METRIC DIFFERENTIATION, MONOTONICITY AND MAPS TO L^1

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Abstract. We present a new approach to the infinitesimal structure of Lipschitz maps into L^1, and as an application, we give an alternative proof of the main theorem of [CK06], that the Heisenberg group does not admit a bi-Lipschitz embedding in L^1. The proof uses the metric differentiation theorem of [Pau01] and the cut metric description in [CK06] to reduce the nonembedding argument to a classification of monotone subsets of the Heisenberg group.

Contents

1. Introduction

In this paper, we continue our investigation of Lipschitz maps of metric spaces into Banach spaces [CK06, CK08a, CK08b], which is motivated by the role of bi-Lipschitz embedding problems in theoretical computer science [LLR95, AR98, Lin02, LN06], earlier developments in the infinitesimal geometry of metric measure spaces [Pan89, Che99], and the geometry of Banach spaces [BL00, Chapters 6-7],[Bou85]. Our main purpose here is to present a new approach to Lipschitz maps into L^1, which gives new insight into both embeddability and non-embeddability questions; as a first application, we give a new proof of a (slightly stronger version of) the main

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result of [CK06]. Other implications will be pursued in subsequent papers, see below for more discussion.

An approach to the infinitesimal structure of Lipschitz maps into L^1. We begin by describing the new approach in general terms, cf [CK06, Section 1.8].

Let X be a metric space equipped with a Borel measure μ, and suppose $f : X \to L^1$ is a Lipschitz map. Under certain assumptions on the pair (X, μ) (e.g. if it is doubling and satisfies a Poincare inequality [HK96]), one can prove a generalization of Kirchheim's metric differentiation theorem [Kir94]. This says, roughly speaking, that for μ-a.e. $x \in X$, if one blows up f near x then it looks more and more like a geodesic map, when restricted to certain curves. Passing to limits, one arrives at a new map $f_\infty : X_\infty \to L^1$, where X_∞ is blow-up of X containing a distinguished class of geodesics called **lines**, and the restriction of f_∞ to each line $L \subset X_\infty$ gives a constant speed parametrization of some geodesic $f_\infty(L) \subset L^1$, i.e. for all x_1, $x_2 \in L$,

$$\|f_\infty(x_1) - f_\infty(x_2)\|_{L^1} = c_L \, d(x_1, x_2) \,, \tag{1.1}$$

where $c_L \in [0, \infty)$ is a constant depending on L. Here f_∞ has Lipschitz constant (respectively bi-Lipschitz constant) no larger than that of f.

Let $\rho_\infty : X_\infty \times X_\infty \to \mathbb{R}$ denote the pseudo-distance given by $\rho_\infty(x_1, x_2) = \|f_\infty(x_1) - f_\infty(x_2)\|_{L^1}$. Appealing to [CK06, Ass80], one obtains a representation of ρ_∞ as a superposition of elementary cut metrics [CK06, Section 3]:

$$\rho_\infty = \int_{\mathrm{Cut}(\mathbb{H})} d_E \, d\Sigma(E), \tag{1.2}$$

where $\mathrm{Cut}(X_\infty)$ is the collection of (equivalence classes of) measurable subsets of X_∞, $d_E : X_\infty \times X_\infty \to \{0, 1\}$ is defined by $d_E(x_1, x_2) = |\chi_E(x_1) - \chi_E(x_2)|$, and Σ is a measure on $\mathrm{Cut}(X_\infty)$.

The geodesic property (1.1) turns out to be equivalent to the condition that Σ-a.e. $E \in \mathrm{Cut}(X_\infty)$ is **monotone**, which means that for almost every line $L \subset X_\infty$, the characteristic function χ_E restricted to L agrees almost everywhere (with respect to linear measure on L) with a monotone function (Proposition 3.5). Thus questions about bi-Lipschitz embedding lead directly to an investigation of monotone sets in blow-up spaces, and some instances this leads to a complete resolution. In this paper we implement this approach when X is the Heisenberg group, and in [CK08b] we use it to exhibit embeddings of Laakso-type spaces into L^1.

Lipschitz maps from the Heisenberg group. Let $\mathbb{H}$ denote the Heisenberg group equipped with the Carnot-Caratheodory metric d. It was shown in [CK06] that metric balls $B \subset \mathbb{H}$ do not bi-Lipschitz embed in L^1. More specifically, it was shown that if $f : B \to L^1$ is any Lipschitz map, then blowing f up at a generic point $x \in B$, one obtains a family of maps which degenerate along cosets of the center of $\mathbb{H}$, which implies that f is not bi-Lipschitz. In this paper we give a shorter and largely self-contained proof of the nonembedding result, as well as a strengthening of the main result of [CK06], using the approach indicated above.

Let $f : \mathbb{H} \to L^1$ be a Lipschitz map, and let $\rho : \mathbb{H} \times \mathbb{H} \to [0, \infty)$ be defined by $\rho(x_1, x_2) = \|f(x_1) - f(x_2)\|_{L^1}$, i.e. ρ is the pullback of the distance on L^1 by f. If $x \in \mathbb{H}$ and $\lambda \in (0, \infty)$, let $\rho_{x,\lambda}$ be the result of dilating ρ at x, and renormalizing:

$$\rho_{x,\lambda}(z_1, z_2) = \frac{1}{\lambda}\, \rho(x\, s_\lambda z_1, x\, s_\lambda z_2) = \frac{1}{\lambda}((s_\lambda)^*(\ell_x)^*\rho)(z_1, z_2)\,,$$

where $s_\lambda : \mathbb{H} \to \mathbb{H}$ is the automorphism which scales distances by the factor λ, and $\ell_x : \mathbb{H} \to \mathbb{H}$ is left translation by x.

Theorem 1.3. *For almost every $x \in \mathbb{H}$, there is a semi-norm $\|\cdot\|_x$ on $\mathbb{R}^2$ such that*

$$\rho_{x,\lambda}(z_1, z_2) \to \|\pi(z_1) - \pi(z_2)\|_x \quad \text{as} \quad \lambda \to 0,$$

uniformly on compact subsets of $\mathbb{H} \times \mathbb{H}$. Here $\pi : \mathbb{H} \to \mathbb{H}/[\mathbb{H}, \mathbb{H}] \simeq \mathbb{R}^2$ is the abelianization homomorphism. In particular, $\rho_{x,\lambda}$ converges to a pseudo-distance which is zero along fibers of π, and hence f is not bi-Lipschitz in any neighborhood of x.

Discussion of the proof. We use the term **line** to refer to a coset $g \exp \mathbb{R}X$ of a horizontal 1-parameter subgroup $\exp \mathbb{R}X \subset \mathbb{H}$, and we refer to a pair of points $(z_1, z_2) \in \mathbb{H} \times \mathbb{H}$ as **horizontal** if it lies on a line, see Section 2.

The first step in the proof of Theorem 1.3 is to invoke the metric differentiation theorem of Pauls [Pau01] (see Theorem 2.5). This guarantees that for almost every $x \in \mathbb{H}$, there is a semi-norm $\|\cdot\|_x$ on $\mathbb{R}^2$ such that the statement of Theorem 1.3 holds provided we restrict to horizontal pairs (z_1, z_2), i.e.

$$\rho_{x,\lambda}(z_1, z_2) \to \|\pi(z_1) - \pi(z_2)\|_x \quad \text{as} \quad \lambda \to 0\,, \tag{1.4}$$

with uniform convergence on compact sets of horizontal pairs. The remainder of the argument is devoted to showing that (1.4) holds for all pairs $(z_1, z_2) \in \mathbb{H} \times \mathbb{H}$, not just horizontal pairs. If this were false, then using the fact that $\rho_{x,\lambda} \leq \mathrm{Lip}(f)\, d$ for all $(x, \lambda) \in \mathbb{H} \times (0, \infty)$, we may apply the Arzela-Ascoli theorem to find a sequence $\{\lambda_k\} \to 0$ such that the sequence of pseudo-metrics $\{\rho_{x,\lambda_k}\}$ converges uniformly on compact subsets of $\mathbb{H} \times \mathbb{H}$ to a pseudo-distance ρ_∞, where:

(1) $\rho_\infty(z_1, z_2) = \|\pi(z_1) - \pi(z_2)\|_x$ for all horizontal pairs (z_1, z_2).

(2) $\rho_\infty(\bar{z}_1, \bar{z}_2) \neq \|\pi(\bar{z}_1) - \pi(\bar{z}_2)\|_x$ for some $(\bar{z}_1, \bar{z}_2) \in \mathbb{H} \times \mathbb{H}$.
(3) $\rho_\infty \leq \mathrm{Lip}(f)\, d$.

Next, we apply ultralimits (or ultraproducts in the Banach space literature) and a theorem of Kakutani [Kak39], to see that ρ_∞ is also induced by a Lipschitz map $f_\infty : \mathbb{H} \to L^1$. Therefore ρ_∞ has a cut metric representation as a superposition of elementary cut metrics (1.2).

Condition (1) implies that the restriction of f_∞ to any line gives a constant speed parametrization of some geodesic in L^1, and as mentioned above, this property of f_∞ is equivalent to the condition that Σ-a.e. $E \in \mathrm{Cut}(\mathbb{H})$ is **monotone**: for almost every line $L \subset \mathbb{H}$, the characteristic function χ_E restricted to L agrees almost everywhere (with respect to linear measure on L) with a monotone function (Proposition 3.5).

Most of the work in the proof goes into Theorem 5.1, which classifies monotone subsets of $\mathbb{H}$. The monotone subsets of $\mathbb{H}$ turn out to be the half-spaces, modulo sets of measure zero. A **half-space** in $\mathbb{H}$ is a connected component of $\mathbb{H} \setminus P$ where P is either a vertical plane (a coset of a subgroup isomorphic to $\mathbb{R}^2$), or a horizontal plane (the union of the lines passing through some point $g \in \mathbb{H}$).

Thus the cut measure Σ is supported on half-spaces. We then show that Σ is in fact supported on vertical half-spaces (Section 7). This involves proving the injectivity of a certain convolution operator on $\mathbb{H}$, and invokes some harmonic analysis results from [Str91]. Finally, for cut measures supported on vertical half-spaces, the cut metric $\rho_\infty(z_1, z_2)$ depends only on the projections $\pi(z_1)$, $\pi(z_2)$, which contradicts (2).

We would like to emphasize that there is a simpler way to conclude the argument which avoids the harmonic analysis in Section 7, if one is only interested in the bi-Lipschitz nonembedding result. We present this alternate endgame in Section 6.

Comparison with [CK06]**.** Both the proof given here and the original proof in [CK06] use the cut metric representation, together with a differentiation argument. Here we use the L^1 cut metric representation [CK06, Section 3], rather than the finer representation using sets of finite perimeter in [CK06, Section 4]. Also, we use the differentiation result of [Pau01], rather than the differentiation results [Amb01, Amb02, FSSC01, FSSC03], which were a key ingredient in [CK06]. This leads to a much stronger restriction on the cuts showing up in the cut representation of the blown-up map as compared with the original map – they are monotone, rather than arbitrary sets of locally finite perimeter. We point out that the classification proof for monotone sets has some similarities with the classification proof for sets with constant normal, an important component of [FSSC01, FSSC03]. The harmonic analysis material appearing in Section 7 does not seem to correspond to anything in [CK06].

Apart from providing a substantially different approach from [CK06] to (generalized) differentiability theory for Lipschitz maps into L^1, the argument here gives a stronger conclusion, and is significantly shorter than [CK06]. The bi-Lipschitz nonembedding proof via the Theorem 6.1 is much shorter than [CK06], and is self-contained, apart from foundational material on L^1-cut measures taken from [CK06, Section 3].

Further results. In a forthcoming paper with Assaf Naor [CKN], we prove a quantitative version of the nonembedding theorem, i.e. for every $\epsilon > 0$ we find an explicit $\delta = \delta(\epsilon)$, such that for every 1-Lipschitz map $f : B \to L^1$, there exist x_1, x_2 with $d(x_1, x_2) > \delta$ and $\|f(x_1) - f(x_2)\|_{L_1} < \epsilon \cdot d(x_1, x_2)$. Central to the argument is the formulation and proof of a quantitative version of the classification of monotone sets given in this paper. It is also necessary to estimate, in terms of ϵ, a scale on which this quantitative classification can be applied; sets of finite perimeter a play a direct role in this step of the argument.

The rough outline of the first part of the proof given here is applicable in much greater generality, in particular to a large family of spaces satisfying Poincare inequalities. In this broader context there is a version of metric differentiation [CK], as well as an associated notion of monotone sets, which can be used to study bi-Lipschitz embedding in L^1; the final conclusions about embeddability or nonembedding depend on the the structure of monotone sets, which varies from example to example. For instance, in contrast to the Heisenberg group, the Laakso spaces bi-Lipschitz embed in L^1, even though they do not bi-Lipschitz embed in Banach spaces satisfying the Radon-Nikodym property, such as the space of sequences ℓ^1, see [CK08b]. We will pursue these ideas elsewhere.

Organization of the paper. Section 2 collects some background material. In Section 3 we relate the geodesic property of maps $X \to L^1$ with the monotonicity of the associated cut measure. In Section 4, we classify precisely monotone subsets of $\mathbb{H}$; this argument has fewer technical complications, but the same outline as the proof Theorem 5.1. In Section 5 we prove Theorem 5.1. In Section 6 we prove Theorem 6.1. In Section 7 we analyze the linear operator $\Sigma \mapsto d_\Sigma$ which assigns a cut metric to a signed cut measure. In Section 8 we complete the proof of Theorem 1.3.

