabolic subgroup is the simplex group generated by the faces adjacent to the truncated one. Its limit set is an Euclidean $(n-2)$-sphere. Let $\mathcal{L}$ be the collection of all proper parabolic limit sets. It follows that $\mathrm{Confdim}(\mathcal{L}) = n-2$ and that $\mathcal{L}$ satisfies the Γ-invariance and separability hypotheses of Theorem 6.6. Moreover one has $\mathcal{F}^g = \mathcal{F}_r(\mathcal{L})$. Therefore the following lemma combined with Theorem 6.6 shows that $\partial\Gamma$ admits the CLP.

Lemma 7.1. *One has $Q_M > n-2$.*

Proof. When $n = 3$ this follows from Proposition 6.5 since in this case $\partial\Gamma$ is homeomorphic to the Sierpinski carpet which doesn't possess any local cut point.

In general we will prove that $\partial\Gamma$ equipped with the induced Euclidean metric satisfies $Q_m > n-2$. Since $Q_m \leq Q_M$ and since Q_M is a quasi-Moebius invariant (see Remark 2 at the end of Section 3), the lemma will follow. For this purpose we will exhibit a collection $\{F_i\}_{i\in\mathbb{N}}$ of pairwise disjoint closed subsets of $\partial\Gamma$ with the following properties :

- the F_i's are uniformly Lipschitz equivalent to the unit Euclidean sphere S^{n-2},
- there exists a constant D with $d_0 \leq D \leq \frac{1}{2}\,\mathrm{diam}(F_i)$, such that for every $i \in \mathbb{N}$ one has $\{\gamma \subset F_i \ ;\ \mathrm{diam}(\gamma) \geq D\} \subset \mathcal{F}^g$.

Existence of such a collection implies that $\lim_{k\to+\infty} \mathrm{Mod}_{n-2}(\mathcal{F}^g, G_k) = +\infty$, which in turn implies that $Q_m > n-2$ thanks to Corollary 6.4.

Pick a wall M of Γ, its limit set in $\partial\mathbb{H}^n$ is contained in a $(n-2)$-sphere. Identifying $\partial\mathbb{H}^n$ with the unit $(n-1)$-Euclidean sphere, we may assume that ∂M is contained in the equator. Let $\mathcal{B}$ be the set of connected components of $\partial\mathbb{H}^n \setminus \partial\Gamma$. Because $\partial\Gamma$ is invariant by the reflection along M, every ball $B \in \mathcal{B}$ with $\overline{B} \cap \partial M \neq \emptyset$ is centered on the equator. Thus adding to ∂M a countable union of disjoint $(n-2)$-half-spheres contained in the north hemisphere, we obtain a subset $F \subset \partial\Gamma$ homeomorphic to S^{n-2}. Since the parabolic subgroups are quasi-convex in Γ, there exists a constant $C > 0$ such that for any distinct $B, B' \in \mathcal{B}$ one has $\Delta(B, B') \geq C$ (see [Bou04] p. 89). One deduces easily from this fact that F is Lipschitz-equivalent to S^{n-2}.

We construct F_i by induction, as a small deformation of F. Let S_0 be a smooth manifold diffeomorphic to S^{n-2}, contained in the north hemisphere, disjoint from F and lying within small Hausdorff distance from F. For every $B \in \mathcal{B}$ with $B \cap S_0 \neq \emptyset$, replace $B \cap S_0$ by the spherical boundary of the connected component of $B \setminus S_0$ which does not contain the center of B. Let F_0 be the resulting subset of $\partial\Gamma$. Still it is a topological $(n-2)$-sphere disjoint from F, which lies within small Hausdorff distance from F. We claim that one can choose S_0 properly in order that F_0 is bi-Lipschitz equivalent to S^{n-2} with controlled Lipschitz constants. Indeed the Lipschitz regularity holds provided the diedral angles of the singular locus in F_0 admit an uniform positive lower bound. Using the previous lower bound estimates for the relative distances between elements of $\mathcal{B}$, one can construct S_0 in such a way that at every location its curvatures are smaller than those of the balls $B \in \mathcal{B}$ with $B \cap S_0 \neq \emptyset$. The control on the diedral angles follows, which in turn implies the claim. We repeat this procedure starting with a manifold S_1 whose Hausdorff distance to F is much smaller than the one between F_0 and F, in order that the resulting subset $F_1 \subset \partial\Gamma$ is disjoint from F_0. The construction of the F_i's is now clear. □

7.3. **Highly symmetric Coxeter groups.** Let L be a finite graph whose girth is greater or equal to 4 and such that the valency of each vertex is at least equal to 3. Let r be an even integer greater or equal to 6. Consider the Coxeter system (Γ, S) with one generating reflection for every vertex in L, and such that the order of the product of two distinct vertices v, w is $r/2$ if $(v, w) \in L^1$, and $+\infty$ otherwise. Its Davis complex is a 2-cell contractible complex X where every 2-cell is isomorphic to the regular right-angled r-gon in $\mathbb{H}^2$, and where the link of every vertex is isomorphic to L. Equipped with the geodesic distance induced by its 2-cells, X is a CAT(-1)-space on which Γ acts properly discontinuously, by isometries, and cocompactly. In particular Γ is a hyperbolic group. Moreover the Cayley graph of (Γ, S) identifies with the 1-skeleton of X. The walls of (Γ, S) identify with totally geodesic subtrees of X, namely those generated by a geodesic segment which joins the middles of two opposite edges in a 2-cell. We shall call them the *walls of* X. Consider X minus the union of its walls, the closures of the connected components are called the *Davis chambers of* X. Each of them is homeomorphic to the cone over L (see [Dav08], [Hag98] for more details).

