Separated nets in Euclidean space and Jacobians of biLipschitz maps

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Abstract

We show that there are separated nets in the Euclidean plane which are not biLipschitz equivalent to the integer lattice. The argument is based on the construction of a continuous function which is not the Jacobian of a biLipschitz map.

1 Introduction

A subset X of a metric space Z is a *separated net* if there are constants a, $b > 0$ such that $d(x, x') > a$ for every pair x, $x' \in X$, and $d(z, X) < b$ for every $z \in Z$. Every metric space contains separated nets: they may be constructed by finding maximal subsets with the property that all pairs of points are separated by some distance $a > 0$. It follows easily from the definitions that two spaces are quasi-isometric if and only if they contain biLipschitz equivalent separated nets. One may ask if the choice of these nets matters, or, in other words, whether any two separated nets in a given space are biLipschitz equivalent. To the best of our knowledge, this problem was first posed by Gromov [Gro93, p.23]. The answer is known to be yes for separated nets in non-amenable spaces (under mild assumptions about local geometry), see [Gro97, Why97]; more constructive proofs in the case of trees or hyperbolic groups can be found in [Pap95, Bog96].

In this paper, we prove the following theorem:

Theorem 1.1 *There exists a separated net in the Euclidean plane which is not biLipschitz equivalent to the integer lattice.*

The proof of Theorem 1.1 is based on the following result:

Theorem 1.2 *Let $I := [0, 1]$. Given $c > 0$, there is a continuous function $\rho : I^2 \to [1, 1 + c]$, such that there is no biLipschitz map $f : I^2 \to \mathbb{E}^2$ with*

$$Jac(f) := Det(Df) = \rho \quad a.e.$$

Remarks

1. Although we formulate and prove these theorems in the 2-dimensional case, the same proofs work with minor modifications in higher dimensional Euclidean spaces as well. We only consider the 2-dimensional case here to avoid cumbersome notation.

2. Theorem 1.2 also works for Lipschitz homeomorphisms; we do not use the lower Lipschitz bound on f. Also, since any modulus of continuity on a map between separated nets implies that it is Lipschitz, the nets described in Theorem 1.1 are not equivalent to the integer lattice even if we work in the larger category of maps with a modulus of continuity.

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3. After the first version of this paper had been written, Curt McMullen informed us that he also had a proof of Theorems 1.1 and 1.2. See [McM97] for a discussion of the the linear analog of Theorem 1.2, and the (homogeneous) Hölder analogs of the mapping problems in Theorems 1.1 and 1.2.

The problem of prescribing Jacobians of homeomorpisms has been studied by several authors. Using the idea of [Mos65], [DM90] proved that every α-Holder continuous function is locally the Jacobian of a $C^{1,\alpha}$ homeomorphism, and they then raised the question of whether any continuous function is (locally) the Jacobian of a C^1 diffeomorphism. [RY94, Ye94] consider the prescribed Jacobian problem in other regularity classes, including the cases when the Jacobian is in L^∞ or in a Sobolev space.

Overview of the proofs

Theorem 1.2 implies Theorem 1.1. Let $\rho : I^2 \to \mathbb{R}$ be measurable with $0 < \inf \rho \leq \sup \rho < \infty$. We will indicate why ρ would be the Jacobian of a biLipschitz map $f : I^2 \to \mathbb{E}^2$ if all separated nets in $\mathbb{E}^2$ were biLipschitz equivalent. Take a disjoint collection of squares $S_i \subset \mathbb{E}^2$ with side lengths l_i tending to infinity, and "transplant" ρ to each S_i using appropriate similarities $\alpha_i : I^2 \to S_i$, i.e. set $\rho_i := \rho \circ \alpha_i^{-1}$. Then construct a separated net $L \subset \mathbb{E}^2$ so that the "local average density" of L in each square S_i approximates ρ_i^{-1}. If $g : L \to \mathbb{Z}^2$ is a biLipschitz homeomorphism, consider "pullbacks" of $g\big|_{S_i}$ to I^2, i.e. pre and post-compose $g\big|_{S_i}$ with suitable similarities so as to get a sequence of uniformly biLipschitz maps $g_i : I^2 \supset Z_i \to \mathbb{E}^2$. Then extract a convergent subsequence of the g_i's via the Arzela-Ascoli theorem, and obtain a limit map $f : I^2 \to \mathbb{E}^2$ with Jacobian ρ.

