Spaces with nonpositive curvature and their ideal boundaries

Christopher B. Croke*
Bruce Kleiner†

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Abstract

We construct a pair of finite piecewise Euclidean 2-complexes with nonpositive curvature which are homeomorphic but whose universal covers have nonhomeomorphic ideal boundaries, settling a question from [8].

1.1 Introduction

The ideal boundary of a locally compact Hadamard space[1] X is a compact metrizable space on which the isometry group of X acts by homeomorphisms. Even though the ideal boundary is a well known construct with many applications in the literature (see for example [10, 4, 2]), the action of the isometry group on the boundary has not been studied closely except in the case of symmetric spaces, Gromov hyperbolic spaces, Euclidean buildings, and a handful of other cases. In the Gromov hyperbolic case[2] the boundary behaves nicely with respect to quasi-isometries: any quasi-isometry $f : X_1 \to X_2$ between Gromov hyperbolic Hadamard spaces induces a boundary homeomorphism $\partial_\infty f : \partial_\infty X_1 \to \partial_\infty X_2$ [7]. This has the consequence that the ideal boundary is "geometry independent":

If a finitely generated group G acts discretely, cocompactly and isometrically on two Gromov hyperbolic Hadamard spaces X_1, X_2, then there is a G-equivariant homeomorphism $\partial_\infty X_1 \to \partial_\infty X_2$.

In [8, p. 136] Gromov asked whether this fundamental property still holds if the hyperbolicity assumption is dropped. Sergei Buyalo [5] and the authors [6] independently answered Gromov's question negatively: [5, 6] exhibit a pair of deck group invariant Riemannian metrics on a universal cover which have ideal boundaries homeomorphic to S^2, such that the deck group actions on the boundaries are topologically

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[1]Following [3] we will call complete, simply connected length spaces with nonpositive curvature Hadamard spaces.

[2]The same statement is true of higher rank irreducible symmetric spaces and Euclidean buildings by [9].

inequivalent. Gromov also asked if $\partial_\infty X_1$ must be (non-equivariantly) homeomorphic to $\partial_\infty X_2$ whenever X_1 and X_2 are Hadamard spaces admitting discrete, cocompact, isometric actions by the same finitely generated group G. In this paper we show that even this can fail:

Theorem 1 *There is a pair $\bar{X}_1$, $\bar{X}_2$ of homeomorphic finite 2-complexes with nonpositive curvature such that the universal covers X_1, X_2 have nonhomeomorphic ideal boundaries.*

We remark that if M_1 and M_2 are closed Riemannian manifolds with nonpositive curvature and $\pi_1(M_1) \simeq \pi_1(M_2)$, then their universal covers will have ideal boundaries homeomorphic to spheres of the same dimension.

Although some basic questions about the boundary have now been answered, a number of related issues are wide open, except in a few special cases. It would be interesting to know exactly which geometric features determine the ideal boundary of a Hadamard space up to (equivariant) homeomorphism. This question has a clean answer (see [6]) in the case of graph manifolds or the 2-complexes considered in this paper. In order to answer the question in any generality, it appears that it will be necessary to develop a kind of "generalized symbolic dynamics" for geodesic flows of nonpositively curved spaces.

1.2 Notation and preliminaries

A reference for the facts recalled here is [3]. If X is a Hadamard space, then we denote the ideal boundary of X by $\partial_\infty X$, the geodesic segment joining x_1, $x_2 \in X$ by $\overline{x_1x_2}$, and the geodesic ray leaving $p \in X$ in the asymptote class of $\xi \in \partial_\infty X$ by $\overline{p\xi}$. If $p \in X$, ξ_1, $\xi_2 \in \partial_\infty X$, then $\angle_p(\xi_1, \xi_2)$ is the angle between the initial velocities of the rays $\overline{p\xi_1}$, $\overline{p\xi_1}$. $\angle_T(\xi_1, \xi_2) := \sup_{p\in X} \angle_p(\xi_1, \xi_2)$ will denote the Tits angle between ξ_1, $\xi_2 \in \partial_\infty X$. If $p \in X$ then $\angle_p(\xi_1, \xi_2) = \angle_T(\xi_1, \xi_2)$ iff the rays $\overline{p\xi_1}$ and $\overline{p\xi_2}$ bound a flat sector.