Assume now that L is a highly symmetric graph in the following sense: for every pair of adjacent vertices $v, w \in L^0$ the pointwise stabilizer of the star of v acts transitively on the set of remaining edges in the star of w. Recall that the star of a vertex is the union of its incident edges. Examples of such graphs include the full bipartite graph with $s+t$ vertices $(s, t \geq 3)$, the Mouffang generalized polygons (see [Ron89]), every 3-transitive trivalent graph (such as the Petersen graph), the odd graphs (see [Big74]), *etc.*

We have

Proposition 7.2. *$\partial\Gamma$ satisfies the CLP.*

When L is a generalized polygon the statement was already known. Indeed in this case $\partial\Gamma$ possesses a self-similar metric δ such that $(\partial\Gamma, \delta)$ is an Loewner space [BP00].

Proof. Let $K \leqq \mathrm{Isom}(X)$ be the pointwise stabilizer of the Davis chamber containing the identity of Γ. Endow $\partial\Gamma$ with the self-similar metric d induced by a visual metric on ∂X ; then K acts on $(\partial\Gamma, d)$ by bi-Lipschitz homeomorphisms.

We will show that the assumptions of Theorem 6.6 are satisfied with $\mathcal{L} = \emptyset$. Since $\partial\Gamma$ does not admit any local cut point – it is homeomorphic to the Menger curve, see [KK00] – Proposition 6.5 implies that $Q_M > 1 = \mathrm{Confdim}(\mathcal{L})$. Let $p > 1$ with $Q_m \leq p \leq Q_M$.

Let $I \subset S$ be such that $\partial\Gamma_I$ is a connected parabolic limit set. We will prove by induction on $|S| - |I|$, that for $\delta, r > 0$ small enough, there exists a constant $C \geq 1$ such that for every $k \in \mathbb{N}$ one has

$$\mathrm{Mod}_p(\mathcal{F}_{\delta,r}(\partial\Gamma_I), G_k) \leq C\,\mathrm{Mod}_p(\mathcal{F}^g, G_k).$$

For $|S| - |I| = 0$ there is nothing to prove. Assume that the property holds for $|S| - |I| \leq N$ and let prove it for $|S| - |I| \leq N + 1$. Pick a curve $\gamma \subset \partial\Gamma_I$ which crosses every ∂M_v for $v \in I$, and such that for δ, r, ϵ small enough $\mathcal{F}_{\delta,r}(\partial\Gamma_I)$ and $\mathcal{U}_\epsilon(\gamma)$ have comparable modulus (see Corollary 4.2). We wish to use the group K to deform γ into a curve contained in a larger parabolic limit set.

Let $v \in I$ and consider a special subgroup Γ_J with $v \in J$. Its Davis complex, denoted by X_J, embeds equivariantly and totally geodesically in X. The wall of v in X_J is equal to $M_v \cap X_J$, where M_v denotes the wall of v in X. Hence we see that either $\partial\Gamma_J \cap \partial M_v$ is of empty interior in ∂M_v, or J contains all the neighbors of v in L. For $v \in S$, let U_v be the open dense subset of ∂M_v whose complement is the union of all empty interior subsets of the form $\partial\Gamma_J \cap \partial M_v$. We now use :

Lemma 7.3. *Every orbit of the K-action on $\Pi_{v\in S}\partial M_v$ is open.*

Assuming the lemma we finish the proof of the proposition. Thanks to the lemma there exists $g \in K$ such that for every $v \in I$ one has $g\gamma \cap U_v \neq \emptyset$. Let $Q \leqq \Gamma$ be a parabolic subgroup such that $g\gamma \subset \partial Q$. Enlarging γ if necessary we may assume that its convex hull contains the identity of Γ. This property remains for $g\gamma$, thus Q is a special subgroup, Γ_J say. Since $g\gamma$ crosses every ∂M_v for $v \in I$, one has $I \subset J$ (see Remark 1 at the end of Section 2). Therefore the above discussion implies that J contains the neighbors in L of every vertex $v \in I$. Hence, assuming $|I| < |S|$, one obtains that $|I| < |J|$. Moreover, since g is bi-Lipschitz, $\mathcal{U}_\epsilon(\gamma)$ and $\mathcal{U}_\epsilon(g\gamma)$ have comparable modulus. Therefore Corollary 4.2 combined with the induction assumption give the expected inequality. □

