CHARACTERIZATION OF THE RADON-NIKODYM PROPERTY IN TERMS OF INVERSE LIMITS

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ABSTRACT. In this paper we clarify the relation between inverse systems, the Radon-Nikodym property, the Asymptotic Norming Property of James-Ho [JH81], and the GFDA spaces introduced in [CK06].

1. INTRODUCTION

A Banach space V is said to have the *Radon-Nikodym Property* (RNP) if every Lipschitz map $f : \mathbf{R} \to V$ is differentiable almost everywhere. By now, there are a number of characterizations of Banach spaces with the RNP, the study of which goes back to Gelfand [Gel38]; for additional references and discussion, see [BL00, Chapter 5], [GM85]. Of particular interest here is the characterization of the RNP in terms of the Asymptotic Norming Property; [JH81, GM85].

In this paper we will show that a variant of the GFDA property introduced in [CK06] is actually equivalent to the Asymptotic Norming property of James-Ho, and hence by [JH81, GM85], is equivalent to the RNP. In addition, we observe that the GFDA spaces of [CK06] are just spaces which are isomorphic to a separable dual space.

Definition 1.1. An inverse system

$$W_1 \xleftarrow{\theta_1} W_2 \xleftarrow{\theta_2} \ldots \xleftarrow{\theta_{i-1}} W_i \xleftarrow{\theta_i} \ldots, \tag{1.2}$$

is *standard* if the W_i's are finite dimensional Banach spaces and the θ_i's are linear maps of norm ≤ 1. We let $\pi_j : \varprojlim W_i \to W_j$ denote the projection map.

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Definition 1.3. Let $\{(W_i, \theta_i)\}$ be a standard inverse system and $V \subset \varprojlim W_i$ be a subspace. The pair $(\varprojlim W_i, V)$ has the *Determining Property* if a sequence $\{v_k\} \subset V$ converges strongly provided the projected sequences $\{\pi_j(v_k)\} \subset W_j$ converge for every j, the sequence $\{\|v_k\|\}$ is bounded, and the convergence $\|\pi_j(v_k)\| \to \|v_k\|$ is uniform in k. A Banach space U has the *Determining Property* if there is a pair $(\varprojlim W_i, V)$ with Determining Property, such that V is isomorphic to U.

We have:

Theorem 1.4. *A separable Banach space has the RNP if and only it has the Determining Property.*

Since a Banach space has the RNP if and only if every separable subspace has the RNP, Theorem 1.4 yields a characterization of the RNP for nonseparable Banach spaces as well.

To prove the theorem, we first observe in Proposition 2.8 that the inverse limit $\varprojlim W_i$ is the dual space of a separable Banach space. Then, by a completely elementary argument, we show that a Banach space has the Determining Property if and only if it has the Asymptotic Norming Property (ANP) of James-Ho [JH81]. Since a separable Banach space U has the RNP if and only if it has the ANP [JH81, GM85], the theorem follows. We remark that there is a simple direct proof that if V has the ANP (or the Determining Property), then every Lipschitz map $f : \mathbf{R} \to V$ is differentiable almost everywhere, see [CK].

Characterizations of the RNP using inverse limits are useful for applications; see [CK06], the discussion below concerning metric measure spaces, and [CK].

Relation with previous work.

In slightly different language, our earlier paper [CK06] also considered pairs $(\varprojlim W_i, V)$, where $\varprojlim W_i$ is the inverse limit of a standard inverse system, and $V \subset \varprojlim W_i$ is a closed subspace. A Good Finite Dimensional Approximation (GFDA) of a Banach space V, a notion introduced in [CK06], is a pair $(\varprojlim W_i, V)$ with the Determining Property such that $\pi_i\big|_V : V \to W_i$ is a quotient map for every i.

It follows immediately from Lemma 3.8 of [CK06] that if $(\varprojlim W_i, V)$ is a GFDA of V, then $V = \varprojlim W_i$. Since such inverse limits are dual spaces by Proposition 2.8, V is a separable dual space in this case. Conversely, using the Kadec-Klee renorming Lemma [Kad59, Kle61], it was shown in [CK06] that every separable dual space is isomorphic to a Banach space which admits a GFDA. Thus, a Banach space admits a GFDA if and only if it is isomorphic to a separable dual space.

