GENERALIZED DIFFERENTIATION AND BI-LIPSCHITZ NONEMBEDDING IN L^1

JEFF CHEEGER AND BRUCE KLEINER

Résumé. We consider Lipschitz mappings $f : X \to V$, where X is a doubling metric measure space which satisfies a Poincaré inequality, and V is a Banach space. We show that earlier differentiability and bi-Lipschitz nonembedding results for maps $f : X \to \mathbb{R}^N$ remain valid when $\mathbb{R}^N$ is replaced by any separable dual space. We exhibit spaces which bi-Lipschitz embed in L^1, but not in any separable dual V. For certain domains, including the Heisenberg group with its Carnot-Caratheodory metric, we establish a new notion of differentiability for maps into L^1. This implies that the Heisenberg group does not bi-Lipschitz embed in L^1, thereby proving a conjecture of J. Lee and A. Naor. When combined with their work, this has implications for theoretical computer science.

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Résumé : Nous considérons des applications lipchitziennes $f : X \to V$, où X est un espace métrique mesuré tel que l'on contrôle le volume des boules par doublement du rayon et qui satisfait à une inégalité de Poincaré, et où V est un espace de Banach. On montre que des résultats antérieurs de différentiabilité et de non plongement bilipschitzien pour des applications $f : X \to \mathbb{R}^N$ restent valables quand on suppose que V est un dual séparable. Nous donnons des exemples d'espaces plongés de manière bilipschitzienne dans L^1, mais qui ne sont plongeables dans aucun dual séparable. Pour certains espaces, dont le groupe d'Heisenberg muni de la métrique de Carnot-Caratheodory, on établit une nouvelle notion de différentiabilité pour des applications dans L^1. Ceci implique que le groupe de Heisenberg ne possède aucun plongement bilipschitzien dans L^1, un résultat conjecturé par J. Lee et A. Naor. Quand il est combiné avec des résultats de ces deux auteurs, notre travail a des applications en informatique théorique.

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1. Introduction

Rademacher's differentiation theorem states that a real valued Lipschitz function on $\mathbb{R}^k$ is differentiable almost everywhere. The literature contains a vast number of variants of this theorem, in which either the domain, the target, or the class of mappings is generalized. In this paper we announce several results in a similar vein, examples showing their sharpness and applications to bi-Lipshitz nonembedding theorems. [1]

2. PI SPACES

Henceforth (X, d^X, μ) will denote a metric measure space satisfying a doubling condition and a Poincaré inequality in the sense of [HK96] (see below for the definition), and we will refer to such spaces as *PI spaces*.

Examples of PI spaces include Gromov-Hausdorff limits of sequences of Riemannian manifolds with a uniform lower Ricci curvature bound, Bourdon-Pajot spaces, Laakso spaces and Carnot groups such as the Heisenberg group $\mathbb{H}$; see [CC97], [BP99], [Laa00], [Gro90]. For our present purposes, the latter two examples are particularly significant; see Theorems 5.1, 6.1.

We now recall some relevant definitions concerning PI spaces. Let $B_r(x)$ denote the metric ball, $\{x' \mid d^X(x, x') < r\}$.

The *doubling condition* is: for all $x \in X$ and $r \leq R$,

$$\mu(B_{2r}(x)) \leq 2^{\kappa(R)} \mu(B_r(x)) \,. \tag{2.1}$$

A function, $g : X \to [0, \infty]$, is called an *upper gradient* for $f : X \to \mathbb{R}$, if for all rectifiable curves, $c : [0, \ell] \to X$, parametrized by arc length, s,

$$|f(c(\ell)) - f(c(0))| \leq \int_0^\ell g(c(s))\, ds \,. \tag{2.2}$$

Given an integrable function, $f : B_r(x) \to \mathbb{R}$, put $\underline{f} = \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f \, d\mu$.

The $(1, p)$-Poincaré inequality asserts the existence of constants, $\tau(R)$, $\lambda(R)$, such that for all $x \in X$, $r \leq R$, and all f, g, with g upper gradient for f,

$$\int_{B_r(x)} |f - \underline{f}| \leq \tau r \left(\int_{B_{\lambda r}(x)} g^p \, d\mu \right)^{\frac{1}{p}} \,. \tag{2.3}$$

[1] Our work was motivated by the conjecture of J. Lee and A. Naor, that the Heisenberg group with its Carnot-Caratheodory metric does not bi-Lipschitz embed in L^1; see Theorem 6.1. We are very much indebted to Naor for telling us of their conjecture and for explaining its significance in connection with the Goemans-Linial conjecture of theoretical computer science; see Remark 6.5.