Theorem 1.2. The observation underlying our construction is that if the Jacobian of $f : I^2 \to \mathbb{E}^2$ oscillates in a rectangular neighborhood U of a segment $\overline{xy} \subset I^2$, then f will be forced to stretch for one of two reasons: either it maps $\overline{xy}$ to a curve which is far from a geodesic between its endpoints, or it maps $\overline{xy}$ close to the segment $\overline{f(x)f(y)}$ but it sends U to a neighborhood of $\overline{f(x)f(y)}$ with wiggly boundary in order to have the correct Jacobian. By arranging that $Jac(f)$ oscillates in neighborhoods of a hierarchy of smaller and smaller segments we can force f to stretch more and more at smaller and smaller scales, eventually contradicting the Lipschitz condition on f.

We now give a more detailed sketch of the proof.

We first observe that it is enough to construct, for every $L > 1$, $\bar{c} > 0$, a continuous function $\rho_{L,\bar{c}} : I^2 \to [1, 1+\bar{c}]$ such that $\rho_{L,\bar{c}}$ is not the Jacobian of an L-biLipschitz map $I^2 \to \mathbb{E}^2$. Given such a family of functions, we can build a new continuous function $\rho : I^2 \to [1, 1+c]$ which is not the Jacobian of any biLipschitz map $I^2 \to \mathbb{E}^2$ as follows. Take a sequence of disjoint squares $S_k \subset I^2$ which converge to some $p \in I^2$, and let $\rho : I^2 \to [1, 1+c]$ be any continuous function such that $\rho\big|_{S_k} = \rho_{k,\min(c,\frac{1}{k})} \circ \alpha_k$ where $\alpha_k : S_k \to I^2$ is a similarity.

Also, note that to construct $\rho_{L,\bar{c}}$, we really only need to construct a measurable function with the same property: if $\rho^k_{L,\bar{c}}$ is a sequence of smoothings of a measurable function $\rho_{L,\bar{c}}$ which converge to $\rho_{L,\bar{c}}$ in L^1, then any sequence of L-biLipschitz maps $\phi_k : I^2 \to \mathbb{E}^2$ with $Jac(\phi_k) = \rho^k_{L,\bar{c}}$ will subconverge to a biLipschitz map $\phi : I^2 \to \mathbb{E}^2$ with $Jac(\phi) = \rho_{L,\bar{c}}$.

We now fix $L > 1$, $c > 0$, and explain how to construct $\rho_{L,c}$. Let R be the rectangle $[0,1] \times [0, \frac{1}{N}] \subset \mathbb{E}^2$, where $N \gg 1$ is chosen suitably depending on L and c, and let $S_i = [\frac{i-1}{N}, \frac{i}{N}] \times [0, \frac{1}{N}]$ be the i^{th} square in R. Define a "checkerboard" function $\rho_1 : I^2 \to [1, 1+c]$ by letting ρ_1 be $1+c$ on the squares S_i with i even and 1 elsewhere. Now subdivide R into M^2N squares using M evenly spaced horizontal lines and MN evenly spaced vertical lines. We call a pair of points *marked* if they are the endpoints of a horizontal edge in the resulting grid.