2. Preliminaries

In this section we recall various facts that will be needed later, and fix notation.

We will use $\mathcal{L}$ as a generic symbol to denote Haar measure on Lie groups and associated homogeneous spaces.

2.1. **Carnot groups.** We recall that a **Carnot group** is a triple $(G, \Delta, \langle \cdot, \cdot \rangle)$, where G is a simply-connected nilpotent Lie group, Δ is a subspace of the Lie algebra of G, $\langle \cdot, \cdot \rangle$ is an inner product on Δ, and there is a decomposition of the Lie algebra of G as a direct sum

$$L(G) \;=\; V_1 \oplus \ldots \oplus V_k,$$

where $V_1 = \Delta$, and $[V_1, V_i] = V_{i+1}$ for all $i \in \{1, \ldots, k-1\}$. For every $\lambda \in (0, \infty)$, there is a unique automorphism $s_\lambda : G \to G$ whose derivative scales V_i by the factor λ^i. The direct sum $V_2 \oplus \ldots \oplus V_k$ is an ideal in the Lie algebra $L(G)$, which is the tangent space of the derived subgroup $[G, G] \subset G$. We denote the canonical epimorphism to the abelianization of G by $\pi : G \to G/[G, G]$; the latter is just a copy of $\mathbb{R}^n$ for $n = \dim \Delta$.

We will also view Δ as a left invariant distribution on G (or left invariant sub-bundle of TG), and refer to it as the **horizontal space**. A C^1 path $c : I \to G$ is **horizontal** if its velocity is tangent to Δ everywhere. A horizontal path c is a **horizontal lift** of a path $\bar{c} : I \to G/[G, G]$ if $\bar{c} = \pi \circ c$. Given a C^1 path $\bar{c} : I \to G/[G, G]$, $t \in I$, and $x \in \pi^{-1}(\bar{c}(t))$, there is a unique horizontal lift c of $\bar{c}$ such that $c(t) = x$. A **line** is the image of a horizontal lift of a straight line in $G/[G, G] \simeq \mathbb{R}^n$, or equivalently, a line is a subset $L \subset G$ of the form

$$L = \{g \, \exp(tX) \mid t \in \mathbb{R}\}$$

for some $g \in G$, $X \in \Delta \setminus \{0\}$, or to put is another way, a line is a left translate of a (nontrivial) horizontal 1-parameter subgroup. We let $\mathbb{L}(G)$ denote the collection of all lines in G; this has a natural smooth structure. A **horizontal pair** is pair of points x_1, $x_2 \in X$ which lie on a line. We let $\mathrm{hor}(G) \subset G \times G$ denote the collection of horizontal pairs; this is a closed subset of $G \times G$.

We equip G with the Carnot-Carathéodory (or sub-Riemannian) distance function d_G associated with the pair $(\Delta, \langle \cdot, \cdot \rangle)$, namely $d_G(p, q)$ is the infimal length of a horizontal path joining p to q.

2.2. **The Heisenberg group.** Recall that the 3-dimensional Heisenberg group $\mathbb{H}$ is the matrix group

$$\left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\},$$

whose Lie algebra of $\mathbb{H}$ has the presentation

$$[X, Y] = Z, \quad [X, Z] = [Y, Z] = 0\,,$$

where

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We will identify X, Y, and Z with left invariant vector fields on $\mathbb{H}$. The Carnot group structure on $\mathbb{H}$ is the triple $(\mathbb{H}, \Delta, \langle \cdot, \cdot \rangle_\Delta)$, where Δ is the 2-dimensional subbundle of the tangent bundle $T\mathbb{H}$ spanned by $\{X, Y\}$, and $\langle \cdot, \cdot \rangle_\Delta$ is the left invariant Riemannian metric on Δ for which $\{X, Y\}$ are orthonormal. The center of $\mathbb{H}$ is the 1-parameter group $\{\exp tZ \mid t \in \mathbb{R}\}$, which is also the derived subgroup $[\mathbb{H}, \mathbb{H}]$. The canonical epimorphism to the abelianization $\pi : \mathbb{H} \to \mathbb{H}/[\mathbb{H}, \mathbb{H}] = \mathbb{H}/\operatorname{Center}(\mathbb{H})$ will be identified with the homomorphism $\pi : \mathbb{H} \to \mathbb{R}^2$ where

$$\pi\left(\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}\right) = (a, b).$$

Lemma 2.1. *If $c : I \to \mathbb{H}$ is a horizontal lift of a loop $\bar{c} : I \to \mathbb{R}^2$, then the endpoints $c(0), c(1)$ both lie in the same fiber $F = \pi^{-1}(c(0)) = \pi^{-1}(c(1))$, and satisfy*

$$c(1) = c(0) \exp(A\, Z), \tag{2.2}$$

where $A \in \mathbb{R}$ is the signed Euclidean area enclosed by the loop $\bar{c}$.

Proof. For readers familiar with connections on principal bundles, the projection $\pi : \mathbb{H} \to \mathbb{R}^2$ defines a principal $\mathbb{R}$-bundle, and horizontal distribution is a connection with curvature form $dx \wedge dy$. The lemma follows from the relation between holonomy and curvature, for abelian principal bundles.

Here is an elementary proof. Let $\{\alpha_X, \alpha_Y, \alpha_Z\}$ be the basis of of left-invariant 1-forms dual to $\{X, Y, Z\}$, so $\alpha_X = \pi^* dx$, $\alpha_Y = \pi^* dy$, and $d\alpha_Z = -\alpha_X \wedge \alpha_Y = -\pi^*(dx \wedge dy)$. Let $\eta : I \to \mathbb{H}$ be a path in the fiber F running from $c(1)$ to $c(0)$. If γ is the concatenation of c and η, then γ is a cycle which bounds a 2-chain ζ, so by Stokes' theorem we have

$$\int_\eta \alpha_Z = \int_\gamma \alpha_Z$$

$$= \int_\zeta d\alpha_Z = -\int_{\pi \circ \zeta} dx \wedge dy = -A,$$

which means that $c(0) = c(1) \exp(-AZ)$, or $c(1) = c(0) \exp\, AZ$.

∎

If $x, y \in \mathbb{H}$, then by Lemma 2.1 any two horizontal paths c_1, c_2 from x to y have projections $\pi \circ c_1$, $\pi \circ c_2$ which enclose zero signed area; conversely, any path $c : I \to \mathbb{H}$ which starts at x, and ends in $\pi^{-1}(y)$ will terminate at y provided the signed area enclosed by c_1 and c is zero.

The geodesics (locally length minimizing paths) in $\mathbb{H}$ are horizontal lifts of circles in $\mathbb{R}^2$. This implies that for every $p \in \mathbb{H}$,

$$d(p, p \exp tZ) = \sqrt{4\pi|t|}\,, \tag{2.3}$$

by the solution to the isoperimetric problem.

A **vertical plane** is a subset $P \subset \mathbb{H}$ of the form $\pi^{-1}(L)$, where L is a line in the plane. The **horizontal plane centered at** $p \in \mathbb{H}$ is the union of the lines passing through p. A **plane** is a vertical or horitzontal plane. A **vertical (respectively horizontal) half-space** is one of the two components of $\mathbb{H} \setminus P$, for some vertical (respectively horizontal) plane.

Two lines L_1, $L_2 \in \mathbb{L}(\mathbb{H})$ are **parallel** if they are tangent to the same horizontal vector field $X \in \Delta \setminus \{0\}$, or equivalently, if $\pi(L_1)$, $\pi(L_2)$ are parallel lines in $\mathbb{R}^2$. Two lines are **skew** if they are disjoint and not parallel.

Lemma 2.4.

(A) *Suppose $L_1, L_2 \in \mathbb{L}(\mathbb{H})$ are parallel but $\pi(L_1) \neq \pi(L_2)$. Then there is a unique fiber $\pi^{-1}(x) \subset H$ lying halfway between $\pi(L_1)$ and $\pi(L_2)$ such that every point in L_1 can be joined to L_2 by a unique line, and this line will pass through $\pi^{-1}(x)$. Moreover every point in $\pi^{-1}(x)$ lies on a unique such line.*

(B) *Suppose $L_1, L_2 \in L$ are skew lines. Then there is a hyperbola $Y \subset \mathbb{R}^2$ with asymptotes L_1 and L_2, such that every tangent line of Y has a unique horizontal lift which intersects both L_1 and L_2, and conversely, if $L \in \mathbb{L}(\mathbb{H})$ and $L \cap L_i \neq \emptyset$ for $i = 1, 2$, then $\pi(L)$ is tangent to Y.*

Proof. (A). Let $\bar{\eta} : [0, 1] \to \mathbb{R}^2$ be a line segment running from $\pi(L_1)$ to $\pi(L_2)$, and let $\eta : [0, 1] \to \mathbb{H}$ be a horizontal lift of $\bar{\eta}$ ending in L_2. Then $x_1 = \eta(0) \exp AZ$ for some $A \in \mathbb{R}$, where $x_1 \in L_1$. Now form a closed quadrilateral $\bar{\eta}$, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ enclosing signed area A, where $\bar{\alpha} \subset \pi(L_2)$ and $\bar{\gamma} \subset \pi(L_1)$. Horizontally lifting this to an open quadrilateral η, α, β, γ, by Lemma 2.1 we have $\gamma \subset L_1$, which implies that β intersects both L_1 and L_2.

Let $\bar{\beta}_1 : [0, 1] \to \mathbb{R}^2$ be a line segment from $\pi(L_1)$ to $\pi(L_2)$ which passes through the midpoint x of $\bar{\beta}$. Then we obtain a closed (self-intersecting) quadrilateral $\bar{\beta}$, $\bar{\delta}_1$, $\bar{\beta}_1$, $\bar{\delta}_2$ enclosing zero signed area, where $\bar{\delta}_i \subset \pi(L_i)$. Horizontally lifting this to $\mathbb{H}$, by Lemma 2.1 we get a closed quadrilateral β, δ_1, β_1, δ_2, and hence β_1 intersects both L_1 and L_2.

Given $A \in \mathbb{R}$, we may choose $\bar{\beta}_1$ so that the area of the triangle enclosed by $\bar{\beta}_1$, $\bar{\beta}$, and $\pi(L_1)$ is A; it follows that β_1 may be chosen to pass through any prescribed point in the fiber $\pi^{-1}(x)$.

(B). Let $x = \pi(L_1) \cap \pi(L_2)$, and $x_i = L_i \cap \pi^{-1}(x)$. Then $x_2 = x_1 \exp AZ$ for some $A \in \mathbb{R} \setminus \{0\}$. Let Γ be the collection of line segments $\bar{\eta} : I \to \mathbb{R}^2$ running from $\pi(L_1)$ to $\pi(L_2)$, such that the oriented triangle with vertices x, $\bar{\eta}(0)$, $\bar{\eta}(1)$ encloses signed area A. By Lemma 2.1, if $\bar{\eta} \in \Gamma$, then the horizontal lift $\eta : I \to \mathbb{H}$ starting in L_1 ends on L_2. The elements of Γ are precisely the segments tangent to a hyperbola with asymptotes $\pi(L_1)$ and $\pi(L_2)$. To see this, apply an area-preserving affine transformation so that $\pi(L_1)$ and $\pi(L_2)$ are the x and y axes, respectively; then by analytic geometry, the tangent lines to the hyperbola defined by $xy = C$ enclose area $2C$.

□

2.3. **Metric differentiation and blow-ups of Lipschitz maps.** Let $f : G \to Y$ be a Lipschitz map from a Carnot group to a metric space, and let $\rho : G \times G \to [0, \infty)$ be the pullback of the distance function, i.e. $\rho(g_1, g_2) = f^* d_Y(g_1, g_2) = d_Y(f(g_1), f(g_2))$. We will need the following metric differentiation theorem of Pauls [Pau01], which generalizes Kirchheim's metric differentiation theorem [Kir94]:

Theorem 2.5. *For almost every $g \in G$, rescalings of ρ at g converge on uniformly on compact subsets of* $\mathrm{hor}(G) \subset G \times G$ *to the left invariant Carnot (pseudo)distance $\alpha : G \times G \to [0, \infty)$ induced by some Finsler semi-norm on the horizontal space. In other words, if $K \subset \mathrm{hor}(G)$ is compact, then*

$$\frac{1}{\lambda} s_\lambda^*(\ell_g^* \rho)\Big|_K \xrightarrow{C^0} \alpha\Big|_K \quad as \quad \lambda \to 0 \,. \tag{2.6}$$

In actuality, suitably formulated, metric differentiation holds whenever the domain is any PI space; see [CK].

