The asymptotic geometry of negatively curved spaces: uniformization, geometrization and rigidity

Bruce Kleiner*

Abstract. This is a survey of recent developments at the interface between quasiconformal analysis and the asymptotic geometry of Gromov hyperbolic groups. The main theme is the extension of Mostow rigidity and related theorems to a broader class of hyperbolic groups, using recently developed analytic structure of the boundary.

Mathematics Subject Classification (2000). Primary 30C65; Secondary 20F57

Keywords. Quasiconformal geometry, geometric group theory, rigidity.

1. Introduction

The celebrated Mostow rigidity theorem [49] states that if X and X' are symmetric spaces of noncompact type with no de Rham factors isomorphic to the hyperbolic plane, and $\Gamma \subset \mathrm{Isom}(X)$, $\Gamma' \subset \mathrm{Isom}(X')$ are uniform lattices, then any isomorphism $\Gamma \to \Gamma'$ between the lattices is the restriction of a Lie group isomorphism $\mathrm{Isom}(X) \to \mathrm{Isom}(X')$. This theorem and its proof have many important implications, and have inspired numerous generalizations and variants (e.g. [53, 47, 32, 59, 57, 27, 35, 51, 44, 2]) most of which concern lattices in semisimple groups. Mostow's proof was based on the asymptotic geometry of symmetric spaces, more specifically the quasiconformal or combinatorial structure of the boundary at infinity. Recent developments in analysis on metric spaces and quasiconformal geometry have begun to create the technical framework needed to implement Mostow's proof in a much more general context, yielding new Mostow-type rigidity theorems. The goal of this article is to survey some of these developments and their group theoretic applications.

Organization of the paper. Section 2 discusses some general issues in geometric group theory. Section 3 covers basic facts about Gromov hyperbolic spaces and their boundaries. Section 4 reviews a selection of recent work related to quasiconformal geometry in a metric space setting, and some applications of this are

*The author gratefully acknowledges support from NSF grants DMS-0224104 and DMS-0505610.

covered in section 5. Quasiconformal uniformization is discussed in section 6, and quasiconformal geometrization in section 7. The last section presents some open problems.

The survey by Mario Bonk in these Proceedings provides a complementary viewpoint on some of the material presented here.

Acknowledgements. I would like to thank Mario Bonk and Juha Heinonen for numerous helpful comments on an earlier version of this article.

2. Rigidity and geometrization in geometric group theory

The ideas presented in this section are strongly influenced by the work of Gromov [32, 31, 34].

The asymptotic viewpoint in geometric group theory. One of the guiding themes in geometric group theory is that one can often understand the algebraic structure of a group by finding the right geometric realization of the group as a group of isometries. For instance it is very constructive to think of nonabelian free groups as groups acting on trees, and lattices in semi-simple Lie groups as groups of isometries acting on the associated symmetric space. Every finitely generated group has a plentiful supply of isometric actions, namely the actions on its Cayley graphs. Recall that if Σ is a finite generating set for a group G, then the **Cayley graph** of (G, Σ), denoted $\mathrm{Cayley}(G, \Sigma)$, is the graph with vertex set G, in which two group elements $g, g' \in G$ are joined by an edge if and only if $g = g'\sigma$ for some $\sigma \in \Sigma$. The action of G on itself by left translation extends to an action $G \curvearrowright \mathrm{Cayley}(G, \Sigma)$ by graph isomorphisms; one may view this as an isometric action by equipping the Cayley graph with the path metric where each edge has length 1. As a tool for understanding the original group G, Cayley graphs have a drawback: there are too many of them. Nonetheless their asymptotic, or large-scale, structures are all the same. To formalize this idea, we now recall a few definitions.

Definition 2.1. A (possibly discontinuous) map $f \colon X \to X'$ between metric spaces is a **quasi-isometry** if there are constants L, A such that for every $x_1, x_2 \in X$,

$$\frac{1}{L} d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq L d(x_1, x_2) + A, \tag{2.1}$$

and every $x' \in X'$ lies within distance at most A from a point in the image of f. Here and elsewhere we will use the generic letter "d" for metric space distance functions when it is clear in which metric space distances are being measured. Two metric spaces are **quasi-isometric** if there exists a quasi-isometry from one to another; this defines an equivalence relation on the collection of metric spaces.

A metric space is **proper** if every closed ball is compact. An isometric action $G \curvearrowright X$ on a metric space X is **discrete** if for every ball $B = B(x,r) \subset X$, the set

$$\{g \in G \mid gB \cap B \neq \emptyset\} \tag{2.2}$$

is finite, and **cocompact** if there is a compact subset $K \subset X$ such that

$$X = \bigcup_{g \in G} gK. \tag{2.3}$$

The fundamental lemma of geometric group theory ties these together:

Lemma 2.2. *Suppose G is a finitely generated group, X, X' are proper geodesic spaces, and $G \overset{\rho}{\curvearrowright} X$, $G \overset{\rho'}{\curvearrowright} X'$ are two discrete, cocompact, and isometric actions of G. Then X is quasi-isometric to X'. Moreover one may find a quasi-isometry $f : X \to X'$ which is "quasi-equivariant" in the sense that there is a constant D such that*

$$d(f(gx), gf(x)) < D \tag{2.4}$$

for all $g \in G$, $x \in X$.

If Σ is a finite generating set for a group G, then the action $G \curvearrowright \mathrm{Cayley}(G, \Sigma)$ is discrete, cocompact, and isometric; therefore the collection of actions covered by the lemma is nonempty for any finitely generated group G. This means that there is a well-defined quasi-isometry class of geodesic metric spaces associated with each finitely generated group G, which we will refer to as the quasi-isometry class of G. Another implication of the lemma is that the universal cover of a compact, connected Riemannian manifold M (equipped with the Riemannian distance function) is quasi-isometric to $\pi_1(M)$; this follows from Lemma 2.2 because the deck group action $\pi_1(M) \curvearrowright \tilde{M}$ is discrete, cocompact, and isometric. Thus in addition to Cayley graphs, one has an abundance of other metric spaces representing the quasi-isometry class of the group.

The asymptotic approach to geometric group theory is to study groups by identifying quasi-isometry invariant structure in their quasi-isometry class.

Some asymptotic problems. One of the first quasi-isometry invariants of a metric space X is its **quasi-isometry group**, denoted $\mathrm{QI}(X)$. This is defined to be the collection of equivalence classes of quasi-isometries $f : X \to X$, where two quasi-isometries f, f' are declared to be equivalent if and only if their supremum distance

$$d(f, f') := \sup_{x \in X} d(f(x), f'(x))$$

is finite, and the group law is induced by composition of quasi-isometries. The group $\mathrm{QI}(X)$ is quasi-isometry invariant because a quasi-isometry $X \to X'$ between two metric spaces induces an isomorphism $\mathrm{QI}(X) \to \mathrm{QI}(X')$ by "quasi-conjugation". By Lemma 2.2 it therefore makes sense to speak of the quasi-isometry group of a finitely generated group G.

We now discuss several problems which are central in geometric group theory.

Question 2.3.

A. (QI classification of groups) What are the finitely generated groups in a given quasi-isometry class?

B. (Classification of QI's) Given a finitely generated group G, what is the quasi-isometry group of G?

C. (Uniformization/Recognition) Given a model group, find a criterion for deciding when another group is quasi-isometric to it.

Question A has led to some remarkable mathematics. In each case where it was resolved successfully, new geometric, analytic, combinatorial, or topological ingredients were brought into play. Here are few examples:

• A finitely generated group is quasi-isometric to $\mathbb{Z}^n$ if and only if it is virtually $\mathbb{Z}^n$, i.e. it contains a finite index subgroup isomorphic to $\mathbb{Z}^n$. The proof relies on Gromov's theorem on groups of polynomial growth [33], Pansu's description of asymptotic cones of nilpotent groups [50], and Bass's formula for the growth of a nilpotent group [1]. We note that a new proof was found recently by Shalom [55]; it uses a simpler approach, but it is still far from elementary.

• A finitely generated group quasi-isometric to a free group F_k is virtually free. This is due to Gromov, and uses Stallings' theorem on ends of groups [56].

• A finitely generated group quasi-isometric to a symmetric space X of non-compact type admits a discrete, cocompact, isometric action on X. This follows from combined work of Sullivan, Gromov, Tukia, Pansu, Kleiner-Leeb, and Gabai, Casson-Jungreis. The proofs involve asymptotic geometry – quasiconformal structure on boundaries, asymptotic cones, and Tits geometry.

Question A is related to Question B because if a group G' is quasi-isometric to G, then there is a homomorphism $G' \to \mathrm{QI}(G') \simeq \mathrm{QI}(G)$, and one can approach Question A by studying subgroups of $\mathrm{QI}(G)$.

