DIFFERENTIABLE STRUCTURES ON METRIC MEASURE SPACES: A PRIMER

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1. INTRODUCTION

1.1. **Overview.** A key result of geometric function theory is Rademacher's theorem: any real-valued Lipschitz function on \mathbb{R}^n is differentiable almost everywhere. In [Che99], Cheeger found a far-reaching generalization of this result in the context of doubling metric measure spaces that satisfy a Poincaré inequality. The goal of this primer is to give a streamlined account of the construction of a measurable differentiable structure on such spaces, in the hopes of providing an accessible introduction to this area of active research. Our exposition is based on Cheeger's work, and incorporates a number of clarifications due to Keith [Kei04a], as well as a few of our own.

The scope of this primer is limited to the foundational results obtained in the first part of Cheeger's paper. For a broader discussion

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of the historical and mathematical context of this result, we refer the reader to the aforementioned papers and to the survey of Heinonen [Hei07].

One of Cheeger's first achievements was to see that it is possible to define a notion of differentiability in a metric space without any additional algebraic structure. A real valued function $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point x_0 if there is a some linear combination L of the coordinate functions $x_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \dots, n$, so that the behavior of f and L near x_0 agree up to first order. In other words, $f(\cdot) - f(x_0) = L(\cdot) - L(x_0) + o(d(\cdot, x_0))$. Cheeger observed that this definition of differentiability with respect to a set of coordinate functions makes sense for real valued functions on general metric measure spaces, where the role of the coordinate functions is played by suitable tuples of real valued Lipschitz functions. Cheeger's version of Rademacher's theorem for metric measure spaces asserts that there is a countable, full measure, disjoint collection of measurable subsets equipped with coordinate functions, so that every Lipschitz function is differentiable almost everywhere with respect to the corresponding coordinate functions.

Of course, there is no reason for this conclusion to hold for every metric measure space. Following Cheeger and Keith, we will show that it does hold when the space admits a *p*-Poincaré inequality.

Throughout this paper, $X = (X, d, \mu)$ denotes a metric measure space with metric d and measure μ . We will assume that μ is Borel regular and doubling: there exists some constant C so that for every $x \in X$ and r > 0, $\mu(B(x, 2r)) \leq C\mu(B(x, r))$.

1.2. Statement of the theorem. The discussion of differentiability given in the overview is formalized by the following definition.

Definition 1.1 (Cheeger, Keith). A measurable differentiable structure on a metric measure space (X, d, μ) is a countable collection of pairs $\{(X_{\alpha}, \mathbf{x}_{\alpha})\}$, called *coordinate patches*, that satisfy the following conditions:

- (1) Each X_{α} is a measurable subset of X with positive measure, and the union of the X_{α} 's has full μ -measure in X.
- (2) Each \mathbf{x}_{α} is a $N(\alpha)$ -tuple of Lipschitz functions on X, for some $N(\alpha) \in \mathbb{N}$, where $N(\alpha)$ is bounded from above independently of α . The maximum of all the $N(\alpha)$ is called the *dimension* of the differentiable structure. (3) For each α , $\mathbf{x}_{\alpha} = (x_{\alpha}^{1}, \dots, x_{\alpha}^{N(\alpha)})$ spans the differentials almost
- everywhere for X_{α} , in the following sense: For every Lipschitz

function $f: X \to \mathbb{R}$, there exists a measurable function $df^{\alpha}: X_{\alpha} \to \mathbb{R}^{N(\alpha)}$ so that for μ -a.e. $x \in X$,

(1.2)
$$\limsup_{y \to x} \frac{|f(y) - f(x) - df^{\alpha}(x) \cdot (\mathbf{x}_{\alpha}(y) - \mathbf{x}_{\alpha}(x))|}{d(x, y)} = 0.$$

Moreover, df^{α} is unique up to sets of measure zero.

(Note that Keith uses the term "strong measurable differentiable structure" for the above object.)

We can now state the main theorem, which gives a sufficient condition for the existence of such a differentiable structure. (See [Kei04a, Theorem 2.3.1] and [Che99, Theorem 4.38].)

Theorem 1.3. If (X, d, μ) is a metric measure space that is doubling and supports a p-Poincaré inequality with constant $L \ge 1$ for some $p \ge 1$ (see Definition 6.1), then X admits a measurable differentiable structure with dimension bounded above by a constant depending only on L and the doubling constant.

1.3. **Examples.** We illustrate some of the possibilities for measurable differentiable structures with the following examples.

- (1) Euclidean spaces: As a consequence of Rademacher's theorem, the metric measure space \mathbb{R}^n (with the usual Euclidean metric and Lebesgue measure), has a measurable differentiable structure given by a single coordinate patch (\mathbb{R}^n, \mathbf{x}), where \mathbf{x} is given by the coordinate functions $\mathbf{x} = (x^1, \dots x^n)$.
- (2) **Carnot groups:** As a specific example of such a space, consider the Heisenberg group \mathbb{H} of unipotent, upper triangular, 3×3 matrices. As a set, \mathbb{H} can be described by $\mathbb{R}^3 = \{(x, y, z)\}$ with a Carnot-Carathèodory metric and the usual Lebesgue measure. As a consequence of a theorem of Pansu [Pan89], this space carries a measurable differentiable structure with a single coordinate patch given by $\mathbf{x} = (x, y)$. In particular, the dimension of the differentiable structure is two, the topological dimension of the space is three, and the Hausdorff dimension of the space is four, showing that all three may differ.
- (3) Glued spaces: Consider the Heisenberg group H = {(x, y, z)} as above, and R⁴ = {(a, b, c, d)} with its usual metric and measure. Note that these are both Ahlfors 4-regular metric measure spaces. Choose an isometrically embedded copy of R¹ in each for example, the x-axis in H, and the a-axis in R⁴ and let X be the space formed by gluing H and R⁴ along these subsets.

There is a natural geodesic path metric d on X, and the measures combine to give an Ahlfors 4-regular measure μ on (X, d). By [HK98, Example 6.19(a)], X admits a *p*-Poincaré inequality for p > 3.

The space (X, d, μ) has a measurable differentiable structure with two coordinate patches,

$$(X_1 = \mathbb{H}, \mathbf{x_1} = (x, y))$$
 and $(X_2 = \mathbb{R}^4, \mathbf{x_2} = (a, b, c, d)).$

Notice that these coordinate patches are of different dimensions.

- (4) Laakso spaces: For every $Q \ge 1$, Laakso builds an Ahlfors Q-regular space that admits a 1-Poincaré inequality [Laa00]. These fractal spaces have topological dimension one.
- (5) **Bourdon-Pajot spaces:** These spaces arise as the boundary at infinity of certain Fuchsian buildings that are important examples in geometric group theory. They are all homeomorphic to the Menger sponge, and admit a 1-Poincaré inequality.
- (6) Limit spaces: The Gromov-Hausdorff limit of a sequence of Riemannian manifolds with Ricci curvature uniformly bounded from below, and diameter uniformly bounded from above, will admit a 1-Poincaré inequality, even though it may no longer be a manifold.