It remains to give the

Proof of Lemma 7.3. Let $\Sigma \subset X$ be the Davis chamber containing the identity of Γ. For a pair of adjacent vertices $v, v' \in L^0$ consider the walls $M, M' \subset X$ of the reflections v, v'. They intersect exactly at the center $o \in X$ of the 2-cell corresponding to the edge $(v, v') \in L^1$. Denote by H_-, H_+ the closed half-spaces bounded by M with $\Sigma \subset H_-$. We will show that the pointwise stabilizer of H_+ in $\mathrm{Isom}(X)$ acts transitively on the set of the edges of $M' \cap H_-$ adjacent to the one containing o. Since v, v' is an arbitrary pair of adjacent vertices, this result combined with the Γ-action on X gives the lemma. To do so we shall use an argument of F. Haglund ([Hag98], démonstration de la Proposition A.3.1). For simplicity assume first that r is a multiple of 4. Every automorphism of L gives rise to a group automorphism of Γ which in turn induces an isometry of X. Let $a \in \mathrm{Aut}(L)$ which pointwise stabilizes the star of v. Then the associated isometry f acts trivially on the wall M (the hypothesis r is a multiple of 4 is used at this point). Therefore one can define $g \in \mathrm{Isom}(X)$ by letting $g = \mathrm{id}$ on H_+ and $g = f$ on H_-. Since L is highly symmetric the result follows.

When r is only a multiple of 2, f is not anymore the identity on M but still it acts trivially on a tree T containing o and which zig-zag in the 2-cells (joining the center to the middles of two edges at even distance). Moreover one can choose T such that H_+ and $M' \cap H_-$ are contained in the closure of two distinct connected components of $X \setminus T$. Hence the previous argument generalizes. $\square$

7.4. **More planar examples.** Let Γ be a hyperbolic Coxeter group with planar connected boundary. Planarity is exploited to define transversal intersection of parabolic limit sets. Some modulus estimates are derived in Proposition 7.5. They are useful to establish the CLP in some examples.

Definition 7.4. Two curves $\gamma_1, \gamma_2 \subset \partial\Gamma$ *intersect tranversely* if there exists $\epsilon > 0$ such that every pair of curves $\eta_1 \in \mathcal{U}_\epsilon(\gamma_1),\ \eta_2 \in \mathcal{U}_\epsilon(\gamma_2)$ intersect. We say that two connected parabolic limit sets $\partial P_1, \partial P_2 \subset \partial\Gamma$ *intersects tranversely* if there exists two curves $\gamma_1 \subset \partial P_1, \gamma_2 \subset \partial P_2$ which intersect transversely, and such that the smallest parabolic limit set containing γ_i is ∂P_i for $i = 1, 2$.

Proposition 7.5. *Let $\partial P_1, \partial P_2$ be two connected parabolic limit sets which intersect transversely and let ∂Q be the smallest parabolic limit set containing $\partial P_1 \cup \partial P_2$. Then for every $p \geq 1$ and for δ, r small enough one has*

$$\inf_{i=1,2} \mathrm{Mod}_p(\mathcal{F}_{\delta,r}(\partial P_i), G_k) \asymp \mathrm{Mod}_p(\mathcal{F}_{\delta,r}(\partial Q), G_k).$$

Proof. Pick a pair of curves $\gamma_1 \subset \partial P_1, \gamma_2 \subset \partial P_2$ as in the definition. Let $\epsilon > 0$ be small enough and assume by contradiction that for $i = 1, 2$ the modulus $\mathrm{Mod}_p(\mathcal{U}_\epsilon(\gamma_i), G_k)$ is large compared to $\mathrm{Mod}_p(\mathcal{F}_{\delta,r}(\partial Q), G_k)$. Let $\rho : G_k \to \mathbb{R}_+$ be a $\mathcal{F}_{\delta,r}(\partial Q)$-admissible function with minimal p-mass. Then for $i = 1, 2$ the family $\mathcal{U}_\epsilon(\gamma_i)$ contains a curve η_i of small ρ-length (see Lemma 3.7). By transversality the curves η_1 and η_2 intersect, thus $\mathcal{F}_{\delta,r}(\partial Q)$ contains a curve of small ρ-length, contradicting the admissibility of ρ. Therefore the moduli $\inf_{i=1,2} \mathrm{Mod}_p(\mathcal{U}_\epsilon(\gamma_i), G_k)$ and $\mathrm{Mod}_p(\mathcal{F}_{\delta,r}(\partial Q), G_k)$ are comparable, so the proposition follows from Corollary 4.2. $\square$

As an illustation consider a 3 dimensional cube and truncate an open neighborhood of every vertex. The resulting polyhedron possesses 8 triangular faces and 6 hexagonal ones. Let $\Sigma \subset \mathbb{H}^3$ be a regular geodesic truncated cube whose diedral angles are submultiples of π, and those of the triangular faces are equal to $\frac{\pi}{2}$. By regular we mean that Σ admits all the cube symmetries. Let Γ be the discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ generated by reflections along the hexagonal faces of Σ. The subset $\Gamma \cdot \Sigma$ is equal to $\mathbb{H}^3$ minus a countable disjoint union of totally geodesic half-spaces. Therefore Γ is a (word) hyperbolic Coxeter group whose boundary is homeomorphic to the Sierpinski carpet. We will check that $\partial\Gamma$ satisfies the CLP.

For this purpose, equip $\partial\Gamma$ with a self-similar metric d such that the symmetries of the cube act on $(\partial\Gamma, d)$ by bi-Lipschitz homeomorphisms.