One may refine the conclusion somewhat by making use of ultralimits, which have been used frequently in geometric group theory see [Gro93, KL97], or earlier in the Banach space literature [DilK72, Hei80, HM82]. If g is as in the theorem above, and $\{\lambda_k\} \subset (0, \infty)$ is a sequence tending to zero, then f defines a sequence of uniformly Lipschitz maps $(\frac{1}{\lambda_k} G, g) \xrightarrow{f} (\frac{1}{\lambda_k} Y, f(g))$ between pointed metric spaces. The ultralimit of this sequence is a Lipschitz mapping $f_\omega : G_\omega \to L^1_\omega$, where G_ω and L^1_ω are ultralimits of G and L^1, respectively. Up to isometry, G_ω may be identifying with G itself, while a theorem of Kakutani [Kak39] implies that L^1_ω is isometric to an L^1 space for some (typically not σ-finite) measure. This gives:

Corollary 2.7. *If the rescaled (pseudo)distance functions in (2.6) converge uniformly on compact subsets of $G \times G$ to a limiting pseudo-distance ρ_∞, then ρ_∞ is the metric induced by a map $f_\omega : G \to L^1$.*

2.4. L^1 **metrics and cut metrics.** We refer the reader to [CK06] for more discussion of the material reviewed in this subsection. We are using a slightly different setup here, working with L^1_{loc} rather than L^1, but the adaptation to this setting is straightforward.

We let (X, μ) denote a locally compact metric measure space, where μ is a Borel measure which is finite on compact subsets of X.

A **cut** in X is an equivalence class of measurable subsets, where two subsets E, E' are equivalent if their symmetric difference has measure zero. We let $\mathrm{Cut}(X)$ denote the collection of cuts in X. We may view $\mathrm{Cut}(X)$ as a subset of $L^1_{\mathrm{loc}}(X)$, by identifying a cut $E \in \mathrm{Cut}(X)$ with its characteristic function $\chi_E \in L^1_{\mathrm{loc}}(X)$. We will endow $\mathrm{Cut}(X)$ with the topology induced by $L^1_{\mathrm{loc}}(X)$ via this embedding.

A **cut measure** on X is a Borel measure Σ on $\mathrm{Cut}(X)$ such that

$$\int_{\mathrm{Cut}(X)} \mu(E \cap K)\, d\Sigma(E) < \infty$$

for every compact subset $K \subset X$.

For every cut measure Σ, there is a **tautological** $\Sigma \times \mu$-measurable function $\Phi : \mathrm{Cut}(X) \times X \to \{0,1\}$ such that for Σ-a.e. cut E, we have $\Phi(E,x) = \chi_E(x)$ for μ-a.e. $x \in X$; this function is unique, up to sets of measure zero by Fubini's theorem. For such a function Φ, if $x \in X$, $E \in \mathrm{Cut}(X)$, we let $\Phi_x = \Phi(\cdot, x)$ and $\Phi_E = \Phi(E, \cdot)$.

If Σ is a cut measure with tautological function Φ, then we obtain an L^1_{loc} mapping $X \to L^1(\mathrm{Cut}(X), \Sigma)$ by sending $x \in X$ to Φ_x. In particular, there is a full measure subset $Z \subset X$ such that if x_1, $x_2 \in Z$, then Φ_{x_i} is Σ-integrable and so we obtain a (pseudo)distance

$$d_\Sigma(x_1, x_2) = \|\Phi_{x_1} - \Phi_{x_2}\|_{L^1(\mathrm{Cut}(X),\Sigma)}\,,$$

which is the **cut metric** associated with the cut measure Σ. Modulo changing Z by a set of measure zero, the cut metric is independent of the choice of tautological function Φ.

A cut $E \in \mathrm{Cut}(X)$ defines an **elementary cut metric** $d_E : X \times X \to [0, \infty)$, where $d_E(x_1, x_2) = |\chi_E(x_1) - \chi_E(x_2)|$. Since $\Phi_E = \chi_E$ for Σ-a.e. $E \in \mathrm{Cut}(X)$, we may view the cut metric d_Σ as a superposition of elementary cut metrics:

$$d_\Sigma(x_1, x_2) = \int_{\mathrm{Cut}(X)} |\Phi_{x_1}(E) - \Phi_{x_2}(E)|\, d\Sigma(E) = \int_{\mathrm{Cut}(X)} d_E(x_1, x_2)\, d\Sigma(E) \tag{2.8}$$

Notice that above discussion of cut measures and associated cut metrics makes perfect sense for signed measures. This leads to the notion of a **signed cut measure**, and the associated **cut metric** which is still given by (2.8), except that it may take negative values. Signed cut measures will appear in Section 7.

Now let $f : (X, \mu) \to L^1(Y, \nu)$ be an L^1_{loc} mapping, where (Y, ν) is a σ-finite measure space, and let $\rho = f^* d_{L^1(Y,\nu)}$ be the pullback distance, $\rho(x_1, x_2) = \|f(x_1) - f(x_2)\|_{L^1(Y,\nu)}$. Then ρ arises from a cut measure:

Theorem 2.9. *There is a cut measure Σ such that for any tautological function $\Phi : \mathrm{Cut}(X) \times X \to \{0, 1\}$ as above, there is a full measure subset $Z \subset X$ such that if $x_1, x_2 \in Z$, then $\rho(x_1, x_2) = d_\Sigma(x_1, x_2)$.*

Remark 2.10. In some respects a more natural setting for the material in this section would be a σ-finite measure space equipped with an exhaustion $X_1 \subset X_2 \subset \ldots$ by finite measure subsets. Since our applications only involve locally compact metric measure spaces, we have chosen this setting.

3. Monotonicity and geodesic maps to L^1

In this section, we show that geodesic maps to L^1 may characterized by a monotonicity property of the cuts in the support of the cut measure.

Geodesic maps from $\mathbb{R}$ to L^1. We begin with the following observation:

Lemma 3.1. *Suppose $f = (f_1, \ldots, f_n) : \mathbb{R} \to \ell^1(\mathbb{R}^n)$ is a continuous map. Then f is a weakly monotonic parametrization of a geodesic in $\ell^1(\mathbb{R}^n)$ if and only if each component $f_i : \mathbb{R} \to \mathbb{R}$ is weakly monotonic.*

Proof. For any $a \le b \le c \in \mathbb{R}$,

$$\begin{aligned}\|f(a) - f(c)\| = \sum_i \ |f_i(a) - f_i(c)| \ &\le \ \sum_i \ (|f_i(a) - f_i(b)| + |f_i(b) - f_i(c)|) \\ &= \|f(a) - f(b)\| + \|f(b) - f(c)\| \ .\end{aligned}$$

Therefore we have equality if and only if $|f_i(a)-f_i(c)| = |f_i(a)-f_i(b)|+|f_i(b)-f_i(c)|$ for all i. The lemma follows. $\square$

It is natural to ask for an equivalent characterization in terms of the associated cut measure. This leads to:

Definition 3.2. A cut (or measurable subset) $E \subset \mathbb{R}$ is **monotone** if it is equivalent to a measurable subset which is connected, and has connected complement.

Every monotone cut $E \subset \mathbb{R}$ may be represented by the empty set, a ray, or $\mathbb{R}$. We use the word "monotone" for this condition, because monotone functions have monotone sublevel/superlevel sets, and the characteristic function of a measurable set is essentially monotone if and only if the set is monotone.

Given a distance function α on a subset $\{a,b,c\} \subset \mathbb{R}$, where $a \le b \le c$, the **excess** of α is the quantity $\operatorname{excess}(\alpha)\{a,b,c\} = d(a,b) + d(b,c) - d(a,c) \ge 0$. Note that if d_E is the elementary cut metric associated with a measurable subset $E \subset \mathbb{R}$, then E is monotone if and only if $\operatorname{excess}(d_E)\{x_1,x_2,x_3\} = 0$ for $\mathcal{L}^3$-a.e. triple (x_1,x_2,x_3). To see this, observe that if $a < b \in \mathbb{R}$ lie in the support of E (respectively $\mathbb{R} \setminus E$), then $(a,b) \setminus E$ (respectively $\mathbb{R} \cap E$) has measure zero.

Lemma 3.3. *Suppose $f : \mathbb{R} \to L^1$ is an L^1_{loc} mapping. Then the following are equivalent:*

(1) *There is a full measure subset $Z \subset \mathbb{R}$ such that if z_1, z_2, $z_3 \in Z$ and $z_1 \le z_2 \le z_3$, then*
$$\|f(z_1) - f(z_3)\|_{L^1} = \|f(z_1) - f(z_2)\|_{L^1} + \|f(z_2) - f(z_3)\|_{L^1}\,.$$
(2) *If Σ is the cut measure associated with f, then Σ-a.e. cut E is monotone.*

Proof. Let Σ be the cut measure on X guaranteed by Theorem 2.9, and let $\rho : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ be the pullback of the distance on L^1 by f, so
$$\rho = \int_{\mathrm{Cut}(\mathbb{R})} d_E \, d\Sigma(E)$$
where d_E is the elementary cut metric associated with E. Then
$$\int_{\mathbb{R}^3} \operatorname{excess}(\rho)\{z_1,z_2,z_3\}\, d\mathcal{L}^3(z_1,z_2,z_3)$$
$$= \int_{\mathbb{R}^3} \operatorname{excess}\left(\int_{\mathrm{Cut}(\mathbb{R})} d_E\, d\Sigma(E)\right)\{z_1,z_2,z_3\}\, d\mathcal{L}^3(z_1,z_2,z_3)$$
$$= \int_{\mathbb{R}^3} \left(\int_{\mathrm{Cut}(\mathbb{R})} \operatorname{excess}(d_E)\{z_1,z_2,z_3\}\, d\Sigma(E)\right) d\mathcal{L}^3(z_1,z_2,z_3)\,.$$
By Fubini's theorem, it follows that the vanishing of the quantity above is equivalent to either (1) or (2). $\square$

L^1-mappings which are geodesic along a family of curves. We now consider an L^1_{loc}-mapping $f : (X, \mu) \to L^1$ with associated cut measure Σ, where (X, μ) is a locally compact space and μ is a Radon measure. We would like to examine the implications for Σ when f is geodesic along certain curves in X. However, both cuts and L^1 mappings are only defined up to sets of measure zero, and since curves typically have measure zero, this relation only becomes meaningful when we consider generic curves belonging to a sufficiently rich family. We formalize this as follows.

Let $\mathbb{P}$ be a locally compact measure space equipped with a Radon measure π, and $\Gamma : \mathbb{R} \times \mathbb{P} \to X$ be a continuous map such that the pushforward measure satisfies $\Gamma_*(\mathcal{L} \times \pi) \leq C\mu$ for some $C \in \mathbb{R}$. If $p \in \mathbb{P}$, we denote the map $t \mapsto \Gamma(t, p)$ by Γ_p.

Definition 3.4. A measurable subset $E \subset X$ is **Γ-monotone** if for π-a.e. $p \in \mathbb{P}$, the inverse image $\Gamma_p^{-1}(E) \subset \mathbb{R}$ is measurable and monotone. By Fubini's theorem, this property is shared by all measurable subsets representing the same cut, and hence we may speak of **Γ-monotone cuts**.

Proposition 3.5. *The following are equivalent:*

- *For π-a.e. $p \in \mathbb{P}$, the composition $f \circ \Gamma_p : \mathbb{R} \to L^1$ is a geodesic map, i.e. it satisfies the conditions of Lemma 3.3.*
- *Σ-a.e. cut E is Γ-monotone.*

Proof. In brief, after sorting out the behavior of cut metrics under composition of maps and slicing, this follows from the previous lemma.

Let Σ, the map $\Phi : \mathrm{Cut}(X) \times X \to \{0, 1\}$, and $Z \subset X$ be as in Theorem 2.9, and let $\rho = f^* d_{L^1(Y,\nu)}$ be the pullback distance.

For π-a.e. $p \in \mathbb{P}$, the map $\Gamma : \mathbb{R} \times \mathbb{P} \to X$ induces a map $\mathrm{Cut}(\Gamma_p) : \mathrm{Cut}(X) \to \mathrm{Cut}(\mathbb{R})$ given by $\mathrm{Cut}(\Gamma_p)(E) = \Gamma_p^{-1}(E)$; pushing Σ forward under $\mathrm{Cut}(\Gamma_p)$ we get an L^1_{loc} cut measure Σ_p on $\mathrm{Cut}(\mathbb{R})$. For such p, we may choose a $(\Sigma_p \times \mathcal{L})$-measurable function $\hat{\Phi}^p : \mathrm{Cut}(\mathbb{R}) \times \mathbb{R} \to \{0, 1\}$ such that for Σ_p-a.e. $E \in \mathrm{Cut}(\mathbb{R})$ we have $\hat{\Phi}^p(E, \cdot) = \chi_E$. Also, for π-a.e. $p \in \mathbb{P}$, we have a well-defined L^1_{loc}-map $f_p = f \circ \Gamma_p : \mathbb{R} \to L^1(Y, \nu)$, and a pullback distance $\rho_p = \Gamma_p^* d_{L^1(Y,\nu)}$.

Lemma 3.6. *For π-a.e. $p \in \mathbb{P}$, and Σ-a.e. $E \in \mathrm{Cut}(X)$,*

$$\Phi(E, \Gamma_p(t)) = \hat{\Phi}^p(\mathrm{Cut}(\Gamma_p)(E), t)$$

for $\mathcal{L}$-a.e. $t \in \mathbb{R}$.