1.4. Organization of the paper. In Section 2 we give an overview of the proof; readers with background in analysis on metric spaces may prefer to skip this, and refer back to it for definitions as needed. The proof of Theorem 1.3 is given in Sections 3-6. In Appendix A we give a simpler proof of the well known result of Semmes [Che99, Appendix A] that a Poincaré inequality on a complete doubling metric space implies that the space is quasiconvex. (That is, for all $x, y \in X$ there is a path joining x to y of length at most Cd(x, y), for some uniform constant C.)

Theorem A.1. Suppose X admits a p-Poincaré inequality (with constant $L \ge 1$) for some $p \ge 1$. Then X is C-quasiconvex, where C depends only on L, p and the doubling constant.

1.5. Acknowledgments. We thank Enrico Le Donne for comments on an earlier draft.

2. Overview of the proof of Theorem 1.3

Our purpose in this section is to give a nontechnical presentation of the proof of Theorem 1.3, providing motivation, and a treatment more accessible to readers from other areas.

2.1. Finite dimensionality yields a measurable differentiable structure. The first step in the proof of Theorem 1.3 is a rather general argument showing that a σ -finite metric measure space has a measurable differentiable structure provided it satisfies a certain finite dimensionality condition. This involves two definitions:

Definition 2.1. An *N*-tuple of functions $\mathbf{f} = (f_1, \ldots, f_N)$, where $f_i : X \to \mathbb{R}$ for $1 \le i \le N$, is dependent (to first order) at $x \in X$ if there exists $\lambda \in \mathbb{R}^n \setminus \{0\}$ so that

(2.2) $\lambda \cdot \mathbf{f}(y) - \lambda \cdot \mathbf{f}(x) = o(d(x, y))$

as y goes to x. We denote the set where **f** is not dependent by $Ind(\mathbf{f})$.

Definition 2.3. We say that in (X, d, μ) the differentials have dimension at most N if every (N+1)-tuple of Lipschitz functions is dependent almost everywhere. We say that the differentials have finite dimension if they have dimension at most N for some $N \in \mathbb{N}$.

With these definitions, the first step of the proof is the following:

Proposition 4.1. If the differentials have dimension at most N_0 , then X admits a measurable differentiable structure whose dimension is at most N_0 .

The proof of Proposition 4.1 is selection argument analogous to the proof that a spanning subset of a vector space contains a basis. It works in considerable generality, e.g. for any σ -finite metric measure space.

2.2. Blow-up arguments, tangent spaces and tangent functions. The remainder of the proof is devoted to showing that under the conditions of Theorem 1.3, the differentials have finite dimension. To do this, one is faced with analyzing the behavior of a tuple (f_1, \ldots, f_N) of Lipschitz functions near a typical point in X, in order to produce nontrivial linear combinations satisfying (2.2). Following [Kei04a], we approach this using a blow-up argument. Blow-up arguments occur in many places in geometry and analysis; the common features are a rescaling procedure which normalizes some quantity of interest, combined with a compactness result which allows one pass to a limiting object which reflects the asymptotic behavior of the rescaled quantity. Then one proceeds by studying the limiting object in order to derive a contradiction, or to establish a desired estimate. We point out that the blow-up argument is not essential to this proof; it is possible to work directly in the space itself. However, in our view, the blow-up argument clarifies and streamlines the proof.

For readers who are unfamiliar with this setting and/or blow-up arguments, we first illustrate the ideas using a single function.

To fix terminology and notation, we recall that a function $f: Y \to Z$ between metric spaces is C-Lipschitz if

$$d_Z(f(p), f(q)) \le C \, d_Y(p, q)$$

for all $p, q \in Y$, while the Lipschitz constant of f

$$\operatorname{LIP}(f) = \sup_{p,q \in Y, p \neq q} \frac{d_Z(f(p), f(q))}{d_Y(p, q)}$$

is the infimal such C. We let LIP(Y) denote the collection of realvalued Lipschitz functions $f: Y \to \mathbb{R}$.

Now suppose $f \in \text{LIP}(X)$ is a Lipschitz function, and $x \in X$. To study the behavior of f near x, we may choose a sequence of scales $\{r_k\}$ tending to 0, and consider the corresponding sequence of rescalings of (X, d), i.e. the sequence of metric spaces $\{(X_k, d_k)\}$, where $X_k = X$ and $d_k = \frac{1}{r_k}d$. One then defines a sequence of functions $\{f_k : X_k \to \mathbb{R}\}$ by rescaling f accordingly: $f_k = \frac{1}{r_k}f$. Then f_k has the same Lipschitz constant as f, and the behavior of f in the ball $B(x, r_k)$ corresponds to the behavior of f_k on the unit ball $B(x, 1) \subset (X_k, d_k)$.

Next, by passing to a subsequence, and using a suitable notion of convergence, we may assume that the metric spaces (X_k, d_k) converge to a (Gromov-Hausdorff) tangent space (X_{∞}, d_{∞}) , and the functions $f_k : X_k \to \mathbb{R}$ converge to a tangent function $f_{\infty} : X_{\infty} \to \mathbb{R}$ which is LIP(f)-Lipschitz. We will suppress the details for now, and refer the reader to Section 3 for the notion of convergence (pointed Gromov-Hausdorff convergence) and the relevant compactness theorems. The space X_{∞} comes with a specified basepoint $x_{\infty} \in X_{\infty}$, and the restriction of f_{∞} to the ball $B(x_{\infty}, R)$ is a limit of the restrictions $f_k|_{B(x_k, R)}$.

2.3. Pointwise Lipschitz constants and tangent functions. The tangent function f_{∞} is LIP(f)-Lipschitz; however, since f_{∞} only reflects the behavior of the original function f near x, one is led to consider

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localized versions of the Lipschitz constant, as in the following definitions.

Definition 2.4. (Variation and pointwise Lipschitz constants) Suppose Y is a metric space, $x \in Y$, and $u \in LIP(Y)$.

(1) The variation of u on a ball $B(x,r) \subset Y$ is defined to be

(2.5)
$$\operatorname{var}_{x,r} u := \sup\left\{\frac{|u(y) - u(x)|}{r} \mid y \in B(x,r)\right\}.$$

We always have $\operatorname{var}_{x,r} u \leq \operatorname{LIP}(u)$.

(2) The lower pointwise Lipschitz constant of u at x is

$$\lim_{x \to 0} u := \liminf_{x \to 0} \operatorname{var}_{x,r} u \,.$$

(3) The upper pointwise Lipschitz constant of u at x is

$$\operatorname{Lip}_{x} u := \limsup_{r \to 0} \operatorname{var}_{x,r} u.$$

For any function $u: Y \to \mathbb{R}$, and $x \in Y$, we have $\lim_x u \leq \lim_x u$. In general, $\lim_x u$ and $\lim_x u$ need not be comparable. However, in the special case of $Y = \mathbb{R}^n$, if x is a point of differentiability of u, observe that $\lim_x u = \lim_x u = |\nabla u(x)|$.

Returning to the tangent function $f_{\infty} : X_{\infty} \to \mathbb{R}$, one observes that the restriction of f_{∞} to the ball $B(x_{\infty}, R) \subset X_{\infty}$ is the limit of the sequence $\{f_k|_{B(x_k,R)}\}$, which, in turn, arises from rescaling $f|_{B(x,Rr_k)}$. This leads to the bound

(2.6)
$$\lim_{x} f \le \operatorname{var}_{x_{\infty},R} f_{\infty} \le \operatorname{Lip}_{x} f$$

for all $R \in [0, \infty)$; in other words, the lower and upper pointwise Lipschitz constants of f at x control the variation of the tangent function f_{∞} on balls centered at x_{∞} .

