The local structure of length spaces with curvature bounded above

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Abstract

We show that a number of different notions of dimension coincide for length spaces with curvature bounded above. We then apply this result, showing that if X is a locally compact $CAT(0)$ space with cocompact isometry group, then the dimension of the Tits boundary and the asymptotic cone(s) of X are determined by the maximal dimension of a flat in X.

1 Introduction

Spaces with curvature bounded above were introduced by Alexandrov in [Ale51]; see [ABN86] for a survey of developments in the subject prior to 1980. Gromov's paper [Gro87], led to an explosion of literature on singular spaces, see the bibliography of [Bal95].

Examples of spaces with curvature bounded above (henceforth CBA spaces) include:

- Complete Riemannian manifolds with sectional curvature bounded above.
- Euclidean, spherical, and hyperbolic Tits buildings, see [Tit74, Ron89, KL97].
- Complexes with piecewise constant curvature. [DJ91, CD95, Ben91, BB94, Ś93] construct examples with interesting geometric and topological properties.
- Limits of Hadamard spaces[1], such as Tits boundaries and asymptotic cones. These have a number of applications, see for example [Mos73, BGS85, KL95, KL97].

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[1]Following [Bal95] we call $CAT(0)$ spaces (complete simply connected length spaces with non-positive curvature) Hadamard spaces.

In this paper we will study the local geometry and topology of CBA spaces. We recall ([Nik95], [KL97, section 2.1.3] or section 2.1) that there is a $CAT(1)$ space, the space of directions $\Sigma_p X$, associated with each point p in a CBA space X. We define the **geometric dimension** of CBA spaces to be the smallest function[2] on the class of CBA spaces such that a) $\mathrm{GeomDim}(X) = 0$ if X is discrete, and b) $\mathrm{GeomDim}(X) \geq 1 + \mathrm{GeomDim}(\Sigma_p X)$ for every $p \in X$. In other words, to find the geometric dimension of a CBA space we look for the largest number of times that we can pass to spaces of directions without getting the empty set. Our results relate this notion of dimension with several others.

Theorem A *Let X be a CBA space. Then the following (possibly infinite) quantities are equal to the geometric dimension of X:*

1. $\sup\{\mathrm{TopDim}(K) \mid K \subset X$ *is compact*$\}$. $\mathrm{TopDim}(Y)$ *denotes the topological dimension of* Y.

2. $\sup\{k \mid H_k(U,V) \neq \{0\}$ *for some open pair* (U,V) *in* $X\}$.

3. $\sup\{k \mid$ *There are sequences* $R_j \to 0$, $S_j \subset X$ *so that* $d(S_j, p) \to 0$ *for some* $p \in X$, *and* $\frac{1}{R_j} S_j$ *converges to the unit ball* $B(1) \subset \mathbb{E}^k$ *in the Gromov-Hausdorff topology*$\}$.

4. $\sup\{k \mid$ *There is a biHölder embedding (with exponent* $\frac{1}{2}$*) of an open set* $U \subset \mathbb{E}^k$ *into* $X\}$.

When the geometric dimension of X is finite, we have somewhat stronger conclusions:

Theorem B *Let X be a CBA space, and suppose* $\mathrm{GeomDim}(X)$ *is finite. Then the following quantities are equal to* $\mathrm{GeomDim}(X)$.

1. $\sup\{k \mid H_{k-1}(\Sigma_p X) \neq \{0\}$ *for some* $p \in X\}$.

2. $\sup\{k \mid H_k(X, X - p) \neq \{0\}$ *for some* $p \in X\}$.

3. $\sup\{k \mid$ *For every* $\epsilon > 0$ *there is a* $(1+\epsilon)$*-biLipschitz embedding of an open set* $U \subset \mathbb{E}^k$ *into* $X\}$.

4. $\sup\{k \mid$ *There is an isometric embedding of the standard sphere* $S^{k-1}(1) \subset \mathbb{E}^k$ *into* $\Sigma_p X$*, for some* $p \in X\}$.