Let L be the graph whose vertices are the generators of Γ and whose edges are the pairs (s, s') with $s \neq s'$ and ss' of finite order. Then L is the 1-skeleton of the octahedron. One sees easily that there is only one type of proper parabolic subgroups with connected non-circular limit set: those whose graph is equal to L minus a vertex and its adjacent edges. By applying an appropriate cube symmetry to such a parabolic P, we get another parabolic P' such that ∂P and $\partial P'$ intersect transversely. Moreover the smallest parabolic limit set containing $\partial P \cup \partial P'$ is $\partial\Gamma$ itself. Hence Proposition 7.5 and the Lipschitz invariance of d, show that $\mathcal{F}_{\delta,r}(\partial P)$ and $\mathcal{F}^g$ have comparable modulus. The CLP comes now from Theorem 6.6 with $\mathcal{L}$ equal to the collection of all circular parabolic limit sets.

Remarks and questions : 1) In [Ben06] Y. Benoist exhibits examples of hyperbolic Coxeter groups with the following properties :

- Their Davis chambers are isomorphic to the product of two simplices of dimension 2,
- They are virtually the fundamental group of a compact locally CAT(-1) 4-dimensional manifold,
- They are not quasi-isometric to $\mathbb{H}^4$,
- They admit a properly discontinuous cocompact projective action on a strictly convex open subset of the real projective space $\mathbb{P}^4$.

An interesting question is to determine whether their boundary satisfies the CLP or even the analytical Loewner property. Previous examples of hyperbolic Coxeter groups enjoying the first three properties above appear in Moussong's thesis (unpublished, see [Dav08] example 12.6.8). In connection with these problems, we remark that F. Esselmann has classified the Coxeter polytopes in $\mathbb{H}^4$ isomorphic to the product of two simplices of dimension 2 (see [Ess96]).

2) Suppose Z is a Sierpinski carpet sitting in the standard 2-sphere. Mario Bonk ([Bon06] Prop. 7.6) has shown that if the peripheral circles of Z are uniformly quasi-Moebius homeomorphic to the standard circle, and have pairwise relative distance bounded away from zero, then Z satisfies a version of the Loewner property for transboundary 2-modulus. Since a Sierpinski carpet Coxeter group boundary $\partial\Gamma$ is quasi-Moebius homeomorphic to such a Z, this should imply that $\partial\Gamma$ satisfies a transboundary variant of the CLP for 2-modulus. It would be interesting to have examples which do not satisfy the (usual) CLP.

8. The Combinatorial Loewner Property for the standard Sierpinski carpet and Menger curve

Let $\mathbb{S} \subset [0,1]^2 \subset R^2$ be the square Sierpinski carpet (see Section 5 for the definition). Recall that the *cubical Menger sponge* is the continuum

$$\mathbb{M} = \bigcap_{i=1}^{3} \pi_i^{-1}(\mathbb{S}),$$

where $\pi_i : R^3 \to R^2$ is the map which forgets the i^{th} coordinate. Both $\mathbb{S}$ and $\mathbb{M}$ are endowed with the induced Euclidean metric.

Many ideas appearing in this paper have their origin in the analysis of the combinatorial modulus on $\mathbb{S}$ and $\mathbb{M}$. This section establishes the CLP for $\mathbb{S}$ and $\mathbb{M}$ (Theorem 8.4); we remark that similar reasoning applies to other families of self-similar examples, see Subsection 8.2.

The proof goes along the same lines as for Coxeter groups, but it is simpler. We think that this exposition may be useful to the reader to understand the Coxeter group case.

8.1. A criterion for a self-similar space to satisfy the CLP.

With the notation of Sections 3.3 and 4, one has:

Proposition 8.1. *Let Z be a linearly connected, approximately self-similar metric space, such that $Q_M > 1$. Assume that for every $p \geq 1$, for every non constant curve $\eta \subset Z$ and for every $\epsilon > 0$, there exists a constant $C = C(p, \eta, \epsilon)$ such that for every $k \in \mathbb{N}$:*

$$\mathrm{Mod}_p(\mathcal{F}_0, G_k) \leq C \,\mathrm{Mod}_p(\mathcal{U}_\epsilon(\eta), G_k).$$

Suppose futhermore that when p belongs to a compact subset of $[1, +\infty)$ the constant C may be choosen independent of p. Then Z satisfies the CLP.

The proof will reuse some previous arguments. In the following two lemmata the hypotheses of the proposition are supposed to be satisfied.

Lemma 8.2. *Write* $M_k := \mathrm{Mod}_p(\mathcal{F}_0, G_k)$ *and* $L_k := M_k^{-1/p}$ *for simplicity. There exist constants* $C \geq 1$ *and* $b \in (0,1)$ *such that for every* $k, \ell \in \mathbb{N}$ *one has :*

$$L_{k+\ell} \leq C \cdot L_k \cdot \sum_{n=0}^{\ell} L_{\ell-n}\, b^n.$$

Morever when p *belongs to a compact subset of* $[1, +\infty)$ *the constants* C *and* b *may be choosen independent of* p.