Using the fact that the measure on X is doubling, one can strengthen this assertion to: For almost every $x \in X$, every tangent function f_{∞} of f at x satisfies $\lim_{x} f \leq \operatorname{var}_{y,r} f_{\infty} \leq \operatorname{Lip}_{x} f$ for every $y \in X_{\infty}$, $r \in [0, \infty)$. The second inequality is equivalent to $\operatorname{LIP}(f_{\infty}) \leq \operatorname{Lip}_{x} f$. However, for a general doubling metric measure space, the quantity $\operatorname{var}_{x,r} f$ can fluctuate wildly as $r \to 0$, which means that one could have $\operatorname{LIP}(f_{\infty}) \ll \operatorname{Lip}_{x} f$. A key observation of Keith – based on a closely related earlier observation of Cheeger – is that when (X, d, μ) satisfies a Poincare inequality, then this bad behavior can only occur when $x \in X$ belongs to a set of measure zero. **Definition 2.7** ([Kei04a, (5)]). We say X is a K-Lip-lip space if for every $f \in LIP(X)$,

(2.8)
$$\operatorname{Lip}_{x} f \leq K \operatorname{lip}_{x} f$$

for μ -a.e. $x \in X$. If X is a K-Lip-lip space for some K > 0, we say that X is a Lip-lip space.

Proposition 6.3. [Kei04a, Prop. 4.3.1] If (X, μ) is doubling, and satisfies a p-Poincaré inequality, then X is a K-Lip-lip space, where K depends only on the constants in the doubling and Poincaré inequalities.

By Proposition 6.3 it suffices to prove that the differentials have finite dimension in any Lip-lip space.

2.4. Tangent functions in Lip-lip spaces, and quasilinearity. By (2.6), if (X, d, μ) is a K-Lip-lip space, and $f \in \text{LIP}(X)$, then for μ -a.e. $x \in X$, every tangent function $f_{\infty} : X_{\infty} \to \mathbb{R}$ of f at x, and every $y \in X_{\infty}, r \in [0, \infty)$, one has

(2.9)
$$\lim_{x} f \leq \operatorname{var}_{y,r} f_{\infty} \leq \operatorname{LIP}(f_{\infty}) \leq \operatorname{Lip}_{x} f \leq K \operatorname{lip}_{x} f$$
,

so in particular

(2.10)
$$\operatorname{var}_{y,r} f_{\infty} \ge \frac{1}{K} \operatorname{LIP}(f_{\infty})$$

Thus for any ball $B(y,r) \subset X_{\infty}$, the variation of f_{∞} on B(y,r) agrees with the global Lipschitz constant $\text{LIP}(f_{\infty})$ to within a factor of K. This leads to:

Definition 2.11. A Lipschitz function $u : Z \to \mathbb{R}$ on a metric space Z is *L*-quasilinear if the variation of u on every ball B(x, r) satisfies

$$\operatorname{var}_{x,r} u \ge \frac{1}{L} \operatorname{LIP}(u).$$

In summary: when X satisfies the K-Lip-lip condition, then for every $f \in \text{LIP}(X)$ and μ -a.e. $x \in X$, every tangent function of f at x is K-quasilinear.

We need another version of the doubling condition appropriate to metric spaces:

Definition 2.12. A metric space Z is C-doubling if every ball can be covered by at most C balls of half the radius. A metric space is doubling if it is C-doubling for some C.

The last key ingredient in the proof is:

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Lemma 5.9. For every K, C there is an $N \in \mathbb{N}$ such that the space of K-quasilinear functions on a C-doubling metric space Z has dimension at most N.

The Gromov-Hausdorff tangent spaces X_{∞} arising from a doubling metric measure space X are all C-doubling for a fixed $C \in [1, \infty)$. Therefore by Lemma 5.9 there is a uniform upper bound on the dimension of any space of K-quasilinear functions on any Gromov-Hausdorff tangent space of X.

A related finite dimensionality result appears in [Che99]. We would like to point out that a similar idea appears in the earlier finite dimensionality theorem of Colding-Minicozzi [CM97], also in the setting of spaces which satisfy a doubling condition and a Poincare inequality (in [CM97] the spaces are Riemannian manifolds, though the smooth structure is not used in an essential way). In their paper, the quasilinearity condition is replaced by a condition which compares the size of a function on a ball (measured in terms of normalized energy) with its size on subballs, and uses this together with the Poincare inequality and doubling property to bound the dimension of a space of harmonic functions.

To complete the proof that the differentials have finite dimension in a K-Lip-lip space, we fix an n-tuple of Lipschitz functions $\mathbf{f} = (f_1, \ldots, f_n)$ for some $n \in \mathbb{N}$. Amplifying the above reasoning, there will be a full measure set of points $x \in X$ such that every set of tangent functions $\mathbf{f}_{\infty} = (f_{1,\infty}, \ldots, f_{n,\infty})$ at x spans a space of K-quasilinear functions. Thus when n is larger than the dimension bound coming from Lemma 5.9, there will be a nontrivial linear relation $\lambda \cdot \mathbf{f}_{\infty} = 0$ for some $\lambda \in \mathbb{R}^n \setminus \{0\}$. This implies that f_1, \ldots, f_n are dependent at x.

3. Preliminaries

3.1. Lipschitz constants. Recall that we work inside a metric measure space (X, d, μ) , where μ is a Borel regular measure on X.

We begin by making some observations about $\lim_{x} f$ and $\lim_{x} f$ (see Definition 2.4).

Lemma 3.1. If $f : X \to \mathbb{R}$ is Lipschitz, then $\lim_x f$ and $\lim_x f$ are Borel measurable functions of x.

Proof. For fixed r > 0, we see that $\operatorname{var}_{x,r} f$ is a lower-semicontinuous function of x. (Note that f is Lipschitz, so the variation over open balls cannot jump up as we approach a point.)

We can rewrite $\operatorname{Lip}_x f$ as follows:

(3.2)
$$\operatorname{Lip}_{x} f = \lim_{r \to 0} \sup \{ \operatorname{var}_{x,s} f \mid s < r \} \\= \lim_{r \to 0} \sup \{ \operatorname{var}_{x,s} f \mid s < r, s \in \mathbb{Q} \}$$

The first equality holds by definition, and the second from the inequalities

$$(s-\epsilon)\operatorname{var}_{x,(s-\epsilon)} f \le s \operatorname{var}_{x,s} f \le (s+\epsilon) \operatorname{var}_{x,(s+\epsilon)} f.$$

A countable supremum of measurable functions is measurable, and a pointwise limit of measurable functions is also measurable. Therefore, by equation (3.2), we see that $\operatorname{Lip}_x f$ is a measurable function of x. An analogous argument gives the same conclusion for $\operatorname{Lip}_x f$.

In fact, for any $x \in X$, $\operatorname{Lip}_{x}(\cdot)$ defines a seminorm on $\operatorname{LIP}(X)$.

Lemma 3.3. If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are Lipschitz, then for all $x \in X$ we have $\operatorname{Lip}_x(f+g) \leq \operatorname{Lip}_x f + \operatorname{Lip}_x g$.