By the Cartan-Hadamard theorem [1, 3], the universal cover of a connected, complete, length space with nonpositive curvature is a Hadamard space with the natural metric. Let Z be a complete, connected space with nonpositive curvature, and let $\pi : \tilde{Z} \to Z$ be the universal cover. If $Y \subset Z$ is a closed, connected, locally convex subset, then the induced length metric on Y has nonpositive curvature, $\pi^{-1}(Y) \subset \tilde{Z}$ is a disjoint union of closed convex components isometric to $\tilde{Y}$, and the induced map $\pi_1(Y) \to \pi_1(Z)$ is a monomorphism.

1.3 Torus complexes

The following piecewise Euclidean 2-complexes were suggested to us by Bernhard Leeb, after a discussion of the graph manifold geometry in [6].

Let T_0, T_1, T_2 be flat two-dimensional tori. For $i = 1, 2$, we assume that there are (primitive) closed geodesics $a_i \subset T_0$ and $b_i \subset T_i$ with $length(a_i) = length(b_i)$, and we glue T_i to T_0 by identifying a_i with b_i isometrically. We assume that a_1 and a_2 lie in distinct free homotopy classes, and intersect once at an angle $\alpha \in (0, \frac{\pi}{2}]$. The resulting

2-complex $\bar{X}$ is nonpositively curved as a length space because gluing of nonpositively curved spaces along locally convex subsets produces a nonpositively curved space [3]. We refer to $\bar{X}$ as a **torus complex**. For $i = 1, 2$ let $\bar{Y}_i := T_0 \cup T_i \subset \bar{X}$. Notice that $\bar{Y}_i$ and T_0 are closed, locally convex subsets of $\bar{X}$. Therefore the inclusions $\bar{Y}_i \subset \bar{X}$ and $T_0 \subset \bar{X}$ induce monomorphisms of fundamental groups.

1.4 The structure of the universal cover

Let $\pi : X \to \bar{X}$ be the universal covering of $\bar{X}$. X is a Hadamard space by the Cartan-Hadamard theorem. A **block** is a connected component of $\pi^{-1}(\bar{Y}_i) \subset X$, and a **wall** is a connected component of $\pi^{-1}(T_0) \subset X$. Let $\mathcal{B}$ and $\mathcal{W}$ denote the (locally finite) collection of blocks and walls in X. Each block (resp. wall) is a closed, connected, locally convex subset of X. Hence by 1.2 each block (resp. wall) is a convex subset of X which is intrinsically isometric to the universal cover of $\bar{Y}_i$ (resp. T_0). If $W \in \mathcal{W}$, $B \in \mathcal{B}$, then either $W \cap B = \emptyset$ or $W \cap B = W$ since $W \cap B$ is open and closed in W; W is contained in precisely two blocks, one covering $\bar{Y}_1$ and the other covering $\bar{Y}_2$. If B_1, $B_2 \in \mathcal{B}$ are distinct blocks and $B_1 \cap B_2 \neq \emptyset$, then (after relabelling if necessary) B_i covers $\bar{Y}_i$ and so $B_1 \cap B_2$ consists of a (convex) union of walls; therefore $B_1 \cap B_2 = W$ for some $W \in \mathcal{W}$. When $B_1 \cap B_2 \neq \emptyset$ we will say that the blocks B_1 and B_2 are adjacent.

$\bar{Y}_i$ is a "flat" S^1 bundle over a bouquet of two circles, so the universal cover Y_i of $\bar{Y}_i$ (and hence each block) is isometric to the metric product of a simplicial tree with R. A **singular geodesic of a block** B is the inverse image of a vertex under the projection of B to its tree factor. Note that singular geodesics of adjacent blocks which lie in the common wall intersect at angle α.