Applications to metric measure spaces.

We will call a metric measure space (X, μ) a *PI space* if the measure is doubling, and a Poincaré inequality holds in the sense of upper gradients [HK98, Che99]. In [CK06], differentiation and bi-Lipschitz non-embedding theorems were proved for maps $f : X \to V$ from PI spaces into GFDA targets V, generalizing results of [Che99] for finite dimensional targets. As explained above, it turns out that these targets are just separable dual spaces, up to isomorphism.

As an application of the inverse limit framework and the equivalence between the ANP and RNP, we will show in [CK] that the differentiation theorem [CK06, Theorem 4.1] and bi-Lipschitz non-embedding theorem [CK06, Theorem 5.1] hold whenever the target has the RNP.

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2. Inverse systems

In this section, we recall some basic facts concerning direct and inverse systems, and the duality between them. Then we show that inverse limits of standard inverse systems are precisely duals of separable spaces.

The following conventions will be in force throughout the remainder of the paper.

Definition 2.1. An *standard direct system* is a sequence of finite dimensional Banach spaces $\{E_i\}$ and 1-Lipschitz linear maps $\iota_i : E_i \to E_{i+1}$.

Definition 2.2. An *standard inverse system* is a sequence of finite dimensional Banach spaces $\{W_i\}$ and 1-Lipschitz linear maps $\theta_i : W_{i+1} \to W_i$.

Definition 2.3. A standard direct system is *isometrically injective* if the maps $\iota_i : E_i \to E_{i+1}$ are isometric injections.

Definition 2.4. A standard inverse system is *quotient* if the maps $\theta_i : W_{i+1} \to W_i$ are quotient maps.

By a *quotient map* of normed spaces, we mean a surjective map $\pi : U \to V$ for which the norm on the target is the quotient norm, i.e. for every $v \in V$,

$$\|v\| = \inf\{\ \|u\| \mid u \in \pi^{-1}(v)\}.$$

We will refer to the maps ι_i and θ_i as *bonding maps.*

There is a duality between the objects in Definitions 2.1 and 2.2, respectively, 2.3 and 2.4: if $\{(E_i, \iota_i)\}$ is a standard direct system, then $\{(E_i^*, \iota_i^*)\}$ is a standard inverse system and conversely; similarly, isometrically injective direct systems are dual to quotient systems. To see this, one uses the facts that the adjoint of a 1-Lipschitz map of Banach spaces is 1-Lipschitz and the the adjoint of an isometric embedding is a quotient map. (This follows from the Hahn-Banach theorem.) In particular, since the spaces in our systems are assumed to be finite dimensional (hence reflexive) every inverse system arises as the dual of its dual direct system and conversely. The same holds for quotient inverse systems.

We now recall the definitions of direct and inverse limits.

Given a standard direct system $\{(E_i, \iota_i)\}$ we form the direct limit Banach space $\varinjlim E_i$ as follows. We begin with the disjoint union $\sqcup_i E_i$, and declare two elements $e \in E_i$, $e' \in E_{i'}$ to be equivalent if their images in E_j coincide for some $j \geq \max\{i, i'\}$. Since the bonding maps are 1-Lipschitz, the set of equivalence classes inherits an obvious vector space structure with a pseudo-norm. The direct limit $\varinjlim E_i$ is defined to be the completion of the quotient of this space by the closed subspace of elements whose pseudo-norm is zero. Clearly, there are 1-Lipschitz maps

$$\tau_i : E_i \to \varinjlim E_i\,,$$

which in the case of isometrically injective direct systems, are isometric injections. The union $\bigcup_i \tau_i(E_i)$ is dense in $\varinjlim E_i$.