Proof. Fix $x \in X$. Suppose we are given $\epsilon > 0$. By equation (3.2) there exists r > 0 so that for all $y \in B(x, r)$ we have

$$\frac{|f(y) - f(x)|}{d(x, y)} \le \operatorname{Lip}_x f + \epsilon \quad \text{and} \quad \frac{|g(y) - g(x)|}{d(x, y)} \le \operatorname{Lip}_x g + \epsilon.$$

We can find $y \in B(x, r)$ so that

$$\operatorname{Lip}_x(f+g) \leq \frac{|(f+g)(y) - (f+g)(x)|}{d(x,y)} + \epsilon,$$

and applying the triangle inequality we see that

$$\operatorname{Lip}_{x}(f+g) \leq (\operatorname{Lip}_{x}f+\epsilon) + (\operatorname{Lip}_{x}g+\epsilon) + \epsilon. \qquad \Box$$

Definition 3.4. Suppose $A \subset X$ is measurable. A point $x \in X$ is a *point of density of* A if

$$\lim_{r \to 0} \frac{\mu(B(x,r) \setminus A)}{\mu(B(x,r))} = 0.$$

A function $f: X \to \mathbb{R}$ is approximately continuous at $x \in X$ if there exists a measurable set A, for which x is a point of density, so that f restricted to A is continuous at x.

Lemma 3.5 (Theorem 2.9.13, [Fed69]). Assume μ is doubling. If $A \in X$ is measurable, then almost every point of A is a point of density for A.

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If $f : X \to \mathbb{R}$ is measurable, then f is approximately continuous almost everywhere.

For the first part of this lemma, see also [Hei01, Theorem 1.8]. The second part follows from Lusin's theorem.

3.2. Gromov-Hausdorff convergence. In this subsection we deal with metric spaces that do not a priori come with a doubling measure; however, they are doubling metric spaces, see Definition 2.12. Every metric measure space with a doubling measure is also a doubling metric space. (For complete metric spaces the converse is also true, but much less obvious.)

Definition 3.6. A sequence $\{(X_i, x_i)\}$ of pointed metric spaces *Gromov-Hausdorff converges* to a pointed metric space (X, x) if there is a sequence of maps $\{\phi_i : X \to X_i\}$, with $\phi_i(x) = x_i$ for all i, such that for all $R \in [0, \infty)$ we have

 $\limsup_{i \to \infty} \left\{ |d_{X_i}(\phi_i(y), \phi_i(z)) - d_X(y, z)| \mid y, z \in B(x, R) \subset X \right\} = 0,$

and

$$\forall \delta > 0, \limsup_{i \to \infty} \left\{ d(y, \phi_i(B(x, R + \delta))) \mid y \in B(x_i, R) \subset X_i \right\} = 0.$$

Such a sequence of maps is called a *Hausdorff approximation*.

Theorem 3.7. Every sequence of C-doubling pointed metric spaces $\{(X_i, x_i)\}$ has a subsequence which Gromov-Hausdorff converges to a complete C-doubling pointed metric space (X, x).

This follows from an Arzelà-Ascoli type of argument. For each $\epsilon > 0$ and radius r > 0 we can approximate $B(x_i, r) \subset X_i$ by a maximal ϵ -separated net whose cardinality is independent of i. By repeatedly choosing subsequences we can ensure that these nets converge in the limit to a net of at most the same cardinality. To finish the proof, take further subsequences as $\epsilon \to 0$ and $r \to \infty$. For more details see [BBI01, Theorem 7.4.15].

Definition 3.8. Let $\{(X_i, x_i)\}$ be a sequence of pointed metric spaces. For a fixed countable index set \mathcal{A} , suppose that $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ is a sequence of collections of functions indexed by \mathcal{A} :

$$\mathcal{F}_i = \{f_{i,\alpha} : X_i \to \mathbb{R}\}_{\alpha \in \mathcal{A}}.$$

Then the sequence of tuples $\{(X_i, x_i, \mathcal{F}_i)\}_{i \in \mathbb{N}}$ Gromov-Hausdorff converges to a tuple (X, x, \mathcal{F}) , where $\mathcal{F} = \{f_\alpha : X \to \mathbb{R}\}_{\alpha \in \mathcal{A}}$, if there is

a Hausdorff approximation $\{\phi_i : X \to X_i\}$ such that for all $x \in X$, $\alpha \in \mathcal{A}$,

$$\lim_{i \to \infty} f_{i,\alpha}(\phi_i(x)) = f_\alpha(x)$$

If $\{(X_i, x_i)\}$ is a sequence of *C*-doubling metric spaces, and $\{\mathcal{F}_i = \{f_{i,\alpha} : X_i \to \mathbb{R}\}_{\alpha \in \mathcal{A}}\}$ is a sequence such that for every $\alpha \in \mathcal{A}$, both the Lipschitz constants of the family $\{f_{i,\alpha}\}$ and the values $\{f_{i,\alpha}(x_i)\}$ are uniformly bounded, then after passing to a subsequence if necessary, the sequence of tuples $\{(X_i, x_i, \mathcal{F}_i)\}_{i \in \mathbb{N}}$ Gromov-Hausdorff converges.

Definition 3.9. Suppose X = (X, d) is a metric space, and $x \in X$.

- (1) A pointed metric space $(X_{\infty}, d_{\infty}, x_{\infty})$ is a *Gromov-Hausdorff* (*GH*) tangent space to X at x if it is the Gromov-Hausdorff limit of the pointed metric spaces $\{(X, d_i, x)\}_{i \in \mathbb{N}}$, where each $d_i = \frac{1}{r_i}d$ is the original metric d rescaled by $r_i > 0$, and the sequence (r_i) converges to zero.
- (2) Suppose now that $\mathcal{F} = \{f_{\alpha} : X \to \mathbb{R}\}_{\alpha \in \mathcal{A}}$ is a (countable) collection of functions on X. Then

$$\mathcal{U} = \{ u_{f_{\alpha}} : X_{\infty} \to \mathbb{R} \}_{\alpha \in \mathcal{A}}$$

is a collection of tangent functions of the functions $f_{\alpha} \in \mathcal{F}$ at $x \in X$ if \mathcal{U} is the Gromov-Hausdorff limit of the sequence of tuples $\{(X, \frac{1}{r_i}d, x, \mathcal{F}_i)\}_{i \in \mathbb{N}}$, where

$$\mathcal{F}_{i} = \{f_{i,\alpha} : (X, d_{i}, x) \to \mathbb{R}\}_{\alpha \in \mathcal{A}}, \text{ and}$$
$$f_{i,\alpha}(\cdot) = \frac{f_{\alpha}(\cdot) - f_{\alpha}(x)}{r_{i}}.$$

Since we used the same Hausdorff approximation and scaling factors for every $f_{\alpha} \in \mathcal{F}$, we say that the tangent functions are *compatible*.

We caution the reader that the terminology used for GH tangent spaces varies: Cheeger calls them tangent cones, and other objects tangent spaces, while Keith just calls them tangent spaces.

In general, the GH tangent spaces and functions one sees are highly dependent on the sequence of scales chosen.

Since rescaling preserves doubling and Lipschitz constants, our previous discussion has the following

Corollary 3.10. (1) Doubling metric spaces have (doubling) GH tangent spaces at every point.