We remark that the Hausdorff dimension of a metric space is always at least as big as its topological dimension [HW69, Chapter VII]; however, there are compact Hadamard spaces X with $\mathrm{GeomDim}(X) = 1$ which have infinite Hausdorff dimension.[3]

The next theorem shows that the dimension of limits of a Hadamard space X is controlled by the dimension of flats occurring in X.

Theorem C *Let X be a locally compact Hadamard space on which $Isom(X)$ acts cocompactly. Then the following are equal:*

[2] Taking values in $\mathbb{N} \cup \infty$.

[3] The completion of a metric simplicial tree which "branches fast enough" will have infinite Hausdorff dimension, although it may still be compact. In this case if we remove $\{x \in X \mid |\Sigma_x X| = 1\}$ we get a countably 1-rectifiable set: a set with Hausdorff dimension 1.

1. $\sup\{k \mid$ *There is an isometric embedding* $\mathbb{E}^k \longrightarrow X\}$. *Note that this is finite.*

2. $\sup\{k \mid$ *There is a quasi-isometric embedding* $\phi : \mathbb{E}^k \longrightarrow X\}$.

3. $1 + \mathrm{GeomDim}(\partial_T X)$. *Here* $\partial_T X$ *denotes the geometric boundary of* X *equipped with the Tits angle metric, see section 2.4 or [KL97, section 2.3.2].*

4. $\sup\{k \mid H_{k-1}(\partial_T X) \neq \{0\}\}$.

5. $\sup\{k \mid$ *There is an isometric embedding of the standard sphere* $S^{k-1}(1) \subset \mathbb{E}^k$ *in* $\partial_T X\}$.

6. *The geometric dimension of any asymptotic cone of* X.

Theorem C may be formulated for arbitrary families of Hadamard spaces, see Theorem 7.1; in particular it may be adapted to a foliated setting. Theorem C extends the main result of [AS86], see also [Gro93, pp.129-30]. Theorems A-C may also be applied to many of the questions posed in [Gro93, pp. 127-133], see section 9 for a discussion.

A key ingredient in the proofs of the theorems is the notion of a barycentric simplex. If X is a Hadamard space and $z = (z_0, \dots, z_n) \in X^{n+1}$, the barycentric simplex determined by z is the singular simplex $\sigma_z : \Delta_n \longrightarrow X$ which maps each $\alpha = (\alpha_0, \dots, \alpha_n) \in \Delta_n$ to the unique minimum of the uniformly convex function $\sum \alpha_i [d(z_i, \cdot)]^2$. Barycentric simplices are Lipschitz, and the restriction to each face of Δ_n is also a barycentric simplex (up to simplicial reparametrization). A remarkable feature of barycentric simplices which is not shared by some other constructions of simplices, such as the iterated coning of [Thu] is that if $n > \mathrm{GeomDim}(X)$ then σ_z is **degenerate**: $\sigma_z(\Delta_n) = \sigma_z(Bdy(\Delta_n))$ (see section 4). Since σ_z is Lipschitz this implies that the image of every barycentric simplex has Hausdorff dimension $\leq \mathrm{GeomDim}(X)$; this leads to the estimate $\mathrm{TopDim}(K) \leq \mathrm{GeomDim}(X)$ for compact subsets $K \subset X$. To obtain biHölder and biLipschitz embeddings of open sets $U \subset \mathbb{E}^k$ into X, we use the "nondegenerate part" of $\sigma_z(\Delta_n)$, i.e. $\sigma_z(\Delta_n) \setminus \sigma_z(Bdy(\Delta_n))$; this turns out to be a topological manifold in spite of the fact that σ_z is typically nowhere locally injective.

In view of Theorem A, we make the following

Conjecture(cf. [Gro93, pp. 133]) *If* X *is a CBA space, then* $\mathrm{TopDim}(X) = \mathrm{GeomDim}(X)$.

When X is separable, the conjecture follows from the method of proof of Theorem A.

