INDUCED QUASI-ACTIONS: A REMARK

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1. Introduction

In this note we observe that the notion of an induced representation has an analog for quasi-actions, and give some applications.

We will use the definitions and notation from [KL01].

1.1. **Induced quasi-actions and their properties.** Let G be a group and $\{X_i\}_{i\in I}$ be a finite collection of unbounded metric spaces.

Definition 1.1. A quasi-action $G \overset{\rho}{\curvearrowright} \prod_i X_i$ *preserves the product structure* if each $g \in G$ acts by a product of quasi-isometries, up to uniformly bounded error. Note that we allow the quasi-isometries $\rho(g)$ to permute the factors, i.e. $\rho(g)$ is uniformly close to a map of the form $(x_i) \mapsto \big(\phi_{\sigma^{-1}(i)}(x_{\sigma^{-1}(i)})\big)$ with a permutation σ of I and quasi-isometries $\phi_i : X_i \mapsto X_{\sigma(i)}$.

Associated to every quasi-action $G \overset{\rho}{\curvearrowright} \prod_i X_i$ preserving product structure is the action $G \overset{\rho_I}{\curvearrowright} I$ corresponding to the induced permutation of the factors; this is well-defined because the X_i's are unbounded metric spaces. For each $i \in I$, the stabilizer G_i of i with respect to ρ_I has a quasi-action $G_i \curvearrowright X_i$ by restriction of ρ. It is well-defined up to equivalence in the sense of [KL01, Definition 2.3].

If the permutation action ρ_I is *transitive*, all factors X_i are quasi-isometric to each other, and the restricted quasi-actions $G_i \curvearrowright X_i$ are quasi-conjugate (when identifying different stabilizers G_i by inner automorphisms of G). The main result of this note is that in this case any of the quasi-actions $G_i \curvearrowright X_i$ determines ρ up to quasi-conjugacy, and moreover any quasi-conjugacy class may arise as a restricted action.

Theorem 1.2. *Let G be a group, H be a finite index subgroup, and $H \overset{\alpha}{\curvearrowright} X$ be a quasi-action of H on an unbounded metric space X.*

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Then there exists a quasi-action preserving product structure, where $G \overset{\beta}{\curvearrowright} \prod_{i \in G/H} X_i$*, where*

(1) *Each factor* X_i *is quasi-isometric to* X.
(2) *The associated action* $G \overset{\beta_{G/H}}{\curvearrowright} G/H$ *is the natural action by left multiplication.*
(3) *The restriction of* β *to a quasi-action of* H *on* X_H *is quasi-conjugate to* $H \overset{\alpha}{\curvearrowright} X$.

Furthermore, there is a unique such quasi-action β *preserving the product structure, up to quasi-conjugacy by a product quasi-isometry. Finally, if* α *is an isometric action, then the* X_i *may be taken isometric to* X *and* β *may be taken to be an isometric action.*

Definition 1.3. Let G, H and $H \curvearrowright X$ be as in Theorem 1.2. The quasi-action β is called the *quasi-action induced by* $H \curvearrowright X$.

As a byproduct of the main construction, we get the following:

Corollary 1.4. *If* $G \overset{\rho}{\curvearrowright} X$ *is an* (L, A)*-quasi-action on an arbitrary metric space* X*, then* ρ *is* $(L, 3A)$*-quasi-conjugate to a canonically defined isometric action* $G \curvearrowright X'$.

1.2. **Applications.** The implication of Theorem 1.2 is that in order to quasi-conjugate a quasi-action on a product to an isometric action, it suffices to quasi-conjugate the factor quasi-actions to isometric actions. We begin with a special case:

Theorem 1.5. *Let* $G \overset{\rho}{\curvearrowright} X$ *be a cobounded quasi-action on* $X = \prod_i X_i$*, where each* X_i *is either an irreducible symmetric space of noncompact type, or a thick irreducible Euclidean building of rank at least two, with cocompact Weyl group. Then* ρ *is quasi-conjugate to an isometric action on* X*, after suitable rescaling of the metrics on the factors* X_i.