(2) Any countable collection \mathcal{F} of uniformly Lipschitz functions on a doubling metric space X has a compatible collection of tangent functions \mathcal{U} at every point of X.

4. FINITE DIMENSIONALITY IMPLIES MEASURABLE DIFFERENTIABLE STRUCTURE

Our goal in this section is to prove

Proposition 4.1 (cf. Prop. 7.3.1, [Kei04a]). If the differentials have finite dimension of at most N_0 (see Definition 2.3), then X admits a measurable differentiable structure whose dimension is at most N_0 .

Proof. We have N_0 fixed by the hypotheses.

Lemma 4.2. We assume the hypothesis of Proposition 4.1. Then. given any measurable $A \subset X$ with positive measure, we can find a measurable $V \subset A$ with positive measure and a function $\mathbf{x}: V \to \mathbb{R}^N$ so that (V, \mathbf{x}) is a coordinate patch.

We now complete the proof, assuming Lemma 4.2. Since X is a doubling metric measure space it is σ -finite, so without loss of generality we may assume it has finite measure. Applying Lemma 4.2, we construct a sequence of coordinate patches $(U_1, \mathbf{x}_1), \ldots, (U_i, \mathbf{x}_i), \ldots$ inductively as follows. Given $i \geq 0$ and charts $(U_1, \mathbf{x}_1), \ldots, (U_i, \mathbf{x}_i)$, if the union $\bigcup_{j\leq i} U_j$ has full measure in X, we stop; otherwise, let \mathcal{C} be the collection of coordinate patches (V, \mathbf{x}) with $V \subset X \setminus \bigcup_{j \leq i} U_j$, and choose $(U_{i+1}, \mathbf{x}_{i+1}) \in \mathcal{C}$ such that $\mu(U_{i+1}) \geq \frac{1}{2} \sup\{\mu(V) \mid (V, \mathbf{x}) \in \mathcal{C}\}$. If the resulting sequence of charts $\{U_i\}$ is infinite, then we have $\mu(U_i) \to 0$ as $j \to \infty$, because $\mu(X) < \infty$. The union $\cup_j U_j$ has full measure, else we could choose a chart (V, \mathbf{x}) where V is a positive measure subset of $X \setminus \bigcup_j U_j$, and this contradicts the choice of the U_j 's.

It remains to prove Lemma 4.2. Before proceeding with this we note that (1.2) can be expressed more concisely as

(4.3)
$$\operatorname{Lip}_{x}\left(f(\cdot) - df^{\alpha}(x) \cdot \mathbf{x}_{\alpha}(\cdot)\right) = 0, \text{ for } \mu\text{-a.e. } x \in X_{\alpha}.$$

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Proof of Lemma 4.2. Consider the maximal N so that there exists some positive measure set $V \subset A$, and some N-tuple of Lipschitz functions **x**, so that $V \subset \text{Ind}(\mathbf{x})$, the set where **x** is not dependent. (Because of finite dimensionality, we have $0 \leq N \leq N_0$.)

We want to show that (V, \mathbf{x}) is a coordinate patch. Take any Lipschitz function $f \in \text{LIP}(X)$, and consider the (N+1)-tuple of functions (\mathbf{x}, f) . By the maximality of N this is dependent almost everywhere in V, so for μ -almost every $x \in V$ there exists $\lambda(x) \in \mathbb{R}$ and $df(x) \in \mathbb{R}^N$ so that

(4.4)
$$\operatorname{Lip}_{x}\left(\lambda(x)f(\cdot) - df(x) \cdot \mathbf{x}(\cdot)\right) = 0.$$

Since $V \subset \text{Ind}(\mathbf{x})$, we know that $\lambda(x) \neq 0$ almost everywhere, so, without loss of generality, we may assume that $\lambda(x) = 1$ everywhere.

The uniqueness of df, up to sets of measure zero, follows from the fact that $\operatorname{Lip}_x(\cdot)$ is a semi-norm on the space of Lipschitz functions (Lemma 3.3). Indeed, suppose that $df_1: V \to \mathbb{R}^N$ and $df_2: V \to \mathbb{R}^N$ both satisfy (4.4) for almost every x. Then

$$\operatorname{Lip}_{x}\left(\left(df_{1}(x) - df_{2}(x)\right) \cdot \mathbf{x}(\cdot)\right)$$

$$\leq \operatorname{Lip}_{x}\left(f(\cdot) - df_{1}(x) \cdot \mathbf{x}(\cdot)\right) + \operatorname{Lip}_{x}\left(f(\cdot) - df_{2}(x) \cdot \mathbf{x}(\cdot)\right)$$

$$= 0, \text{ for } \mu\text{-a.e. } x.$$

So, if df_1 and df_2 differed on a set of positive measure, then **x** would be dependent on that same set, but this is not possible. Therefore $df_1 = df_2$ almost everywhere.

It only remains to show that df is measurable. This follows if $df^{-1}(K)$ is measurable for each compact $K \subset \mathbb{R}^N$. We fix such a K for the remainder of the proof.

Consider the function $h_x : \mathbb{R}^n \to \mathbb{R}$ given by

$$h_x(\lambda) := \operatorname{Lip}_x(f(\cdot) - \lambda \cdot \mathbf{x}(\cdot)).$$

The triangle inequality for $\operatorname{Lip}_x(\cdot)$ (Lemma 3.3) implies that h_x is continuous; in fact, for $\lambda, \lambda' \in \mathbb{R}^N$,

$$|h_x(\lambda) - h_x(\lambda')| \leq \operatorname{Lip}_x((\lambda - \lambda') \cdot \mathbf{x})$$

$$\leq \sum_{1 \leq i \leq N} |\lambda_i - \lambda'_i| \operatorname{Lip}_x(x_i)$$

$$\leq \left(N \max_{1 \leq i \leq N} \operatorname{LIP}(x_i) \right) |\lambda - \lambda'|$$

Now set

$$E := \{ x \in V \mid \exists \lambda \in K \text{ s.t. } h_x(\lambda) = 0 \}.$$

As we have seen, df is uniquely defined up to a set of measure zero, so $df^{-1}(K)$ equals E less a set of measure zero. Consequently, it suffices

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to show that E is measurable. Fix a dense countable subset K' of K, and observe that

$$E = \{ x \in V \mid \exists (\lambda_n)_{n \in \mathbb{N}} \subset K', \lambda \in K \text{ s.t. } h_x(\lambda_n) \to 0, \lambda_n \to \lambda \}$$
$$= \bigcap_{n \in \mathbb{N}} \bigcup_{\lambda \in K'} \{ x \in V \mid h_x(\lambda) < \frac{1}{n} \}.$$

The first equality follows from the continuity of h_x and the density of K' in K. The second equality follows from the compactness of K. Note that $h_x(\lambda)$ is a measurable function of x for fixed $\lambda \in \mathbb{R}^N$ (applying Lemma 3.1). Therefore, E is a measurable set, and we are done. \Box

We note one consequence of the above proof.

Lemma 4.5. Suppose (X, d, μ) is a Borel regular metric measure space, and that **x** is an N-tuple of real-valued Lipschitz functions on X. Then $Ind(\mathbf{x})$, the set where **x** is not dependent to first order, is a measurable set.