In most of the cases when Question B has been answered satisfactorily, the approach is to identify a "canonical" or "optimal" metric space X which is quasi-isometric to G, and then argue that the homomorphism $\mathrm{Isom}(X) \to \mathrm{QI}(X)$ is an isomorphism, i.e. that every quasi-isometry of X lies at finite distance from a unique isometry [51, 44, 12]. When this happens, one says that the metric space X is **quasi-isometrically rigid**. This leads to:

Question 2.4. *When can one find a quasi-isometrically rigid space in the quasi-isometry class of a given finitely generated group G?*

In general, an answer to this question has two parts. The first part is geometrization: finding a candidate for the optimal/rigid metric space in the given quasi-isometry class. In most of the earlier rigidity theorems, geometrization was easy because there was an obvious model space to consider. The second part is to show that the candidate space is actually rigid, which requires one to exploit appropriate asymptotic structure.

Quasi-isometric rigidity may fail because the quasi-isometry group is too large, which is what happens for $\mathbb{Z}^n$, free groups F_k, or lattices in $\mathrm{Isom}(\mathbb{H}^n)$. Nonetheless in many cases a weaker form of rigidity survives: there is a proper metric space X in the quasi-isometry class such that any group in this class admits – possibly after passing to a finite index subgroup – a discrete, cocompact, isometric action on X. To put it another way: any group quasi-isometric to X is virtually a lattice in $\mathrm{Isom}(X)$, when one topologizes $\mathrm{Isom}(X)$ using the compact-open topology. So although individual quasi-isometries are not rigid, sufficiently large groups of quasi-isometries may turn out to be rigid. When this weaker form of rigidity holds, one may ask if the analog of Mostow rigidity is true.

Question C has a satisfactory answer for only a few classes of groups, such as free groups, free abelian groups, and surface groups. For instance, a finitely generated group G is virtually free if for every Cayley graph $\mathcal{G}$ there are constants r, R such that any two points $x, y \in \mathcal{G}$ at distance at least R lie in distinct components of the complement of some r-ball $B(z, r) \subset \mathcal{G}$. An important unresolved case of Question C, which is tied to conjectures in 3-manifold topology, is to find a characterization of groups quasi-isometric to hyperbolic 3-space $\mathbb{H}^3$. This is discussed in section 6.

In the remainder of this paper, we will focus on these questions and related mathematics in the context of negatively curved manifolds, or more generally Gromov hyperbolic spaces.

3. Gromov hyperbolic spaces and their boundaries

In this section we review some facts about Gromov hyperbolicity, see [31, 30, 16, 34, 41]. Gromov hyperbolic spaces form a robust class of metric spaces to which much of the theory of negatively curved Riemannian manifolds applies. They have a boundary at infinity, which plays an essential role in rigidity applications. Much of their asymptotic structure is encoded in the quasiconformal (or quasi-Möbius) structure of the boundary; this fact enables one to exploit the analytic theory of quasiconformal homeomorphisms.

We recall that a geodesic metric space X is **Gromov hyperbolic** if there is a constant δ such that every geodesic triangle in X is **δ-thin**, in other words, each side lies within the δ-neighborhood of the union of the other two sides. Gromov hyperbolicity is a quasi-isometry invariant property for geodesic metric spaces [16]. A finitely generated group is **Gromov hyperbolic** if its quasi-isometry class is Gromov hyperbolic, see Lemma 2.2 and the ensuing commentary.

Except when it is explicitly stated to the contrary, for the remainder of this paper X will denote a proper Gromov hyperbolic geodesic metric space and δ its hyperbolicity constant.

Prime examples of Gromov hyperbolic spaces are complete simply-connected Riemannian manifolds of sectional curvature bounded above by a constant $\kappa < 0$, equipped with their Riemannian distance functions, and more generally, simply-connected Alexandrov spaces of curvature $\leq \kappa$ [16]. Metric trees and piecewise Euclidean polyhedra satisfying appropriate link conditions also provide many examples of group theoretic interest [31, 30, 16].

Two geodesic rays $\gamma_1, \gamma_2 : [0,\infty) \to X$ are **asymptotic** if the Hausdorff distance between their images is finite; this defines an equivalence relation on the collection of geodesic rays in X. The **boundary** of X, denoted ∂X, is the collection of equivalence classes. Pick $p \in X$. Given $[\gamma] \in \partial X$, there is a unit speed geodesic ray starting from p which is asymptotic to γ; thus one may identify ∂X with the collection of asymptote classes of geodesics rays starting at p. Given unit speed geodesic rays $\gamma_1, \gamma_2 : [0,\infty) \to X$ starting from p, their **Gromov overlap** is defined to be

$$\langle \gamma_1 \mid \gamma_2 \rangle := \lim_{t\to\infty} \frac{1}{2}\left(2t - d(\gamma_1(t), \gamma_2(t))\right) \in [0,\infty]. \tag{3.1}$$

To within an additive error comparable to the hyperbolicity constant δ, the overlap of two rays is the infimal $t \in \mathbb{R}$ such that the distance from $\gamma_1(t)$ to $\gamma_2(t)$ is $> \delta$.

Visual metrics. A **visual metric** on ∂X is a metric ρ such that for some constants $a > 0$ and C (a is called the visual parameter of ρ), and some $p \in X$,

$$\frac{1}{C} e^{-a\langle \gamma_1|\gamma_2\rangle} \le \rho(\gamma_1, \gamma_2) \le C e^{-a\langle \gamma_1|\gamma_2\rangle} \tag{3.2}$$

for every pair of rays γ_1 and γ_2 starting at p. This condition is independent of the choice of basepoint used to define the overlap. When a is small compared to $1/\delta$, visual metrics with visual parameter a always exist.

Henceforth when we refer to the boundary ∂X, we will mean the set ∂X equipped with some visual metric, unless otherwise stated. Here are some examples of visual metrics (note that the assertions apply to a particular choice of visual metric):

- The boundary of $\mathbb{H}^n$ is bi-Lipschitz homeomorphic to the sphere S^{n-1} equipped with the usual metric.

- The boundary of complex hyperbolic space $\mathbb{CH}^n$ is bi-Lipschitz homeomorphic to S^{2n-1} equipped with the usual Carnot metric. We recall that this metric is defined as follows. Let Δ be the usual contact structure on S^{2n-1} induced by the embedding of S^{2n-1} into $\mathbb{C}^n$. The Carnot distance between two points $p, q \in S^{2n-1}$ is the infimum of the lengths of integral curves of Δ which join p to q. This metric on the $(2n-1)$-sphere has Hausdorff dimension $2n$.

- Let T be a trivalent simplicial tree where each edge has length 1. Then ∂T is a Cantor set.

- Let M be a compact Riemannian 3-manifold of constant sectional curvature -1 with nonempty totally geodesic boundary, and let X be its universal cover equipped with the Riemannian distance function. Then X isometrically embeds in $\mathbb{H}^3$ as a convex subset C bounded by a countable collection of disjoint totally geodesic planes, and ∂X may be identified with the limit set of C in the sphere at infinity of $\mathbb{H}^3$. The limit set is obtained by removing a countable disjoint collection of round spherical caps from the 2-sphere, and is homeomorphic to the Sierpiński carpet.

There is not much one can say about boundaries of Gromov hyperbolic spaces in general since every compact doubling metric space is isometric to the boundary of a proper Gromov hyperbolic space equipped with a visual metric, see [13, Section 2]. However, if X admits a discrete, cocompact, isometric action, then the boundary structure is much more restricted – it is **approximately self-similar** in the following sense. There are constants L, $D > 0$ such that if $B(p,r) \subset \partial X$ is a ball with $0 < r \leq \operatorname{diam}(\partial X)$, then there is an open subset $U \subset \partial X$ of diameter $> D$ which is L-bi-Lipschitz homeomorphic to the rescaled ball $\frac{1}{r}B(p,r)$. This approximate self-similarity has many implications for the topology and geometry of ∂X. Before stating them, we need several definitions.

Definition 3.1. A metric space Z is **Ahlfors Q-regular** if there is a constant C such that the Q-dimensional Hausdorff measure of any r-ball B satisfies

$$\frac{1}{C} r^Q \leq \mathcal{H}^Q(B) \leq C r^Q, \tag{3.3}$$

provided $r \ \leq\ \operatorname{diam}(Z)$.

Note that an Ahlfors Q-regular space has Hausdorff dimension Q. Examples of Ahlfors regular spaces are $\mathbb{R}^n$ and S^{2n-1} equipped with standard Carnot metric.

The next definitions are scale-invariant, quantitative versions of standard topological conditions.