Proof. By Fubini's theorem and the defining properties of Φ and $\hat{\Phi}^p$, for π-a.e. $p \in \mathbb{P}$, and Σ-a.e. $E \in \mathrm{Cut}(X)$,

$$\Phi(E, \Gamma(p,t)) = \chi_{\mathrm{Cut}(\Gamma_p)(E)}(t) = \hat{\Phi}^p(\mathrm{Cut}(\Gamma_p)(E), t)$$

for $\mathcal{L}$-a.e. $t \in \mathbb{R}$. □

Now for π-a.e. $p \in \mathbb{P}$, there is a full measure set $T_p \subset \mathbb{R}$ such that $\Gamma_p(T_p) \subset Z$, and therefore for every t_1, $t_2 \in T_p$ we get

$$\begin{aligned}\rho_p(t_1, t_2) &= \int_{\mathrm{Cut}(X)} |\Phi_{\Gamma_p(t_1)}(E) - \Phi_{\Gamma_p(t_2)}(E)|\, d\Sigma(E) \\ &= \int_{\mathrm{Cut}(\mathbb{R})} |\hat{\Phi}^p(t_1)(E) - \hat{\Phi}^p(t_2)(E)|\, d\Sigma_p(E)\end{aligned}$$

by Lemma 3.6. Lemma 3.3 implies that ρ_p satisfies the conditions of the lemma for π-a.e. $p \in \mathbb{P}$ if and only if $\Phi(E, \Gamma_p(\cdot))$ is the characteristic function of a monotone set for π-a.e. $p \in \mathbb{P}$ and Σ-a.e. $E \in \mathrm{Cut}(X)$.

□

Next we apply Proposition 3.5 to a Carnot group G. We recall that $\mathbb{L}(G)$ denotes the family of (horizontal) lines in G; we let $\mathbb{P}$ be the family of unit speed parametrized lines in G. Here $\mathbb{L}(G)$ and $\mathbb{P}$ have natural smooth structures such that the tautological map $\mathbb{P} \to \mathbb{L}(G)$ is a smooth fibration with fibers diffeomorphic to the Lie group $\mathrm{Isom}(\mathbb{R})$. We endow $\mathbb{P}$ and the space of line $\mathbb{L}(G)$ with smooth measures with positive density. If $\Gamma : \mathbb{R} \times \mathbb{P} \to G$ is defined by $\Gamma(t,p) = p(t)$, then Fubini's theorem implies that a measurable set $E \subset G$ is Γ-monotone if and only if the intersection $E \cap L$ is a monotone subset of $L \simeq \mathbb{R}$ for almost every $L \in \mathbb{L}(G)$.

Definition 3.7. A measurable subset (or cut) $E \subset G$ is **monotone** if $E \cap L$ is a monotone subset of $L \simeq \mathbb{R}$ for almost every $L \in \mathbb{L}(G)$.

With this definition, Proposition 3.5 yields:

Corollary 3.8. *Let $f : G \to L^1$ be an L^1_{loc}-mapping with associated cut measure Σ, such that $f\big|_L$ is a geodesic map for almost every $L \in \mathbb{L}(G)$. Then Σ-a.e. $E \in \mathrm{Cut}(G)$ is monotone.*

In Section 5 we will show that nontrivial monotone subsets of $\mathbb{H}$ are half-spaces, modulo sets of measure zero.

4. The classification of precisely monotone sets

Throughout this section ∂E will denote the topological frontier (boundary) of a subset E.

In Section 5 we will classify monotone subsets of the Heisenberg group. Before doing this, we first consider the easier task of classifying precisely monotone sets:

Definition 4.1. Let X be either $\mathbb{R}^n$ or $\mathbb{H}$. A subset $E \subset X$ is **precisely monotone** if every line $L \in \mathbb{L}(X)$ intersects both E and its complement in a connected set.

Thus in the $\mathbb{R}^n$ case, a precisely monotone set is a convex set with convex complement.

Lemma 4.2. *If $E \subset \mathbb{R}^n$ is precisely monotone, then either $E = \emptyset$, $E = \mathbb{R}^n$, or $C \subset E \subset \overline{C}$ for some (open) half-space $C \subset \mathbb{R}^n$.*

This follows immediately by looking at a supporting half-space for E, assuming both E and its complement are nonempty.

We now focus on precisely monotone subsets of the Heisenberg group:

Theorem 4.3. *If $E \subset \mathbb{H}$ is a precisely monotone subset, then either $E = \emptyset$, $E = \mathbb{H}$, or $C \subset E \subset \overline{C}$ for some half-space $C \subset \mathbb{H}$.*

For the remainder of this section, we fix a precisely monotone subset $E \subset \mathbb{H}$, and let $E^c = \mathbb{H} \setminus E$ be its complement. Note that a subset of $\mathbb{H}$ is precisely monotone if and only if its complement is precisely monotone, so the roles of E and E^c will be symmetric throughout.

The proof will proceed in the following steps:

(1) Lemma 4.8: If $L \in \mathbb{L}(\mathbb{H})$ contains more than one point of ∂E, then $L \subset \partial E$.

(2) Lemma 4.9: ∂E is a union of lines.

(3) Lemma 4.11: Either ∂E is contained in a plane, or $\partial E = \mathbb{H}$.

(4) Lemma 4.12: The case $\partial E = \mathbb{H}$ does not occur.

(5) Lemma 4.13: If ∂E is nonempty, then it is a plane and $C \subset E \subset \bar{C}$, where C is a connected component of $\mathbb{H} \setminus \partial E$.

We now proceed with the steps of the proof.

If a monotone (or more generally convex) subset $Y \subset \mathbb{R}^n$ contains a subset Σ and a point p, then it also contains the cone over Σ with vertex at p. We begin with an analogous statement in the Heisenberg group; it is more subtle than the Euclidean case, due to the fact that the lines in $\mathbb{H}$ passing through a point $x \in \mathbb{H}$ lie in a horizontal plane, which has empty interior. To implement the argument, we will use piecewise horizontal curves.

Definition 4.4. For $x \in \mathbb{H}$, v_1, $v_2 \in \Delta$, let $\gamma(x, v_1, v_2)$ be the unit speed path which starts at x, moves along a horizontal curve in the direction v_1 a distance $|v_1|$, and then along a horizontal curve in the direction v_2 a distance $|v_2|$. Thus $\gamma(x, v_1, v_2)$ is a broken horizontal line with vertices x, $x \exp(v_1)$, and $x \exp(v_1) \exp(v_2)$. Using this, we may define a map $\Gamma : \mathbb{H} \times \Delta^2 \to \mathbb{H}$ by letting $\Gamma(x, v_1, v_2)$ be the other endpoint of $\gamma(x, v_1, v_2)$, i.e. $\Gamma(x, v_1, v_2) = \gamma(x, v_1, v_2)(|v_1| + |v_2|)$. For $x \in \mathbb{H}$, we define $\Gamma_x : \Delta^2 \to \mathbb{H}$ by $\Gamma_x(v_1, v_2) = \Gamma(x, v_1, v_2)$.

Lemma 4.5. *The map Γ is smooth. For all x, the map Γ_x is a submersion near any pair $(v_1, v_2) \in \Delta^2$ with $v_1 + v_2 \neq 0$ (recall that we are viewing Δ as a subspace of the Lie algebra of $\mathbb{H}$).*

Proof. The smoothness of Γ is immediate from the smoothness of the group operation.

Let $\pi : \mathbb{H} \to \mathbb{R}^2$ be the abelianization map. Pick $x \in \mathbb{H}$, $(v_1, v_2) \in \Delta^2$ such that $v_1 + v_2 \neq 0$. Then $\pi(\Gamma_x(v_1, v_2))$ is the point $y := x + \pi(v_1) + \pi(v_2)$. Define a smooth path $\eta : \mathbb{R} \to \Delta^2$ by $t \mapsto (v_1 + tw, v_2 - tw)$ where w is a nonzero vector orthogonal to $v_1 + v_2$. Then $\Gamma_x \circ \eta$ has a nonzero velocity tangent to the fiber $\pi^{-1}(y)$ (this follows by using (2.2)). Evidently $D(\pi \circ \Gamma_x)(v_1, v_2)$ is onto, which implies that $D\Gamma_x(v_1, v_2)$ is onto as well. $\square$

Proposition 4.6. *Suppose $L \in \mathbb{L}(\mathbb{H})$, $p \in L$, and $\Sigma \subset E$ is a surface intersecting L transversely at a point $q \in L \setminus \{p\}$. Then:*

1. *If $p \notin \operatorname{Int}(E^c)$, then the open segment $(p, q) \subset L$ is contained in $\operatorname{Int}(E)$.*
2. *If $p \in E^c$, then connected component of $L \setminus \{q\}$ not containing p lies in $\operatorname{Int}(E)$.*

The same statements hold with the roles of E and E^c exchanged.

Remark 4.7. The first assertion still holds if one merely assumes that E is **precisely convex**, i.e. its intersection with any $L \in \mathbb{L}(\mathbb{H})$ is connected.

Proof. We orient the line L in the direction from p to q. Choose a point $y \in L$ which is separated from p by q. Since L intersects Σ transversely at q, any path close to the segment $\overline{py} \subset L$ will intersect Σ.

Choose $z \in (p, q)$, and define $\bar{v} \in \Delta$ by $z = p \exp 2\bar{v}$. Thus $z = \Gamma_p(\bar{v}, \bar{v})$.

We claim that there is an $\epsilon > 0$ such that if $x \in E$, v_1, $v_2 \in \Delta$ satisfy

$$\max(d^{\mathbb{H}}(x,p), \|v_1 - \bar{v}\|, \|v_2 - \bar{v}\|) < \epsilon\,,$$

then $\Gamma_x(v_1, v_2) \in E$. To see this, note that when ϵ is sufficiently small, we may choose $\rho_1 \in (1,\infty)$ such that $\gamma(x, \rho_1 v_1, 0)$ is a segment ending near y, and by precise monotonicity the subsegment $\gamma(x, v_1, 0)$ lies in E. Similarly, we can choose $\rho_2 \in (1,\infty)$ such that $\gamma(x, v_1, \rho_2 v_2)$ is a path ending near y, so precise monotonicity implies that $\gamma(x, v_1, v_2) = \Gamma_x(v_1, v_2)$ is contained in E.

By Lemma 4.5, the map Γ_p restricts to a submersion on a ball $B \subset \Delta^2$ centered at $(\bar{v}, \bar{v})$; by shrinking B, we may assume that it is contained in the set $\{(v_1, v_2) \in \Delta^2 \mid \|v_i - \bar{v}\| < \epsilon\}$. Therefore by the implicit function theorem, if we choose $x \in E \cap B(p, \epsilon)$ sufficiently close to p, then Γ_x will map B onto a neighborhood U of z; by the preceding paragraph we have $U \subset E$. Since z was an arbitrary point in (p, q), we have $(p,q) \subset \mathrm{Int}(E)$.

The proof of part (2) is similar, except that one considers paths $\gamma(p, v_1, v_2)$ which cross Σ, and the component of $\gamma(p, v_1, v_2) \setminus \Sigma$ lying on the other side of Σ. □

Proposition 4.6 implies:

Lemma 4.8. *If $L \in \mathbb{L}(\mathbb{H})$, and $L \cap \partial E$ contains more than one point, then $L \subset \partial E$.*

Proof. Suppose $p, q \in L \cap \partial E$ are distinct points, and $x \in L \setminus \partial E$. Then either x is in the interior of E, or E^c; without loss of generality we may assume that $x \in \mathrm{Int}(E)$. Then there is a surface $\Sigma \subset \mathrm{Int}(E)$ intersecting L transversely at x. By Proposition 4.6, the open segment of L lying between p and x lies in $\mathrm{Int}(E)$; therefore x lies between p and q. Hence any point $y \in L \setminus \{p, q\}$ separated from p by q belongs to ∂E. Repeating the above reasoning with p replaced by some p' between p and q gives a contradiction. □

Lemma 4.9. *∂E is the union of the lines it contains.*

Proof. Pick $x \in \partial E$.

Suppose every line $L \in \mathbb{L}(\mathbb{H})$ which passes through x intersects ∂E only at x. Then the union of the lines passing through x is a horizontal plane P, and $(P \setminus \{x\}) \cap \partial E = \emptyset$. Since $P \setminus \{x\}$ is connected, it follows that either $P \setminus \{x\}$ is entirely contained in $\mathrm{Int}(E)$ or $\mathrm{Int}(E^c)$; without loss of generality we assume the former. Because $x \in \partial E$, there is a sequence $\{x_k\} \subset E^c$ which converges to x. For each k, choose a line L_k passing through x_k, and (by precise monotonicity) a ray $\eta_k \subset L_k \cap E^c$ containing x_k. Then the sequence $\{\eta_k\}$ will accumulate on some point in $P \setminus \{x\}$ contradicting the fact that $P \setminus \{x\} \subset \mathrm{Int}(E)$.

□

Our next goal is:

Lemma 4.10. *Suppose $\mathcal{G} \subset \mathbb{H}$ has the property that if $L \in \mathbb{L}(\mathbb{H})$ and $L \cap \mathcal{G}$ contains more than one point, then $L \subset \mathcal{G}$. If $\mathcal{G}$ contains either a pair of skew lines or a pair of parallel lines with distinct projection, then $\mathcal{G} = \mathbb{H}$.*

Proof. Observe that if $\mathcal{G}$ contains a pair of parallel lines with distinct projections, then by part A of Lemma 2.4 and the hypothesis on $\mathcal{G}$, there will be a pair of skew lines contained in $\mathcal{G}$; therefore we may assume that $\mathcal{G}$ contains a pair of skew lines.