Proof. The assumptions on Z allow one to check that the proofs of Lemmata 6.1, 6.2 and of Proposition 6.3 apply verbatim to Z, with M_k instead of m_k. Therefore the inequality holds. □

Lemma 8.3. *For* $p = Q_M$ *the sequence* $\{M_k\}_{k\in\mathbb{N}}$ *admits a positive lower bound, and it doesn't tend to* $+\infty$.

Proof. The first part of the statement follows from Corollary 3.16(ii). The previous lemma shows that the set of p such that $\{M_k\}_{k\in\mathbb{N}}$ tends to $+\infty$ is an open subset of $[1, +\infty)$ — see the proof of Corollary 6.4 for more details. Therefore the second part of the statement holds. □

Proof of Proposition 8.1. It is enough to prove that for $p = Q_M$ one has $M_k \asymp 1$. The p-combinatorial Loewner property will then come from Lemma 6.1 and from Propositions 3.9 and 3.13. Note that Proposition 3.13 requires that $p > 1$, at this point the hypothesis $Q_M > 1$ is needed.

Thanks to Corollary 8.3 the sequence $\{M_k\}_{k\in\mathbb{N}}$ admits a positive lower bound. To obtain the upper bound, one argues exactly as in the second part of the proof of Theorem 6.6, using the fact that $\{M_k\}_{k\in\mathbb{N}}$ does not tend to $+\infty$. □

Proposition 8.1 will be useful to obtain the

Theorem 8.4. $\mathbb{S}$ *and* $\mathbb{M}$ *satisfy the combinatorial Loewner property.*

The proof requires some preliminary materials. For the rest of the subsection Z will denote $\mathbb{S}$ or $\mathbb{M}$. By construction, for every $k \in \mathbb{N}$, the space Z is tesselated by homothetic copies of itself of size 3^{-k}. Let G_k be the incidence graph of this covering. The collection $\{G_k\}_{k\in\mathbb{N}}$ is a κ-approximation of Z ([3]). As usual we identify every vertex of G_k with the corresponding subset in Z.

Definition 8.5. A *partial isometry* of Z is a map $s : D \subset Z \to s(D) \subset Z$ which is the restriction of an Euclidean isometry. It is said to be *at level* ℓ if the subsets D and $s(D)$ are unions of elements of G_ℓ^0. The set of level ℓ partial isometries of Z is denoted by $\mathcal{I}_\ell$. For a subset $E \subset Z$ and a partial isometry $s : D \subset Z \to s(D) \subset Z$, we denote by $s(E)$ the image $s(E \cap D)$.

The following lemma is an analogue of Proposition 2.10.

Lemma 8.6. *For every $\epsilon > 0$ and $d > 0$ there exists an integer $\ell = \ell(\epsilon, d)$, such that for every curves $\eta, \gamma \subset Z$ with $\operatorname{diam}(\gamma) \geq d$, the subset $\bigcup_{s \in \mathcal{I}_\ell} s(\gamma) \subset Z$ contains a curve which approximates η to within ϵ with respect to the C^0 distance.*

Sketch of proof. The main observation is the following. Let θ be a curve in the unit square (resp. cube) joining two opposite faces F_1, F_2, and let F_3 be a third face different from F_1 and F_2. Then there is an appropriate (diagonal) reflection s of the square (resp. cube), such that $\theta \cup s(\theta)$ is an arwise connected subset joining F_1, F_2, F_3 and the face opposite to F_3. The rest of the proof is left to the reader as an exercice. $\square$

Proof of Theorem 8.4. Apart from the modulus inequality and the assumption $Q_M > 1$, the hypotheses of Proposition 8.1 are clearly satisfied. The modulus inequality is derived from the previous lemma as in the second paragraph of the proof of Theorem 4.1.

To see that $Q_M > 1$, observe that Z contains an isometric copy of the product space $K \times [0,1]$, where K is the tryadic Cantor set. Let $\mathcal{F}$ be the curve family consisting of the segments $k \times [0,1] \subset Z$, with $k \in K$. One can compute the modulus of $\mathcal{F}$ (using *e.g.* the Beurling criterion [Haib] Prop. 2.1). Since $\operatorname{Mod}_p(\mathcal{F}, G_k) \leq \operatorname{Mod}_p(\mathcal{F}_0, G_k)$, this computation yields the inequality $Q_M > 1$. $\square$

[3] More rigourously there is a subsequence of $\{G_k\}_{k \in \mathbb{N}}$ which is a κ-approximation of Z. Indeed any interval of the form $[\frac{1}{2^k}, \frac{3}{2^k}]$ contains a power of $\frac{1}{3}$.

8.2. **Other examples.** Similar reasoning establishes the CLP for the following examples:

1) *Higher dimensional Menger spaces.* For $n \geq 2$, the Sierpinski carpet may be generalized to a space $\mathbb{S}_n$, by iterating the subdivision of the unit cube $[0,1]^n$ into 3^n subcubes of side length $\frac{1}{3}$, and removing the central open subcube. Then for $\ell \geq n$, we get an analog of the Menger space by letting

$$\mathbb{M}_{\ell,n} = \cap\{\pi_I^{-1}(\mathbb{S}_n) \mid I \subset \{1,\ldots,\ell\}, |I| = n\},$$

where $\pi_I : [0,1]^\ell \to [0,1]^n$ is the projection map which retains the coordinates indexed by elements of I. These examples can be further generalized by subdividing into k^n subcubes instead of 3^n, where k is an odd integer, or by removing a symmetric pattern of subcubes at each stage, instead of just the central cube, etc.