The key step in the proof (Lemma 3.2) is to show that any biLipschitz map $f : I^2 \to \mathbb{E}^2$ with Jacobian ρ_1 must stretch apart a marked pair quantitatively more than it stretches apart the pair $(0,0), (1,0)$; more precisely, there is a $k > 0$ (depending on L, c) so that $\frac{d(f(p),f(q))}{d(p,q)} > (1+k)d(f(0,0), f(1,0))$ for some marked pair p, q. If this weren't true, then we would have an L-biLipschitz map $f : I^2 \to \mathbb{E}^2$ which stretches apart all marked pairs by a factor of at most $(1+\epsilon)d(f(0,0), f(0,1))$, where $\epsilon \ll 1$. This would mean that f maps horizontal lines in R to "almost taut curves". Using triangle inequalities one checks that this forces f to map most marked pairs p, q so that vector $f(q) - f(p)$ is $\approx d(p,q)(f(1,0) - f(0,0))$; this in turn implies that for some $1 \leq i \leq N$, *all* marked pairs p, q in the adjacent squares S_i, S_{i+1} are mapped by f so that

$f(q) - f(p) \approx d(p,q)(f(1,0) - f(0,0))$. Estimates then show that $f(S_i)$ and $f(S_{i+1})$ have nearly the same area, which contradicts the assumption that $Jac(f) = \rho_1$, because ρ_1 is 1 on one of the squares and $1+c$ on the other.

Our next step is to modify ρ_1 in a neighborhood of the grid in R: we use thin rectangles (whose thickness will depend on L, c) containing the horizontal edges of our grid, and define $\rho_2 : I^2 \to [1, 1+c]$ by letting ρ_2 be a "checkerboard" function in each of these rectangles and ρ_1 elsewhere. Arguing as in the previous paragraph, we will conclude that some suitably chosen pair of points in one of these new rectangles will be stretched apart by a factor $> d(f(0,0), f(1,0))(1+k)^2$ under the map f. Repeating this construction at smaller and smaller scales, we eventually obtain a function which can't be the Jacobian of an L-biLipschitz map.

The paper is organized as follows. In Section 2 we prove that Theorem 1.2 implies Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2.

2 Reduction of Theorem 1.1 to Theorem 1.2

Recall that every biLipschitz map is differentiable a.e., and the area of the image of a set is equal to the integral of the Jacobian over this set. We formulate our reduction as the following Lemma:

Lemma 2.1 *Let $\rho : I^2 \to [1, 1+c]$ be a measurable function which is not the Jacobian of any biLipschitz map $f : I^2 \to \mathbb{E}^2$ with*

$$Jac(f) := det(Df) = \rho \quad a.e. \tag{2.2}$$

Then there is a separated net in $\mathbb{E}^2$ which is not biLipschitz homeomorphic to $\mathbb{Z}^2$.

Proof. In what follows, the phrase "subdivide the square S into subsquares will mean that S is to be subdivided into squares using evenly spaced lines parallel to the sides of S. Let $\mathcal{S} = \{S_k\}_{k=1}^{\infty}$ be a disjoint collection of square regions in $\mathbb{E}^2$ so that each S_k has integer vertices, sides parallel to the coordinate axes, and the side length l_k of S_k tends to ∞ with k. Choose a sequence $m_k \in (1, \infty)$ with $\lim_{k\to\infty} m_k = \infty$ and $\lim_{k\to\infty} \frac{m_k}{l_k} = 0$. Let $\phi_k : I^2 \to S_k$ be the unique affine homeomorphism with scalar linear part, and define $\rho_k : S_k \to [1, 1+c]$ by $\rho_k = (\frac{1}{\rho}) \circ \phi_k^{-1}$. Subdivide S_k into m_k^2 subsquares of side length $\frac{l_k}{m_k}$. Call this collection $\mathcal{T}_k = \{T_{ki}\}_{i=1}^{m_k^2}$. For each i in $\{1, \ldots, m_k^2\}$, subdivide T_{ki} into n_{ki}^2 subsquares U_{kij} where n_{ki} is the integer part of $\sqrt{\int_{T_{ki}} \rho_k d\mathcal{L}}$. Now construct a separated net $X \subset \mathbb{E}^2$ by placing one point at the center of each integer square not contained in $\cup S_k$, and one point at the center of each square U_{kij}.