The inverse limit $\varprojlim W_i$ of a standard inverse system $\{(W_i, \theta_i)\}$ is defined as follows. The underlying set consists of the collection of elements $(w_i) \in \prod_i W_i$ which are compatible with the bonding maps, i.e. $\theta_i(w_i) = w_{i-1}$ for all i, and which satisfy $\sup_i \|w_i\| < \infty$. This is equipped with the obvious vector space structure and the norm

$$\|\{w_i\}\| := \lim_{j\to\infty} \|w_j\| . \tag{2.5}$$

The map

$$\pi_j : \varprojlim W_i \to W_j \tag{2.6}$$

given by

$$\pi_j(\{w_i\}) = w_j$$

is 1-Lipschitz, and

$$\lim_{j\to\infty} \|\pi_j(\{w_i\})\| = \|\{w_i\}\| .$$

An inverse limit $\varprojlim W_i$ has a natural *inverse limit topology*, namely the weakest topology such that every projection map $\pi_j : \varprojlim W_i \to W_j$ is continuous. Thus a sequence $\{v_k\} \subset \varprojlim W_i$ converges in the inverse limit topology to $v \in \varprojlim W_i$ if and only if for every i, we have $\pi_i(v_k) \to \pi_i(v)$ as $k \to \infty$.

If $\{v_k\} \subset \varprojlim W_i$ and $\{v_k\} \xrightarrow{invlim} v \in \varprojlim W_i$, then

$$\|v\| \ \le\ \liminf_k \|v_k\| . \tag{2.7}$$

Also, every norm bounded sequence $\{v_k\} \subset \varprojlim W_i$ has a subsequence which converges with respect to the inverse limit topology; this follows from a diagonal argument, because $\{\pi_i(v_k)\}$ is contained in a compact subset of W_i, for all i.

Proposition 2.8. *Given a standard inverse system $\{(W_i, \theta_i)\}$, there is an isometric isomorphism*

$$C : \varprojlim\, W_i \equiv (\varinjlim W_i^*)^* . \tag{2.9}$$

In particular, $\varprojlim W_i$ is the dual of the separable Banach space $\varinjlim W_i^*$.

Proof. Pick a compatible sequence $(x_i) \in \varprojlim W_i$. We get a map

$$\sqcup W_j^* \to \mathbf{R}$$

by sending $\phi \in W_j^*$ to $\phi(x_j)$; because (x_i) is compatible with bonding maps and

$$|\phi(x_j)| \leq \|\phi\| \, \|x_j\| \leq \|\phi\| \, \|\{x_j\}\|,$$

this defines a linear functional of norm $\leq \|\{x_j\}\|$ on $\varinjlim W_i^*$. Therefore we get a 1-Lipschitz map

$$C : \varprojlim W_i \longrightarrow \left(\varinjlim W_i^*\right)^* .$$

We now verify that C is an isometry.

Pick $(x_i) \in \varprojlim W_i$, and choose $n \in \mathbb{N}$ such that $\|x_n\| \geq \|(x_i)\| - \epsilon$. If $\phi \in W_n^*$ has norm 1 and $\phi(x_n) = \|x_n\|$, then

$$\|C((x_i))\| \, \|\tau_n(\phi)\| \geq C((x_i))(\tau_n(\phi)) = \phi(x_n) = \|x_n\| \geq \|(x_i)\| - \epsilon,$$

where $\tau_n : W_n^* \to \varinjlim W_i^*$ is the canonical 1-Lipschitz map described above. This shows that C is an isometric embedding.

If $\Phi \in (\varinjlim W_i^*)^*$, then we define $\Phi_i \in W_i^{**} = W_i$ to be the composition

$$W_i^* \longrightarrow \varinjlim W_i^* \xrightarrow{\Phi} \mathbf{R}.$$

This defines a compatible sequence $(\Phi_i) \in \varprojlim W_i$, such that $\|(\Phi_i)\| = \|\Phi\|$ and $C((\Phi_i)) = \Phi$. Hence C is onto. $\square$

Corollary 2.10.