The nerve of $\mathcal{B}$ (the simplicial complex recording (multiple) intersections of blocks) is a simplicial tree. (This is just the Bass-Serre tree of the amalgamated free product decomposition $\pi_1(\bar{X}) = \pi_1(\bar{Y}_1) *_{\pi_1(T_0)} \pi_1(\bar{Y}_2)$.) To see this note that if $\epsilon > 0$ is sufficiently small and $\mathcal{B}_\epsilon$ is the collection of (open) ϵ-tubular neighborhoods of blocks, then $Nerve(\mathcal{B}_\epsilon)$ is isomorphic to $Nerve(\mathcal{B})$. Using a partition of unity subordinate to this cover of $|Nerve(\mathcal{B}_\epsilon)|$ one gets a continuous map $\phi : X \to |Nerve(\mathcal{B}_\epsilon)|$. Any map $\gamma : S^1 \to |Nerve(\mathcal{B})|$ can be "lifted" to X up to homotopy: there is a $\hat{\gamma} : S^1 \to X$ so that $p \circ \hat{\gamma}$ is homotopic to γ. Since $\pi_1(X)$ is trivial, this implies that $\pi_1(|Nerve(\mathcal{B})|)$ is trivial. In particular, every wall separates X. We will say that a wall (resp. block) separates two blocks B_1, $B_2 \in \mathcal{B}$ if the edge (resp. vertex) of $|Nerve(\mathcal{B})|$ corresponding to the wall (resp. vertex) lies between the vertices of $|Nerve(\mathcal{B})|$ corresponding to B_1 and B_2.

Our plan is to show that the subspace $\cup_{B \in \mathcal{B}} \partial_\infty B \subset \partial_\infty X$ can be characterized purely topologically[3], and that its topology is different depending on whether $\alpha = \frac{\pi}{2}$ or not. It will then follow that a torus complex with $\alpha < \frac{\pi}{2}$ and a torus complex with $\alpha = \frac{\pi}{2}$ have universal covers with nonhomeomorphic ideal boundaries.

[3]At first glance one might think that $\cup_{B \in \mathcal{B}} \partial_\infty B$ is a path component of $\partial_\infty X$, but this turns out not to be the case. It is a "safe" path component, see 1.7.

1.5 Itineraries

For each $p \in X \setminus \cup_{W \in \mathcal{W}} W$, $\xi \in \partial_\infty X$, we get a sequence of blocks B_i called the p**-itinerary** (simply the **itinerary** if the basepoint p is understood) of ξ, as follows. Let B_i be the i^{th} block that the ray $\overline{p\xi}$ enters; the ray **enters** a block B if it reaches a point in $B \setminus \cup_{W \in \mathcal{W}} W$. We will denote the p-itinerary of $\overline{p\xi}$ by $Itin(\overline{p\xi})$ or $Itin(\xi)$.

Lemma 2 *The itinerary of any $\xi \in \partial_\infty X$ is the sequence of successive vertices of a geodesic segment or geodesic ray in the simplical tree $Nerve(\mathcal{B})$.*

Proof. Blocks are convex, so a geodesic cannot revisit any block which it left. The topological frontier of any $B \in \mathcal{B}$ is the union of the walls contained in B, so a geodesic segment which leaves B must arrive at a wall $W \subset B$, and then enter the block $B' \in \mathcal{B}$ adjacent to B along W. The collection $\mathcal{B}$ is locally finite, so the lemma follows. $\square$

Note that $\xi \in \partial_\infty X$ has a finite itinerary iff $\xi \in \partial_\infty B$ for some $B \in \mathcal{B}$.

1.6 Local components of $\partial_\infty X$

Since each block B is isometric to the product of simplicial tree with R, $\partial_\infty B$ is homeomorphic to the suspension of a Cantor set. A **pole** of B is one of the two suspension points in $\partial_\infty B$.

Lemma 3 *If $B_1, B_2 \in \mathcal{B}$, then one of the following holds:*

1. $\partial_\infty B_1 \cap \partial_\infty B_2 = \emptyset$.

2. $B_1 \cap B_2 = W \in \mathcal{W}$ and $\partial_\infty B_1 \cap \partial_\infty B_2 = \partial_\infty W$.