We first claim that if L_1 and L_2 are skew lines contained in $\mathcal{G}$, then the fiber

$$\pi^{-1}(\pi(L_1) \cap \pi(L_2))$$

is contained in $\mathcal{G}$.

By Lemma 2.4, there is a hyperbola $Y \subset \mathbb{R}^2$ with asymptotes $\pi(L_1)$ and $\pi(L_2)$ such that every tangent line of Y has a unique lift $L \in \mathbb{L}(\mathbb{H})$ which intersects both L_1 and L_2, and which is therefore contained in $\mathcal{G}$. Thus we can find a pair of parallel lines L_3, $L_4 \in \mathbb{L}(\mathbb{H})$ which intersect both L_1 and L_2, such that $\pi(L_3)$ and $\pi(L_4)$ are distinct tangent lines of the hyperbola Y.

By Lemma 2.4 the collection $\mathcal{C}$ of lines which intersect both L_3 and L_4 contains L_1 and L_2, and their union contains

$$\pi^{-1}(\pi(L_1) \cap \pi(L_2)).$$

Since $\cup_{L \in \mathcal{C}} L \subset \mathcal{G}$, the claim is established.

Note that the hyperbola Y separates $\mathbb{R}^2$ into three connected components, and let U be the one whose closure contains Y. Let $V := U \setminus \{L_1 \cup L_2\}$. Every point in V is the intersection point of two tangent lines of Y, and the corresponding lifts will be skew lines contained in $\mathcal{G}$. Therefore by the claim, we have $\pi^{-1}(V) \subset \mathcal{G}$.

Now if $x \in \mathbb{H}$, there is an $L \in \mathbb{L}(\mathbb{H})$ containing x which passes through $\pi^{-1}(V)$, and such an L will intersect $\mathcal{G}$ in more than one point, forcing $x \in \mathcal{G}$. Thus $\mathcal{G} = H$. □

Using this lemma, we get:

Lemma 4.11. *Either ∂E is contained in a plane, or $\partial E = \mathbb{H}$.*

Proof. Assume that $\partial E \neq \mathbb{H}$, but that ∂E is not contained in a plane.

By Lemma 4.9, we know that ∂B is a union of lines; since ∂E is not contained in a plane, it must therefore contain at least two lines L_1, L_2.

If L_1, L_2 have parallel projection, then $p(L_1) = p(L_2)$; otherwise by Lemma 4.10 we would contradict our assumption that $\partial E \neq \mathbb{H}$. Furthermore, any third line $L \subset \partial E$ must also have the same projection, since otherwise ∂E would contain a pair of skew lines or a pair a parallel lines with distinct projection. So in this case ∂E is contained in a vertical plane.

Therefore we may assume that ∂E does not contain distinct lines with parallel projection. Hence every pair of lines contained in ∂E must intersect. Since any triple of lines which intersect pairwise must have a common intersection point, it follows that all lines contained in ∂E pass through a single point $x \in \mathbb{H}$, and hence ∂E is contained in a horizontal plane.

□

Lemma 4.12. *Either E or E^c has nonempty interior; equivalently, $\partial E \neq \mathbb{H}$.*

Proof. Let $L_1, L_2 \in \mathbb{L}(\mathbb{H})$ be skew lines. By Lemma 2.4, for $i \in \{1, 2\}$, we can find open intervals $(a_i, b_i) \subset L_i$ and smooth parametrizations $x_i : I \to (a_i, b_i)$ such that for all $t \in I$, the points $x_1(t) \in L_1$ and $x_2(t) \in L_2$ lie in a line L_t. By monotonicity, after passing to subintervals if necessary, we may assume that for $i \in \{1, 2\}$, the characteristic function χ_E is constant on (a_i, b_i), i.e. it lies entirely in E, or entirely in E^c.

If $(a_1, b_1) \cup (a_2, b_2) \subset E$ (respectively E^c), then by monotonicity for all $t \in I$, the segment $[x_1(t)x_2(t)] \subset L_t$ lies in E (respectively E^c). Hence $\cup_{t \in I}\ (x_1(t), x_2(t))$ is a relatively open subset U of a ruled surface which is contained in E (respectively E^c). If $(a_1, b_1) \subset E$ and $(a_2, b_2) \subset E^c$ (or vice-versa), then $L_t \setminus [x_1(t), x_2(t)]$ is a union two rays, one of which lies in E, and the other lies in E^c. Therefore in this case, we also obtain a relatively open subset U of a ruled surface, which lies in E, or in E^c.

Choose a line L which intersects the surface U transversely at some point p, and pick $x \in L \setminus U$. Then Proposition 4.6 implies that either E or E^c has nonempty interior.

□

Lemma 4.13. *If E and E^c are both nonempty, then ∂E is a plane, and $C \subset E \subset \bar{C}$, where C is a component of $\mathbb{H} \setminus \partial E$.*

Proof. By assumption, ∂E is nonempty. By Lemmas 4.11 and 4.12, it follows that ∂E is contained in a plane P. If $\partial E \neq P$, then $\mathbb{H} \setminus \partial E$ is connected, and hence $\mathbb{H} \setminus \partial E$ is contained in E, or in E^c. We may therefore assume that it is contained in E. But then every point $x \in P$ lies on a line L transverse to P, so $L \setminus \{x\}$ is contained in E, and by precise monotonicity, we get $L \subset E$. Therefore $E = \mathbb{H}$. This is a contradiction, so $\partial E = P$. The assertion that $C \subset E \subset \bar{C}$ follows from the

Jordan separation theorem (or by the elementary fact that a horizontal or vertical plane separates into two components). □

5. The classification of monotone sets

The goal of this section is:

Theorem 5.1. *If $E \subset \mathbb{H}$ is a monotone set, then modulo a null set, either $E = \emptyset$, $E = \mathbb{H}$, or E is a half-space.*

For the remainder of this section, E will denote a fixed monotone set $E \subset \mathbb{H}$.

Measure theoretic preparation. The proof of Theorem 5.1 will follow the proof of Theorem 4.3 closely. The main difference stems from the fact that the monotonicity condition only holds for almost every line L, and up to a null set within L; this forces one to modify the proof of Theorem 4.3 by considering positive measure families of certain configurations (such as piecewise horizontal curves), instead of single configurations.

We begin with a measure-theoretic replacement for the boundary and interior. These were chosen so that the proof of the classification of monotone sets closely parallels the proof for precisely monotone sets.

Definition 5.2. The **support** of a measurable set E, denoted $\mathrm{spt}(E)$, is the support of its characteristic function. The **(measure-theoretic) boundary** $\partial_\mu E$ of $E \subset \mathbb{H}$ is $\mathrm{spt}(E) \cap \mathrm{spt}(E^c)$, i.e. the set of points $x \in \mathbb{H}$ such that $\min(\mu(B_r(x) \cap E), \mu(B_r(x) \cap E^c)) > 0$ for all $r > 0$. The **measure-theoretic interior** $\mathrm{Int}_\mu(E)$ of E is $\mathbb{H} \setminus \mathrm{spt}(E^c)$.

Note that $\partial_\mu E$ is a closed set, and $\mathbb{H} \setminus \partial E$ is the disjoint union of $\mathrm{Int}_\mu(E)$ and $\mathrm{Int}_\mu(E^c)$. Similar defintions apply to subsets of $\mathbb{R}$. Also, a subset of $\mathbb{R}$ is monotone iff its measure-theoretic boundary contains at most one point.

Definition 5.3. A line $L \in \mathbb{L}(\mathbb{H})$ is **monotone** if $E \cap L$ is a monotone subset of $L \simeq \mathbb{R}$. A pointed line (L, p) is **monotone** if $p \in L \in \mathbb{L}(\mathbb{H})$, the line L is monotone, and p does not lie in the measure-theoretic boundary of $E \cap L$ (relative to L). A direction $v \in \Delta \setminus \{0\}$ is **monotone** if almost every line tangent to v is monotone.

Note that L is monotone if $L \cap E$ is a measurable subset of L, and the measure theoretic boundary of $E \cap L$ (in L) contains at most one point.

Lemma 5.4.

(1) *Almost every direction $v \in \Delta$ is monotone.*

(2) *Almost every pointed line is monotone.*

Proof. (1) Let $\mathcal{P}(\Delta)$ denote the projectivization of the horizontal space Δ. Then there is a smooth fibration $\mathbb{L}(\mathbb{H}) \to \mathcal{P}(\Delta)$ which sends a line L tangent to the direction $[v] \in \mathcal{P}(\Delta)$ to $[v]$, whose fibers are the lines parallel to a given direction. Therefore by Fubini's theorem, for almost every $[v] \in \mathcal{P}(\Delta)$, the set of monotone elements in the fiber over $[v]$ is a measurable subset of full measure.

(2) The space of pointed lines fibers over $\mathbb{L}(\mathbb{H})$, with fiber $\mathbb{R}$. Since there is a full measure set $Y \subset \mathbb{L}(\mathbb{H})$ consisting of monotone lines, and almost every point p in the fiber over a monotone line L yields a monotone pointed line (L, p), the statement follows from Fubini's theorem. □

If $x \in \mathbb{H}$, $v \in \Delta \setminus \{0\}$, we will use $L_{x,v}$ to denote the line passing through x tangent to the horizontal left invariant vector vield v.

Lemma 5.5. *For a.e. triple* $(x, v_1, v_2) \in \mathbb{H}\times\Delta\times\Delta$*, the pairs* (L_{x,v_1}, x), $(L_{x,v_1}, x \exp v_1)$, $(L_{x \exp v_1, v_2}, x \exp v_1)$*, and* $(L_{x \exp v_1, v_2}, x \exp v_1 \exp v_2)$ *are monotone.*

Proof. Suppose $v_1, v_2 \in \Delta \setminus \{0\}$ are monotone directions. Then by Fubini's theorem, the set $M_i \subset \mathbb{H}$ of points $w \in \mathbb{H}$ such that (L_{w,v_i}, w) is monotone has full measure. Since the maps $x \mapsto x \exp v_1$, $x \mapsto x \exp v_1 \exp v_2$ are diffeomorphisms, it follows that for a.e. $x \in \mathbb{H}$, $x \in M_1$, $x \exp v_1 \in M_1$, $x \exp v_1 \in M_2$, and $x \exp v_1 \exp v_2 \in M_2$.

Since a.e. direction $v \in \Delta$ is monotone, the lemma follows. □

Definition 5.6. Let X, M, and A be smooth manifolds. An **admissible family of submanifolds in** X is a smooth submersion $\Phi : M \times A \to X$ such that $\Phi(\cdot, \alpha)$ is an embedding for all $\alpha \in A$. We let M_α denote the image of $\Phi\big|_{M\times\{\alpha\}}$, and refer simply to the resulting family of submanifolds $\{M_\alpha\}_{\alpha\in A}$.

Lemma 5.7. *Let* $\{M_\alpha\}_{\alpha\in}$*,* $\{N_\beta\}_{\beta\in B}$ *be two admissible families of manifolds in a smooth manifold* X*, and* $S \subset X$ *be a measurable subset. We equip* A*,* B*,* X*, and each of the submanifolds* M_α*,* N_β *with smooth measures with positive density. Assume that*

- *For all* $\alpha \in A$*,* $\beta \in B$*, the submanifolds* M_α *and* N_β *intersect transversely in a single point.*
- *For a positive measure set* $A_1 \subset A$*, for all* $\alpha \in A_1$ *the submanifold* M_α *interesects* S *in a full measure subset of* M_α*.*

Then for a full measure set of $\beta \in B$*, the submanifold* N_β *intersects* S *in positive measure subset of* N_β*.*

Proof. Let $I : A \times B \to X$ be the map which sends $(\alpha, \beta) \in A \times B$ to the unique intersection point of M_α and N_β. By the transversality assumption and the definition of admissible families of submanifolds, the map I is a smooth submersion from $A \times B$

onto an open subset of X. Also, for each $\alpha \in A$ (respectively $\beta \in B$), the restriction of I to $\{\alpha\} \times B$ (respectively $A \times \{\beta\}$) is a submersion onto a relatively open subset of M_α (respectively N_β).

Let $\hat{S} := I^{-1}(S)$.

Suppose $\alpha \in A_1$. Since $I\Big|_{\{\alpha\}\times B} : \{\alpha\} \times B \to M_\alpha$ is a submersion, it follows that $\hat{S} \cap (\{\alpha\} \times B)$ has full measure in $\{\alpha\} \times B$.

We claim that there is a full measure subset $B_1 \subset B$ such that for every $\beta \in B_1$, the intersection $\hat{S} \cap (A \times \{\beta\})$ has positive measure in $A \times \{\beta\}$. To see this, note that otherwise there would be a positive measure subset $B_0 \subset B$ such that for every $\beta \in B_0$, the intersection $\hat{S} \cap (A \times \{\beta\})$ has zero measure in $A \times \{\beta\}$. But then by Fubini's theorem, $\hat{S} \cap (A \times B_0)$ has measure zero, which contradicts the fact that $\hat{S} \cap (A \times B_0)$ intersects a positive measure set of fibers $\{\alpha\} \times B$ in a set of positive measure.

Now for every $\beta \in B_1$, the intersection $S \cap N_\beta$ has positive measure in N_β, since its inverse image under $I\Big|_{A\times\{\beta\}}$ has positive measure in $A \times \{\beta\}$.