It is interesting to compare Alexandrov spaces with curvature bounded above with Alexandrov spaces with curvature bounded below (CBB spaces). The papers [BGP92, Per94] show that a CBB space X has very restricted structure provided its Hausdorff or topological dimension is finite: [Per94] shows in particular that X is locally homeomorphic to its tangent cone at each point, and that X is a stratified manifold. This implies that the links of a polyhedron with a CBB metric are homotopy equivalent to spaces with curvature ≥ 1; and this is a very strong restriction on the polyhedron. In contrast to this, Berestovskii's result [Ber83] shows that an upper curvature bound does not impose any restriction on topology, at least if one works in the setting of polyhedra. Also, a CBA space with finite Hausdorff dimension need not have manifold points: build an $\mathbb{R}$-tree by completing an increasing union $T_1 \subset T_2 \subset \dots$ of metric simplicial trees where $Length(T_k) \leq 1$ and every branch point free segment $\sigma \subset T_k$ has length $\leq \frac{1}{k}$; the resulting space has Hausdorff dimension 1. There are other natural conditions that one can impose on a CBA space. If a CBA space X is locally compact and has extendible geodesics[4], then one can prove statements analogous to [BGP92, 5.4,10.6] about X; however there is a 2-dimensional locally compact Hadamard space with extendible geodesics which is not a stratified manifold, even though it is a Gromov-Hausdorff limit of a sequence of 2-dimensional CBA polyhedra which satisfy the same conditions. The natural applications of CBB and CBA spaces are also different. Finite dimensional CBB spaces arise as Gromov-Hausdorff limits of Riemannian manifolds with a lower bound on curvature and an upper bound on dimension, and are a powerful tool for studying (pre)compactness for families of Riemannian manifolds. There are two main classes of "interesting" singular CBA spaces. The first one is locally finite polyhedra with nonpositively curved metrics; these provide numerous examples with interesting topology and fundamental groups. The second is Tits boundaries and asymptotic cones – these are limit spaces associated with a Hadamard space, and are usually "large", i.e. nonseparable. The Tits boundary of a Hadamard space X is an important tool for studying the isometry group and the geodesic flow of X. Asymptotic cones arise in compactness arguments, for instance when studying degeneration of hyperbolic structures [MS84, Pau88, RS94, Bes88], or when studying quasi-isometries [KL95, KL97, KKL98].

We recall that a complete length space (X, d) is **convex** [Gro78] if for every pair $\gamma_1 : [a_1, b_1] \to X$ and $\gamma_2 : [a_2, b_2] \to X$ of constant speed geodesic segments, the function $d \circ (\gamma_1, \gamma_2) : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ is convex. It turns out that a slightly weaker version of Theorem C holds for the more general class of convex length spaces.

Theorem D *Let X be a locally compact convex length space with cocompact isometry group. Then the following are equal:*

1. $\sup\{k \mid$ *There is an isometric embedding of a k-dimensional Banach space in* $X\}$. *Note that this is finite.*

2. $\sup\{k \mid$ *There is a quasi-isometric embedding* $\phi : \mathbb{E}^k \to X\}$.

3. $\sup\{k \mid H_k(U, V) \neq \{0\}$ *for some open pair* $V \subseteq U \subseteq C_T X\}$. *$C_T X$ denotes the Tits cone of X, see section 10 for the definition.*

4. $\sup\{k \mid$ *There is a k-dimensional Banach space* $(\mathbb{R}^k, \|\cdot\|)$, *sequences* $R_j \to 0$, $S_j \subset C_T X$ *so that* $\frac{1}{R_j} S_j$ *converges to the unit ball in* $(\mathbb{R}^k, \|\cdot\|)$ *in the Gromov-Hausdorff topology*$\}$.

5. $\sup\{k \mid$ *There is an asymptotic cone X_ω of X, and an open pair* $V \subseteq U \subseteq X_\omega$ *such that* $H_k(U, V) \neq \{0\}\}$.

6. $\sup\{k \mid$ *There is a k-dimensional Banach space* $(\mathbb{R}^k, \|\cdot\|)$, *sequences* $R_j \to 0$, $S_j \subset X_\omega$ *so that* $\frac{1}{R_j} S_j$ *converges to the unit ball in* $(\mathbb{R}^k, \|\cdot\|)$ *in the Gromov-Hausdorff topology*$\}$.