Definition 3.2. A metric space Z is **linearly locally contractible** if there is a constant $\lambda > 0$ such that for all $x \in Z$, $0 < r \leq \operatorname{diam}(Z)$, the inclusion

$$B(x, \lambda r) \to B(x, r) \tag{3.4}$$

is null-homotopic. Linear local contractibility excludes examples where the metric topology looks worse and worse at smaller and smaller scales. For instance, a metric on a two sphere which has a sequence of "fingers" of smaller and smaller diameter, for which the ratio of the diameter to the circumference tends to infinity will not be linearly locally contractible.

A metric space Z is **uniformly perfect** if there is a constant $\lambda > 0$ such that if $p \in Z$, $0 < r \leq \operatorname{diam}(Z)$, then $B(p,r) \setminus B(p, \lambda r)$ is nonempty.

A metric space Z is **LLC** (not be to be confused with "linear locally contractible") if there is an L such that for all $p \in Z$, $0 < r \leq \operatorname{diam}(X)$, the inclusions

$$B(p,r) \to B(p, Lr), \quad X \setminus B(p,r) \longrightarrow X \setminus B\left(p, \frac{r}{L}\right) \tag{3.5}$$

induce the zero homomorphism on reduced 0-dimensional homology. This is a standard type of condition in quasiconformal geometry, and is a quantitative version of local connectedness and absence of local cut points.

Theorem 3.3 (Structure of the boundary). *If X is a proper, geodesic, Gromov hyperbolic space which admits a discrete, cocompact, isometric action, then:*

1. *∂X is either empty, has two elements, or is uniformly perfect.*

2. ∂X is Ahlfors Q-regular for some Q ([26]) .

3. If ∂X is connected, it is LLC ([4, 15, 58, 7]).

4. If ∂X is homeomorphic to S^k, then ∂X is linearly locally contractible.

5. If X is a contractible n-manifold, then ∂X is a homology $(n-1)$-manifold with the homology of the $(n-1)$-sphere [4, 3]. When $n \leq 3$ then ∂X is a sphere.

Induced homeomorphisms between boundaries. In applications to group theory and rigidity, a key property of the boundary is that quasi-isometries between Gromov hyperbolic spaces induce homeomorphisms between their boundaries. To formulate this more precisely we need the following definition, which is due to Väisälä.

Definition 3.4 ([63]). The **cross-ratio**, $[z_1, z_2, z_3, z_4]$, of a 4-tuple of distinct points (z_1, z_2, z_3, z_4) in a metric space Z is the quantity

$$[z_1, z_2, z_3, z_4] := \frac{d(z_1, z_3)d(z_2, z_4)}{d(z_1, z_4)d(z_2, z_3)}. \tag{3.6}$$

This is a metric space version of the familiar cross-ratio in complex analysis. A homeomorphism $\phi : Z \to Z'$ between metric spaces is **quasi-Möbius** if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for every quadruple of distinct points $(z_1, z_2, z_3, z_4) \in Z$,

$$[\phi(z_1), \phi(z_2), \phi(z_3), \phi(z_4)] \leq \eta([z_1, z_2, z_3, z_4]). \tag{3.7}$$

Such a homeomorphism η is called a **distortion function** for the homeomorphism ϕ. Intuitively, a homeomorphism is quasi-Möbius if it distorts cross-ratios in a controlled way.

Compositions and inverses of quasi-Möbius homeomorphisms are quasi-Möbius, so every metric space Z has an associated group of quasi-Möbius homeomorphisms, which we denote by $\mathrm{QM}(Z)$.

Theorem 3.5 ([52]). *Every quasi-isometry $f : X \to X'$ between Gromov hyperbolic spaces induces a quasi-Möbius homeomorphism $\partial f : \partial X \to \partial X'$ between their boundaries. The distortion function η for ∂f can be chosen to depend only on the the hyperbolicity constants of X and X', the constants for the visual metrics on ∂X and $\partial X'$, and the quasi-isometry constants of f.*

In particular, the full isometry group of a Gromov hyperbolic space acts on its boundary as a group of quasi-Möbius homeomorphisms with uniform distortion function.

The proof of Theorem 3.5 has two ingredients. The first is the "Morse Lemma", which implies that f maps each geodesic segment (respectively ray) in X to within controlled Hausdorff distance of a geodesic segment (respectively ray) in X'. This yields the induced set-theoretic bijection $\partial f : \partial X \to \partial X'$. To verify that ∂f is quasi-Möbius, the idea is to associate to each 4-tuple of points in ∂X a configuration in X consisting of a pair of geodesics and a shortest geodesic segment connecting them. The cross-ratio is determined by the length of the connecting geodesic segment, and the latter is preserved to within a factor by the quasi-isometry.

Under mild assumptions, for instance if X is quasi-isometric to a nonelementary hyperbolic group (a hyperbolic group which does not contain a finite index cyclic subgroup), the homomorphism $\mathrm{QI}(X) \to \mathrm{QM}(\partial X)$ given by taking boundary homeomorphisms is an isomorphism. Thus one can translate questions about quasi-isometries into questions about quasi-Möbius homeomorphisms of the boundary. This has the additional advantage of eliminating some ambiguity (an equivalence class of quasi-isometries is replaced by a single quasi-Möbius homeomorphism), as well as providing extra structure – the group $\mathrm{QM}(Z)$ has a natural topology.

Questions A–C in section 2 translate to:

A'. What are the hyperbolic groups whose boundary is quasi-Möbius homeomorphic to a given metric space Z?

B' What is the group of quasi-Möbius homeomorphisms of the metric space ∂G?

C'. Given a boundary ∂G, how can one tell if another boundary (or space) is quasi-Möbius homeomorphic to it?

The latter two questions make perfect sense and are interesting for spaces other than boundaries, e.g. self-similar spaces like the standard square Sierpinski carpet or the Menger sponge.

4. Quasiconformal homeomorphisms

This section presents some recent results on quasiconformal homeomorphisms. The material was selected for its applicability to rigidity and group theoretic problems, and does not represent a balanced overview. See [38, 37, 61, 25] for more discussion. The somewhat separate topic of uniformization is discussed in section 6.

We begin with some definitions.

Definition 4.1. Let $f : X \to X'$ be a homeomorphism between metric spaces, and $p \in X$. The **dilatation of f at p** is

$$H(f,p) := \limsup_{r \to 0} \frac{\sup\{d(f(x), f(p)) \mid x \in B(p,r)\}}{\inf\{d(f(x), f(p)) \mid x \notin B(p,r)\}}. \tag{4.1}$$

The homeomorphism f is C**-quasiconformal** if $H(f,p) \le C$ for every $p \in X$, and **quasiconformal** if it is C-quasiconformal for some C.

Heuristically, a homeomorphism is quasiconformal if it maps infinitesimal balls to sets of controlled eccentricity.

Definition 4.2. Let (Z, μ) be a metric space equipped with a Borel measure μ, and $p \geq 1$. If Γ is a collection of paths in Z (i.e. continuous maps $[0,1] \to Z$), then a Borel measurable function $\rho : Z \to [0, \infty]$ is **Γ-admissible** if

$$\int_\gamma \rho \, ds \geq 1 \tag{4.2}$$

for every rectifiable path $\gamma \in \Gamma$, where ds denotes the arclength measure. The **p-modulus of** Γ is the infimum of the quantities

$$\int_Z \rho^p \, d\mu \tag{4.3}$$

where ρ ranges over all Γ-admissible functions on Z. If E, F are subsets of Z, then the **p-modulus of** (E, F), denoted $\mathrm{Mod_p}(E, F)$, is the p-modulus of the collection of paths running from E to F. When Z is an Ahlfors Q-regular space and no measure is specified, we will be using Q-dimensional Hausdorff measure by default.

Modulus is an old and important tool in conformal and quasiconformal geometry, due to its conformal invariance and the fact that it permits one to relate infinitesimal with global behavior of homeomorphisms. One checks by a change of variable computation that a conformal diffeomorphism $M \to M'$ between Riemannian manifolds preserves modulus of curve families, and that a diffeomorphism which preserves modulus is conformal. More generally, the effect of a diffeomorphism on modulus can be controlled by its dilatation.

Quasiconformal homeomorphisms between domains in $\mathbb{R}^n$ have a number of important regularity properties.