Remarks

- Theorem 1.5 was stated incorrectly as Corollary 4.5 in [KL01]. The proof given there was was only valid for quasi-actions which do not permute the factors.
- Rescaling of the factors is necessary, in general: if one takes the product of two copies of $\mathbb{H}^2$ where the factors are scaled to have different curvature, then a quasi-action which permutes the factors will not be quasi-conjugate to an isometric action. .

We now consider a more general situation. Let $G \overset{\alpha}{\curvearrowright} \prod_{i\in I} X_i$ be a quasi-action, where each X_i is one of the following four types of spaces:

(1) An irreducible symmetric space of noncompact type.
(2) A thick irreducible Euclidean building of rank/dimension ≥ 2, with cocompact Weyl group.
(3) A bounded valence bushy tree in the sense of [MSW03]. We recall that a tree is *bushy* if each of its points lies within uniformly bounded distance from a vertex having at least three unbounded complementary components.
(4) A quasi-isometrically rigid Gromov hyperbolic space which is of coarse type I in the sense of [KKL98, sec. 3] (see the remarks below). A space is *quasi-isometrically rigid* if every (L, A)-quasi-isometry is at distance at most $D = D(L, A)$ from a unique isometry.

By [KKL98, Theorem B], the quasi-action preserves product structure, and hence we have an induced permutation action $G \curvearrowright I$. Let $J \subset I$ be the set of indices $i \in I$ such that X_i is either a real hyperbolic space $\mathbb{H}^k$ for some $k \geq 4$, a complex hyperbolic space $\mathbb{CH}^l$ for some $l \geq 2$, or a bounded valence bushy tree. Generalizing Theorem 1.5 we obtain:

Theorem 1.6. *If the quasi-action $G_j \curvearrowright X_j$ is cobounded for each $j \in J$, then α is quasi-conjugate by a product quasi-isometry to an isometric action $G \overset{\alpha'}{\curvearrowright} \prod_{i\in I} X'_i$, where for every i, X'_i is quasi-isometric to X_i, and precisely one of the following holds:*

(1) *If X_i is not a bounded valence bushy tree, then X'_i is isometric to $X_{i'}$ for some i' in the G-orbit $G(i)$ of i.*
(2) *If X_i is a bounded valence bushy tree, then so is X'_i.*

As in the previous corollary, it is necessary to permit X'_i to be nonisometric to X_i. Moreover, there may be factors X_i and X_j of type (4) lying in the same G-orbit, but which are not even homothetic, so it is not sufficient to allow rescaling of factors.

Proof. We first assume that the action $G \curvearrowright I$ is transitive. Pick $n \in I$. Then the quasi-action $G_n \curvearrowright X_n$ is quasi-conjugate to an isometric action $G_n \curvearrowright X'_n$, where X'_n is isometric to X_n unless X_n is a bounded valence bushy tree, in which case X'_n is a bounded valence bushy tree but not necessarily isometric to X_n; this follows from:

• [Hin90, Gab92, CJ94, Mar06] when X_n is $\mathbb{H}^2$. Note that any quasi-action on $\mathbb{H}^2$ is quasi-conjugate to an isometric action.

• [Sul81, Gro, Tuk86, Pan89, Cho96] when X_n is a rank 1 symmetric space other than $\mathbb{H}^2$. Note that Sullivan's theorem implies that any quasi-action on $\mathbb{H}^3$ is quasi-conjugate to an isometric action. Also, the proof given in Chow's paper on the complex hyperbolic case covers arbitrary cobounded quasi-actions, even though it is only stated for discrete cobounded quasi-actions.

• [KL97, Lee00] when X_n is an irreducible symmetric space or Euclidean building of rank at least 2.