$\square$

The proof of Theorem 5.1. With our measure-theoretic preparations complete, we will now prove the theorem using essentially the same outline as the proof of Theorem 4.3:

(1) If $L \in \mathbb{L}(\mathbb{H})$ contains more than one point of $\partial_\mu E$, then $L \subset \partial_\mu E$.

(2) $\partial_\mu E$ is a union of lines.

(3) Either $\partial_\mu E$ is contained in a plane, or $\partial_\mu E = \mathbb{H}$.

(4) Lemma 4.12: The case $\partial_\mu E = \mathbb{H}$ does not occur.

(5) Lemma 4.13: If $\partial_\mu E$ is nonempty, then it is a plane and $C \subset E \subset \bar{C}$, where C is a connected component of $\mathbb{H} \setminus \partial_\mu E$.

Proposition 5.8. *Suppose $L \in \mathbb{L}(\mathbb{H})$, $p \in L$, and $\{\Sigma_\alpha\}_{\alpha \in A}$ is an admissible family of surfaces. Assume that*

- *For all $\alpha \in A$, the surface Σ_α intersects L transversely in a single point.*
- *There is a measurable subset $A_1 \subset A$ such that for every a.e. $\alpha \in A_1$, the surface Σ_α intersects E in a set of full measure in Σ_α.*

- *For some* $q \in L \setminus \{p\}$*, there is an* $\alpha_0 \in \operatorname{spt}(A_1)$ *such that* $\Sigma_{\alpha_0} \cap L = \{q\}$*.*

Then:

(1) *If* $p \in \operatorname{spt}(E)$*, then the open segment* $(p,q) \subset L$ *is contained in* $\operatorname{Int}_\mu(E)$ *.*

(2) *If* $p \in \operatorname{spt}(E^c)$*, then connected component of* $L \setminus \{q\}$ *not containing* p *lies in* $\operatorname{Int}_\mu(E)$*.*

The same statements hold with the roles of E *and* E^c *exchanged.*

Proof. (Compare the proof of Proposition 4.6) Pick $z \in (p,q) \subset L$, where $z = p \exp 2\bar{v}$, $\bar{v} \in \Delta$. Hence $z = \Gamma_p(\bar{v},\bar{v})$.

Choose $\epsilon \in (0,\infty)$, and let $\mathcal{G}$ be the set of triples $(x,v_1,v_2) \in \mathbb{H} \times \Delta \times \Delta$ such that $\max\{d^{\mathbb{H}}(x,p), \|v_1 - \bar{v}\|, \|v_2 - \bar{v}\|\} < \epsilon$.

Using the reasoning from the proof Proposition 4.6, to prove (1) of Proposition 5.8, it suffices to show that when ϵ is sufficiently small, for almost every $x \in B_\epsilon(p) \cap E$, the point $\Gamma_x(v_1,v_2)$ belongs to E. To establish this, we need the following:

Lemma 5.9. *For a.e. triple* $(x,v_1,v_2) \in \mathcal{G}$*, the following statements hold:*

(1) *The following pairs are monotone:* (L_{x,v_1}, x)*,* $(L_{x,v_1}, x\exp v_1)$*,* $(L_{x\exp v_1, v_2}, x \exp v_1)$*, and* $(L_{x\exp v_1,v_2}, x\exp v_1 \exp v_2)$*.*

(2) *There is a point* $w_1 \in L_{x,v_1} \cap E$*, close to* q*, such that the pair* (L_{x,v_1}, w_1) *is monotone.*

(3) *There is a point* $w_2 \in L_{x\exp v_1,v_2} \cap E$ *close to* q*, such that the pair* $(L_{x\exp v_1}, w_2)$ *is monotone.*

Proof. The first assertion follows from Lemma 5.5, so we focus on (2) and (3).

Pick $\delta \in (0,\infty)$. We may shrink the surfaces $\{\Sigma_\alpha\}_{\alpha \in A}$ and choose a small neighborhood B of L in $\mathbb{L}(\mathbb{H})$, such that every $L \in B$ intersects every surface Σ_α transversely in a single point lying in $B_\delta(q)$.

By Lemma 5.7, for a.e. $L' \in B$, the intersection $L' \cap E \cap B_\delta(q)$ has positive measure in L'. Therefore, if ϵ is sufficiently small, for a.e. $v_1 \in B_\epsilon(\bar{v})$, the direction v_1 is monotone, and for a.e. $x \in B_\epsilon(p)$ the pair (L_{x,v_1}, x) is monotone and the line L_{x,v_1} intersects $E \cap B_\delta(q)$ in a positive measure set. It follows that for some $w_1 \in E \cap B_\delta(q) \cap L_{x,v_1}$ the pair (L_{x,v_1}, w_1) is monotone. This implies that (2) holds for a.e. triple (x,v_1,v_2).

By the same token, for ϵ' sufficiently small, there is a full measure subset $M \subset B_{\epsilon'}(z)$ such that for every $y \in M$ and a.e. $v_2 \in B_\epsilon(\bar{v})$, there is a $w_2 \in E \cap B_\delta(q) \cap L_{y,v_2}$ such that (L_{y,v_2}, y) and (L_{y,v_2}, w_2) are both monotone. The map $(x,v_1) \mapsto x\exp v_1)$

being a submersion, if ϵ is sufficiently small, we conclude that for a.e. $x \in B_\epsilon(p)$, $v_1 \in B_\epsilon(\bar{v})$, the point $x \exp v_1$ lies in M, and hence for a.e. $v_2 \in B_\epsilon(\bar{v})$ the desired point $w_2 \in L_{x \exp v_1, v_2}$ exists.

□

To prove (2) of Proposition 5.8, we combine Lemma 5.9 and the argument of (2) in Proposition 4.6.

□

Step (1) of the outline follows, by using Proposition 5.8 in place of Proposition 4.6.

Steps (2), (3), and (5) then follow using essentially the same reasoning in the precisely monotone case.

To implement step (4) for monotone sets, it suffices to produce an admissible family of surfaces $\{\Sigma_\alpha\}_{\alpha \in A}$ as in the statement Proposition 5.8 (or the version with E^c replacing E). To that end we have the following:

Lemma 5.10. *For a.e. pair $(L_1, L_2) \in \mathbb{L}(\mathbb{H}) \times \mathbb{L}(\mathbb{H})$, a.e. line L intersecting both L_1 and L_2 is monotone, and the pairs $(L_i, L_i \cap L)$ and $(L, L \cap L_i)$ are monotone for $i \in \{1, 2\}$.*

Proof. Consider the manifold $\mathcal{F}$ of pointed lines (L, p), and the two submersions $\pi_{\mathbb{L}} : \mathcal{F} \to \mathbb{L}(\mathbb{H})$ and $\pi_{\mathbb{H}} : \mathcal{F} \to \mathbb{H}$, where $\pi_{\mathbb{L}}(L, p) = L$ and $\pi_{\mathbb{H}}(L, p) = p$. We know that a.e. $(L, p) \in \mathcal{F}$ is monotone, because a.e. $L \in \mathbb{L}(\mathbb{H})$ is monotone, and a.e. pointed line in the fiber $\pi_{\mathbb{L}}^{-1}(L)$ over a monotone line L, is monotone.

To any pair of skew lines (L_1, L_2), we may associate two 1-dimensional submanifolds $M^1_{L_1,L_2}, M^2_{L_1,L_2} \subset \mathcal{F}$, namely $M^i_{L_1,L_2}$ is the set of pointed lines (L, p) where L intersects both L_1 and L_2, and $p = L \cap L_i$. The implicit function theorem implies that for $i \in \{1, 2\}$, the family

$$\{M^i_{L_1,L_2} \mid (L_1, L_2) \text{ are skew lines}\}$$

is admissible in the sense of Definition 5.6.

It follows that for a.e. pair (L_1, L_2) of skew lines, a.e. pointed line (L, p) on $M^i_{L_1,L_2}$ is monotone. Since a.e. line is monotone, the lemma follows. □

Using the lemma, we may imitate the construction of Lemma 4.12 using a family of pairs, in the measurable setting. This yields for a.e. (L_1, L_2), open intervals $(a_i, b_i) \subset L_i$ and smooth parametrizations $x_i : I \to (a_i, b_i)$ as in Lemma 4.12, which vary measurably with (L_1, L_2). By Lusin's theorem, after passing to a subset of nonzero measure, we may arrange that the interval (a_i, b_i) and maps x_i varying continuously with (L_1, L_2). Then the rest of the lemma may be implemented as in Lemma 4.12.

6. The proof of a weak version of Theorem 1.3

In this section we prove a weak version of the main theorem, which is enough to imply the nonexistence of bi-Lipschitz embeddings $\mathbb{H} \to L^1$. We include this result here because it follows easily from the work we have done so far, and avoids the harmonic analysis in Section 7.

Theorem 6.1. *Let Z be the infinitesimal generator of* $\mathrm{Center}(\mathbb{H})$*, so* $\mathrm{Center}(\mathbb{H}) = \{\exp tZ \mid t \in \mathbb{R}\}$*. If $f : \mathbb{H} \to L^1$ is a Lipschitz map, then for a full measure set of points $p \in \mathbb{H}$*

$$\liminf_{t\to 0} \frac{d(f(p), f(p\exp tZ))}{d(p, p\exp tZ)} = 0\,. \tag{6.2}$$

Proof. Suppose the theorem were false. Then the set of points $x \in \mathbb{H}$ such that

$$\liminf_{t\to 0} \frac{d(f(x), f(x\exp tZ))}{d(x, x\exp tZ)} > 0$$

is measurable, and has positive measure. By countable additivity, it follows that there is a measurable set Y of positive measure, and constants $\lambda \in (0,1)$, $r \in (0,1)$, such that if $p \in Y$ and $q = p\, \exp(tZ)$ for some $|t| < r$, then $d(f(p), f(q)) \ge \lambda d(p,q)$.

Let p be a density point of Y, where in addition the conclusion of Theorem 2.5 holds. We now blow up the pullback distance at p. Take a sequence $\lambda_k \to 0$, define $\rho_k : \mathbb{H} \times \mathbb{H} \to [0,\infty)$ by

$$\rho_k = \frac{1}{\lambda_k}\,(f \circ \ell_p \circ s_{\lambda_k})^*\, d_{L^1}\,,$$

where $\ell_p : G \to G$ is left translation by p, and $s_{\lambda_k} : G \to G$ is the automorphism which scales by λ_k. Since f is Lipschitz, for some $C \in (0,\infty)$ we get $\rho_k \le Cd$, and therefore by Arzela-Ascoli we may assume, after passing to a subsequence if necessary, that the sequence $\{\rho_k\}$ converges uniformly on compact subsets of $G \times G$ to a pseudo-distance ρ_∞. Since p was a density point of Y, it follows that for any $x \in G$, $t \in \mathbb{R}$,

$$\rho_\infty(x, x\, \exp\, tZ) \ge \lambda\, d(x, x\, \exp\, tZ)\,. \tag{6.3}$$

By Corollary 2.7, the pseudo-distance ρ_∞ is induced by a map $f_\omega : \mathbb{H} \to L^1$, which by the choice of p, restricts to a geodesic map on every line $L \in \mathbb{L}(\mathbb{H})$. If Σ is the cut measure associated with f_ω, then Proposition 3.5 implies that Σ-a.e. cut $E \in \mathrm{Cut}(\mathbb{H})$ is monotone; then Theorem 5.1 gives that Σ a.e. $E \in \mathrm{Cut}(\mathbb{H})$ is a half-space.

For almost every $z \in \mathbb{R}^2$, the restriction of the cut measure Σ to the fiber $\pi^{-1}(z)$ is well-defined, and supported on monotone cuts (since the intersection of a fiber with a half-space is monotone). Therefore if $x_1, x_2, x_3 \in \pi^{-1}(z)$ are in linear order, then $\rho_\infty(x_1, x_3) = \rho_\infty(x_1, x_2) + \rho_\infty(x_2, x_3)$. Combining this with (6.3), for $n \in \mathbb{N}$ we get

$$\rho_\infty(x, x\,\exp nZ) = n\rho_\infty(x, x\,\exp Z) \geq n\lambda\, d(x, x\,\exp Z)\,.$$

This contradicts the Lipschitz condition, since $d(x, x\exp nZ) \simeq \sqrt{n}$, see (2.3).

□

7. Uniqueness of cut measures

Our main goal in this section is to show that under appropriate conditions, there is a unique cut measure inducing a given cut metric. Since the assignment $\Sigma \mapsto d_\Sigma$ is linear, it is natural to investigate injectivity in a linear framework, and for this reason we will work with signed measures in this section.

The setup. Throughout this section, Σ will be a signed cut measure on $\mathbb{H}$ supported on half-spaces. (Recall from Subsection 2.4 that the definitions of cut measure and cut metric adapt directly to signed measures.) We let d_Σ^{hor} denote the restriction of the cut metric d_Σ to the set of horizontal pairs $\mathrm{hor}(\mathbb{H}) \subset \mathbb{H} \times \mathbb{H}$. Let $\Sigma = \Sigma_+ - \Sigma_-$ be the decomposition of Σ into its positive and negative parts, and let $|\Sigma| = \Sigma_+ + \Sigma_-$ be the absolute value of Σ. Since opposite half-spaces yield the same elementary cut metric up to sets of measure zero, we may symmetrize Σ so that it is invariant under interchange of opposite components, without affecting the associated cut metric d_Σ. An alternative way to view this is to pushforward Σ under the 2-to-1 map $\mathrm{HS}(\mathbb{H}) \to \mathbb{P}$ from the space of half-spaces to planes, which sends each connected component of $\mathbb{H} \setminus P$ to P. This pushforward operation induces a bijection between symmetrized cut measures and measures on $\mathbb{P}$. We will often find it more convenient to work with measures on $\mathbb{P}$ rather than symmetrized cut measures.