3. Differentiation theory for separable dual space targets

A Banach space V is said to have the *Radon-Nikodym property* if the conclusion of Rademacher's differentiation theorem holds for every Lipschitz map $\mathbb{R}^k \to V$. Separable dual spaces and reflexive spaces (separable or not) have the Radon-Nikodym property, [Gel38]. It is difficult to construct spaces with the Radon-Nikodym property which are not isomorphic to a subspace of some separable dual space; see p. 124 and also pp. 121–124 of [BL00].

An extension of the original Rademacher theorem to PI spaces was obtained in [Che99]. In [CK05], we show that the differentiation theory of [Che99] extends to separable dual space targets.

Let $a(x,x')$, $b(x,x')$, take values in V and let $\|\cdot\|$ denote the norm of V. The notation,

$$a(x,x') = b(x,x') + o(d^X(x,x'))\,,$$

means

$$\lim_{d^X(x,x')\to 0} \frac{\|a(x,x') - b(x,x')\|}{d^X(x,x')} = 0.$$

Theorem 3.1. ([CK05]) *Let X denote a PI space with associated constants κ, λ, τ. Then there exists a countable covering of X by measurable sets, $X = \bigcup_\alpha A_\alpha$, and Lipschitz maps,*

$$u_\alpha : A_\alpha \to \mathbf{R}^{k_\alpha}\,,$$

$u_\alpha = (u_{1,\alpha}, \dots, u_{k_\alpha,\alpha})$, with $k_\alpha \le c(\kappa,\tau,\lambda)$, with the following property: For any Lipschitz map, $f : X \to V$, with V a separable dual space, there exist μ-a.e. unique, bounded measurable functions,

$$\frac{\partial f(x')}{\partial u_{i,\alpha}} : A_\alpha \to V\,,$$

such that for μ-a.e. $x, x' \in A_\alpha$,

$$f(x) = f(x') + \sum_{i=1}^{k_\alpha} \frac{\partial f(x')}{\partial u_{i,\alpha}} \cdot (u_{i,\alpha}(x) - u_{i,\alpha}(x')) + o(d^X(x,x'))\,. \tag{3.2}$$

Remark 3.3. The countable covering, $X = \bigcup_\alpha A_\alpha$, and Lipschitz maps, $u_\alpha : A_\alpha \to \mathbb{R}^{k_\alpha}$, would determine an atlas of charts for a connected Lipschitz manifold, if we added the conditions that k_α is independent of α, the measurable sets, A_α, are open, and the Lipschitz maps, u_α, are bi-Lipschitz onto their images. If in such a case, μ denotes Lebesgue measure, then (3.2) reduces to the almost everywhere defined first order Taylor expansion of the Lipschitz function, f, in local coordinates u_α.

The space, X, also has a *measurable cotangent bundle*, T^*X, which is a measurable vector bundle whose fiber at μ-a.e. $x \in A_\alpha$ has dimension k_α. Every real valued Lipschitz function, $f : X \to \mathbb{R}$, has a differential, df, which is a bounded measurable section of T^*X. Likewise, if V is a separable dual space, then any Lipschitz map into V has a differential, which is a section of $T^*X \otimes V$.

Remark 3.4. An earlier differentiation theory for Lipschitz maps between Carnot groups was developed by Pansu; [Pan89]. For $\mathbb{R}^N$ targets, it coincides with that of [Che99]. The theory of [Pan89], which uses curve families, has a rather straightforward extension to targets having the Radon-Nikodym property; [CK05], [LN06]. We do not know if this holds for Theorem 3.1.

4. Bi-Lipschitz nonembedding in separable dual spaces

Generalized differentiation theorems for Lipschitz maps, $f : X \to V$, can be used to show that under certain conditions on X, f cannot be a bi-Lipschitz embedding. An early such result of Semmes, based on the differentiation theory of Pansu, states that $\mathbb{H}$ does not bi-Lipchitz embed in any finite dimensional Banach space; see [Sem96].

In [Che99], the differentiation theory developed there was used to prove results on bi-Lipschitz nonembedding for finite dimensional Banach space targets. This unified and extended previously known nonembedding theorems for certain classes of PI spaces, including in particular, Laakso spaces, Carnot groups and Bourdon-Pajot spaces.

In light of the differentiation result, Theorem 3.2, the bi-Lipschitz nonembedding theorems, Theorems 14.1–14.3 of [Che99], and their proofs, extend directly to separable dual targets; see, for instance, Theorem 4.1 below.