Proof. This follows from the same argument that we used to prove that E was measurable in the previous lemma. Notice that

$$X \setminus \operatorname{Ind}(\mathbf{x}) = \left\{ x \in X \mid \exists \lambda \in \mathbb{R}^N \setminus \{0\} \text{ s.t. } \operatorname{Lip}_x(\lambda \cdot \mathbf{x}) = 0 \right\}$$
$$= \bigcup_{n \in \mathbb{N}} E_n,$$

where

$$E_n = \left\{ x \in X \mid \exists \lambda \in \mathbb{R}^N, \text{ s.t. } \frac{1}{n} \le |\lambda| \le n, \text{ and } \operatorname{Lip}_x(\lambda \cdot \mathbf{x}) = 0 \right\}.$$

Since the annulus $\{\lambda \in \mathbb{R}^N \mid \frac{1}{n} \leq |\lambda| \leq n\}$ is compact, the argument at the end of the proof of Lemma 4.2 shows that E_n is measurable, and this completes the proof.

5. A Lip-lip inequality implies finite dimensionality

In this section we prove the following statement, which perhaps is the heart of the theorem. Throughout this section, (X, d, μ) is a doubling metric measure space with a K-Lip-lip bound, for fixed K > 0.

Proposition 5.1 (Prop. 7.2.2, [Kei04a]). There exists an N_0 , depending only on K and the doubling constant, so that any $(N_0 + 1)$ -tuple **f** of Lipschitz functions is dependent almost everywhere.

In other words, (X, d, μ) is finite dimensional.

Suppose we fix N Lipschitz functions $\mathbf{f} = (f_1, \ldots, f_N)$. By Lemma 4.5, we know that $\text{Ind}(\mathbf{f})$, the set of points where \mathbf{f} is not dependent, is measurable, and we assume that it has positive measure. The proposition will be proved if we can find a bound $N \leq N_0$.

Let \mathcal{F} be the countable collection of all rational linear combinations

$$\mathcal{F} = \{\lambda \cdot \mathbf{f} \mid \lambda \in \mathbb{Q}^N\} \subset \mathrm{LIP}(X).$$

The rough idea is that we can take tangents to X and \mathcal{F} at a suitable point to get a vector space of uniformly *quasilinear* functions that is, Lipschitz functions whose variation on any ball is comparable to their Lipschitz constant. The doubling condition then provides an an upper bound for the size of this vector space, and hence of N.

5.1. Finding good tangent functions.

Definition 5.2. If f is a Lipschitz function and $\epsilon > 0$, a subset $Y \subset X$ is ϵ -good for f if there is an $r_0 \in (0, \infty)$ such that if $r \in (0, r_0)$ and $x \in Y$, then

(5.3)
$$\frac{1}{K}\operatorname{Lip}_{x} f - \epsilon \leq \operatorname{lip}_{x} f - \epsilon \leq \operatorname{var}_{x,r} f \leq \operatorname{Lip}_{x} f + \epsilon.$$

The set Y is good for f if it is ϵ -good for f, for all $\epsilon > 0$. If \mathcal{F} is a collection of functions, then the set Y is ϵ -good for \mathcal{F} (respectively good for \mathcal{F}) if it is ϵ -good (respectively good) for every $f \in \mathcal{F}$.

Lemma 5.4. Suppose $Y_0 \subset X$ is a measurable subset of finite measure and $\epsilon > 0$. Given a Lipschitz function f, for all $\delta > 0$ there exists $Y \subset Y_0$ so that $\mu(Y_0 \setminus Y) < \delta$ and Y is ϵ -good for f.

Consequently, given a countable collection of Lipschitz functions \mathcal{F} , neglecting a set of arbitrarily small measure we can find $Y \subset Y_0$ so that Y is good for \mathcal{F} .

Proof of Lemma 5.4. The first inequality of (5.3) follows, almost everywhere, from the Lip-lip inequality (2.8).

We saw $\operatorname{Lip}_x f$ was a measurable function of x using the pointwise convergence of functions in equation (3.2). (A similar equation holds for $\operatorname{lip}_x f$.) By Egoroff's theorem, after neglecting a subset of arbitrarily small measure, we may obtain a measurable set $Y \subset Y_0$ where the convergence is uniform. This completes the proof of (5.3).

As in the introduction to this section, we fix N Lipschitz functions f_1, \ldots, f_N , and let \mathcal{F} be the countable collection of all rational linear combinations of these functions.

Let $Y_0 \subset X$ be a finite measure subset. By the above reasoning, and Lusin's theorem, after neglecting a subset of arbitrarily small measure, we may obtain a measurable subset $Y_1 \subset Y_0$ such that

- for all $f \in \mathcal{F}$, the function $\operatorname{Lip}_x f$ (viewed as a function of x) is continuous on Y_1 , and
- the set Y_1 is good for \mathcal{F} .

Lemma 5.5. Suppose $x \in Y$ is a density point of the above set Y_1 . Let X_{∞} denote a tangent of X at x, and $\{u_f : X_{\infty} \to \mathbb{R} \mid f \in \mathcal{F}\}$ denote a compatible collection of tangent functions. Then

- (1) $\operatorname{LIP} u_f \leq \operatorname{Lip}_x f.$
- (2) For every $p \in X_{\infty}$, and every $r \in (0, \infty)$,

 $\operatorname{Lip}_{x} f \leq K \operatorname{var}_{p,r} u_{f}$.

Thus the functions u_f are uniformly quasilinear (Definition 2.11), and have global Lipschitz constant comparable to $\operatorname{Lip}_x f$.

Proof. Fix a Hausdorff approximation

$$\{\phi_i: (X_\infty, d_\infty, x_\infty) \to (X, d_i, x)\}_{i \in \mathbb{N}},\$$

where $d_i = \frac{1}{r_i}d$ and $r_i \to 0$. As x is a point of density for Y, and μ is doubling, we can find maps

$$\{\phi'_i: (X_\infty, d_\infty, x_\infty) \to (Y, d_i, x)\}_{i \in \mathbb{N}},\$$

so that $d_i(\phi_i(\cdot), \phi'_i(\cdot))$ converges to zero uniformly on compact sets.

Suppose we fix $p \neq q$ in X_{∞} , $f \in \mathcal{F}$, and $\epsilon > 0$. Let $p_i = \phi'_i(p), q_i = \phi'_i(q) \in Y$. Notice that $d(p_i, q_i) \to 0$ as $i \to \infty$.

For all sufficiently large i we have, using the fact that f is Lipschitz,

(5.6)
$$\frac{|u_f(p) - u_f(q)|}{d_{\infty}(p,q)} \le \frac{|\frac{1}{r_i}f(p_i) - \frac{1}{r_i}f(q_i)|}{\frac{1}{r_i}d(p_i,q_i)} + \epsilon.$$

Since Y is ϵ -good for f, there exists r_0 so that (5.3) holds. To prove (1), use (5.6) to see that

$$\frac{|u_f(p) - u_f(q)|}{d_{\infty}(p, q)} \le \operatorname{var}_{p_i, (1+\epsilon)d(p_i, q_i)} f + \epsilon$$
$$\le \operatorname{Lip}_{p_i} f + 2\epsilon, \text{ by } (5.3)$$

Since $\operatorname{Lip}_x f$ is continuous on Y, and $p_i \to x$ in the metric d, we see that

$$\frac{|u_f(p) - u_f(q)|}{d_{\infty}(p,q)} \le \operatorname{Lip}_x f + 2\epsilon,$$

but ϵ was arbitrary, and so were p and q, so (1) is proved.