Definition 7.1. Let Σ be a signed cut measure supported on half-spaces. Then Σ is **Lipschitz** if there is a constant $C \in [0, \infty)$ such that $d_{|\Sigma|} \leq C\, d$. The **Lipschitz constant of** Σ is the infimum $\mathrm{Lip}(\Sigma)$ of such constants C.

Note that the inequality is to be interpreted in accordance with the definition of d_Σ, i.e. there should be a full measure subset $Z \subset \mathbb{H}$ such that $d_\Sigma(x_1, x_2) \leq C\, d(x_1, x_2)$ for every $x_1,\, x_2 \in Z$.

Our main objective in this section is:

Theorem 7.2. *The linear map $\Sigma \mapsto d_\Sigma^{\mathrm{hor}}$ is injective on symmetrized Lipschitz signed cut measures supported on half-spaces.*

Let $\Sigma = \Sigma_v + \Sigma_h$ be the decomposition into the parts supported on vertical and horizontal half-spaces, respectively. To simply terminology slightly, we will use **horizontal cut measure** (respectively **vertical cut measure** to refer to a measure supported on horizontal (resp. vertical) half-spaces. We will first treat the injectivity question for the restricted operators $\Sigma_v \mapsto d^{\mathrm{hor}}_{\Sigma_v}$ and $\Sigma_h \mapsto d^{\mathrm{hor}}_{\Sigma_h}$, before demonstrating injectivity in the general case.

Estimates on horizontal cut measures. We now fix a symmetrized horizontal cut measure Σ, i.e. Σ is supported on horizontal half-spaces. We will view Σ as a Radon measure on the manifold $\mathbb{P}_h$ of horizontal planes, which we identify with $\mathbb{H}$ by the diffeomorphism $\mathbb{H} \to \mathbb{P}_h$ which sends $x \in \mathbb{H}$ to the unique horizontal plane centered at x.

From now until Theorem 7.9 below, we will assume that Σ is absolutely continuous with respect to $\mathcal{L}$, so $\Sigma = u\,\mathcal{L}$ where $u : \mathbb{H} \to \mathbb{R}$ is a locally integrable function.

For $x \in \mathbb{H}$, let $P_x \subset \mathbb{H}$ denote the horizontal plane centered at x. If x_1 and x_2 are distinct points lying on a line $L \in \mathbb{L}(\mathbb{H})$, we define P_{x_1,x_2} to be the union of the horizontal planes P_x, where x ranges over the interval $(x_1, x_2) \subset L$. Since $x \in P_y$ if and only if $y \in P_x$, we have $P_{x_1,x_2} = \{y \in \mathbb{H} \mid P_y \cap (x_1, x_2) \neq \emptyset\}$. The complement of the union $P_{x_1} \cup P_{x_2}$ has four wedge-shaped connected components, precisely two of which are “horizontal”, in the sense that they intersect each coset of the center in an interval. Neglecting the intersection with L, the set P_{x_1,x_2} coincides with the union of these two horizontal components.

Lemma 7.3. *There is a full measure subset $Z \subset \mathbb{H}$ such that if $(x_1,\, x_2) \in Z \times Z$ is a horizontal pair, then*

$$d_\Sigma(x_1, x_2) \,=\, \Sigma(P_{x_1,x_2}) \,=\, \int_{P_{x_1,x_2}} u\,d\mathcal{L} \,\leq\, \mathrm{Lip}(\Sigma)\,d(x_1, x_2)\,.$$

Proof. From the definition of the cut metric, the distance $d_\Sigma(x_1, x_2)$ is given by

$$\int_{\mathrm{Cut}(\mathbb{H})} |\chi_E(x_1) - \chi_E(x_2)|\,d\Sigma(E)\,,$$

when x_1 and x_2 belong to a full measure subset of $\mathbb{H}$. If P is a plane disjoint from a horizontal pair (x_1, x_2), and E is a half-space component of $\mathbb{H} \setminus P$, then $|\chi_E(x_1) - \chi_E(x_2)|$ is nonzero precisely when $P \cap (x_1, x_2) \neq \emptyset$. Since the set of planes passing through x_1 or x_2 has $|\Sigma|$-measure zero, it follows that $d_\Sigma(x_1, x_2) = \Sigma(P_{x_1,x_2})$. ∎

For $x \in \mathbb{H}$, let $s_x : \mathbb{R}^2 \to \mathbb{H}$ be the inverse of $\pi\big|_{P_x}$, and let $d_{\pi(x)} : \mathbb{R}^2 \to [0,\infty)$ be the distance from $\pi(x) \in \mathbb{R}^2$.

Lemma 7.4. *Suppose in addition that Σ is Lipschitz. There is a constant C depending only on $\mathbb{H}$, with the following property.*

(1) *For almost every $x \in \mathbb{H}$, and for almost every line $L \in \mathbb{L}(\mathbb{H})$ passing through x, the composition $u \circ s_x : \mathbb{R}^2 \to \mathbb{R}$ is measurable. Moreover, if $d_{\pi(L)} : \mathbb{R}^2 \to [0,\infty)$ is the distance from $\pi(L)$, then*
$$\int_{\mathbb{R}^2} |u \circ s_x|\, d_{\pi(L)}\, d\mathcal{L} \,\le\, \mathrm{Lip}(\Sigma)\,.$$
(2) *For almost every $x \in \mathbb{H}$, the composition $u \circ s_x : \mathbb{R}^2 \to \mathbb{R}$ is measurable,*
$$\int_{\mathbb{R}^2} |u \circ s_x|\, d_{\pi(x)}\, d\mathcal{L} \,\le\, C\ \mathrm{Lip}(\Sigma)\,.$$
(3) *If in addition $d_\Sigma = 0$, then for almost every $x \in \mathbb{H}$ and almost every line L passing through x,*
$$\int_{\mathbb{R}^2} (u \circ s_x)\, d_{\pi(L)}\, d\mathcal{L} \,=\, \int_{\mathbb{R}^2} (u \circ s_x)\, d_{\pi(x)}\, d\mathcal{L} \,=\, 0\,.$$

Proof. Let $Z \subset \mathbb{H}$ be as in Lemma 7.3. By Fubini's theorem, for almost every $L \in \mathbb{L}(\mathbb{H})$, the intersection $L \cap Z$ has full (linear) measure in L. Now let $\gamma : \mathbb{R} \to L$ be a unit speed parametrization, and let $S : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{H}$ be defined by $S(y,t) = s_{\gamma(t)}(y)$. By Lemma 2.1, we have $s_{\gamma(t)}(y) = s_x(y)\exp(A(y)Z)$, where $A(y)$ is the signed area enclosed by the triangle with vertices $\pi(x)$, $\pi(\gamma(t))$, $\pi(y)$; this implies that the Jacobian of $S : (\mathbb{R}^2 \times \mathbb{R}, \mathcal{L}) \to (\mathbb{H}, \mathcal{L})$ is $J(y,t) = d_{\pi(L)}(y)$. Therefore by Lemma 7.3 and the change of variables formula,

$$d_\Sigma(\gamma(t_1), \gamma(t_2)) = \int_{P_{\gamma(t_1),\gamma(t_2)}} u\, d\mathcal{L} \tag{7.5}$$
$$= \int_{t_1}^{t_2} \int_{\mathbb{R}^2} (u \circ S)(y,t)\, d_{\pi(L)}(y) d\mathcal{L}(y) d\mathcal{L}(t)\,.$$

Since $d_{|\Sigma|}(\gamma(t_1), \gamma(t_2)) \le \mathrm{Lip}(\Sigma)\,|t_1 - t_2|$, by Fubini's theorem and the Fundamental Theorem of Calculus, it follows that for $\mathcal{L}$-a.e. $t \in \mathbb{R}$, we have
$$\int_{\mathbb{R}^2} |u \circ s_{\gamma(t)}(y,t)|\, d_{\pi(L)}(y)\ d\mathcal{L}(y) \le \mathrm{Lip}(\Sigma)\,.$$
Applying Fubini's theorem once again, we get (1).

Part (2) of the lemma follows from part (1) by integrating over the lines passing through x.

Part (3) follows from (7.5), Fubini's theorem, and the Fundamental Theorem of Calculus.

$\square$

Remark 7.6. Part (3) of Lemma 7.4 generalizes to a statement about arbitrary Lipschitz signed horizontal cut measures. The metric differentiation result of Pauls generalizes to signed Lipschitz distance functions such as d_Σ. It gives rise to a measurable function on the collection $\{(L,x) \in \mathbb{L}(\mathbb{H}) \times \mathbb{H} \mid x \in L\}$ of pointed lines, which compares the infinitesimal behavior of d_Σ along L, near x, with that of d. For almost every $x \in \mathbb{H}$ and almost every line $L \in \mathbb{L}(\mathbb{H})$ passing through x, the integral $\int_{\mathbb{R}^2} (u \circ s_x)\, d_{\pi(L)}\, d\mathcal{L}$ agrees with this function.

Lemma 7.7. *There is a constant C which depends only on the geometry of $\mathbb{H}$, such that if Σ is Lipschitz, then:*

(1) *For every $L \in \mathbb{L}(\mathbb{R}^2)$, $r \in (0,\infty)$, the strip $S = \{y \in \mathbb{R}^2 \mid d(y,L) < r\} \subset \mathbb{R}^2$ satisfies*
$$|\Sigma|(\pi^{-1}(S)) \;\le\; \operatorname{width}(S)\operatorname{Lip}(\Sigma) \;=\; 2r\operatorname{Lip}(\Sigma)\,.$$
(2) *If $\phi : \mathbb{H} \to \mathbb{R}$ is a measurable function and*
$$\sup_{x\in\mathbb{H}} \;|\phi(x)|(1+|\pi(x)|)^p \;<\; \infty$$
for some $p > 1$, then $\phi\, u \in L^1(\mathbb{H},\mathcal{L})$.

Proof. Suppose x, $y \in \mathbb{H}$ are points lying on a line $L \in \mathbb{L}(\mathbb{H})$ at distance $d(x,y) = R > 0$, and L_x, $L_y \in \mathbb{L}(\mathbb{H})$ are the lines intersecting L orthogonally at x and y, respectively. Now let y_1, $y_2 \in L_y$ be the two points in L_y at distance r from y, and choose $x' \in L_x$.

Lemma 2.1 implies that the intersection of P_{y_1,y_2} with the fiber $\pi^{-1}(\pi(x')) = x' \exp(\mathbb{R}Z)$ is of the form $x'\exp(a,b)Z$, where (a,b) is an interval of length Rr shifted by the signed area of the triangle with vertices $\pi(x), \pi(x'), \pi(y)$, i.e. $\pm\frac{1}{2}d(x,x')R$. Therefore if $r > |d(x,x')|$, if we hold x and L fixed while letting R tend to infinity, the set P_{y_1,y_2} will contain more and more of the fiber $\pi^{-1}(\pi(x')) = x'\exp\mathbb{R}Z$.

Consider the strip $S = \{p \in \mathbb{R}^2 \mid d(p,\pi(L)) < r\}$. If $K \subset \mathbb{H}$ is any compact subset of $\pi^{-1}(S)$, then the discussion above implies that when R is sufficiently large, P_{y_1,y_2} will contain K. If we choose a sequence of horizontal pairs $\{(y_1^k, y_2^k)\}$ which converge to (y_1,y_2), such that the conclusion of Lemma 7.3 holds for each of the pairs (y_1^k, y_2^k), then we conclude that
$$|\Sigma|(P_{y_1,y_2}) \le \liminf_{k\to\infty}\; |\Sigma|(P_{y_1^k,y_2^k}) \;\le\; \liminf_{k\to\infty}\; \operatorname{Lip}(\Sigma)\, d(y_1^k,y_2^k) \;=\; \operatorname{Lip}(\Sigma)\, 2r\,.$$

In this case we get

$$|\Sigma|(K) \le |\Sigma|(P_{y_1,y_2}) \le 2\,\mathrm{Lip}(\Sigma)\,r \;=\; \mathrm{Lip}(\Sigma)\,\mathrm{width}(S)\,.$$

As K was arbitrary (1) follows.

To prove (2), for $k \in \mathbb{N}$, let S_k be the double strip

$$\{(x,y) \in \mathbb{R}^2 \mid y \in (-k,-(k-1)) \cup (k-1,k)\}\,.$$

Then (1) gives

$$\int_{\pi^{-1}(S_k)} |\phi\, u|\, d\mathcal{L} \;\le\; 2\,(1+(k-1))^{-p}\ \mathrm{Lip}(\Sigma)\,,$$

so

$$\int_{\mathbb{H}} |\phi\, u|\, d\mathcal{L} \;=\; \sum_{k=1}^{\infty} \int_{\pi^{-1}(S_k)} |\phi\, u|\, d\mathcal{L} \;\le\; 2\,\mathrm{Lip}(\Sigma)\, \sum_{k=1}^{\infty} (1-(k-1))^{-p} \;<\; \infty\,.$$

□

Injectivity for horizontal cut measures. Now suppose $\Sigma = u\,\mathcal{L}$ is a Lipschitz signed horizontal cut measure such that $d_\Sigma^{\mathrm{hor}} = 0$, and let K be the distribution on $\mathbb{H}$ defined by the linear functional

$$\phi \mapsto \int_{\mathbb{R}^2} (\phi \circ s_e)\, d_{\pi(e)}\, d\mathcal{L} \;=\; \int_{\mathbb{R}^2} (\phi \circ s_e)(x)\,|x|\, d\mathcal{L}(x)\,.$$

By Lemma 7.4, the convolution $u*K$ is well-defined, and equals zero. The convolution operator $\phi \mapsto \phi * K$ was studied in [Str91]. Before proceeding, we briefly summarize the relevant conclusions from that paper.