The doubling condition, (2.1), together with Gromov's compactness theorem, implies that any sequence of pointed PI spaces with rescaled metrics, $\{X, x, c_i \cdot d^X, \mu\}$, where $c_i \to \infty$, has a subsequence which converges to $(X_x, x_\infty, d_\infty, \mu_\infty)$ in the pointed measured Gromov-Hausdorff sense. The space X_x is called a *tangent cone* of X at x. Let TX be the *tangent bundle* of X, i.e. the dual of (T^*X), equipped with the dual norm. Then there is a *surjective* Lipschitz map $X_x \to TX_x$; see Section 13 of [Che99]. Thus,

$$\dim X_x \geq \dim TX_x \,.$$

Here and below, dim denotes Hausdorff dimension.

Theorem 4.1. ([CK05]) *Let X denote a PI space and let $G \subset X$ denote the set of points, x, such that for some tangent cone, X_x,*

$$\dim X_x > \dim TX_x \,.$$

If $\mu(G) > 0$, then X does not bi-Lipschitz embed in any separable dual space.

The targets covered by Theorem 4.1 include L^p, $1 < p < \infty$ and ℓ^p, $1 \leq p < \infty$.

In view of Remark 3.4, for the special case, $\mathbb{H}$, we also have the following generalization of Semmes' result; see also [LN06].

Theorem 4.2. ([CK05]) *A Carnot group does not bi-Lipschitz embed in any Banach space with the Radon-Nikodym property.*

5. Bi-Lipschitz embedding in L^1

The Lipschitz map, $I : (0, \infty) \to L^1$, given by $I(t) = \chi_{(0,t)}$, is nowhere differentiable, [Aro76]. Thus, L^1 does not have the Radon-Nikodym property. This leads one to ask if the nonembedding result Theorem 4.1 also breaks down when the target is L^1. Indeed this is the case:

Theorem 5.1 ([CK06b])**.** *There exist PI spaces, including Laakso spaces, which satisfy the hypothesis of Theorem 4.1, and hence do not bi-Lipschitz embed in any separable dual space, but which do bi-Lipschitz embed in L^1.*

Remark 5.2. As far as we know, the PI spaces in Theorem 5.1 provide the first examples of spaces which bi-Lipschitz embed in L^1 but not in ℓ^1.

6. Generalized differentiation for L^1 and bi-Lipschitz nonembedding

In [CK06a], it is shown that there is a generalized sense in which Lipschitz maps, $\mathbb{R}^n \to L^1$, or $\mathbb{H} \to L^1$, are differentiable. As a direct consequence, it follows that $\mathbb{H}$ does not bi-Lipschitz embed in L^1. This was conjectured by J. Lee and A. Naor.

Theorem 6.1 ([CK06a])**.** *If $U \subset \mathbb{H}$ is an open subset, and $f : U \to L^1$ is a Lipschitz map, then for almost every point, $x \in \mathbb{H}$, the map collapses in the direction of the center of $\mathbb{H}$ i.e.*

$$\lim_{g \to e} \frac{\|f(gx) - f(x)\|_{L^1}}{d^{\mathbb{H}}(gx, x)} = 0 \,, \qquad g \in \mathrm{Center}(\mathbb{H}) \,. \tag{6.2}$$

Corollary 6.3 ([CK06a])**.** *There is a compact doubling metric space which does not bi-Lipschitz embed in L^1.*

Let W denote some Cayley graph for the subgroup of the Heisenberg group consisting of matrices with integer entries. Give W the length space metric such that each edge has length 1. For $k \geq 0$, let $W_k := B_k(e) \subset W$ denote the ball of radius k in W, with center the identity element.

Corollary 6.4. ([CK06a]) *The sequence, $\{W_k\}$, is a sequence of uniformly doubling finite graphs with uniformly bounded valence, which do not admit embeddings into L^1 with uniformly bounded bi-Lipschitz distortion.*

Remark 6.5. A metric space, (Z, ρ), is said to be of *negative type* if $(Z, \rho^{\frac{1}{2}})$ isometrically embeds in Hilbert space. The Goemans-Linial conjecture asserts that if (Z, ρ) is of negative type, then (Z, ρ) bi-Lipschitz embeds in L^1. In [LN06], Lee and Naor show that $(\mathbb{H}, d^{\mathbb{H}})$ is bi-Lipschitz to a metric space of negative type. Therefore, by Theorem 6.1, $(\mathbb{H}, d^{\mathbb{H}})$ is bi-Lipschitz equivalent to a doubling metric space which does not satisfy the conclusion of the Goemans-Linial conjecture. For a prior recent counter example which is not doubling, see [KV05].

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