[4] These assumptions are natural when studying the geodesic flow.

7. $\sup\{k \mid$ *There is a k-dimensional Banach space* $(\mathbb{R}^k, \|\cdot\|)$*, sequences* $R_j \to \infty$*,* $S_j \subset X$ *so that* $\frac{1}{R_j} S_j$ *converges to the unit ball in* $(\mathbb{R}^k, \|\cdot\|)$ *in the Gromov-Hausdorff topology*$\}$.

The proof of Theorem D is somewhat different from the proof of Theorem C because barycentric simplices do not behave well in convex spaces (squared distance functions are no longer uniformly convex). Instead we use the differentiation theory for Lipschitz maps into metric spaces of Korevaar-Schoen [KS93] (see Theorem 10.7, Corollary 10.9, and Proposition 10.18).

The paper is organized as follows. In section 2 we briefly recall background material and introduce notation. In section 3 we use the geometric dimension to bound topological and homological dimensions from below. In section 4 we define barycentric simplices, and prove that they degenerate above the geometric dimension (Corollary 4.10). In section 5 we use the results of section 4 to prove that the geometric dimension is at least as big as the topological and homological dimensions. In section 6 we construct biHölder and biLipschitz embeddings of open sets $U \subseteq \mathbb{E}^n$ into CBA spaces. In section 7 we construct flats in Hadamard spaces X starting with objects in $\partial_T X$ or in an asymptotic cone of X. In section 8 we combine the earlier sections to prove Theorems A, B, and C. In section 9 we discuss questions posed in [Gro93]. In section 10 we discuss convex length spaces, giving a short proof of (a special case of) the metric differentiation result of [KS93] and applications to convex length spaces.

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2 Preliminaries

We refer the reader to [KL97] for a more detailed discussion of the material in this section.

2.1 $CAT(\kappa)$ spaces, spaces of directions, tangent cones, etc

Let M^2_κ denote the two-dimensional model space with constant curvature κ, and let $D(\kappa)$ denote the diameter of M^2_κ. A complete metric space X is a ***CAT(κ)* space** if

1. $Diam(X) \leq D(\kappa)$.
2. Every two points $x_1, x_2 \in X$ with $d(x_1, x_2) < D(\kappa)$ are joined by a geodesic segment of length $d(x_1, x_2)$. Note that some other definitions in the literature require X to be a length space; this is inconvenient in induction arguments since spaces of directions can be disconnected.
3. Every geodesic triangle in X with perimeter $< 2D(\kappa)$ is at least as thin as a geodesic triangle in M^2_κ with the corresponding side lengths. In particular, every two points $x_1, x_2 \in X$ with $d(x_1, x_2) < D(\kappa)$ are joined by a unique geodesic segment with length $< D(\kappa)$ (up to reparametrization), which we will confuse with its image $\overline{x_1 x_2}$.