To conform with the notation from [Str91], we let $z = \pi : \mathbb{H} \to \mathbb{R}^2 \simeq \mathbb{C}$, and $t : \mathbb{H} \to \mathbb{R}$ be the function given by $p = s_e(\pi(p))\, \exp(tZ) = s_e(z(p))\, \exp(tZ)$. Strichartz works with the general Heisenberg group of dimension $2n+1$, so we are in the $n=1$ case. For $\lambda > 0$, k a nonnegative integer, and $\epsilon = \pm 1$, let

$$\phi_{\lambda,k,\epsilon}(z,t) \tag{7.8}$$

$$= (2\pi)^{n+1} \frac{\lambda^n}{(n+2k)^{n+1}}\ \exp\left(-\frac{i\epsilon\lambda t}{n+2k}\right)\ \exp\left(-\frac{\lambda|z|^2}{4(n+2k)}\right)\ L_k^{n-1}\left(\frac{\lambda|z|^2}{2(n+2k)}\right)\,,$$

where L_k^{n-1} is a Laguerre polynomial. The exponential decay in $|z|$, together with Lemmas 7.4 and 7.7 are sufficient to justify the calculations that arise below.

It was shown in [Str91] that convolution operators with radial kernels (i.e. kernels that are functions of $|z|$ and t) commute, so for instance we have

$$*K * \phi_{\lambda,k,\epsilon} \;=\; *\phi_{\lambda,k,\epsilon} * K\,.$$

The convolution operator with K as above was considered in Section 5, Example 2, with $n = 1$, $\alpha = -1$ and $\beta = 1$. (Although there it was assumed that $0 < \operatorname{Re} \alpha < 2n$, the calculations are valid when $\alpha = -1$.) It is shown there that for all $(\lambda, k, \epsilon) \in (0, \infty) \times \mathbb{Z}_{\geq 0} \times \{\pm 1\}$,

$$\phi_{\lambda,k,\epsilon} * K \;=\; \kappa(\lambda, k, \epsilon)\, \phi_{\lambda,k,\epsilon}\,,$$

where $\kappa(\lambda, k, \epsilon)$ is $m(\lambda(n+2k), \epsilon\lambda)$ in the notation of [Str91, (2.35),(2.36), (5.9)]. His calculations in (5.10), (5.13') imply that $\kappa(\lambda, k, \epsilon)$ is nonzero for all (λ, k, ϵ). From this and the commutativity mentioned above, it follows that

$$u * \phi_{\lambda,k,\epsilon} \;=\; \frac{1}{\kappa(\lambda, k, \epsilon)}\; u * \phi_{\lambda,k,\epsilon} * K \;=\; \frac{1}{\kappa(\lambda, k, \epsilon)} (u * K) * \phi_{\lambda,k,\epsilon} \;=\; 0\,,$$

since $u * K = 0$. Therefore, using the exponential decay in (7.8), for any smooth compactly supported function $v : (0, \infty) \to (0, \infty)$, convolution of u with

$$V \;=\; \int_{(0,\infty)} \sum_{k=1}^{N} \sum_{\epsilon=\pm 1} \phi_{\lambda,k,\epsilon}\, v(\lambda)\, d\lambda$$

is also zero. It is not hard to see that such functions V are dense among radial functions in the Schwartz space $\mathcal{S}(\mathbb{H})$ of rapidly decreasing functions, and this implies that $u = 0$.

Theorem 7.9. *If Σ is a Lipschitz signed horizontal cut measure such that $d_\Sigma^{\mathrm{hor}} \;=\; 0$, then $\Sigma \;=\; 0$.*

The theorem follows from the above discussion when Σ is absolutely continuous with respect to $\mathcal{L}$. The general case follows from this, by an approximation argument:

Lemma 7.10. *There is a sequence of smooth functions $\{\rho_k : \mathbb{H} \to \mathbb{R}\}$ such that the sequence of measures $\{\Sigma_k = \rho_k\, \mathcal{L}\}$ converges weakly to Σ, and each Σ_k is a Lipschitz signed cut measure with vanishing cut metric.*

Proof. Due to the $\mathbb{H}$-invariance of the setup, for any $g \in \mathbb{H}$, the pushforward of Σ under left translation $(\ell_g)_*\Sigma$ is also a Lipschitz horizontal cut measure with vanishing cut metric on horizontal pairs. Therefore the same will be true of any convolution $\phi * \Sigma$, where ϕ is a compactly supported continuous function. Now let $\Sigma_k = \phi_k * \Sigma$, where $\{\phi_k\}$ is an appropriate sequence of smooth compactly supported functions converging weakly to a Dirac mass. □

Corollary 7.11. *If Σ is a Lipschitz signed horizontal cut measure, and the restriction of d_Σ to horizontal pairs is invariant under translation by the center, then $\Sigma = 0$.*

Proof. By the same smoothing argument as in Lemma 7.10, it suffices to treat the case when Σ is absolutely continuous with respect to $\mathcal{L}$, so $\Sigma = u\,\mathcal{L}$.

Pick a central element $g \in \exp \mathbb{R}Z$. Define Σ' to be the different of signed measures $\Sigma - (\ell_g)_*\Sigma$. By linearity $d_{\Sigma'}^{\rm hor} = 0$, and so by Theorem 7.9 we have $\Sigma' = 0$. Therefore Σ is invariant under translation by the center $\exp \mathbb{R}Z$. On the other hand, by Lemma 7.7, the pushforward $\pi_*\Sigma$ is a Radon measure on $\mathbb{R}^2$. This forces $\Sigma = 0$. □

Injectivity for vertical cut measures. We now suppose Σ is a symmetrized Lipschitz signed cut measure supported on vertical half-spaces.

Lemma 7.12. *If* $d_\Sigma^{\rm hor} = 0$*, then* $\Sigma = 0$.

Remark 7.13. This lemma is not needed in the proof of the main theorem, so some readers may prefer to skip it. It does, however, give some additional information about the situation of the theorem.

Proof. As with horizontal cut measures, we prefer to work with a measure on the space of vertical planes $\mathbb{P}_v$, rather than a symmetric cut measure. Moreover, since vertical planes are in obvious bijection with lines in $\mathbb{R}^2$, we may reformulate this as a question about a signed cut measure σ on the space of lines $\mathbb{L}(\mathbb{R}^2)$, and the associated cut metric d_σ on $\mathbb{R}^2$. By a smoothing argument as in Lemma 7.10, we may assume that σ is of the form $u\,\mathcal{L}$, where $u : \mathbb{L}(\mathbb{R}^2) \to \mathbb{R}$ is a smooth function, and $\mathcal{L}$ is Haar measure on $\mathbb{L}(\mathbb{R}^2)$ (when viewed as a homogeneous space of the isometry group $\mathrm{Isom}(\mathbb{R}^2)$). Note that the asymptotic behavior of the cut metric d_σ near a point $x \in \mathbb{R}^2$ agrees with the one induced by a translation invariant cut measure σ_x on $\mathbb{R}^2$, where σ_x is determined by the density function u restricted to the set of lines passing through x; hence we are reduced to the case when σ is translation invariant.

A translation invariant cut measure σ is obtained as follows. For each direction v in $\mathbb{R}^2$, there is a unique translation invariant measure τ_v on $\mathbb{L}(\mathbb{R}^2)$ supported on the set of lines parallel to v, normalized such that the d_{τ_v}-distance between two lines L_1, L_2 parallel to v is the same as their Euclidean distance. A general translation invariant cut measure σ is obtained as a superposition of the τ_v's, or as the pushforward of a signed Radon measure μ on $\mathbb{R}P^1$ under the map $v \mapsto \tau_v$. A calculation shows that the induced distance on $\mathbb{R}^2$ is homogeneous of degree 1, and that if ξ is a unit vector, then

$$d_\sigma(0,\xi) \;=\; \int_{\mathbb{R}P^1} |\sin\angle(\xi,v)|\, d\mu(v)\,.$$

Thus the operator $\sigma \mapsto d_\sigma$ is equivalent to the convolution operator $\mu \mapsto \mu * |\sin\theta|$ on $\mathbb{R}P^1$. Direct calculation shows that this is injective for complex exponentials $\exp(ik\theta) : \mathbb{R}P^1 \to \mathbb{R}$ for k even, and this implies injectivity in general. □

Proof of Theorem 7.2. Suppose $\Sigma = \Sigma_v + \Sigma_h$ is a symmetrized Lipschitz signed cut measure and $d_\Sigma^{\mathrm{hor}} = 0$. Since both d_Σ^{hor} and $d_{\Sigma_v}^{\mathrm{hor}}$ are invariant under translation by the center, the same is true of $d_{\Sigma_h}^{\mathrm{hor}} = d_\Sigma^{\mathrm{hor}} - d_{\Sigma_v}^{\mathrm{hor}}$. By Corollary 7.11, we have $\Sigma_h = 0$. Then $d_{\Sigma_v} = 0$, and Lemma 7.12 implies that $\Sigma_v = 0$ as well.

8. Cut measures which are standard on lines

When Σ is the cut measure arising from a map $\mathbb{H} \to L^1$ which comes from Pauls' metric diffentiation theorem, then we know that the restriction of d_Σ to lines is a constant multiple of the Heisenberg metric. Using the injectivity statement in Theorem 7.2, we get:

Theorem 8.1. *Suppose Σ is a cut measure on $\mathbb{H}$ such that:*

- *The cut metric d_Σ is bounded by a multiple of d: $d_\Sigma \leq C\, d$.*
- *The restriction of d_Σ to almost every line $L \in \mathbb{L}(\mathbb{H})$ is a constant multiple of d:*
$$d_\Sigma\Big|_L \;=\; c_L\, d\Big|_L\,.$$

Then Σ is supported on vertical half-spaces, and moreover, its symmetrization is translation invariant.

Proof. Applying Proposition 3.5, it follows that Σ is supported on monotone cuts, and hence by Theorem 5.1 on vertical and horizontal half-spaces. We may assume that Σ is symmetric.

The Lipschitz condition implies that if L_1 and L_2 are parallel lines, then the multiples c_{L_1} and c_{L_2} coincide, because the lines diverge sublinearly. Thus d_Σ^{hor} is invariant under translation. Then $d_{\Sigma_h}^{\mathrm{hor}} = d_\Sigma^{\mathrm{hor}} - d_{\Sigma_v}^{\mathrm{hor}}$ is invariant under vertical translation, and Corollary 7.11 gives $\Sigma_h = 0$. Therefore $d_{\Sigma_v}^{\mathrm{hor}}$ is $\mathbb{H}$-invariant, so Lemma 7.12 implies that $\Sigma = \Sigma_v$ is $\mathbb{H}$-invariant. □

Proof of Theorem 1.3. Let $f : \mathbb{H} \to L^1$ be a Lipschitz map, and let $\rho_{x,\lambda}$ be as in the statement of Theorem 1.3.

By Theorem 2.5, for almost every point $x \in \mathbb{H}$, there is a norm $\|\cdot\|_x$ on $\mathbb{R}^2$, such that $\rho_{x,\lambda}\Big|_{\mathrm{hor}(\mathbb{H})}$ converges uniformly on compact sets to the pseudo-metric $(z_1, z_2) \mapsto \|\pi(z_1) - \pi(z_2)\|_x$, for all $(z_1, z_2) \in \mathrm{hor}(\mathbb{H})$. We claim that the same statement holds for arbitrary pairs. If not, since the family $\{\rho_{x,\lambda}\}_{\lambda\in(0,\infty)}$ is uniformly Lipschitz, by the

Arzela-Ascoli theorem we may find a sequence $\{\lambda_k\} \to 0$ such that ρ_{x,λ_k} converges uniformly on compact subsets of $\mathbb{H} \times \mathbb{H}$ to a limiting pseudo-distance ρ_∞, such that

$$\rho_\infty(z_1, z_2) \neq \|\pi(z_1) - \pi(z_2)\|_x \tag{8.2}$$

for some $(z_1, z_2) \in \mathbb{H} \times \mathbb{H}$, while $\rho_\infty(z_1, z_2) \neq \|\pi(z_1) - \pi(z_2)\|_x$ for all $(z_1, z_2) \in \mathrm{hor}(\mathbb{H})$.

By Corollary 2.7, the pseudo-distance ρ_∞ is induced by a map $f_\omega : \mathbb{H} \to L^1$. If Σ is the cut measure associated with f_ω, then Σ satisfies the hypotheses of Theorem 8.1. Therefore Σ is supported on vertical half-spaces, which means that d_Σ is the pullback of a metric from $\mathbb{R}^2$ by the projection map π. This contradicts (8.2).

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