Theorem 4.3 ([62]). *Let $f : U \to U'$ be a quasiconformal homeomorphism between domains in $\mathbb{R}^n$, $n \geq 2$. Then f belongs to the Sobolev space $W^{1,n}_{\mathrm{loc}}(U, U')$, it is differentiable almost everywhere, and maps sets of measure zero to sets of measure zero. Furthermore, f is ACL; this means that for every direction $v \in \mathbb{R}^n$, for almost every line L parallel to v, the restriction of f to $U \cap L$ is absolutely continuous with respect to 1-dimensional Hausdorff measure $\mathcal{H}^1$. Furthermore, inverses and compositions of quasiconformal homeomorphism between such domains are quasiconformal.*

The following theorem shows that there are many characterizations of quasiconformal homeomorphisms. This is important because different characterizations are useful in different situations, and also because it demonstrates that this notion is robust.

Theorem 4.4 (See [62]). *Let M and M' be Riemannian manifolds of dimension $n \geq 2$. Then the following conditions on a homeomorphism $f : M \to M'$ are quantitatively equivalent:*

1. *f is C-quasiconformal.*
2. *f distorts n-modulus of curve families by a factor at most L.*

3. f belongs to the Sobolev space $W^{1,n}_{\mathrm{loc}}(M, M')$ and the distributional derivative Df satisfies $|Df|^n \leq C\,\mathrm{Jac}(f)$ almost everywhere, for some C, where $\mathrm{Jac}(f)$ denotes the Jacobian of f with respect to the Riemannian structures on M and M' (the infinitesimal volume distortion factor).

If one assumes in addition that M and M' are compact, then one may add quasi-Möbius to this list of quantitatively equivalent conditions:

4. f is η-quasi-Möbius.

These two theorems show that quasiconformal homeomorphisms are very nicely behaved in a Riemannian context, when $n \geq 2$. Unfortunately, for general metric spaces, or even general Ahlfors regular metric spaces, Definition 4.1 is not necessarily equivalent to the quasi-Möbius condition, and does not lead to a useful theory. Examples show that it is necessary to impose further conditions on the structure of the metric space in order to prove anything akin to Theorem 4.4. After earlier work on Carnot-Carathéodory geometry [49, 45, 48], the breakthrough paper [38] introduced natural conditions for precisely this purpose, and by now they have been shown to imply most of the properties above. The essential requirement on the space is a quantitative link between infinitesimal structure and global structure, which is expressed by a relation between moduli of pairs (E, F) and their relative distance. The **relative distance** between two subsets E, F of a metric space is

$$\Delta(E, F) := \frac{\mathrm{dist}(E, F)}{\min\left(\mathrm{diam}(E), \mathrm{diam}(F)\right)}, \tag{4.4}$$

where

$$\mathrm{dist}(E, F) := \inf\{d(x, y) \mid x \in E,\ y \in F\}. \tag{4.5}$$

This is a simple scale invariant measure of the separation between two subsets.

Definition 4.5. Let Z be an Ahlfors Q-regular metric space. Then Z is **Q-Loewner** if there is a positive decreasing function $\phi : (0, \infty) \to (0, \infty)$ such that

$$\mathrm{Mod}_{\mathrm{Q}}(E, F) \geq \phi(\Delta(E, F)), \tag{4.6}$$

whenever $E, F \subset Z$ are disjoint nondegenerate continua. We recall that a **continuum** is a compact, connected, subset, and a nondegenerate continuum is one of positive diameter. Recall that for Q-regular spaces, Q-dimensional Hausdorff measure is the default measure used to calculate modulus.

We remark that for Ahlfors Q-regular spaces, one always has an upper bound on modulus of the following form. There is a function

$$\psi : [0, \infty) \to [0, \infty] \tag{4.7}$$

with $\lim_{t\to\infty} \psi(t) = 0$ such

$$\mathrm{Mod}_{\mathrm{Q}}(E, F) \leq \psi(\Delta(E, F)) \tag{4.8}$$

for any pair of closed subsets E, F. Combining (4.6) and (4.8), one can say that for a Q-Loewner space, the modulus for a pair of disjoint continua is quantitatively controlled by their relative distance.

The paper [38] introduced Definition 4.5, and together with subsequent work [61, 39, 25], most of the facts in Theorems 4.3 and 4.4 have now been proved for quasiconformal homeomorphisms between Loewner spaces.

Theorem 4.6. *Let Z, Z' be compact Q-Loewner for $Q > 1$.*

1. Quasiconformal homeomorphisms $Z \to Z'$ are absolutely continuous, i.e. map sets of measure zero to sets of measure zero.

2. Every quasiconformal homeomorphism $f : Z \to Z'$ is ACL in the following sense. There is a collection of paths Γ of Q-modulus zero such that if $\gamma : [0,1] \to Z$ is a path, $\gamma \notin \Gamma$, then $f \circ \gamma : [0,1] \to Z$ is absolutely continuous with respect to 1-dimensional Hausdorff measure on the target.

3. Every quasiconformal homeomorphism $Z \to Z'$ belongs to the Sobolev space $W^{1,Q}(Z, Z')$.

4. For a homeomorphism $f : Z \to Z'$, conditions 1,2, and 4 of Theorem 4.4 are quantitatively equivalent.

5. The inverse of a quasiconformal homeomorphism is quasiconformal.

In the paper [61] it was shown that quasi-Möbius homeomorphisms between compact Q-regular spaces distort Q-modulus in a controlled way; in particular the Q-Loewner property is invariant under quasi-Möbius homeomorphisms.

The paper [38] also showed that when $Q > 1$, for compact Ahlfors Q-regular spaces, the Q-Loewner condition is equivalent to a $(1, Q)$-Poincaré inequality. This is an analytic condition which relates upper gradients to mean oscillation. We refer the reader to [37] for more on this topic.

Another breakthrough was the paper [25] which established a notion of differentiability for Lipschitz functions on doubling metric measure spaces satisfying a Poincaré inequality, in particular for compact Q-Loewner spaces. This has the remarkable implication that there is a **cotangent bundle** T^*Z for such metric measure spaces, which is a measurable vector bundle equipped with a canonical measurably varying fiberwise norm. Any bi-Lipschitz homeomorphism $(Z, \mu) \to (Z', \mu')$ preserving measure classes induces a derivative mapping $T^*Z' \to T^*Z$, which is a measurable bundle isomorphism which distorts the norms by a factor controlled by the bi-Lipschitz constant. The paper [39] extended this assertion to Sobolev functions, and showed that quasiconformal homeomorphisms $Z \to Z'$ between Q-regular, Q-Loewner spaces also have a well-defined Cheeger derivative, when $Q > 1$. Using this the author showed [43] that under the same assumptions, the dilatation of a quasiconformal homeomorphism is controlled quantitatively by the dilatation of its derivative mapping $T^*Z' \to T^*Z$; the latter is defined relative to the canonical fiberwise norms on T^*Z and T^*Z'.

5. Applications to rigidity

We now discuss applications of the analytic results in the previous section to rigidity theorems.

The first is Mostow rigidity in the negatively curved case:

Theorem 5.1. *Suppose X and X' are rank 1 symmetric spaces of noncompact type other than $\mathbb{H}^2$, and G and G' are cocompact lattices in $\mathrm{Isom}(X)$ and $\mathrm{Isom}(X)$ respectively. Then any isomorphism $G \to G'$ extends to a Lie group isomorphism $\mathrm{Isom}(X) \to \mathrm{Isom}(X')$.*

The outline of the proof is as follows. Identifying the two groups G and G' using the isomorphism, one obtains discrete, cocompact, isometric actions of $G \curvearrowright X$ and $G \curvearrowright X'$. These induce uniformly quasi-Möbius (in fact Möbius in the $\mathbb{H}^n$ case) boundary actions $G \curvearrowright \partial X$, $G \curvearrowright \partial X'$. By Lemma 2.2 one gets a "quasi-equivariant" quasi-isometry

$$f : X \to X', \tag{5.1}$$

which has a quasi-Möbius boundary homeomorphism $\partial f : \partial X \to \partial X'$. The quasi-equivariance of f implies that ∂f is equivariant with respect to the actions $G \curvearrowright \partial X$ and $G \curvearrowright \partial X'$. By using the equivariance and the dynamics of the group action, one then argues that the derivative of ∂f, which is defined almost everywhere because ∂X and $\partial X'$ are Carnot spaces [49, 45, 48, 51], must actually be conformal almost everywhere. This implies that $\partial f = \partial h$, for a unique isometry $h : X \to X'$. It follows readily that h is G-equivariant, and induces the desired isomorphism $\mathrm{Isom}(X) \to \mathrm{Isom}(X')$.