3. There is a $B \in \mathcal{B}$ such that $B \cap B_i = W_i \in \mathcal{W}$ and $\partial_\infty B_1 \cap \partial_\infty B_2$ is the set of poles of B.

Proof. Suppose B_1, $B_2 \in \mathcal{B}$ are distinct blocks, $\xi \in \partial_\infty B_1 \cap \partial_\infty B_2$, and $W \in \mathcal{W}$ is a wall separating B_1 from B_2. Choose basepoints $b_i \in B_i$, $w \in W$. If $x_k \in \overline{b_1\xi}$ is a sequence tending to infinity, and $y_k \in \overline{b_2\xi}$ is a sequence with $d(y_k, x_k) < C$, then we can find a $z_k \in \overline{x_k y_k} \cap W$ since W separates B_1 from B_2. Therefore $\overline{wz_k} \subset W$ converges, and the limit ray $\overline{w\xi}$ lies in W. Hence $\xi \in \partial_\infty W$.

Note that if W_1, $W_2 \subset B \in \mathcal{B}$, then $\partial_\infty W_1 \cap \partial_\infty W_2$ is just the set of poles of B; and $\xi \in \partial_\infty X$ cannot be a pole of two adjacent blocks simultaneously.

The lemma follows, since $\partial_\infty B_1 \cap \partial_\infty B_2 \neq \emptyset$ now implies that the combinatorial distance between B_1 and B_2 in $Nerve(\mathcal{B})$ is ≤ 2. $\square$

Lemma 4 *Suppose ξ lies on the ideal boundary of a block $B \in \mathcal{B}$, and assume ξ is not a pole of any block other than B. Then the path component of ξ in a suitable neighborhood Ω of ξ is contained in $\partial_\infty B$.*

Proof. Case I: $\xi \in \partial_\infty B$ is a pole of B. Choose $p \in B \setminus \cup_{W \in \mathcal{W}} W$. Recall (see section 1.3) that α is the angle between singular geodesics of adjacent blocks lying in the common wall, so α is the minimum Tits angle between ξ and any pole of a block adjacent to B. Let $\Omega := \{\xi' \in \partial_\infty X \mid \angle_p(\xi', \xi) < \frac{\alpha}{2}\}$, where $\angle_p(\xi, \xi')$ is the angle between the initial velocities of the two rays $\overline{p\xi}$, $\overline{p\xi'}$. We define an **exit from** B to be a singular geodesic $E \subset B$ of a block adjacent to B. A ray $\overline{p\xi'}$ **exits from** B **via** E if $\overline{p\xi'} \cap B$ is a geodesic segment ending at E, and the ray $\overline{p\xi'}$ continues into the block containing E. For each exit E from B, let Ω_E be the set of $\xi' \in \Omega$ such that $\overline{p\xi'}$ exits B via E.

Sublemma 5 *Ω_E is an open and closed subset of Ω.*

Proof. Openness. If $\xi' \in \Omega_E$, then $\overline{p\xi'} \cap B$ is a segment ending at some $e \in E$, and $\overline{p\xi'}$ enters the block B' adjacent to B which contains E. But then any sufficiently nearby (in the cone topology) ray $\overline{p\xi''}$ also leaves B at a point close to e; clearly this point must lie on E as the collection of exits is discrete. Therefore Ω_E is open in $\partial_\infty X$.