Let X be a $CAT(\kappa)$ space. If p, x, $y \in X$ and $d(p, x) + d(x, y) + d(p, y) < 2D(\kappa)$, then there is a well-defined geodesic triangle Δpxy. The **comparison angle** of the triangle Δpxy at p is defined to be the angle of the comparison triangle (in M^2_κ) for Δpxy at the vertex corresponding to p; this angle is denoted $\widetilde{\angle}_p(x, y)$. The thin triangle condition implies that if $x' \in \overline{px} - \{p\}$ and $y' \in \overline{py} - \{p\}$ then $\widetilde{\angle}_p(x', y') \leq \widetilde{\angle}_p(x, y)$. Therefore if we let $x' \in \overline{px}$, $y' \in \overline{py}$ tend to p then $\widetilde{\angle}_p(x', y')$ has a limit; we call this the **angle** between $\overline{px}$ and $\overline{py}$ at p, and denote it by $\angle_p(x, y)$. If we let $y' \in \overline{py} - \{p\}$ tend to p, then $\widetilde{\angle}_p(x, y')$ also tends to $\angle_p(x, y)$. The function $p \mapsto \angle_p(x, y)$ is upper semicontinuous. $\angle_p$ defines a pseudo-metric (see definition 2.7) on the collection of geodesic segments leaving p. We define $\Sigma_p^* X$ to be the associated quotient metric space obtained by collapsing zero diameter subsets to points. The **space of directions at *p*** is the completion of $\Sigma_p^* X$, and is denoted $\Sigma_p X$. $\Sigma_p X$ is a $CAT(1)$ space [Nik95] (see also [KL97, section 2.1.3]). The **tangent cone at *p*** is the Euclidean cone $C(\Sigma_p X)$ over $\Sigma_p X$; it is a $CAT(0)$-space and is denoted by $C_p X$. We will often identify $\Sigma_p X$ (as a set) with the points in $C_p X$ at unit distance from the vertex of the cone. We have a map $\log_{\Sigma_p X} : B(p, D(\kappa)) - \{p\} \longrightarrow \Sigma_p X$ which takes $x \in B(p, D(\kappa)) - \{p\}$ to the direction of the segment $\overline{px}$. We will often use the notation $\overrightarrow{px}$ for $\log_{\Sigma_p X} x$. There is also a map $\log_{C_p X} : B(p, D(\kappa)) \longrightarrow C_p X$ which takes $x \in B(p, D(\kappa))$ to the unique point on the ray $C(\log_{\Sigma_p X}(x))$ at distance $d(p, x)$ from the vertex. It follows from comparison inequalities that $\log_{C_p X}$ is 1-Lipschitz when $\kappa \leq 0$; more generally, if $r < D(\kappa)$ then $\log_{C_p X}\Big|_{B(p,r)}$ is $L(r, \kappa)$-Lipschitz.

Lemma 2.1 *If X is a $CAT(\kappa)$ space, $p \in X$, and $K \subset C_p X$ is a compact subset, then there are sequences $R_j \in \mathbb{R}$, $S_j \subset X$ so that $\lim_{j \to \infty} R_j = 0$, $\lim_{j \to \infty} d(S_j, p) = 0$, and $\frac{1}{R_j} S_j$ converges to K in the Gromov-Hausdorff topology.*

Proof. First assume $K = \log_{C_p X}(Y)$ where $Y \subset B(p, D(\kappa))$ is a finite set. For each $y \in Y$ let $\gamma_y : [0, 1] \to X$ be the constant speed geodesic with $\gamma_y(0) = p$, $\gamma_y(1) = y$. Then

$$\lim_{t \to 0} \frac{d(\gamma_{y_1}(t), \gamma_{y_2}(t))}{t} = d(\log_{C_p X}(y_1), \log_{C_p X}(y_2),$$

so if we set $S_j := \{\gamma_y(\frac{1}{j}) \mid y \in Y\}$ then $\frac{1}{R_j} S_j \to K$ in the Gromov-Hausdorff topology.

Since $\Sigma_p^* X$ is dense in $\Sigma_p X$, the Euclidean cone $C(\Sigma_p^* X)$ is dense in $C_p X := C(\Sigma_p X)$; therefore if $K \subset C_p X$ is an arbitrary compact set then K is a Hausdorff limit of a sequence $K_i \subset C(\Sigma_p^* X) \subseteq C_p X$ of finite subsets. For each i there is a $\lambda_i \in (0, \infty)$ so that $\lambda_i K_i$ (the image of K_i under the λ_i homothety of the cone $C_p X$) is in $Im(\log_{C_p X})$. Applying the preceding paragraph to each K_i and a diagonal construction we obtain the desired sequences.

□

A metric space X has **curvature bounded above by κ** if for every $p \in X$ there is an $r > 0$ so that the closed ball $\overline{B_p(r)}$ is a $CAT(\kappa)$ space; a space X has **curvature bounded above** (or is a CBA space) if it has curvature bounded above by κ for some κ. If X is a CBA space, $p \in X$ and $r > 0$ is small enough that $\overline{B_p(r)} \subseteq X$ is a $CAT(\kappa)$ space, then we may apply all the constructions of the previous paragraph to $\overline{B_p(r)}$. One observes that $\Sigma_p X$ and $C_p X$ are independent of the choice of r, κ. We define the **injectivity radius at *p*, *Inj(p)***, to be the supremum of the radii r so that $\overline{B_p(r)}$ is a $CAT(\kappa)$ space for some κ with $D(\kappa) \geq r$. We have well-defined logarithm maps $\log_{\Sigma_p X} : B_p(Inj(p)) - \{p\} \to \Sigma_p X$ and $\log_{C_p X} : B_p(Inj(p)) \to C_p X$.