To see (2), fix $\epsilon > 0$ and take p_i as before. Now choose $a_i \in B(p_i, (r - \epsilon)r_i) \subset (X, d)$ so that

$$\operatorname{var}_{p_i,(r-\epsilon)r_i} f \le \frac{|f(p_i) - f(a_i)|}{(r-\epsilon)r_i} + \epsilon.$$

For sufficiently large i, at a cost of adding another ϵ to the right hand side, we can assume that $a_i \in Y$, and that $a_i = \phi'_i(v_i)$, for some $v_i \in B(p, r)$. Furthermore, since $f \circ \phi'_i : X_{\infty} \to \mathbb{R}$ converges to u_f pointwise, and these functions are uniformly Lipschitz, the convergence is uniform on compact sets. Therefore for sufficiently large i,

(5.7)
$$\operatorname{var}_{p_i,(r-\epsilon)r_i} f \leq \frac{|u_f(p) - u_f(v_i)|}{r-\epsilon} + 3\epsilon \leq \frac{r}{r-\epsilon} \operatorname{var}_{p,r} u_f + 3\epsilon.$$

But by the continuity of $\operatorname{Lip}_{x} f$ on Y and equation (5.3),

(5.8)
$$\operatorname{Lip}_{x} f = \lim_{i \to \infty} \operatorname{Lip}_{p_{i}} f \leq \lim_{i \to \infty} K \left(\operatorname{var}_{p_{i}, (r-\epsilon)r_{i}} f + \epsilon \right).$$

Since $\epsilon > 0$ was arbitrary, after combining (5.7) and (5.8), we are done.

5.2. Bounding the dimension of the space of tangent functions. We say that $T \subset X$ is a *c*-net if the *c*-neighborhood of T is X. If in addition every two distinct points of T are at least *c* apart, we say that T is a *(maximal) c*-separated net.

Lemma 5.9. Suppose V is a linear space of K-quasilinear functions on a metric space Z.

- (1) If some r-ball in Z contains a finite $\frac{r}{4K}$ -net T, then dim $V \leq |T|$.
- (2) If Z is C-doubling, then dim $V \leq (AK)^{\log_2 C}$, where A is a universal constant.

Proof of (1). After rescaling, we may assume that r = 1. Let B = B(x, r) = B(x, 1), and let $T \subset B$ be a maximal $\frac{1}{4K}$ -separated net.

Suppose $u \in V$ is in the kernel of the restriction map $V \subset L^{\infty}(B) \to L^{\infty}(T)$. If $x \in B$, there is a $t \in T$ with $d(t, x) < \frac{1}{4K}$, so

$$\begin{aligned} u(x)| &= |u(x) - u(t)| \le \operatorname{LIP}(u) \, d(x, t) \\ &\le K(\operatorname{var}_B u) \cdot \frac{1}{4K} \le \frac{1}{2} \left\| u \right\|_B \right\|_{L^{\infty}} \end{aligned}$$

This implies that

$$\left\| u \right\|_{B} \right\|_{L^{\infty}} \leq \frac{1}{2} \left\| u \right\|_{B} \right\|_{L^{\infty}},$$

forcing $||u||_B||_{L^{\infty}} = 0$. By quasilinearity, we get $u \equiv 0$. Thus the restriction map is injective, and dim $V \leq \dim L^{\infty}(T) = |T|$.

Proof of (2). The C-doubling condition implies that if $B \subset Z$ is a unit ball, there is a $\frac{1}{4K}$ -net $T \subset B$ with $|T| \leq (16K)^{\log_2 C}$. Then Part (1) applies.

5.3. Bounding the dimension of the differentials. As stated in the introduction to this section, we assume that Ind(f) is a measurable set of positive measure.

Using Lemmas 5.4 and 5.5 (applied to $Y_0 = \text{Ind}(\mathbf{f})$) we can take a GH tangents to X and \mathcal{F} at some $x \in \text{Ind}(\mathbf{f})$ to find a GH tangent space $Z = X_{\infty}$ with a compatible family of tangent functions $\{u_f \mid f \in \mathcal{F}\}$. Note that this family is the span over \mathbb{Q} of $\{u_{f_1}, \ldots, u_{f_N}\}$. Since these are all K-quasilinear for a fixed K, the same is true of the span over \mathbb{R} of $\{u_{f_1}, \ldots, u_{f_N}\}$.

We suppose for a contradiction that $N > (AK)^{\log_2 C}$. By Lemma 5.9, the functions $\{u_{f_1}, \ldots, u_{f_N}\}$ satisfy a nontrivial linear relation $\sum_i b_i u_{f_i} = 0$ with real coefficients. Approximating the vector $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$ with a sequence of rational vectors $(a_{1,k}, \ldots, a_{N,k}) \in \mathbb{Q}^n$, we get that the sequence of linear combinations $v_k := \{\sum_i a_{i,k} u_{f_i}\}$ tends to zero uniformly on bounded subsets of X_{∞} . From the construction of the u_f 's, this means that $\operatorname{Lip}_x(\sum_i a_{i,k} f_i) \to 0$. But then

$$\begin{split} \operatorname{Lip}_{x}\left(\sum_{i} b_{i} f_{i}\right) \\ &\leq \limsup_{k \to \infty} \left(\operatorname{Lip}_{x}\left(\sum_{i} a_{i,k} f_{i}\right) + \operatorname{Lip}_{x}\left(\sum_{i} (b_{i} - a_{i,k}) f_{i}\right)\right) \\ &\leq \limsup_{k \to \infty} \left(\operatorname{Lip}_{x}\left(\sum_{i} a_{i,k} f_{i}\right) + \sum_{i} |b_{i} - a_{i,k}| \operatorname{LIP} f_{i}\right) = 0. \end{split}$$

Hence the f_i 's are dependent to first order at x, contradicting our assumption.

6. A POINCARÉ INEQUALITY IMPLIES A Lip-lip INEQUALITY

Definition 6.1. Fix $p \ge 1$. A metric measure space (X, μ) admits a *p*-Poincaré inequality (with constant $L \ge 1$) if every ball in X has positive and finite measure, and for every $f \in \text{LIP}(X)$ and every ball B = B(x, r)

(6.2)
$$\int_{B} |f - f_B| d\mu \le Lr \left(\int_{LB} (\operatorname{lip}_x f)^p d\mu(x) \right)^{1/p}.$$

This is equivalent to the usual definition of a Poincaré inequality (see [Che99, (4.3)], and [Kei04b]). (Note that $\lim_{x} f$ is an upper gradient for f.)

The goal of this section is the following proposition.

Proposition 6.3 (Prop. 4.3.1, [Kei04a]). Suppose X admits a p-Poincaré inequality (with constant $L \ge 1$) for some $p \ge 1$. (See Section 6 for the definition.) Then X has a K-Lip-lip bound (2.8), where K depends only on L and the doubling constant of X.