2) *Higher dimensional snowspheres, cf.* [Mey10]. Fix $n \geq 2$. Let Q_k denote the n-dimensional polyhedron obtained from the boundary of the cube $[0, 3^{-k}]^{n+1}$ by removing the interior of one n-dimensional face. Construct a sequence $\{P_k\}_{k\in\mathbb{Z}_+}$, where P_k is a metric polyhedron consisting of Euclidean n-cubes of side length 3^{-k}, as follows. Let P_0 be the boundary of the unit cube $[0,1]^n$, and inductively construct P_k from P_{k-1} by subdividing each n-cube face of P_{k-1} into 3^n subcubes, removing the central open subcube, and gluing on a copy of Q_k along the boundary. If we endow P_k with the path metric, then the sequence $\{P_k\}$ Gromov-Hausdorff converges to a self-similar space Z. As with Menger spaces, there are further generalizations of these examples.

3) *Pontryagin manifolds.* We modify slightly the above construction to obtain some Pontryagin manifolds. Fix $n \geq 2$, and let T^n be the standard n-torus obtained by identification of the opposite faces of the unit cube $[0,1]^n$. Let Q_k denote the polyhedron obtained as follows. Tesselate T^n by 3^n equal subcubes, remove the interior of one of them, and normalize the metric so that the side length of every subcube is 3^{-k}. Define $P_0 = T^n$ and inductively construct P_k from P_{k-1} by subdividing each n-cube face of P_{k-1} into 3^n subcubes, removing the central open subcube, and gluing on a copy of Q_k along the boundary. Then, as above, $\{P_k\}$ Gromov-Hausdorff converges to a self-similar space Z.

9. ℓ_p-Equivalence relations

This section covers some applications of the previous results to the ℓ_p-equivalence relations. These equivalence relations are of great interest because of their invariance by quasi-Moebius homeomorphisms. Moreover they can be used to provide examples of spaces which do not admit the CLP (see the remark at the end of the section).

We start by defining the ℓ_p-equivalence relation, it requires some preliminary materials. Let Z be a compact doubling metric space. Assume in addition that it is *uniformly perfect* , *i.e.* there exists a constant $0 < \lambda < 1$ such that for every ball $B(z, r)$ of Z with $0 < r \leq \operatorname{diam} Z$ one has $B(z, r) \setminus B(z, \lambda r) \neq \emptyset$.

We will associate to Z a Gromov hyperbolic graph G such that ∂G – namely its boundary at infinity – identifies canonically with Z. For this purpose fix a constant $\kappa \geq 1$, and pick for each $k \in \mathbb{N}^*$ a finite covering $\mathcal{U}_k$ of Z with the following properties :

– $\forall v \in \mathcal{U}_k,\ \exists z_v \in Z$ with $B(z_v, \kappa^{-1}2^{-k}) \subset v \subset B(z_v, \kappa 2^{-k})$,
– $\forall v, w \in \mathcal{U}_k,\ v \neq w :\ B(z_v, \kappa^{-1}2^{-k}) \cap B(z_w, \kappa^{-1}2^{-k}) = \emptyset$,
– $\forall z \in Z,\ \exists v \in \mathcal{U}_k$ with $B(z, \kappa^{-1}2^{-k}) \subset v$.

Let $\mathcal{U}_0 = \{Z\}$ be the trivial cover, and let G be the graph whose vertex set is $\cup_{k\in\mathbb{N}}\mathcal{U}_k$ and whose edges are defined as follows : two distinct vertices v and w are joined by an edge if

– v and w both belong to $\mathcal{U}_k$, $(k \in \mathbb{N})$, and $v \cap w \neq \emptyset$, or if
– one belongs to $\mathcal{U}_k$ the other one belongs to $\mathcal{U}_{k+1}$ and $v \cap w \neq \emptyset$.

Then G is a Gromov hyperbolic graph with bounded valency. The metric space Z is bi-Lipschitz equivalent to ∂G equipped with a visual metric [BP03].

For a countable set E and for $p \in [1, \infty)$, we denote by $\ell_p(E)$ the Banach space of p-summable real functions on E. The *first ℓ_p-cohomology group* of G is

$$\ell_pH^1(G) = \{f : G^0 \to \mathbb{R}\ ;\ df \in \ell_p(G^1)\}/\ell_p(G^0) + \mathbb{R},$$

where df is the function on G^1 defined by

$$\forall a \in G^1,\ df(a) = f(a_+) - f(a_-),$$

and where $\mathbb{R}$ denotes the set of constant fonctions on G^0. Equipped with the semi-norm induced by the ℓ_p-norm of df the topological vector space $\ell_pH^1(G)$ is a Banach space. It is a quasi-isometric invariant of

G and a quasi-Moebius invariant of Z. In addition $\ell_pH^1(G)$ injects in $\ell_qH^1(G)$ for $1 \leq p \leq q < +\infty$. See [Gro93], [BP03] for a proof of these results.