1) *A separable Banach space Y is isomorphic to the direct limit of an isometrically injective direct system (E_i, ι_i).*

2) *The dual space Y^* of the separable Banach space Y (as in 1)) is isometric to the inverse limit $\varprojlim E_i^*$ of the a quotient inverse system $\{(E_i^*, \iota_i^*)\}$.*

Proof. To see that 1) holds, start with a countable increasing sequence $E_1 \subset E_2 \subset \cdots \subset Y$ of finite dimensional subspaces whose union is dense in Y, and take the bonding maps $\iota_i : E_i \to E_{i+1}$ to be the inclusions. Clearly the inclusion maps $E_i \to Y$ induce an isometry $\varinjlim E_i \to Y$.

Assertion 2) follows from 1) and Proposition 2.8. $\square$

Let C be the isometry in Proposition 2.8.

Lemma 2.11.

1) *Suppose* $\{v_k\} \subset \varprojlim W_i$ *is a sequence such that* $\{C(v_k)\} \subset (\varinjlim W_i^*)^*$ *weak* converges to some* $y \in (\varinjlim W_i^*)^*$. *Then* $\{v_k\}$ *is convergent with respect to the inverse limit topology, and its limit* $v_\infty \in \varprojlim W_i$ *satisfies* $C(v_\infty) = y$; *in particular,* $y \in C(\varprojlim W_i)$.

2) *If* $\{v_k\} \subset \varprojlim W_i$ *converges in the inverse limit topology, and has uniformly bounded norm, then* $\{C(v_k)\}$ *is weak* convergent.*

Proof. Assertions 1) and 2) follow readily from the assumption that the W_i are finite dimensional together with the density of compatible sequences in $\varprojlim W_i$. □

3. The proof of Theorem 1.4

The proof of Theorem 1.4 is based on the Asymptotic Norming Property, which we now recall.

Let Y denote a separable Banach space and $V \subset Y^*$ a separable subspace of its dual. (Here Y^* need not be separable.)

Definition 3.1. The pair (Y^*, V) has the *Asymptotic Norming Property* (ANP) if a sequence $\{v_k\} \subset V$ converges strongly provided it is weak* convergent and the sequence of norms $\{\|v_k\|\}$ converges to the norm of the weak* limit.

A Banach space U is said to have the *Asymptotic Norming Property* if there is a pair (Y^*, V) with the ANP such that U is isomorphic to V.

Theorem 3.2 ([JH81, GM85])**.** *For separable Banach spaces, the RNP is equivalent to the ANP.*

Hence to prove Theorem 1.4, it suffices to show that for separable Banach spaces, the ANP is equivalent to the Determining Property. By Corollary 2.10, every separable Banach space Y is isometric to the direct limit of a standard direct system, and Y^* is isometric to the inverse limit of the dual inverse system. Hence the proof of Theorem 1.4 reduces to:

Proposition 3.3. *Let $\{(W_i, \theta_i)\}$ be a standard inverse system, and V be a closed separable subspace of $\varprojlim W_i$. Then the pair $(\varprojlim W_i, V)$ has the ANP if and only if it has the Determining Property. Here we are identifying $\varprojlim W_i$ with the dual of $\varinjlim W_i^*$, see Proposition 2.8.*

Proof. Let $\{v_k\} \subset V$ be a sequence with bounded norm. By Lemma 2.11, the sequence $\{v_k\}$ is weak* convergent if and only if it converges in the inverse limit topology. Therefore, to prove the equivalence of the ANP and the Determining Property for the pair $(\varprojlim W_i, V)$, it suffices to show that when

$$v_k \xrightarrow{w^*} w \in \varprojlim W_i\,, \tag{3.4}$$

the sequence of norms $\{\|v_k\|\}$ converges to the $\|w\|$ if and only if the convergence $\|\pi_j(v_k)\| \to \|v_k\|$ is uniform in k. Although this is completely elementary, we will write out the details.