We now prove the lemma by contradiction. Suppose $g : X \to \mathbb{Z}^2$ is an L-biLipschitz homeomorphism. Let $X_k = \phi_k^{-1}(X) \subset I^2$, and define $f_k : X_k \to \mathbb{E}^2$ by

$$f_k(x) = \frac{1}{l_k}(g \circ \phi_k(x) - g \circ \phi_k(\star_k)) \tag{2.3}$$

where $\star_k$ is some basepoint in X_k. Then f_k is an L-biLipschitz map from X_k to a subset of $\mathbb{E}^2$, and the f_k's are uniformly bounded. By the proof of the Arzela-Ascoli theorem we may find a subsequence of the f_k's which "converges uniformly" to some biLipschitz map $f : I^2 \to \mathbb{E}^2$. By the construction of X, the counting measure on X_k (normalized by the factor $\frac{1}{l_k^2}$) converges weakly to $\frac{1}{\rho}$ times Lebesgue measure, while the (normalized) counting measure on $f_k(X_k)$ converges weakly to Lebesgue measure. It follows that $f_*((\frac{1}{\rho})\mathcal{L}) = \mathcal{L}\big|_{f(I^2)}$, i.e. $Jac(f) = \rho$. $\square$

3 Construction of a continuous function which is not a Jacobian of a biLipschitz map

The purpose of this section is to prove Theorem 1.2. As explained in the introduction, Theorem 1.2 follows from

Lemma 3.1 *For any given L and $c > 1$, there exists a continuous function $\rho : S = I^2 \to [1, 1+c]$, such that there is no L-biLipschitz homeomorphism $f : I^2 \to \mathbb{E}^2$ with*

$$Jac(f) = \rho \quad a.e.$$

Proof of Lemma 3.1. From now on, we fix two constants L and c and proceed to construct of a continuous function $\rho : I^2 \to [1, 1+c]$ which is not a Jacobian of an L-biLipschitz map. By default, all functions which we will describe, take values between 1 and $1 + c$.

We say that two points $x, y \in I^2$ are A-stretched (under a map $f : I^2 \to \mathbb{E}^2$) if $d(f(x), f(y)) \geq Ad(x, y)$.

For $N \in \mathbb{N}$, R_N be the rectangle $[0, 1] \times [0, \frac{1}{N}]$ and define a "checkerboard" function $\rho_N : R_N \to [1, 1+c]$ by $\rho_N(x, y) = 1$ if $[Nx]$ is even and $1 + c$ otherwise. It will be convenient to introduce the squares $S_i = [\frac{i-1}{N}, \frac{i}{N}] \times [0, \frac{1}{N}]$, $i = 1, \ldots, N$; ρ_N is constant on the interior of each S_i.

The cornerstone of our construction is the following lemma:

Lemma 3.2 *There are $k > 0$, M, μ, and N_0 such that if $N \geq N_0$, $\epsilon \leq \frac{\mu}{N^2}$ then the following holds: if the pair of points $(0, 0)$ and $(1, 0)$ is A-stretched under an L-biLipschitz map $f : R_N \to \mathbb{E}^2$ whose Jacobian differs from ρ_N on a set of area no bigger than ϵ , then at least one pair of points of the form $((\frac{p}{NM}, \frac{s}{NM}), (\frac{q}{NM}, \frac{s}{NM}))$ is $(1 + k)A$-stretched (where p and q are integers between 0 and NM and s is an integer between 0 and M).*

Proof of Lemma 3.2. We will need constants k, l, m, $\epsilon \in (0, \infty)$ and M, $N \in \mathbb{N}$, which will be chosen at the end of the argument. We will assume that $N > 10$ and $c, l < 1$. Let $f : R_N \to \mathbb{E}^2$ be an L-biLipschitz map such that $Jac(f) = \rho_N$ off a set of measure ϵ. Without loss of generality we assume that $f(x) = (0, 0)$ and $f(y) = (z, 0)$, $z \geq A$.