Pansu used a similar outline to show that for rank 1 symmetric spaces other than the hyperbolic and complex hyperbolic spaces an even stronger rigidity result holds:

Theorem 5.2 ([51]). *Suppose X is a quaternionic hyperbolic space or the Cayley hyperbolic plane, and X' is a rank 1 symmetric space of noncompact type. Then any quasi-isometry $X \to X'$ is at bounded distance from a unique isometry.*

The proof of this theorem also uses boundary geometry. The boundary homeomorphism $\partial f : X \to X'$ for a quasi-isometry is quasiconformal, and Pansu shows that for the spaces in question, the derivative is forced to be conformal even without invoking an equivariance assumption as in Mostow's proof.

This kind of argument was used in a "non-classical" setting in work of Bourdon, Bourdon-Pajot, and Xie, proving a Pansu-type rigidity result for Fuchsian buildings:

Theorem 5.3 ([8, 11, 12, 9, 64]). *Every quasi-isometry between Fuchian buildings is at finite distance from an isomorphism.*

Here a **Fuchsian building** X is a special kind of 2-dimensional polyhedral complex. It is a union of subcomplexes called **apartments**, each of which is isomorphic to the Coxeter complex associated with a fixed Coxeter group acting on $\mathbb{H}^2$. We refer the reader to the papers above for the precise definition. The proof of this rigidity result also uses the quasiconformal structure on the boundary, but in this case the boundary is a Loewner space homeomorphic to the Menger sponge, and much of the theory in section 4 is brought into play.

Another result in a spirit similar to Mostow rigidity is:

Theorem 5.4 ([57, 32, 59, 51, 22]). *Suppose G is a finitely generated group quasi-isometric to a rank 1 symmetric space X other than $\mathbb{H}^2$. Then G admits a discrete, cocompact, isometric action on X.*

The outline of the proof goes as follows. The group G acts by isometries on a Cayley graph $\mathrm{Cayley}(G, \Sigma)$, and a quasi-isometry $\mathrm{Cayley}(G, \Sigma) \to X$ allows one to "conjugate" this isometric action to a "quasi-action" by quasi-isometries $G \curvearrowright X$. Passing to the boundary, one obtains an action $G \curvearrowright \partial X$ by uniformly quasi-Möbius homeomorphisms, in particular uniformly quasiconformal homeomorphisms. By a lemma of Sullivan, this action $G \curvearrowright \partial X$ is actually conformal with respect to some bounded measurable Riemannian structure on $T^*\partial X$; recall that ∂X is a Loewner space and therefore has a Cheeger cotangent bundle $T^*\partial X$ whose fiberwise norm can be used to express the boundedness condition on the Riemannian structure. By using the dynamics of the action and a rescaling argument, one shows that modulo quasi-Möbius conjugation, this Riemannian metric can be taken to be standard. This means that the action is actually conformal in the usual sense, and is therefore induced by an isometric action $G \curvearrowright X$.

Now suppose Z and Z' are compact Q-Loewner metric spaces, where $Q > 1$. The differentiation theory for quasiconformal homeomorphisms enables one to make the following definition:

Definition 5.5. Suppose $\langle \cdot, \cdot \rangle$, $\langle \cdot, \cdot \rangle'$ are measurable Riemannian structures on T^*Z and T^*Z'. A homeomorphism $f : Z \to Z'$ is **conformal** with respect to these structures if it is quasiconformal and its derivative

$$Df(z) : (T^*_{f(z)}Z', \langle \cdot, \cdot \rangle') \to (T^*_z Z, \langle \cdot, \cdot \rangle) \tag{5.2}$$

is conformal for almost every $z \in Z$. The **conformal group of** $(Z, \langle \cdot, \cdot \rangle)$, denoted $\mathrm{Conf}(Z, \langle \cdot, \cdot \rangle)$, is the group of conformal homeomorphisms

$$(Z, \langle \cdot, \cdot \rangle) \to (Z, \langle \cdot, \cdot \rangle). \tag{5.3}$$

By the lemma of Sullivan quoted above, any countable group of uniformly quasiconformal homeomorphisms of Z is conformal with respect to some bounded measurable Riemannian structure on T^*Z. In particular, if G is a Gromov hyperbolic group whose boundary is quasi-Möbius homeomorphic to a Q-Loewner space for $Q > 1$, then G may be viewed as a group of conformal homeomorphisms in this sense. For such a group, the full conformal group $\mathrm{Conf}((Z, \langle \cdot, \cdot \rangle)$ provides a natural substitute for the ambient Lie group that one has in the case of lattices in rank 1 Lie groups. The following result shows that in this case the homomorphism $G \to \mathrm{Conf}((Z, \langle \cdot, \cdot \rangle)$ is canonically attached to G:

Theorem 5.6 (Mostow rigidity for Loewner groups [43]). *Suppose G is a Gromov hyperbolic group, and*

$$G \overset{\rho}{\curvearrowright} (Z, \langle \cdot, \cdot \rangle), \quad G \overset{\rho'}{\curvearrowright} (Z', \langle \cdot, \cdot \rangle') \tag{5.4}$$

are conformal actions of G on Loewner spaces which are topologically conjugate to the action of G on its boundary ∂G. Then ρ is conformally equivalent to ρ'.

The proof is identical to that of Theorem 5.4 until the last step, which requires one has to exploit delicate infinitesimal structure of the Loewner space.

6. Uniformization

The uniformization problem for spheres. We recall the following extension of the Koebe uniformization theorem to measurable conformal structures:

Theorem 6.1 (The measurable Riemann mapping theorem). *If g is a bounded measurable Riemannian metric on the 2-sphere, then g is conformally equivalent to the standard metric g_0, i.e. there is a quasiconformal homeomorphism $f : S^2 \to S^2$ such that the derivative*

$$Df(x) : (T_xS^2, g) \to (T_xS^2, g_0) \tag{6.1}$$

is conformal almost everywhere. Moreover the uniformizing homeomorphism is unique up to post-composition with Möbius transformation.

A Riemannian metric g is **bounded** if there is a C such that

$$\frac{1}{C}g_0(v,v) \leq g(v,v) \leq Cg_0(v,v) \quad \text{for all} \quad v \in TS^2. \tag{6.2}$$

Theorem 6.1 and a version for parametrized families of Riemannian metrics are fundamental tools in Kleinian groups and complex dynamics.

It is very tempting to extend Theorem 6.1 to a more general setting. An approach based on a type of coverings ("shinglings") was introduced by Cannon [17], and further developed in [19, 20, 21, 18]. In a metric space setting, one is naturally led to the following quasi-Möbius uniformization problem:

Question 6.2. *When is a metric n-sphere quasi-Möbius homeomorphic to the standard n-sphere $\mathbb{S}^n$?*

Here a **metric n-sphere** means a metric space homeomorphic to the n-sphere.

One arrives at the $n = 2$ case of this question starting with Question C from section 2, in the $\mathbb{H}^3$ case, since a geodesic space quasi-isometric to $\mathbb{H}^3$ is Gromov hyperbolic and has boundary quasi-Möbius homeomorphic to S^2. The question is also tied to one approach to:

Conjecture 6.3 (Thurston's Hyperbolization Conjecture). *Every closed, aspherical, irreducible, atoroidal 3-manifold M admits a Riemannian metric of constant curvature -1.*

The relation with Question 6.2 is as follows. Gabai-Meyerhoff-Thurston [29] reduced Conjecture 6.3 to showing that $\pi_1(M)$ is isomorphic to the fundamental group of a hyperbolic manifold (a closed Riemannian manifold of constant curvature -1). When $\pi_1(M)$ is Gromov hyperbolic, [4] implies that the boundary of $\pi_1(M)$ is homeomorphic to the 2-sphere. Therefore the Gromov hyperbolic case of Conjecture 6.3 is implied by:

Conjecture 6.4 (Cannon). *If G is a Gromov hyperbolic group and ∂G is homeomorphic to the 2-sphere S^2, then G admits a discrete, cocompact, isometric action on hyperbolic 3-space $\mathbb{H}^3$.*

By Theorem 5.4, it follows readily that Cannon's conjecture is equivalent to the following case of Question 6.2:

Conjecture 6.5. *If G is a Gromov hyperbolic group and ∂G is homeomorphic to S^2, then it is quasi-Möbius homeomorphic to the standard 2-sphere.*

Although Thurston's conjecture appears to have been solved by Perelman, Conjecture 6.4 remains very interesting – it is logically independent of the Hyperbolization Conjecture, and moreover it provides an approach to an old unsolved problem due to Wall: Is every 3-dimensional Poincaré duality group is a 3-manifold group?