• [MSW03] when X_n is a bounded valence bushy tree.

By Theorem 1.2, the associated induced quasi-action of G is quasi-conjugate to the original quasi-action $G \curvearrowright \prod_{i\in I} X_i$ by a product quasi-isometry, and we are done.

In the general case, for each orbit $G(i) \subset I$ of the action $G \curvearrowright I$, we have a well-defined associated quasi-action $G \curvearrowright \prod_{j\in G(i)} X'_j$ for which the theorem has already been established, and we obtain the desired quasi-action $G \curvearrowright \prod_{i\in I} X'_i$ by taking products of these. □

Corollary 1.7. *Let $\{X_i\}_{i\in I}$ be as above, and suppose G is a finitely generated group quasi-isometric to the product $\prod_{i\in I} X_i$. Then G admits a discrete, cocompact, isometric action on a product $\prod_{i\in I} X'_i$, where for every i, X'_i is quasi-isometric to X_i, and precisely one the following holds:*

(1) *X_i is not a bounded valence bushy tree, and X'_i is isometric to $X_{i'}$ for some i' in the G-orbit $G(i) \subset I$ of i.*
(2) *Both X_i and X'_i are bounded valence bushy trees.*

Proof. Such a group G admits a discrete, cobounded quasi-action on $\prod_{i\in I} X_i$. Theorem 1.6 furnishes the desired isometric action $G \curvearrowright \prod_i X'_i$. □

Remarks.

- Corollary 1.7 refines earlier results [Ahl02, KL01, MSW03].
- A proper Gromov hyperbolic space with cocompact isometry group is of coarse type I unless it is quasi-isometric to $\mathbb{R}$ [KKL98, Sec. 3].
- The classification of the four different types of spaces above is quasi-isometry invariant, with one exception: a space of type (1) will also be a space of type (4) iff it is a quasi-isometrically

rigid rank 1 symmetric space (i.e. a quaternionic hyperbolic space or the Cayley hyperbolic plane [Pan89]). See Lemma 3.1.
- Two irreducible symmetric spaces are quasi-isometric iff they are isometric, up to rescaling [Mos73, Pan89, KL97]. Two Euclidean buildings as in (2) above are quasi-isometric iff they are isometric up to rescaling [KL97, Lee00].

2. The construction of induced quasi-actions

The construction of induced quasi-actions is a direct imitation of one of the standard constructions of induced representations. We now review this for the convenience of the reader.

Let H be a subgroup of some group G, and suppose $\alpha : H \curvearrowright V$ is a linear representation. Then we have an action $H \curvearrowright G \times V$ where $(h,(g,v)) = (gh^{-1}, hv)$. Let $E := (G \times V)/H$ be the quotient. There is a natural projection map $\pi : E \to G/H$ whose fibers are copies of V; this would be a vector bundle over the discrete space G/H if V were endowed with a topology. The action $G \curvearrowright G \times V$ by left translation on the first factor descends to E, and commutes with the projection map π. Moreover, it preserves the linear structure on the fibers. Hence there is a representation of G on the vector space of sections $\Gamma(E)$, and this is the representation of G induced by α.

We use the terminology of [KL01, Sec. 2]. (However, we replace *quasi-isometrically conjugate* by the shorter and more accurate term *quasi-conjugate*.)

We will work with generalized metrics taking values in $[0, +\infty]$. A *finite component* of a generalized metric space is an equivalence class of points with pairwise finite distances. Clearly, quasi-isometries respect finite components.

Let $\{X_i\}_{i \in I}$ be a finite collection of unbounded metric spaces in the usual sense, i.e. the metric on each X_i takes only finite values. On their product $\prod_{i\in I} X_i$ we consider the natural (L^2-)product metric. On their disjoint union $\sqcup_{i \in I} X_i$ we consider the generalized metric which induces the original metric on each component X_i and gives distance $+\infty$ to any pair of points in different components.