We will use the notation $x^i_{pq} := (\frac{p+M(i-1)}{NM}, \frac{q}{NM})$, where i is an integer between 1 and N, and p and q are integers between 0 and M. We call these points *marked.* Note that the marked points in S_i are precisely the vertices of the subdivision of S_i into M^2 subsquares. The index i gives the number of the square S_i, and p and q are "coordinates" of x^i_{pq} within the square S_i.

We will prove Lemma 3.2 by contradiction: we assume that all pairs of the form x^i_{pq}, x^j_{sq} are no more than $(1 + k)A$-stretched.

If $x^i_{pq} \in S_i$ is a marked point, we say that $x^{i+1}_{pq} \in S_{i+1}$ is the *marked point corresponding to* x^i_{pq}; corresponding points is obtained by adding the vector $(\frac{1}{N}, 0)$, where $\frac{1}{N}$ is the side length of the square S_i. We are going to consider vectors between the images of marked points in S_i and the images of corresponding marked points in the neighbor square S_{i+1}. We denote these vectors by $W^i_{pq} := f(x^{i+1}_{pq}) - f(x^i_{pq})$. We will see that most of the W^i_{pq}'s have to be extremely close to the vector $W := (A/N, 0)$, and, in particular, we will find a square S_i where W^i_{pq} is extremely close to W for *all* $0 \leq p, q \leq M$. This will mean that the areas of $f(S_i)$ and $f(S_{i+1})$ are very close, since $f(S_{i+1})$ is very close to a translate of $f(S_i)$. On the other hand, except for a set of measure ϵ, the Jacobian of f is 1 in one of the square S_i, S_{i+1} and $1 + c$ in the other. This allows us to estimate the difference of the areas of their images from below and get a contradiction.

If $l \in (0, 1)$, we say that a vector $W^i_{pq} = f(x^{i+1}_{pq}) - f(x^i_{pq})$, (or the marked point x^i_{pq}), is *regular* if the length of its projection to the x-axis is greater than $\frac{(1-l)A}{N}$. We say that a square S_i is *regular* if all marked points x^i_{pq} in this square are regular.

Claim 1. There exist $k_1 = k_1(l) > 0$, $N_1 = N_1(l)$, such that if $k \leq k_1$, $N \geq N_1$, there is a regular square.

Proof. Reasoning by contradiction, we assume that all squares are irregular. By the pigeon-hole principle, there is a value of s (between 0 and M) such that there are at least $\frac{N}{2M+2}$ irregular vectors $W^{i_j}_{p_j s}$, $j = 1, 2, \ldots J \geq \frac{N}{2M+2}$, where i_j is an increasing sequence with a fixed parity. This means that we look for l-irregular vectors between marked points in the same row s and only in squares S_i's which have all indices i's even or all odd. The latter assumption guarantees that the segments $[x^{i_j}_{p_j s}, x^{i_j+1}_{p_j s}]$ do not overlap. We look at the polygon with marked vertices

$$(0, 0), x^0_{0s} = (0, s/MN), x^{i_1}_{p_1 s}, x^{i_1+1}_{p_1 s}, x^{i_2}_{p_2 s}, x^{i_2+1}_{p_2 s} \ldots, x^{i_J}_{p_J s}, x^{i_J+1}_{p_J s}, x^N_{Ms} = (1, s/MN), (1, 0)$$

The image of this polygon under f connects $(0,0)$ and $(z,0)$ and, therefore, the length of its projection onto the x-axis is at least $z \geq A$. On the other hand, estimating this projection separately for the images of l-irregular segments $[x^{i_j}_{p_j s}, x^{i_j+1}_{p_j s}]$, the "horizontal" segments $[x^{i_j+1}_{p_j s}, x^{i_{j+1}}_{p_{j+1} s}]$ and the two "vertical" segments $[(0,0), x^0_{0s}]$ and $[x^N_{Ms}, (1,0)]$, one gets that the lengths of this projection is no bigger than

$$(\frac{N}{2M+2})(\frac{(1-l)A}{N}) + (\frac{(1+k)A}{N})(N - \frac{N}{2M+2}) + 2\frac{L}{N}. \tag{3.3}$$

The first term in (3.3) bounds the total length of projections of images of irregular segments by the definition of irregular segments and total number of them. The second summand is maximum stretch factor $(1+k)A$ between marked points times the total length of remaining horizontal segments. The third summand estimates the lengths of images of segments $[(0,0), x^0_{0s}]$ and $[x^N_{Ms}, (1,0)]$ just by multiplying their lengths by our fixed bound L on the Lipschitz constant.