We will use the following:

Lemma 6.4. The space (X, μ) is given as above. Suppose $A < \infty$ and $\epsilon > 0$ are fixed constants. If $u : X \to \mathbb{R}$ is a Lipschitz function, and $x \in X$ is an approximate continuity point for lip $u : X \to \mathbb{R}$, then there exists $r_0 = r_0(u, x, A, \epsilon) > 0$ such that if $r \leq r_0, y, y' \in B(x, Ar) \subset X$ and $d(y, y') \leq r$, then

(6.5)
$$\left| \oint_{B} u - \oint_{B'} u \right| \le C_1 r \left(\lim_x u + \epsilon \right).$$

where B := B(y, r), B' := B(y', r), and where $C_1 = C_1(X, \mu) < \infty$ is a suitable constant.

Proof. Set $\hat{B} := B(y, 2r)$, so $B, B' \subset \hat{B}$. Then we have

(6.6)
$$C_2 \left| f_B u - f_{B'} u \right| \le f_{\hat{B}} |u - u_{\hat{B}}| \le 2Lr \left(f_{L\hat{B}} (\operatorname{lip} u)^p \right)^{\frac{1}{p}},$$

where $C_2 > 0$ depends only on the doubling constant for μ , and the second inequality comes from the Poincaré inequality for (X, μ) . Since $\lim u \leq \operatorname{LIP}(u)$ everywhere, and x is an approximate continuity point of $\lim u$, when r is sufficiently small we have

(6.7)
$$\left(\oint_{L\hat{B}} (\operatorname{lip} u)^p \right)^{\frac{1}{p}} \le \operatorname{lip}_x u + \epsilon.$$

Combining (6.6) and (6.7) gives the lemma.

Proof of Proposition 6.3. Since lip f is Borel it is approximately continuous almost everywhere. Let $x \in X$ be an approximate continuity point for lip f, and fix $\lambda \in (0, 1)$, $\epsilon \in (0, 1)$.

Since (X, μ) is doubling, its completion \bar{X} equipped with the measure $\bar{\mu}$ defined by $\bar{\mu}(Y) = \mu(Y \cap X)$ is also doubling. If $\bar{x} \in \bar{X}$, $r \in (0, \infty)$, we may find $x \in B(\bar{x}, r) \cap X$. Then using the doubling property of $\bar{\mu}$ and the *p*-Poincare inequality for X, we get

$$\begin{aligned} \int_{B(\bar{x},r)} |f - f_{B(\bar{x},r)}| d\bar{\mu} &\leq C \oint_{B(x,2r)} |f - f_{B(x,2r)}| d\bar{\mu} \\ &\leq 2CLr \left(\oint_{B(x,2Lr)} (\operatorname{lip}_x f)^p d\mu(x) \right)^{\frac{1}{p}} \leq 2CLr \left(\oint_{B(\bar{x},(2L+1)r)} (\operatorname{lip}_x f)^p d\bar{\mu}(x) \right)^{\frac{1}{p}} \end{aligned}$$

where C depends only on the doubling constant of μ . Hence $(X, \bar{\mu})$ also satisfies a p-Poincare inequality, and is quasiconvex by Theorem A.1. Therefore, given r > 0 and $y \in B(x, r)$, by the quasiconvexity of \bar{X} , there is a chain of points $x = p_1, \ldots, p_k = y$ in X, where $d(p_i, p_{i+1}) \leq \lambda r$ and $k \leq \frac{Q}{\lambda}$, for some Q that depends only on X. Set $B_i := B(p_i, \lambda r)$. Then

(6.8)
$$|f(y) - f(x)| \le |f(x) - f_{B_1}f| + \sum_{1 \le i < k} |f_{B_{i+1}}f - f_{B_i}f| + |(f_{B_k}f) - f(y)|.$$

By the lemma, when r is sufficiently small we have

$$\left| \oint_{B_{i+1}} f - \oint_{B_i} f \right| \le C_1 \lambda r(\lim_x f + \epsilon),$$

 \mathbf{SO}

$$|f(y) - f(x)| \le \left(\frac{Q}{\lambda}\right) (C_1 \lambda r(\lim_x f + \epsilon)) + 2\lambda r \operatorname{LIP}(f)$$

= $(QC_1(\lim_x f + \epsilon) + 2\lambda \operatorname{LIP}(f)) r.$

Thus $\operatorname{Lip}_x f \leq QC_1 \operatorname{lip}_x f + QC_1 \epsilon + 2\lambda \operatorname{LIP}(f)$ and, since $\lambda, \epsilon > 0$ were arbitrary, this proves the proposition.

APPENDIX A. A POINCARÉ INEQUALITY IMPLIES QUASICONVEXITY

As mentioned in the introduction, in this appendix we give a simpler proof of the following theorem of Semmes [Che99, Appendix A]. A similar argument can be found in [Kei03, Section 6].

Theorem A.1. Let (X, d, μ) be a complete, doubling metric measure space satisfying a Poincaré inequality. Then X is λ -quasiconvex, where λ depends only on the data (doubling constant and constants in the PI).

The main step in the proof of Theorem A.1 is:

Lemma A.2. There is a constant $C \in (0, \infty)$ such that if $p, q \in X$, and r = d(p,q), then there is a path of length at most C r from $B(p, \frac{r}{4})$ to $B(q, \frac{r}{4})$.

Assuming the lemma, the proof goes as follows. Pick $x, x' \in X$, and apply the lemma to obtain a path γ of length at most Cd(x, x'), such that "total gap" $d(x, \gamma) + d(\gamma, x')$ is at most $\frac{1}{2}d(x, x')$. Now apply the lemma to each of the gaps, to get two new paths, and so on. The total gap at each step is at most half the total gap at the previous step, and the total additional path produced is at most *C*-times the gap left after the previous step. The closure of the union of the resulting collection of paths contains a path from p to q of length at most 2C d(x, x').

Before proving Lemma A.2, we make the following definition:

Definition A.3. An ϵ -path in a metric space X is a sequence of points $x_0, \ldots, x_k \in X$ such that $d(x_{i-1}, x_i) < \epsilon$ for all $i \in \{1, \ldots, k\}$; the length of the ϵ -path is $\sum_i d(x_{i-1}, x_i)$.

To prove Lemma A.2, we will show that for all $\epsilon \in (0, \infty)$, there is an ϵ -path from $B(p, \frac{r}{4})$ to $B(q, \frac{r}{4})$ of length at most $C \ d(p, q)$; then a variant of the Arzelà-Ascoli theorem applied to a sequence of discrete paths implies that there is a path of length at most $C \ d(p, q)$ from $B(p, \frac{r}{4})$ to $B(q, \frac{r}{4})$.

Fix $\epsilon \in (0, \infty)$, and define $u : X \to [0, \infty]$ by setting u(x) equal to the infimal length of an ϵ -path from $B(p, \frac{r}{4})$ to x. For $A \in (0, \infty)$, let $u_A := \min(u, A)$. Then u_A is is a continuous function which is locally 1-Lipschitz; in particular the constant function $\rho \equiv 1$ is an upper gradient for u_A . The Poincaré inequality applied to u_A and $B(p, \frac{5r}{4})$ implies that u_A is $\leq Cr$ somewhere in $B(q, \frac{r}{4})$, where C depends only on the data of X. Since this is true for A > Cr, the desired ϵ -path exists. \Box

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