We observe that a quasi-isometry $\prod_{i\in I} X_i \to \prod_{i\in I} X'_i$ preserving the product structure gives rise to a quasi-isometry $\sqcup_{i\in I} X_i \to \sqcup_{i\in I} X'_i$, well-defined up to bounded error, and vice versa. Thus equivalence classes of quasi-actions $\alpha : G \curvearrowright \prod_{i\in I} X_i$ preserving the product structure correspond one-to-one to quasi-actions $\beta : G \curvearrowright \sqcup_{i\in I} X_i$. In what follows we will prove the disjoint union analog of Theorem 1.2. (The index of H can be arbitrary from now on.)

Lemma 2.1. *Suppose that Y is a generalized metric space and that $G \curvearrowright Y$ is a quasi-action such that G acts transitively on the set of finite components of Y. Let Y_0 be one of the finite components and H its stabilizer in G. Then the restricted action $H \curvearrowright Y_0$ determines the action $G \curvearrowright Y$ up to quasi-conjugacy.*

Proof. If $G \curvearrowright Y'$ is another quasi-action, Y_0' is a finite component with stabilizer H, then any quasi-conjugacy between $H \curvearrowright Y_0$ and $H \curvearrowright Y_0'$ extends in a straightforward way to a quasi-conjugacy between $G \curvearrowright Y$ and $G \curvearrowright Y'$. $\square$

We will now show how to recover the G-quasi-action from the H-quasi-action by quasifying the construction of induced actions as described above.

Definition 2.2. An (L, A)-*coarse fibration* $(Y, \mathcal{F})$ consists of a (generalized) metric space Y and a family $\mathcal{F}$ of subsets $F \subset Y$, the *coarse fibers*, with the following properties:

(1) The union $\cup_{F\in\mathcal{F}} F$ of all fibers has Hausdorff distance $\leq A$ from Y.
(2) For any two fibers $F_1, F_2 \in \mathcal{F}$ holds
$$d_H(F_1, F_2) \leq L \cdot d(y_1, F_2) + A \qquad \forall\, y_1 \in F_1.$$

We also say that $\mathcal{F}$ is a coarse fibration of Y.

Note that the coarse fibers are not required to be disjoint.

It follows from part (2) of the definition that $d_H(F_1, F_2) < +\infty$ if and only if F_1 and F_2 meet the same finite component of Y. We will equip the "base space" $\mathcal{F}$ with the Hausdorff metric.

Lemma 2.3. *If $H \curvearrowright Y$ is an (L, A)-quasi-action then the collection of quasi-orbits $O_y := H \cdot y$ forms an $(L, 3A)$-coarse fibration of Y.*

Proof. For $h, h_1, h_2 \in H$ and $y_1, y_2 \in Y$ we have
$$d\big(hy_1, (hh_1^{-1}h_2)y_2\big) \leq d\big((hh_1^{-1})(h_1y_1), (hh_1^{-1})(h_2y_2)\big) + 2A \leq L\cdot d(h_1y_1, h_2y_2) + 3A$$
and so
$$d(O_{y_1}, O_{y_2}) \leq L \cdot d(h_1y_1, O_{y_2}) + 3A.$$

□

Let $(Y, \mathcal{F})$ and $(Y', \mathcal{F}')$ be coarse fibrations. We say that a map $\phi : Y \to Y'$ *quasi-respects* the coarse fibrations if the image of each fiber $F \in \mathcal{F}$ is uniformly Hausdorff close to a fiber $F' \in \mathcal{F}'$, $d_H(\phi(F), F') \leq C$. The map ϕ then induces a map $\bar{\phi} : \mathcal{F} \to \mathcal{F}'$ which is well-defined up to bounded error $\leq 2C$. Observe that if ϕ is an (L, A)-quasi-isometry then $\bar{\phi}$ is an $(L, A + 2C)$-quasi-isometry.