Recalling that this projection is at least z, which in its turn is no less than A, we get

$$(\frac{N}{2M+2})(\frac{(1-l)A}{N}) + (\frac{(1+k)A}{N})(N - \frac{N}{2M+2}) + 2\frac{L}{N} \geq A.$$

One easily checks that this is impossible when k is sufficiently small and N is sufficiently large. This contradiction proves Claim 1. $\square$

Let $W = (\frac{A}{N}, 0)$.

Claim 2. Given any $m > 0$, there is an $l_0 = l_0(m) > 0$ such that if $l \leq l_0$ and $k \leq l$, then $|W - W^i_{pq}| \leq \frac{m}{N}$ for every regular vector W^i_{pq}.

Proof. Consider a regular vector $W^i_{pq} = (X, Y)$. Since W^i_{pq} is regular, $X \geq \frac{(1-l)A}{N}$. On the other hand, $X^2 + Y^2 \leq \frac{(1+k)^2 A^2}{N^2}$ and $X \leq \frac{(1+k)A}{N}$. Thus the difference of the x-coordinates of W^i_{pq} and W is bounded by $\frac{(l+k)A}{N} < \frac{2lA}{N}$. Substituting the smallest possible value $\frac{(1-l)A}{N}$ for X into $X^2 + Y^2 \leq \frac{(1+k)^2 A^2}{N^2}$, we get $Y^2 \leq \frac{2(l+k)A^2}{N^2} \leq \frac{4lA^2}{N^2}$. This implies that

$$N|W - W^i_{pq}| \leq 2A\sqrt{l^2 + l} \leq 2L\sqrt{l^2 + l}. \tag{3.4}$$

The right-hand side of (3.4) tends to zero with l, so Claim 2 follows. $\square$

Claim 3. There are $m_0 > 0$, M_0 such that if $m < m_0$ and $M > M_0$, then the following holds: if for some $1 \leq i \leq N$ and every p, q we have $|W - W^i_{pq}| \leq \frac{m}{N}$, then

$$|Area(f(S_{i+1})) - Area(f(S_i))| < \frac{c}{2N^2}. \tag{3.5}$$

Proof. We assume that i is even and therefore ρ takes the value 1 on S_i and $1+c$ on S_{i+1}; the other case is analogous. We let $Q := f(S_i)$ and $R = f(S_{i+1})$.

Q is bounded by a curve (which is the image of the boundary of S_i). Consider the result $\tilde{R} := Q + W$ of translating Q by the vector $W = (A/N, 0)$. The area of $\tilde{R}$ is equal to the area of Q.

The images of the marked points on the boundary of S_i form an $\frac{L}{NM}$-net on the boundary of Q, and the images of marked points on the boundary of S_{i+1} form an $\frac{L}{NM}$-net on R. By assumption the difference between W and each vector W^i_{pq} joining the image of a marked point on the boundary of S_i and the image of the corresponding point on the boundary of S_{i+1} is less than $\frac{m}{N}$. We conclude that the boundary of $\tilde{R}$ lies within the $\frac{m}{N} + \frac{2L}{MN}$-neighborhood of the boundary of R. Since f is L-Lipschitz, the length of the boundary of R is $\leq 4L/N$. Using a standard estimate for the area of a neighborhood of a curve, we obtain:

$$|Area(R) - Area(Q)| = |Area(R) - Area(\tilde{R})| \leq \frac{2L}{N}(\frac{m}{N} + \frac{2L}{MN}) + \pi(\frac{m}{N} + \frac{2L}{MN})^2.$$