Necessary conditions for uniformization. The uniformization problem above was discussed in [60], where two necessary conditions were identified. A metric n-sphere which is quasi-Möbius homeomorphic to the standard n-sphere must be doubling and linearly locally contractible. Recall that a metric space is **doubling** if there is a constant N such that every ball can be covered by at most N balls of half the radius. We note that if the boundary of a hyperbolic group is homeomorphic to a sphere, then it satisfies both of these conditions since Ahlfors regular spaces are always doubling, see section 3. When $n = 1$ these two necessary conditions are sufficient:

Theorem 6.6 ([60]). *A doubling, linearly locally contractible metric circle is quasi-Möbius homeomorphic to the standard circle.*

However, when $n \geq 2$, the conditions are not sufficient. One can show that $\mathbb{R}^2$ with the homogeneous distance function

$$d((x_1, y_1), (x_2, y_2)) := |x_1 - x_2| + |y_1 - y_2|^{\frac{1}{2}} \tag{6.3}$$

is not locally quasi-Möbius homeomorphic to $\mathbb{R}^2$, and it is possible to construct a doubling linearly locally contractible metric on S^2 which is locally isometric to the metric (6.3) near some point.

Sufficient conditions for uniformization. Motivated by considerations relating analytic properties of a space and the existence of good parametrizations, Semmes made the following conjecture:

Conjecture 6.7 ([40]). *If Z is an Ahlfors 2-regular, linearly locally contractible 2-sphere, then Z is quasi-Möbius homeomorphic to the standard 2-sphere.*

This conjecture was proven in [5]. Recall that the Hausdorff dimension of any metric space is always greater than or equal to its topological dimension. For linearly locally contractible 2-spheres, one can strengthen this to the quantitative assertion that every r-ball has 2-dimensional Hausdorff measure at least comparable to r^2, for $r \leq \operatorname{diam}(Z)$. Thus the Ahlfors 2-regularity condition in the hypothesis of

Conjecture 6.7 provides a competing upper bound on the Hausdorff measure, and in some sense this tension is the key to the proof. We remark that this result is optimal in several respects. First, Semmes had shown in [54] that the analogous assertion is false in higher dimensions: for every $n \geq 3$ there are Ahlfors n-regular, linearly locally contractible metric n-spheres which are not quasi-Möbius homeomorphic to the standard n-sphere. Also, the conclusion cannot be strengthed to bi-Lipschitz homeomorphism, due to examples of Laakso [46]. Finally, the Ahlfors 2-regularity condition cannot be relaxed to Q-regularity because by using metrics as in (6.3) one can get examples which are Ahlfor 3-regular.

Nonetheless, one can relax the 2-regularity condition if one imposes a Loewner condition:

Theorem 6.8 ([5]). *If Z is a Q-Loewner metric 2-sphere, then $Q = 2$ and Z is quasi-Möbius homeomorphic to the standard 2-sphere.*

In particular, as indicated above, Cannon's conjecture is true for those hyperbolic groups whose boundary is quasi-Möbius homeomorphic to a Loewner 2-sphere. The higher dimensional analog of Theorem 6.8 is false, because the Carnot metric on S^3 is 4-Loewner but not quasi-Möbius homeomorphic to S^3.

Quasi-Möbius characterizations of the 2-sphere. In [5], the proofs of Theorem 6.8 and Conjecture 6.7 invoked a more general necessary and sufficient condition. To formulate this, we require a combinatorial version of modulus, and the related definitions.

Suppose $\mathcal{G}$ is a graph with vertex set $V = V(\mathcal{G})$, and Γ is a collection of subsets of V; in our context, the elements $\gamma \in \Gamma$ will be the vertex sets of certain connected subgraphs of Γ. A function $\rho : V(\mathcal{G}) \to [0, \infty)$ is **Γ-admissible** if for every $\gamma \in \Gamma$,

$$\sum_{v \in \gamma} \rho(v) \geq 1. \tag{6.4}$$

If $Q \geq 1$, the **Q-modulus of** Γ is the infimum of

$$\sum_{v \in \mathrm{Vertex}(\mathcal{G})} \rho^Q(v) \tag{6.5}$$

where ρ ranges over all Γ-admissible functions.

Pick $Q \geq 1$. Now suppose Γ is a path family in a metric space Z, and $\mathcal{U}$ is an open cover of Z. Then the **Q-modulus of Γ with respect to $\mathcal{U}$,** denoted $\mathrm{Mod_Q}(\Gamma, \mathcal{U})$, is defined as follows. We let $\mathcal{G}(\mathcal{U})$ be the nerve of the open cover $\mathcal{U}$, which is the graph with vertex set $\mathcal{U}$ whose edges correspond to pairs $U, U' \in \mathcal{U}$ with nonempty intersection. Then we let $\Gamma(\mathcal{G})$ be the collection of subsets of the vertex set $V(\mathcal{G})$ of the form

$$\{U \in \mathcal{U} \mid U \cap \mathrm{Im}\,\gamma \neq \emptyset\},$$

where γ ranges over all paths $\gamma \in \Gamma$. Note that each element of $\Gamma(\mathcal{G})$ is the 0-skeleton of a connected subgraph if $\mathcal{G}$. We then define $\mathrm{Mod_Q}(\Gamma, \mathcal{U})$ to be the Q-modulus of $\Gamma(\mathcal{G})$ in $\mathcal{G}$. Finally, if $E, F \subset Z$ are subsets, we let $\mathrm{Mod_Q}(E, F; \mathcal{U}) := \mathrm{Mod_Q}(\Gamma(E, F), \mathcal{U})$, where $\Gamma(E, F)$ denotes the family of paths joining E to F.

Theorem 6.9 ([5]). *Let Z be a doubling, linearly locally contractible metric 2-sphere, and let $\{r_i\}$ be a sequence of positive numbers converging to 0. For each i, let V_i be a maximal r_i-separated subset of Z, and let*

$$\mathcal{U}_i := \{B(v, r_i)\}_{v \in V_i} \tag{6.6}$$

be the corresponding open ball cover. Then the following conditions are equivalent.

- *Z is quasi-Möbius homeomorphic to $\mathbb{S}^2$.*
- *There is a function $\psi : [0, \infty) \to [0, \infty]$ tending to zero at infinity, and a number L, such that if $E, F \subset Z$ are closed subsets, then*

$$\operatorname{Mod}_2(E, F; \mathcal{U}_i) \leq \psi(\Delta(E, F)) \tag{6.7}$$

for every i satisfying

$$\min(\operatorname{diam}(E), \operatorname{diam}(F)) \geq L\, r_i.$$

Here $\Delta(E, F)$ denotes the relative distance as before.

- *There is a positive decreasing function $\phi : [0, \infty) \to (0, \infty)$ and a number M such that if $E, F \subset Z$ are continua, then*

$$\operatorname{Mod}_2(E, F; \mathcal{U}_i) \geq \phi(\Delta(E, F)), \tag{6.8}$$

for every i satisfying

$$\operatorname{dist}(E, F) \geq M\, r_i.$$

The theorem says that Z is quasi-Möbius equivalent to the standard 2-sphere if and only if, for a sequence of combinatorial approximations, the combinatorial 2-modulus behaves as in $\mathbb{S}^2$, i.e. it can be bounded below or above by functions of relative distance. The idea of the proof is to associate, for each i, a topological triangulation $\mathcal{T}_i$ of Z whose 1-skeleton is quasi-isometric to the nerve of $\mathcal{U}_i$ (with quasi-isometry constants independent of i). Then one can apply classical uniformization to the equilateral polyhedron associated with $\mathcal{T}_i$ to produce a map $f_i : V_i \to \mathbb{S}^2$. The crux of the argument is to show that the maps f_i, when appropriately normalized, are uniformly quasi-Möbius, and hence subconverge to a quasi-Möbius homeomorphism by the Arzela-Ascoli theorem.

Other uniformization problems. One may formulate uniformization problems for spaces other than spheres. The case of Sierpinski carpets is especially interesting, where there are remarkable uniformization and rigidity results. We refer the reader to Mario Bonk's article in these Proceedings for a treatment of this topic.

7. Geometrization

Minimizing Hausdorff dimension. Geometrization – the problem of finding optimal or canonical geometric structures – appears in many contexts in mathematics: for example, the Yamabe problem (finding conformally equivalent metrics of constant scalar curvature), Thurston's Geometrization conjecture for 3-manifolds, Thurston's characterization of rational maps, and the Calabi conjecture. For each of these problems there are cases where there is no solution: the Yamabe problem for the "teardrop" 2-orbifold has no solution, and any closed 3-manifold whose prime or torus decomposition is nontrivial does not admit a Thurston geometry. The goal is to show that geometrization is always possible unless some alternate structure appears, on which the failure can be blamed (such as a bad isometry group in the case of the teardrop, or an essential decomposition in the 3-manifold geometrization problem).