Recall that Z being a compact, doubling, uniformly perfect metric space, it admits a *doubling measure*, that is a finite measure μ such that for every ball $B \subset Z$ of positive radius one has : $0 < \mu(2B) \leq C\mu(B)$, with $C \geq 0$ independent of B (see [Hei01]). By a result of R. Strichartz [Str83] (see also [Pan89a], [BP03]) there is a continuous monomorphism

$$\begin{array}{rcl} \ell_pH^1(G) & \hookrightarrow & L^p(Z,\mu)/\mathbb{R} \\ [f] & \longmapsto & f_\infty \mod \mathbb{R}, \end{array}$$

where f_∞ is defined μ-almost everywhere as follows : for $z \in Z$ and for any geodesic ray r_z of G with endpoint z,

$$f_\infty(z) = \lim_{t\to+\infty} f(r_z(t)).$$

Following M. Gromov ([Gro93] p. 259, see also [Ele97] and [Bou04]) we set

$$B^0_p(Z) := \{u : Z \to \mathbb{R} \text{ continuous } ;\ u = f_\infty \text{ with } [f] \in \ell_pH^1(G)\},$$

and we define the ℓ_p-*equivalence relation* on Z by :

$$x \sim_p y \Longleftrightarrow \forall u \in B^0_p(Z),\ u(x) = u(y).$$

This is a closed equivalence relation which is invariant by the group of quasi-Moebius homeomorphisms of Z.

Proposition 9.1. *The cosets of the ℓ_p-equivalence relation are continua.*

Proof. We will use the following obvious properties of the space $B^0_p(Z)$:

(i) if u_1 and u_2 belong to $B^0_p(Z)$ then $\max\{u_1,u_2\}$ does too ;
(ii) let $u_1, u_2 \in B^0_p(Z)$, and let U_1, U_2 be open subsets of Z with $U_1 \cup U_2 = Z$. Assume that $u_1 = u_2$ on $U_1 \cap U_2$. Then the function u defined by $u = u_1$ on U_1 and $u = u_2$ on U_2, belongs to $B^0_p(Z)$.

First we claim that for any coset F and any compact subset $K \subset Z$ disjoint from F, there exists $u \in B^0_p(Z)$ such that $u(F) = 0$ and $u(K) = 1$. Indeed for any $z \in K$ there exists a function $u_z \in B^0_p(Z)$ with $u_z(F) = 0$ and $u_z(z) > 1$. Extract a finite cover of K from the open subsets $\{u_z > 1\}$. Let $U_1, ..., U_n$ be such a cover and let $u_1, ..., u_n$ be the corresponding functions. Then the function

$$v = \sum_{i=1}^{n} \max\{0, u_i\}$$

belongs to $B_p^0(Z)$ and satisfies $v(F) = 0$ and $v \geq 1$ on K. Letting $u = \min\{1, v\}$ the claim follows.

Assume now by contradiction that a coset F is a disjoint union of two non-empty compact subsets F_1 and F_2. Let $r > 0$. By the previous claim there exists a function $u \in B_p^0(Z)$ such that $u(F) = 0$ and $u(Z \setminus N_r(F)) = 1$. For r small enough $N_r(F)$ is the disjoint union of $N_r(F_1)$ and $N_r(F_2)$. Define a function v on Z by letting $v(z) = 1$ for $z \in N_r(F_2)$, and $v(z) = u(z)$ otherwise. Then the above property (ii) applied to the open subsets $Z \setminus \overline{N}_r(F_i)$ shows that the function v belongs to $B_p^0(Z)$. Moreover it satisfies $v(F_1) = 0$ and $v(F_2) = 1$. Contradiction. $\square$

The following result relates the ℓ_p-equivalence relation with the combinatorial p-modulus.

Proposition 9.2. *Let $\{G_k\}_{k\in\mathbb{N}}$ be a κ-approximation of Z. Assume $x, y \in Z$ satisfy $x \nsim_p y$, then there exist open subsets $U, V \subset Z$ containing respectively x and y such that*

$$\lim_{k\to+\infty} \mathrm{Mod}_p(U, V, G_k) = 0.$$

Proof. Recall that Z being a doubling metric space, up to a multiplicative constant the G_k-combinatorial p-modulus does not depend on the κ-approximation (see Proposition 3.2). Consider the graph G associated to a family of covers $\{\mathcal{U}_k\}_{k\in\mathbb{N}}$ as described at the beginning of the section. For $k \in \mathbb{N}$, let G_k be the subgraph of G which is the incidence graph of the covering $\mathcal{U}_k$. The family $\{G_k\}_{k\in\mathbb{N}}$ is a κ-approximation of Z. In addition G_k^0 identifies with the sphere in G of radius k centered at the unique vertex of G_0.