Therefore (3.5) holds if m is sufficiently small and M is sufficiently large. $\square$

Proof of Lemma 3.2 concluded. Now assume $m < m_0$, $M > M_0$, $l \leq l_0(m)$, $k \leq \min(l, k_1(l))$, $N \geq N_1(l)$, and $\epsilon \leq \frac{c}{8N^2L^2}$. Combining claims 1, 2, and 3, we find a square S_i so that (3.5) holds. On the other hand, since $Jac(f)$ coincides with ρ off a set of measure ϵ, $Area(f(S_i)) \leq 1/N^2 + \epsilon L^2$ and $Area(f(S_{i+1}) \geq (1+c)(1/N^2 - \epsilon)$. Using the assumption that $\epsilon \leq \frac{c}{8N^2L^2}$ we get

$$Area(R) - Area(Q) \geq \frac{c}{2N^2},$$

contradicting (3.5). This contradiction proves Lemma 3.2. □

Proof of Lemma 3.1 continued. We will use an inductive construction based on Lemma 3.2. Rather than dealing with an explicit construction of pairs of points as in Lemma 3.2, it is more convenient to us to use the following lemma, which is an obvious corollary of Lemma 3.2. (To deduce this lemma from Lemma 3.2, just note that all properties of interest persist if we scale our coordinate system.)

Lemma 3.6 *There exists a constant $k > 0$ such that, given any segment $\overline{xy} \subset I^2$ and any neighborhood $\overline{xy} \subset U \subset I^2$, there is a measurable function $\rho : U \to [1, 1+c]$, $\epsilon > 0$ and a finite collection of non-intersecting segments $\overline{l_k r_k} \subset U$ with the following property: if the pair x, y is A-stretched by an L-biLipschitz map $f : U \to \mathbb{E}^2$ whose Jacobian differs from ρ on a set of area $< \epsilon$, then for some k the pair l_k, r_k is $(1+k)A$-stretched by f. The function ρ may be chosen to have finite image.*

We will prove Lemma 3.1 by induction, using the following statement. (It is actually even slightly stronger than Lemma 3.1 since it not only guarantees non-existence of L-biLipschitz maps with a certain Jacobian, but also gives a finite collection of points, such that at least one distance between them is distorted more than by factor L.)

Lemma 3.7 *For each integer i there is a measurable function $\rho_i : I^2 \to [1, 1+c]$, a finite collection $\mathcal{S}_i$ of non-intersecting segments $\overline{l_k r_k} \subset I^2$, and $\epsilon_i > 0$ with the following property: For every L-biLipschitz map $f : I^2 \to \mathbb{E}^2$ whose Jacobian differs from ρ_i on a set of area $< \epsilon_i$, at least one segment from $\mathcal{S}_i$ will have its endpoints $\frac{(1+k)^i}{L}$-stretched by f.*

Proof. The case $i = 0$ is obvious. Assume inductively that there are ρ_{i-1}, ϵ_{i-1}, and a disjoint collection of segments $\mathcal{S}_{i-1} = \{\overline{l_k r_k}\}$ which satisfy the conditions of the lemma. Let $\{U_k\}$ be a disjoint collection of open sets with $U_k \supset \overline{l_k r_k}$ and with total area $< \frac{\epsilon_{i-1}}{2}$. For each k apply Lemma 3.6 to U_k to get a function $\hat{\rho}_k : U_k \to [1, 1+c]$, $\hat{\epsilon}_k > 0$, and a disjoint collection $\hat{\mathcal{S}}_k$ of segments. Now let $\rho_i : I^2 \to [1, 1+c]$ be the function which equals $\hat{\rho}_k$ on each U_k and equals ρ_{i-1} on the complement of $\cup U_k$; let $\mathcal{S}_i = \cup \hat{\mathcal{S}}_k$, and $\epsilon_i = \min \hat{\epsilon}_k$. The required properties follow immediately. □

Lemma 3.1 and (Theorem 1.2) follows from Lemma 3.7.

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