In the metric space context, a natural geometrization problem is to minimize the Hausdorff dimension in an attempt to optimize shape. This leads to the following notion, which is a minor variant of a definition of Pansu:

Definition 7.1. The **conformal dimension of a metric space** Z is the infimal Hausdorff dimension of the Ahlfors regular metric spaces quasi-Möbius homeomorphic to it. We denote this by $\mathrm{Confdim}(Z)$.

Visual metrics on boundaries of hyperbolic groups are Ahlfors regular, so

$$\mathrm{Confdim}(\partial G) < \infty$$

for every hyperbolic group G. Likewise

$$\mathrm{Confdim}(Z) < \infty$$

for every self-similar space Z. Every Q-Loewner metric space (Definition 4.5), and more generally Q-regular metric spaces with nontrivial Q-modulus, are solutions to the geometrization problem, because they minimize Hausdorff dimension in their quasi-Möbius homeomorphism class:

Theorem 7.2. *[Bonk-Tyson, see [37, Thm. 15.10]] Suppose Z is a compact Ahlfors Q-regular metric space which carries a family of nonconstant paths of positive Q-modulus. Then any metric space quasi-Möbius homeomorphic to Z has positive Q-dimensional Hausdorff measure, and in particular $Q = \mathrm{Confdim}(Z)$.*

The following theorem of Keith-Laakso shows that the converse is nearly true:

Theorem 7.3 ([42]). *If Z is an Ahlfors Q-regular metric space where*

$$Q = \mathrm{Confdim}(Z) > 1,$$

then some weak tangent of Z carries a nontrivial curve family of positive Q-modulus.

Here a **weak tangent** is a pointed Gromov-Hausdorff limit of a sequence of rescalings of Z, see [42, 6]. If one adds the hypothesis that Z is self-similar, or approximately self-similar like a visual metric on the boundary of a hyperbolic group, then one can map open subsets of a weak tangent to Z itself, and thereby conclude that Z itself has a nontrivial curve family of positive Q-modulus. Therefore Theorem

7.2 has a converse in the self-similar case. The idea of the proof of Theorem 7.3 is to show that if the conclusion of the theorem fails for some Q-regular space Z, then one can use a construction of Semmes (a "Semmes deformation") to produce an Ahlfors regular metric of strictly smaller Hausdorff dimension.

For spaces quasi-Möbius homeomorphic to boundaries of groups, a much stronger statement holds:

Theorem 7.4 ([6]). *If Z is an Ahlfors Q-regular metric space where*

$$Q = \mathrm{Confdim}(Z) > 1,$$

and Z is quasi-Möbius homeomorphic to the boundary of some hyperbolic group G, then Z is Q-Loewner.

The proof of this theorem uses Theorem 7.3, work of Tyson [61], and a dynamical argument to show that for a large supply of ball pairs $B \subset B'$ the pair $(B, \overline{Z \setminus B'})$ has Q-modulus bounded away from zero. The main work consists in showing that this "ball-Loewner" condition implies the usual Loewner property in Definition 4.5.

Theorem 7.4 connects the problem of realizing the conformal dimension with results such as Theorems 5.6 and 6.8, since one would like to know for which hyperbolic groups the boundary is quasi-Möbius homeomorphic to a Loewner space.

Examples where the conformal dimension is (not) realized. Suppose G is an infinite hyperbolic group and $Q = \mathrm{Confdim}(\partial G)$ can be realized by an Ahlfors Q-regular metric. If $Q < 1$ then one can argue that G must be virtually infinite cyclic. If $Q = 1$, then it is not difficult to deduce that ∂G is homeomorphic to a circle, and hence by [23, 28] it is virtually a surface group. If $Q > 1$, then Theorem 7.4 implies that ∂G is quasi-Möbius homeomorphic to a Loewner space, which implies that it is connected and has no local cut points. These two conditions are equivalent by [4, 15, 58, 14] to saying that G does not virtually split over a virtually cyclic group.

Any hyperbolic group G which splits over a finite group has disconnected boundary, so unless G is virtually cyclic, the previous paragraph implies that the conformal dimension of ∂G cannot be realized. A free group of rank at least two is such an example.

Pansu showed that if one takes two copies of a surface of genus 2 and glues them along a homotopically nontrivial simple closed curve, the fundamental group of the resulting 2-complex is a hyperbolic group G where $\mathrm{Confdim}(\partial G) = 1$. Since G is not a virtual surface group, this group provides another example where the conformal dimension cannot be realized. In the case of self-similar spaces (not arising as boundaries of group), it was shown by Laakso that the Sierpinski gasket has conformal dimension 1, but it cannot be realized.

Much deeper examples were constructed by Bourdon and Pajot [13, 10]. These are hyperbolic groups which do not virtually split over virtually cyclic groups, and whose boundaries are not quasi-Möbius homeomorphic to Loewner spaces. These examples are very intriguing, and have led Marc Bourdon to speculate that the nonexistence of Loewner structure implies with the presence G-invariant fibration-like structure in ∂G.

A further necessary condition. Suppose Z is a compact doubling metric space, and $Q > 1$. Then Z satisfies the **combinatorial Q-Loewner property** if the following combinatorial analog of (4.6) and (4.8) holds. There are constants C, $\lambda > 0$, and functions $\phi : [0, \infty) \to (0, \infty)$, $\psi : [0, \infty) \to [0, \infty]$ with the following properties:

- ϕ is a positive decreasing function, $\psi(t) \to 0$ as $t \to \infty$, and $\phi \le \psi$.
- For every $0 < r \le \operatorname{diam}(Z)$, if V is a maximal r-separated net in Z, $\mathcal{U}$ is the corresponding r-ball cover, and $E, F \subset Z$ are disjoint nondegenerate continua where $r \le \lambda \min(\operatorname{diam}(E), \operatorname{diam}(F))$, then

$$\phi(\Delta(E, F)) \le \mathrm{Mod_Q}(E, F, \mathcal{U}) \le \psi(\Delta(E, F)), \tag{7.1}$$

where $\mathrm{Mod_Q}(E, F, \mathcal{U})$ is the combinatorial modulus defined just before Theorem 6.9.

Using arguments from [38, 5] is not hard to see that if Z is a Q-Loewner space, then Z satisfies the combinatorial Q-Loewner property. Based on current evidence, the following seems plausible:

Conjecture 7.5. *Suppose Z is a self-similar space, or quasi-Möbius homeomorphic to the boundary of a hyperbolic group. If Z satisfies the combinatorial Loewner property, then Z is quasi-Möbius homeomorphic to a Loewner space.*

Currently there is no example of a compact doubling space satisfying the combinatorial Loewner property, which is known not to be quasi-Möbius homeomorphic to a Loewner space.

The conjecture is intriguing, because the author has shown that several examples, including the standard square Sierpinski carpet, the standard Menger sponge (obtained from the unit cube in $\mathbb{R}^3$), and boundaries of certain hyperbolic Coxeter groups satisfy the combinatorial Loewner property. Therefore the conjecture would provide new examples of Loewner spaces.

8. Open problems

We conclude with some open problems. These are questions which seem to be key to making further progress with the central themes of this article.

Question 8.1. *Let Z be the boundary of a hyperbolic group, or more generally an "approximately self-similar" space. When is the conformal dimension of Z realized?*

Question 8.2. *Suppose G is a Gromov hyperbolic group. Suppose G does not virtually split over a virtually cyclic group, or equivalently, that ∂G is connected and has no local cut points. Is every quasiconformal homeomorphism of ∂G quasi-Möbius?*

Question 8.3. *Is the standard square Sierpinski carpet quasi-Möbius homeomorphic to a Loewner space? What is its conformal dimension?*

The author and independently [42] have shown that usual metric does not realize the conformal dimension. The author has shown that for a particular choice of exponent Q, the square carpet satisfies the combinatorial Loewner property, see section 7.

Question 8.4. *If* G *is a random hyperbolic group, is the homomorphism* $G \to \mathrm{QI}(G)$ *an isomorphism?*

See [24, 41] for discussion of random groups.

Question 8.5. *What is the quasi-isometry group of the Gromov-Thurston examples [36]?*

References

[1] H. Bass. The degree of polynomial growth of finitely generated nilpotent groups. *Proc. London Math. Soc. (3)*, 25:603–614, 1972.

[2] G. Besson, G. Courtois, and S. Gallot. Volumes, entropies et rigidités des espaces localement symétriques de courbure strictement négative. *C. R. Acad. Sci. Paris Sér. I Math.*, 319(1):81–84, 1994.

[3] M. Bestvina. Local homology properties of boundaries of groups. *Michigan Math. J.*, 43(1):123–139, 1996.

[4] M. Bestvina and G. Mess. The boundary of negatively curved groups. *J. Amer. Math. Soc.*, 4(3):469–481, 1991.