Closedness. Let $E' \subset E$ be the set of "exit points" for elements of Ω_E: the endpoints of segments $\overline{p\xi'} \cap B$, where $\xi' \in \Omega_E$. E' is bounded, for otherwise we could find a sequence $e_k \in E'$ with $\lim_{k\to\infty} d(e_k, p) = \infty$, and get a limit ray $\overline{pe_\infty} \subset B$ with $e_\infty \in \partial_\infty E \subset \partial_\infty B \cap \partial_\infty B'$, and $\angle_p(\xi, e_\infty) \leq \frac{\alpha}{2}$; this is absurd since e_∞ is a pole of B' and so $\angle_p(e_\infty, \xi) = \angle_T(e_\infty, \xi) \geq \alpha$. Now suppose $\xi'_k \in \Omega_E$ and $\lim_{k\to\infty} \xi'_k = \xi'_\infty \in \Omega$. We have, after passing to a subsequence if necessary, that $\overline{p\xi'_k} \cap B = \overline{pe_k}$ where $e_k \in E$ and $\lim_{k\to\infty} e_k = e_\infty \in E$. Then $\overline{p\xi'_\infty} \cap B$ contains $\overline{pe_\infty}$; if $\overline{p\xi'_\infty} \cap B \neq \overline{pe_\infty}$ then clearly $\overline{p\xi'_\infty}$ contains a segment of E, forcing $\overline{pe_\infty} \subset E$, which contradicts the choice of p. Thus we have $\xi'_\infty \in \Omega_E$. □

It follows that the connected (or path) component of ξ in Ω is contained in $\partial_\infty B$, since any subset $C \subseteq \Omega$ containing ξ and intersecting Ω_E admits a separation $C = (C \cap \Omega_E) \cup (C \setminus \Omega_E)$ into open subsets of C, and any $\xi' \in \Omega \setminus \partial_\infty B$ lies in Ω_E for some E.

Case II: $\xi \in \partial_\infty W$ where W is the wall separating two adjacent blocks B_1, B_2, and ξ is not a pole. Pick $p \in W$ not lying on a singular geodesic. Let ψ be the minimum Tits distance between ξ and a pole of B_i, $i = 1, 2$, and set

$$\Omega := \{\xi' \in \partial_\infty X \mid \angle_p(\xi', \xi) < \frac{\psi}{2}\}.$$

Let E be a singular geodesic of B_1 or B_2 which is contained in W. We say that the ray $\overline{p\xi'}$ **exits** W **via** E if $\overline{p\xi'} \cap W$ ends at a point in E, and $\overline{p\xi'}$ then immediately enters the block corresponding to E. Let Ω_E be the set of $\xi' \in \Omega$ so that $\overline{p\xi'}$ exits W via E. One checks as in case I that Ω_E is closed and open in Ω, so we conclude that the connected component of ξ in Ω is contained in $\partial_\infty W$.

Case III: $\xi \in \partial_\infty B$ does not lie in the boundary of any block other than B. Let ϕ be the minimum Tits angle between ξ and a pole of B, and set

$$\Omega := \{\xi' \in \partial_\infty X \mid \angle_p(\xi', \xi) < \frac{\phi}{2}\}.$$

Pick $p \in B \setminus \cup_{W \in \mathcal{W}} W$. Since ξ is not a pole of B, the ray $\overline{p\xi}$ determines an isometrically embedded Euclidean half-plane $H \subset B$, the intersection of the flat planes in B containing it. Let $\mathcal{B}'$ be the collection of blocks adjacent to B. If $B' \in \mathcal{B}'$ then

$B' \cap H$ ($= W \cap H$ where $W = B \cap B'$ is the wall between B and B') is either empty, a singular geodesic of B, or a flat strip with finite width bounded by singular geodesics, for otherwise we would have $\xi \in \partial_\infty B'$. Removing the singular geodesics and $\cup_{B' \in \mathcal{B}'} B'$ from H, we get a subset H^0 whose connected components are a countably infinite collection of open strips. If $S \subset H^0$ is such a strip, we let Ω_S be the set of $\xi' \in \Omega$ so that $\overline{p\xi'} \cap S \neq \emptyset$. As in cases I and II, Ω_S is closed and open in Ω. This forces the connected component of ξ in Ω to be contained in $\partial_\infty H \subset \partial_\infty B$, as desired. □

1.7 Vertices and safe paths

We say that $\xi \in \partial_\infty X$ is a **vertex** if there is a neighborhood U of ξ such that the path component of ξ in U is homeomorphic to the cone over a Cantor set, with ξ corresponding to the vertex of the cone. By Lemma 4 the set of vertices in $\cup_{B \in \mathcal{B}} \partial_\infty B$ is precisely the set of poles in $\cup_{B \in \mathcal{B}} \partial_\infty B$ (a priori there may be other vertices in $\partial_\infty X$).