2.2 Convex functions

Definition 2.2 *Let X be a complete metric space. If $C \in \mathbb{R}$, a function $f : X \longrightarrow \mathbb{R}$ is C-**convex** if for every unit speed geodesic $\gamma : [a, b] \longrightarrow X$, the function $t \mapsto f \circ \gamma(t) - \frac{C}{2}t^2$ is convex.*

In particular, if f is C-convex, then $f \circ \gamma$ has left and right derivatives for every geodesic γ.

Lemma 2.3 *Let X be a complete metric space such that every two points in X are joined by a geodesic segment. Suppose $C > 0$, and $f : X \to \mathbb{R}$ is a continuous C-convex function which is bounded below. Then f has a unique minimum $\bar{x}$ and any sequence $x_k \in X$ with $\lim(f(x_k)) = \inf f$ converges to $\bar{x}$.*

Proof. Suppose x_k is a sequence with $f(x_k) \to \inf f$ as $k \to \infty$. For every k, l let z_{kl} be the midpoint of a geodesic segment joining x_k to x_l. Then C-convexity of f gives

$$f(z_{kl}) \leq \frac{1}{2}f(x_k) + \frac{1}{2}f(x_l) - \frac{\epsilon}{8}[d(x_k, x_l)]^2.$$

Hence $\lim_{k,l\to\infty} d(x_k, x_l) = 0$ and x_k is Cauchy. Then $\lim_{k\to\infty} x_k$ is the unique minimum of f. □

Lemma 2.4 (Directional derivatives of convex functions) *Let X be a space with curvature bounded above, and let $\phi : X \longrightarrow \mathbb{R}$ be an L-Lipschitz function on X which is C-convex for some $C \in \mathbb{R}$. For every $p \in X$ there is a unique L-Lipschitz function $D\phi : C_pX \longrightarrow \mathbb{R}$ such that for every $x \in X - \{p\}$ in the domain of $\log_{C_pX}$, $D\phi(\log_{C_pX}(x)) = (\phi \circ \gamma)'(0)$ where $\gamma : [0, 1] \to X$ is the constant speed parametrization of the segment $\overline{px}$. Moreover $D\phi$ is convex and homogeneous of degree 1: $(D\phi)(\lambda v) = \lambda(D\phi)(v)$ for every $\lambda \in [0, \infty)$.*

Proof. Replacing X with a convex ball centered at p, we may assume that $\log_{C_pX}$ is defined on all of X. Define $\psi : X \longrightarrow \mathbb{R}$ by setting $\psi(x) = (\phi \circ \gamma_x)'(0)$ where $\gamma_x : [0, 1] \to X$ is the constant speed geodesic segment with $\gamma_x(0) = p$, and $\gamma_x(1) = x$. If x_1, $x_2 \in X$, then

$$\begin{aligned}
|\psi(x_1) - \psi(x_2)| &= |(\phi \circ \gamma_{x_1})'(0) - (\phi \circ \gamma_{x_2})'(0)| \\
&= |\lim_{t\to 0} \frac{(\phi \circ \gamma_{x_1})(t) - (\phi \circ \gamma_{x_2})(t)}{t}| \\
&\leq L|\lim_{t\to 0} \frac{d(\gamma_{x_1}(t), \gamma_{x_2}(t))}{t}| \\
&= Ld(\log_{C_pX}(x_1), \log_{C_pX}(x_2)).
\end{aligned}$$

Therefore ψ descends to a unique L-Lipschitz function $D\phi : C_pX \to \mathbb{R}$ which is homogeneous of degree 1. The convexity of $D\phi$ follows from the C-convexity of ϕ. □