[5] M. Bonk and B. Kleiner. Quasisymmetric parametrizations of two-dimensional metric spheres. *Invent. Math.*, 150(1):127–183, 2002.

[6] M. Bonk and B. Kleiner. Conformal dimension and Gromov hyperbolic groups with 2-sphere boundary. *Geom. Topol.*, 9:219–246, 2005.

[7] M. Bonk and B. Kleiner. Quasi-hyperbolic planes in hyperbolic groups. *Proc. Amer. Math. Soc.*, 133(9):2491–2494, 2005.

[8] M. Bourdon. Immeubles hyperboliques, dimension conforme et rigidité de Mostow. *Geom. Funct. Anal.*, 7(2):245–268, 1997.

[9] M. Bourdon. Sur les immeubles fuchsiens et leur type de quasi-isométrie. *Ergodic Theory Dynam. Systems*, 20(2):343–364, 2000.

[10] M. Bourdon. Cohomologie l_p et produits amalgamés. *Geom. Dedicata*, 107:85–98, 2004.

[11] M. Bourdon and H. Pajot. Poincaré inequalities and quasiconformal structure on the boundary of some hyperbolic buildings. *Proc. Amer. Math. Soc.*, 127(8):2315–2324, 1999.

[12] M. Bourdon and H. Pajot. Rigidity of quasi-isometries for some hyperbolic buildings. *Comment. Math. Helv.*, 75(4):701–736, 2000.

[13] M. Bourdon and H. Pajot. Cohomologie l_p et espaces de Besov. *J. Reine Angew. Math.*, 558:85–108, 2003.

[14] B. H. Bowditch. Cut points and canonical splittings of hyperbolic groups. *Acta Math.*, 180(2):145–186, 1998.

[15] B. H. Bowditch. Connectedness properties of limit sets. *Trans. Amer. Math. Soc.*, 351(9):3673–3686, 1999.

[16] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1999.

[17] J. W. Cannon. The combinatorial Riemann mapping theorem. *Acta Math.*, 173(2):155–234, 1994.

[18] J. W. Cannon, W. J. Floyd, R. Kenyon, and W. R. Parry. Constructing rational maps from subdivision rules. *Conform. Geom. Dyn.*, 7:76–102, 2003.

[19] J. W. Cannon, W. J. Floyd, and W. R. Parry. Conformal modulus: the graph paper invariant or the conformal shape of an algorithm. In *Geometric group theory down under (Canberra, 1996)*, pages 71–102. de Gruyter, Berlin, 1999.

[20] J. W. Cannon, W. J. Floyd, and W. R. Parry. Sufficiently rich families of planar rings. *Ann. Acad. Sci. Fenn. Math.*, 24(2):265–304, 1999.

[21] J. W. Cannon, W. J. Floyd, and W. R. Parry. Finite subdivision rules. *Conform. Geom. Dyn.*, 5:153–196, 2001.

[22] J. W. Cannon and E. L. Swenson. Recognizing constant curvature discrete groups in dimension 3. *Trans. Amer. Math. Soc.*, 350(2):809–849, 1998.

[23] A. Casson and D. Jungreis. Convergence groups and Seifert fibered 3-manifolds. *Invent. Math.*, 118(3):441–456, 1994.

[24] C. Champetier. Propriétés statistiques des groupes de présentation finie. *Adv. Math.*, 116(2):197–262, 1995.

[25] J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.*, 9(3):428–517, 1999.

[26] M. Coornaert. Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov. *Pacific J. Math.*, 159:241–270, 1993.

[27] K. Corlette. Archimedean superrigidity and hyperbolic geometry. *Ann. of Math. (2)*, 135(1):165–182, 1992.

[28] D. Gabai. Convergence groups are Fuchsian groups. *Ann. of Math. (2)*, 136(3):447–510, 1992.

[29] D. Gabai, G. R. Meyerhoff, and N. Thurston. Homotopy hyperbolic 3-manifolds are hyperbolic. *Ann. of Math. (2)*, 157(2):335–431, 2003.

[30] É. Ghys and P. de la Harpe, editors. *Sur les groupes hyperboliques d'après Mikhael Gromov*, volume 83 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1990.

[31] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263.

[32] M. Gromov. Hyperbolic manifolds, groups and actions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 183–213.

[33] M. Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981.

[34] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[35] M. Gromov and R. Schoen. Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. *Inst. Hautes Études Sci. Publ. Math.*, (76):165–246, 1992.

[36] M. Gromov and W. Thurston. Pinching constants for hyperbolic manifolds. *Invent. Math.*, 89(1):1–12, 1987.

[37] J. Heinonen. *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001.

[38] J. Heinonen and P. Koskela. Quasiconformal maps in metric spaces with controlled geometry. *Acta Math.*, 181(1):1–61, 1998.

[39] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. T. Tyson. Sobolev classes of Banach space-valued functions and quasiconformal mappings. *J. Anal. Math.*, 85:87–139, 2001.

[40] J. Heinonen and S. Semmes. Thirty-three yes or no questions about mappings, measures, and metrics. *Conform. Geom. Dyn.*, 1:1–12, 1997.

[41] I. Kapovich and N. Benakli. Boundaries of hyperbolic groups. In *Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001)*, volume 296 of *Contemp. Math.*, pages 39–93.

[42] S. Keith and T. Laakso. Conformal Assouad dimension and modules. *Geom. Funct. Anal.*, 14(6):1278–1321, 2004.

[43] B. Kleiner. in preparation, 2005.

[44] B. Kleiner and B. Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Inst. Hautes Études Sci. Publ. Math.*, (86):115–197 (1998), 1997.

[45] A. Korányi and H. M. Reimann. Quasiconformal mappings on the Heisenberg group. *Invent. Math.*, 80(2):309–338, 1985.

[46] T. J. Laakso. Plane with A_∞-weighted metric not bi-Lipschitz embeddable to $\mathbb{R}^N$. *Bull. London Math. Soc.*, 34(6):667–676, 2002.

[47] G. A. Margulis. Arithmeticity of the irreducible lattices in the semisimple groups of rank greater than 1. *Invent. Math.*, 76(1):93–120, 1984.

[48] G. A. Margulis and G. D. Mostow. The differential of a quasi-conformal mapping of a Carnot-Carathéodory space. *Geom. Funct. Anal.*, 5(2):402–433, 1995.

[49] G. D. Mostow. *Strong rigidity of locally symmetric spaces.* Princeton University Press, Princeton, N.J., 1973. Annals of Mathematics Studies, No. 78.

[50] P. Pansu. Croissance des boules et des géodésiques fermées dans les nilvariétés. *Ergodic Theory Dynam. Systems*, 3(3):415–445, 1983.

[51] P. Pansu. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)*, 129(1):1–60, 1989.

[52] F. Paulin. Un groupe hyperbolique est déterminé par son bord. *J. London Math. Soc. (2)*, 54(1):50–74, 1996.

[53] G. Prasad. Strong rigidity of $\mathbf{Q}$-rank 1 lattices. *Invent. Math.*, 21:255–286, 1973.

[54] S. Semmes. Good metric spaces without good parameterizations. *Rev. Mat. Iberoamericana*, 12(1):187–275, 1996.

[55] Y. Shalom. Harmonic analysis, cohomology, and the large-scale geometry of amenable groups. *Acta Math.*, 192(2):119–185, 2004.

[56] J. R. Stallings. On torsion-free groups with infinitely many ends. *Ann. of Math. (2)*, 88:312–334, 1968.

[57] D. Sullivan. Discrete conformal groups and measurable dynamics. *Bull. Amer. Math. Soc. (N.S.)*, 6(1):57–73, 1982.

[58] G. A. Swarup. On the cut point conjecture. *Electron. Res. Announc. Amer. Math. Soc.*, 2(2):98–100, 1996.

[59] P. Tukia. Homeomorphic conjugates of Fuchsian groups. *J. Reine Angew. Math.*, 391:1–54, 1988.

[60] P. Tukia and J. Väisälä. Quasisymmetric embeddings of metric spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 5(1):97–114, 1980.

[61] J. Tyson. Quasiconformality and quasisymmetry in metric measure spaces. *Ann. Acad. Sci. Fenn. Math.*, 23(2):525–548, 1998.

[62] J. Väisälä. *Lectures on n-dimensional quasiconformal mappings*. Springer-Verlag, Berlin, 1971. Lecture Notes in Mathematics, Vol. 229.

[63] J. Väisälä. Quasi-Möbius maps. *J. Analyse Math.*, 44:218–234, 1984/85.

[64] X. Xie. Mostow rigidity for fuchsian buildings. to appear in *Topology*.

Mathematics Department
Yale University
E-mail: bruce.kleiner@yale.edu