We have

$$\|v_k\|-\|w\| = (\|v_k\|-\|\pi_i(v_k)\|)+(\|\pi_i(v_k)\|-\|\pi_i(w)\|)+(\|\pi_i(w)\|-\|w\|)\,. \tag{3.5}$$

Assume first that $\lim_{k\to\infty} \|v_k\| = \|w\|$. Given $\epsilon > 0$, there exists I_1 such that $\|w\| - \|\pi_i(w)\| < \epsilon/3$, for $i \geq I_1$. By (3.4) there exists K_1 such that $\|\pi_{I_1}(v_k) - \pi_{I_1}(w)\| < \epsilon/3$, for $k \geq K_1$. Also, there exists K_2 such that $\big|\|v_k\| - \|w\|\big| < \epsilon/3$, if $k \geq K_2$. Set $K = \max(K_1, K_2)$.

From (3.5), with $i = I_1$, we get $\|v_k\| - \|\pi_{I_1}(v_k)\| < \epsilon$, for all $k \geq K$. Since, $\|v_k\| - \|\pi_i(v_k)\|$ is a nonnegative decreasing function of i, this implies, $\|v_k\| - \|\pi_i(v_k)\| < \epsilon$, for all $i \geq I_1$, $k \geq K$.

Finally, there exists I_2 such that $\|v_k\| - \|\pi_i(v_k)\| < \epsilon$ for all $i \geq I_2$, $k = 1, \ldots, K-1$, Thus, if $i \geq \max(I_1, I_2)$ then $\|v_k\| - \|\pi_i(v_k)\| < \epsilon$, for all k.

Conversely, suppose the convergence $\|\pi_i(v_k)\| \to \|v_k\|$ is uniform in k. Given $\epsilon > 0$, there exists I such that $\|v_k\| - \|\pi_i(v_k)\| < \epsilon/3$, for $i \geq I$ and all k. Also, there exists I_1 such that $\|w\| - \|\pi_i(w)\| < \epsilon/3$, for $i \geq I_1$. Set $I' = \max(I, I_1)$. By (3.4), there exists K such that $\|\pi_{I'}(v_k) - \pi_{I'}(w)\| < \epsilon/3$.

From (3.5), with $i = I'$, we get $\big|\|v_k| - \|w\|\big| < \epsilon$, for all $k \geq K$. $\square$

4. A variant of the Determining Property

In this section we discuss a variant of the Determining Property, which was introduced in [CK06] (with a different name). A compactness argument implies that it is equivalent to Definition 1.3, see Proposition 4.6.

For the remainder of this section, we fix a standard inverse system $\{(W_i, \theta_i)\}$ and a closed subspace $V \subset \varprojlim W_i$.

Definition 4.1. A positive nonincreasing finite sequence $1 \geq \rho_1 \geq \ldots \geq \rho_N$ is *ϵ-determining* if for any pair $v, v' \in V$, the conditions

$$\|v\| - \|\pi_i(v)\| < \rho_i \cdot \|v\|, \qquad \|v'\| - \|\pi_i(v')\| < \rho_i \cdot \|v'\|, \qquad 1 \leq i \leq N\,, \tag{4.2}$$

and

$$\|\pi_N(v) - \pi_N(v')\| < N^{-1} \cdot \max(\|v\|, \|v'\|)\,, \tag{4.3}$$

imply

$$\|v - v'\| < \epsilon \cdot \max(\|v\|, \|v'\|)\,. \tag{4.4}$$

Observe that by dividing by $\max(\|v\|, \|v'\|)$, it suffices to consider pairs v, v' for which $\max(\|v\|, \|v'\|) = 1$.

This leads to the alternate definition of the Determining Property:

Definition 4.5. The pair $(\varprojlim W_i, V)$ has the *Determining Property* if for every $\epsilon > 0$ and every infinite nonincreasing sequence

$$1 \geq \rho_1 \geq \ldots \geq \rho_i \geq \ldots$$

with $\rho_i \to 0$, some finite initial segment $\rho_1 \geq \ldots \geq \rho_N$ is ϵ-determining.