A computation of Hessians shows that if p is in the model space M^2_κ, then the distance function $d_p := d(p, \cdot)$ is $C_1(r, \kappa)$-convex on $B_p(r)$ when $r < D(\kappa)$. By triangle comparison this implies that if X is a $CAT(\kappa)$ space and $p \in X$, then d_p is a $C_1(r,\kappa)$-convex function on $B_p(r)$ when $r < D(\kappa)$. The directional derivative of d_p at $x \in B_p(D(\kappa)) - \{p\}$ is given by

$$(Dd_p)(v) = -\langle v, \overrightarrow{xp}\rangle = -\langle v, \log_{\Sigma_x X} p\rangle \tag{2.5}$$

where the "inner product" is defined by $\langle v, w\rangle_x := |v||w|\cos\angle_x(v,w)$. Similarly, there is a function $C_2(r,\kappa) > 0$ so that the squared distance function d_p^2 is $C_2(r,\kappa)$-convex on $B_p(r)$ when $r < \frac{D(\kappa)}{2}$.

2.3 Ultralimits and asymptotic cones

A **nonprincipal ultrafilter on** $\mathbb{N}$ is a finitely additive probability measure ω on $\mathbb{N}$ so that $\omega(S) \in \{0,1\}$ for every subset $S \subseteq \mathbb{N}$, and $\omega(S) = 0$ when $S \subset \mathbb{N}$ is finite. If K is a compact metric space and $f : \mathbb{N} \to K$, then there is a unique point $p \in K$ with the property that for every neighborhood U of p, $\omega(f^{-1}(U)) = 1$; this point is called the ω-limit of f and is denoted $f(\omega)$ or ω-$\lim f$. If $\{(X_i, d_i, \star_i)\}_{i\in\mathbb{N}}$ is a sequence of pointed metric spaces, we define a pseudo-metric (see definition 2.7) $\hat{d}_\omega$ on

$$\{(x_i) \in \prod_i X_i \mid \sup_i d_i(x_i, \star_i) < \infty\} \subseteq \prod_i X_i$$

by $\hat{d}_\omega((x_i),(y_i)) := \omega\text{-}\lim d_i(x_i, y_i)$. We let X_ω denote the associated quotient metric space with distance function d_ω, and let $\star_\omega \in X_\omega$ denote the image of $(\star_i)$ under the projection. The **ultralimit of $(X_i, d_i, \star_i)$** is the pointed metric space $(X_\omega, d_\omega, \star_\omega)$; we will also use the notation ω-$\lim(X_i, d_i, \star_i)$ and sometimes suppress d_i when it is clear from the context.

Properties of ultralimits:

1. Ultralimits are complete metric spaces.

2. If $(X_i, d_i, \star_i)$ is a Gromov-Hausdorff precompact sequence of spaces, then $(X_\omega, d_\omega, \star_\omega)$ is a limit point of the sequence.

3. If $\{(X_i, d_i, \star_i)\}_{i\in\mathbb{N}}$ is a sequence of pointed metric spaces and every closed ball in (X_ω, d_ω) is compact, then there is a subset $S \subseteq \mathbb{N}$ with $\omega(S) = 1$ so that the corresponding subsequence of $\{(X_i, d_i, \star_i)\}_{i\in\mathbb{N}}$ converges in the Gromov-Hausdorff topology to ω-$\lim(X_i, d_i, \star_i)$.

4. If $\phi_i : (X_i, d_i, \star_i) \to (\bar{X}_i, \bar{d}_i, \bar{\star}_i)$ is a sequence of (L_i, A_i) quasi-isometric embeddings (resp. quasi-isometries), $\omega\text{-}\lim d_i(\phi_i(\bar{\star}_i), \star_i) < \infty$, $\omega\text{-}\lim L_i = L < \infty$, $\omega\text{-}\lim A_i = A < \infty$, then we get induced quasi-isometric embeddings (resp. quasi-isometries) $\phi_\omega : (X_\omega, d_\omega) \to (\bar{X}_\omega, \bar{d}_\omega)$; when $A = 0$ then ϕ_ω is uniquely determined.