Let $u \in B_p^0(Z)$ with $u(x) \neq u(y)$. Changing u to $au + b$ ($a, b \in \mathbb{R}$) if necessary, we can assume that there exist open subsets U, V of Z with $x \in U, y \in V$ and $u \leq 0$ on U, $u \geq 1$ on V. Pick a function $f : G^0 \to \mathbb{R}$ such that $df \in \ell_p(G^1)$ and $f_\infty = u$. Choosing f properly one can ensure that for k large enough and for every $v \in G_k^0$ the following holds :

$$f(v) \leq 1/3 \text{ if } v \cap U \neq \emptyset,\ \ f(v) \geq 2/3 \text{ if } v \cap V \neq \emptyset,$$

(see [BP03] preuve du Th. 3.4). Let $\rho_k : G_k^0 \to \mathbb{R}_+$ be defined by

$$\rho_k(v) = 3 \max_{(v,w)\in G_k^1} |f(v) - f(w)|.$$

Obviously it is an $\mathcal{F}(U, V)$-admissible function. In addition its p-mass satisfies

$$M_p(\rho_k) \leq 3^p \sum_{a\in G_k^1} |df(a)|^p.$$

For $df \in \ell_p(G^1)$ the right handside term tends to 0 when k tends to $+\infty$. □

Combining several previous results we now collect some applications to hyperbolic Coxeter groups.

Corollary 9.3. *Let Γ be a hyperbolic Coxeter group. Then each coset of the ℓ_p-equivalence relation on $\partial\Gamma$ is either a point or a connected parabolic limit set.*

Proof. It is a straitforward consequence of Proposition 9.1 and Corollary 2.6. □

Corollary 9.4. *Let Γ be a hyperbolic Coxeter group with connected boundary. Let $p \geq 1$ and suppose that $\sim_p$ admits a coset different from a point and the whole $\partial\Gamma$. Then $\sim_p$ admits a coset F with* $\mathrm{Confdim}(F) = \mathrm{Confdim}(\partial\Gamma)$.

Proof. Let p be as in the statement and let $\mathcal{L}$ be the collection of the cosets of $\sim_p$ which are different from a point. From the above corollary its elements are connected proper parabolic limit sets. Obviously the Γ-invariance and separation hypotheses of Corollary 4.4 are satisfied. In addition, for every $r > 0$, Proposition 9.2 shows that $\mathrm{Mod}_p(\mathcal{F}_r(\mathcal{L}), G_k)$ tends to 0 when k tends to $+\infty$. The same holds for every $q \geq p$ by monotonicity of the function $q \mapsto \mathrm{Mod}_q$. Assume by contradiction that $\mathrm{Confdim}(\mathcal{L}) < \mathrm{Confdim}(\partial\Gamma)$. Since $\mathrm{Confdim}(\partial\Gamma) = Q_M$ (see [KK] and the remark 2 at the end of Section 3), one can apply Corollary 4.4 with exponent Q_M. One obtains that $\mathrm{Mod}_{Q_M}(\mathcal{F}_0, G_k)$ tends to 0 when k tends to $+\infty$, contradicting Corollary 3.16(ii). □

Finally we return to the general approximately self-similar metric spaces and to the combinatorial Loewner property. The second part of the following corollary is the combinatorial analogue of Theorem 0.3 in [BP03].

Corollary 9.5. *Let Z be an approximately self-similar metric space. Assume Z is connected and let $p \geq 1$. Then $p > Q_M$ if and only if $(Z/\sim_p) = Z$. If in addition Z satisfies the CLP, then for $1 \leq p \leq Q_M$ the quotient $Z/\sim_p$ is a singleton.*

Proof. The second part of the statement follows from Proposition 9.2 and from the monotonicity of $p \mapsto \mathrm{Mod}_p$. To establish the "only if" part of the first, one invokes that Q_M is equal to the Ahlfors regular conformal dimension of Z [KK], and the fact that $B_p^0(Z)$ separates the points of Z for p strictly larger than the Alhfors-regular conformal dimension [BP03]. Conversely if $(Z/\sim_p) = Z$, then Proposition 9.2 implies that $\mathrm{Mod}_p(\mathcal{F}_0, G_k)$ tends to 0 when k tends to $+\infty$. With Proposition 3.16(ii) we get that $p > Q_M$. □

Remarks and questions : Corollary 9.5 may be used to produce examples of spaces for which the CLP fails. Indeed suppose that for a given hyperbolic group Γ, the family of quotient spaces $\partial\Gamma/\sim_p$, $(p \in [1, +\infty))$, contains intermediate states between the singleton and the whole $\partial\Gamma$. Then, according to Corollary 9.5, $\partial\Gamma$ does not admit the CLP.

Currently all known examples of hyperbolic groups for which the CLP fails are of two types. Either their boundary admits local cut points – in which case the CLP fails "trivially" (see Proposition 3.5.2) – or they decompose as $\Gamma = A \star_C B$ and there exists $p \in [1, Q_M]$ with :

$$x \sim_p y \Longleftrightarrow x = y \text{ or } \exists g \in \Gamma \text{ such that } x, y \in g\partial B.$$

Examples of the second type, including some Coxeter groups, are described in [Bou04]. In the Coxeter group case, Corollary 9.4 shows that $\mathrm{Confdim}(\partial B) = \mathrm{Confdim}(\partial\Gamma)$.

It would be desirable to have a better understanding of the relations between the combinatorial modulus and the ℓ_p-cohomology.

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