A path $c : [0,1] \to \partial_\infty X$ is **safe** if $c(t)$ is a vertex for only finitely many $t \in [0,1]$. Since being joinable by a safe path is an equivalence relation on pairs of points, and since $\partial_\infty B_1 \cup \partial_\infty B_2$ is safe path connected when B_1 is adjacent to B_2, it follows that $\cup_{B \in \mathcal{B}} \partial_\infty B$ is safe path connected.

Lemma 6 *$\cup_{B \in \mathcal{B}} \partial_\infty B$ is a safe path component of $\partial_\infty X$.*

Proof. First note that if $c : [0,1] \to \partial_\infty X$ is a path, $c(t)$ is not a vertex when $t \in (0,1)$, $B \in \mathcal{B}$, and $c(0) \in \partial_\infty B$ is not a pole of any block other than B, then $c([0,1]) \subset \partial_\infty B$. This follows from Lemma 4, the fact that $\partial_\infty B$ is closed in $\partial_\infty X$, and a continuity argument.

Now if $B_0 \in \mathcal{B}$, $c : [0,1] \to \partial_\infty X$ is a safe path starting in $\partial_\infty B_0$, and $0 = t_0 < t_2 \ldots < t_k = 1$ are chosen so that $c(t)$ is a vertex only if $t = t_i$ for some i, then one proves by induction on i that the intervals $[t_{i-1}, t_i]$ are mapped into $\cup_{B \in \mathcal{B}} \partial_\infty B$. □

Lemma 7 *Pick $B_0 \in \mathcal{B}$ and $p \in B_0 \setminus \cup_{W \in \mathcal{W}} W$. Let $c : [0,1] \to \partial_\infty X$ be a path, and suppose $c(0)$ has an infinite p-itinerary. Then either $c(t)$ has the same p-itinerary as $c(0)$ for all $t \in I$, or there is a $\bar{t} \in I$ so that $c(\bar{t})$ has a finite itinerary. In particular, by Lemma 6, if c is a safe path then $c(t)$ has the same p-itinerary as $c(0)$ for all $t \in I$.*

Proof. Suppose $\xi_k \in \partial_\infty X$ is a sequence with $\lim_{k \to \infty} \xi_k = \xi \in \partial_\infty X$, and a certain block B is in the itinerary of $\overline{p\xi_k}$ for every k. Then either

1. $Itin(\xi)$ contains B

or

2. $Itin(\xi)$ is finite and only contains blocks lying between B_0 and B.

To see this, suppose B' is in $Itin(\xi)$ and $x \in \overline{p\xi} \cap Int(B')$. Then $x = \lim_{j \to \infty} x_j$ where $x_j \in \overline{p\xi_j} \cap Int(B')$ for sufficiently large j, so B' is in $Itin(\xi_j)$ for sufficiently large j. This means that B' lies between B_0 and B, for otherwise B would have to lie between B_0 and B', forcing $B \in Itin(\xi)$.

The lemma now follows, since if B is in $Itin(c(0))$ but not in $Itin(c(t))$ for all $t \in [0,1]$, then setting $t_0 := \inf\{t \mid B \notin Itin(c(t))\}$ we get a ray $\overline{pc(t_0)}$ with finite itinerary by the reasoning of the preceding paragraph. □

Corollary 8 *There is a unique safe path component of $\partial_\infty X$ which is dense, namely $\cup_{B\in\mathcal{B}}\partial_\infty B$.*

Proof. By Lemma 6 we know that $\cup_{B\in\mathcal{B}}\partial_\infty B$ forms a safe path component. $\cup_{B\in\mathcal{B}}\partial_\infty B$ is dense in $\partial_\infty X$ since any initial segment $\overline{px}$ of a ray $\overline{p\xi}$ may be continued as a ray $\overline{p\xi'} = \overline{px} \cup \overline{x\xi'}$ where the continuation $\overline{x\xi'}$ lies in a block (one of at most two) containing x.