We say that a quasi-action $\rho : G \curvearrowright Y$ *quasi-respects* a coarse fibration $\mathcal{F}$ if all maps $\rho(g)$ quasi-respect $\mathcal{F}$ with uniformly bounded error. The quasi-action ρ then descends to a quasi-action $\bar{\rho} : G \curvearrowright \mathcal{F}$ which is unique up to equivalence (cf. [KL01, Definition 2.3]).

We apply these general remarks to the following situation in order to obtain our main construction.

Let G be a group, $H < G$ a subgroup (of arbitrary index) and $H \overset{\alpha}{\curvearrowright} X$ an (L, A)-quasi-action. Let $Y = G \times X$ where G is given the metric $d(g_1, g_2) = +\infty$ unless $g_1 = g_2$. That is, Y consists of $|G|$ finite components each of which is a copy of X. The quasi-action α gives rise to a product quasi-action $H \overset{\rho_H}{\curvearrowright} Y$ via

$$\rho_H\big(h, (g, x)\big) = (gh^{-1}, hx).$$

We denote by $\mathcal{F}_H$ the coarse fibration of Y by H-quasi-orbits. The isometric G-action given by

$$\tilde{\rho}_G\big(g', (g, x)\big) = (g'g, x)$$

commutes with ρ_H. As a consequence, $\tilde{\rho}_G$ descends to an isometric action

$$\hat{\beta} := \bar{\rho}_G : G \curvearrowright \mathcal{F}_H. \tag{2.4}$$

If $H = G$ then α is quasi-conjugate to $\hat{\beta}$ via the quasi-isometry $x \mapsto \rho_H(H) \cdot (e, x)$.

In general, the finite components of $\mathcal{F}_H$ correspond to the left H-cosets in G. More precisely, gH corresponds to $\cup_{x \in X} \rho_H(H) \cdot (g, x)$, that is, to the union of ρ_H-quasi-orbits contained in $gH \times X$. H stabilizes the finite component $\cup_{x \in X} \rho_H(H) \cdot (e, x)$. The action of H on this component is quasi-conjugate to α.

As remarked in the beginning of this section, $\hat{\beta}$ is the unique G-quasi-action up to quasi-conjugacy such that G acts transitively on finite components and such that H is the stabilizer of a finite component and the restricted H-quasi-action is quasi-isometrically conjugate to α.

Passing back from disjoint unions to products we obtain Theorem 1.2.

3. Quasi-isometries and the classification into types (1)-(4)

We now prove:

Lemma 3.1. *Suppose Y and Y' are spaces of one of types (1)-(4) as in Theorem 1.6. If Y is quasi-isometric to Y', then they have the same type, unless one is a quasi-isometrically rigid rank* 1 *symmetric space, and the other is of type (4).*

Proof. First suppose one of the spaces is not Gromov hyperbolic. Since Gromov hyperbolicity is quasi-isometry invariant, both spaces must be higher rank space of either of type (1) or (2). But by [KL97], two irreducible symmetric spaces or Euclidean buildings of rank at least two are quasi-isometric iff they are homothetic. Thus in this case they must have the same type.

Now assume both spaces are Gromov hyperbolic. Then ∂Y and $\partial Y'$ are homeomorphic.

If Y is a bounded valence bushy tree, then it is well-known that Y is quasi-isometric to a trivalent tree, and ∂Y is homeomorphic to a Cantor set. Therefore Y cannot be quasi-isometric to a space of type (1), since the boundary of a Gromov hyperbolic symmetric space is a sphere. Also, the quasi-isometry group of a trivalent tree T has an induced action on the space of triples in ∂T which is not proper, and hence it cannot be quasi-isometric to a space of type (4).

If Y is a hyperbolic or complex hyperbolic space, then the induced action of $\mathrm{QI}(X)$ on the space of triples in ∂X is not proper, and hence Y cannot be quasi-isometric to a space of type (4).

The lemma follows. □

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