Proposition 4.6. *The pair $(\varprojlim W_i, V)$ satisfies Definition 1.3 if and only if it satisfies Definition 4.5.*

Proof. First we show that the property in Definition 4.5 implies the property in Definition 1.3. So assume that the sequence $\{\|v_k\|\}$ is bounded and the convergence, $\|\pi_i(v_k)\| \to \|v_k\|$ is uniform in k.

Suppose that there exists a sequence, a positive sequence, $\rho_i \searrow 0$, such that $\|v_k\| - \|\pi_i(v_k)\| \leq \rho_i$. By applying the condition in Definition 4.5 to this sequence and using convergence in the inverse limit topology together with (4.3) it is clear from (4.4) that we obtain strong convergence.

Without loss of essential loss of generality, we can assume $\|v_k\| \le 1$ for all k. Since the convergence, $\|\pi_i(v_k)\| \to \|v_k\|$ is uniform in k, it follows that there exists a strictly increasing sequence, $N_1 < N_2 < \ldots$, such that for all k, we have

$$\|v_k\| - \|\pi_{N_\ell}(v_k)\| < \frac{1}{\ell}\,.$$

Then $\|v_k\| - \|\pi_i(v_k)\| \le \rho_i$, for the sequence, ρ_i given by

$$\rho_i = \frac{1}{\ell} \qquad (N_\ell \le i < N_{\ell+1})\,.$$

Conversely, suppose that the property in Definition 1.3 holds, but not the property in Definition 4.5. Then for some decreasing sequence $\{\rho_i\} \subset (0,\infty)$ with $\rho_i \to 0$, and some $\epsilon > 0$, there are sequences $\{v_k\}, \{v_k'\} \subset V$, such that for all $k < \infty$,

$$\|v_k\|,\, \|v_k'\| \le 1\,, \tag{4.7}$$

$$\max\left(\|v_k\| - \|\pi_i(v_k)\|,\;\; \|v_k'\| - \|\pi_i(v_k')\|\right) < \rho_i \;\text{ for } 1 \le i \le k\,, \tag{4.8}$$

$$\|\pi_i(v_k) - \pi_i(v_k')\| < \frac{1}{k}\,, \tag{4.9}$$

$$\|v_k - v_k'\| \ge \epsilon\,. \tag{4.10}$$

By the Banach-Alaoglu theorem, we can pass to weak* convergent subsequences, with respective limits v_∞ and v_∞'. From (4.9), it follows that $v_\infty = v_\infty'$.

It follows from (4.7), (4.8), that the sequences, $\|v_k\|$, $\|v_k'\|$, are bounded and the convergence $\|\pi_i(v_k)\| \to \|v_k\|$, $\|\pi_i(v_k')\| \to \|v_k'\|$ is uniform in k. Since we assume the property in Definition 1.3, it follows $v_k \to v_\infty$, $v_k' \to v_\infty'$, is actually strong. Since, $v_\infty = v_\infty'$, this contradicts (4.10). □

We remark that proof of the implication Definition 1.3 $\Longrightarrow$ Definition 4.5 is similar to the proof of Proposition 3.11 in [CK06].

5. GFDA versus ANP

We conclude with some remarks about the relation between the ANP and GFDA's.

Suppose Y is a separable Banach space and (Y^*, V) has the ANP. By Lemma 2.10, we may realize Y^* – up to isometry – as the inverse limit of a quotient system $\{(W_i, \theta_i)\}$.

Viewing V as a subspace of $\varprojlim W_i$, one might be tempted to modify the inverse system to produce a GFDA of V. For instance, one could restrict the projection maps $\pi_j : \varprojlim W_i \to W_j$ to V, and replace W_j with $\pi_j(V) \subset W_j$. However, the resulting maps $\pi_j\big|_V : V \to \pi_j(V)$ will usually not be quotient maps. One could also try renorming the spaces $\pi_j(V) \subset W_j$ so that the restrictions $\pi_j\big|_V : V \to \pi_j(V)$ become quotient maps. This will typically destroy the Determining Property, however. In any case, V will not admit any GFDA unless it is a separable dual space, whereas many Banach spaces with the RNP are not separable dual spaces.

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