By Lemma 7, if $\xi \in \partial_\infty X$ has an infinite p-itinerary, then any safe path starting at ξ consists of points with the same p-itinerary. Clearly the collection of points with a given p-itinerary isn't dense in $\partial_\infty X$. The corollary follows. $\square$

1.8 Detecting block boundaries

Call an arc $I \subset \cup_{B\in\mathcal{B}}\partial_\infty B$ an **edge** if its endpoints are both vertices, but no interior point of I is vertex of $\partial_\infty X$. Edges are contained in the boundary of a single block $B \in \mathcal{B}$ (see the proof of Lemma 6). Clearly the endpoints of an edge $I \subset \cup_{B\in\mathcal{B}}\partial_\infty B$ are either the poles of a single block, or $I \subset \partial_\infty W$ where $W = B_1 \cap B_2$ and the endpoints of I are poles of B_1 and B_2. So two points in $\cup_{B\in\mathcal{B}}\partial_\infty B$ are the poles of a single block (resp. adjacent blocks) iff they are the endpoints of more than one edge (resp. a unique edge). A subset of $\cup_{B\in\mathcal{B}}\partial_\infty B$ is the boundary of a block $B \in \mathcal{B}$ iff it is the union of all edges intersecting the poles of B.

1.9 Limiting behavior of poles

Pick $B \in \mathcal{B}$, and consider the set $\mathcal{P}$ of poles of blocks adjacent to B. If $\eta \in \partial_\infty B$ is a pole of B, then we have $\angle_T(\xi,\eta) \in \{\alpha, \pi-\alpha\}$ for every $\xi \in \mathcal{P}$. Let $\mathcal{P}_\alpha := \{\xi \in \mathcal{P} \mid \angle_T(\xi,\eta) = \alpha\}$, and $\mathcal{P}_{\pi-\alpha} := \{\xi \in \mathcal{P} \mid \angle_T(\xi,\eta) = \pi-\alpha\}$. Call each arc of $\partial_\infty B$ joining the poles of B a **longitude**.

Lemma 9 *Each longitude of $\partial_\infty B$ intersects $\bar{\mathcal{P}}_\alpha$ (resp. $\bar{\mathcal{P}}_{\pi-\alpha}$) in a single point ξ with $\angle_T(\xi,\eta) = \alpha$ (resp $\angle_T(\xi,\eta) = \pi-\alpha$).*

Proof. Pick $p \in B$, $\xi \in \partial_\infty B$ with $\angle_T(\xi,\eta) = \alpha$. Any initial segment $\overline{px}$ of the ray $\overline{p\xi}$ may be extended to a segment $\overline{py} = \overline{px} \cup \overline{xy}$ so that $\overline{py} \cap W = \{y\}$ for some wall $W \subset B$. Then $\overline{py}$ may be extended as a ray $\overline{p\xi'} = \overline{py} \cup \overline{y\xi'}$ where $\overline{y\xi'} \subset W$ and $\xi' \in \mathcal{P}_\alpha$. Therefore $\xi \in \bar{\mathcal{P}}_\alpha$. Since $\angle_T(\cdot,\eta)$ is a continuous function on $\partial_\infty B$, each longitude intersects $\mathcal{P}_\alpha$ in a single point. Similar reasoning applies to $\mathcal{P}_{\pi-\alpha}$. $\square$

From the lemma we see that any longitude l of $\partial_\infty B$ intersects $\bar{\mathcal{P}}$ in two points if $\alpha < \frac{\pi}{2}$ and one point if $\alpha = \frac{\pi}{2}$.

1.10 Distinguishing torus complexes

Let $\bar{X}_1$ be a torus complex with $\alpha < \frac{\pi}{2}$, and let $\bar{X}_2$ be a torus complex with $\alpha = \frac{\pi}{2}$. Let X_1 and X_2 be their respective universal covers. A homeomorphism $f : \partial_\infty X_1 \to \partial_\infty X_2$ would carry safe path components to safe path components, block boundaries to block boundaries (Corollary 8 and section 1.8), poles to poles, and longitudes to longitudes. But then section 1.9 gives a contradiction.

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