5. If $K \subset (X_\omega, d_\omega, \star_\omega)$ is a compact subset, then there is a sequence $(K_i, \bar{d}_i, \bar{\star}_i)$ of pointed finite metric spaces and a sequence of isometric embeddings $\phi_i : (K_i, \bar{d}_i, \bar{\star}_i) \to (X_i, d_i, \star_i)$ so that $\phi_\omega : K_\omega \to X_\omega$ maps K_ω isometrically onto K.

6. An ultralimit of a sequence of $CAT(\kappa)$ spaces is a $CAT(\kappa)$ space.

Let (X, d) be a metric space. An **asymptotic cone** of X is an ultralimit of the form ω-$\lim(X, \lambda_i d, \star_i)$ (sometimes written $(\lambda_i X, \star_i)$ when the metric is clear from the context) where $\star_i \in X$, $\lambda_i > 0$, and $\omega\text{-}\lim \lambda_i = 0$.

Properties of asymptotic cones:

1. If $\phi : X \to \bar{X}$ is an (L, A) quasi-isometric embedding (resp. quasi-isometry), then the ultralimit of the sequence $\phi : (X, \lambda_i d, \star_i) \to (\bar{X}, \lambda_i \bar{d}, \phi(\star_i))$ gives an L-biLipschitz embedding (resp. L-biLipschitz homeomorphism) between the asymptotic cones.

2. If the isometry group of X acts with cobounded orbits on X, then the isometry group of any asymptotic cone of X acts transitively.

2.4 The Geometric and Tits boundaries

Let X be a Hadamard space. Two unit speed geodesics[5] $\gamma_1 : [0, \infty) \to X$, $\gamma_2 : [0, \infty) \to X$ are **equivalent (or asymptotic)** if the Hausdorff distance between $\gamma_1([0, \infty))$ and $\gamma_2([0, \infty))$ is finite. Given a geodesic ray γ_1 and $p \in X$, there is a unique geodesic ray $\gamma_2 : [0, \infty) \to X$ asymptotic to γ_1 with $\gamma_2(0) = p$. The **geometric boundary** of X, $\partial_\infty X$, is the set of equivalence classes of geodesic rays in X topologized by viewing it as the collection of geodesic rays leaving some $p \in X$ endowed with the compact-open topology. If $p \in X$, $\xi \in \partial_\infty X$, we let $\overline{p\xi}$ denote the image of the geodesic ray leaving p in the class ξ.

The **Tits angle** between two geodesic ray γ_1, γ_2 leaving $p \in X$ is $\angle_T(\gamma_1, \gamma_2) := \lim_{t \to \infty} \widetilde{\angle}_p(\gamma_1(t), \gamma_2(t))$. $\angle_T$ defines a metric on $\partial_\infty X$, and we denote the resulting metric space by $\partial_T X$.

Properties of $\partial_T X$:

1. $\partial_T X$ is $CAT(1)$-space.
2. $\angle_T : \partial_\infty X \times \partial_\infty X \to [0, \pi]$ is a lower-semicontinuous function.
3. If $p \in X$, $\xi_1, \xi_2 \in \partial_T X$, then $\angle_T(\xi_1, \xi_2) = \sup_{x \in X} \angle_x(\xi_1, \xi_2) = \lim_{x \in \overline{p\xi_1}, d(x,p) \to \infty} \angle_x(\xi_1, \xi_2)$.

2.5 Miscellany

Definition 2.6 *An* ϵ**-Hausdorff approximation** *is a map* ϕ *between metric spaces* (X, d_X) *and* (Y, d_Y) *so that* $|\phi^* d_Y - d_X| < \epsilon$ *and* $d_Y(y, \phi(X)) < \epsilon$ *for every* $y \in Y$. *A sequence of metric spaces* X_k *converges to a metric space* X *in the Gromov-Hausdorff topology iff there is a sequence of* ϵ_k*-Hausdorff approximations* $\phi_k : X \to X_k$ *with* $\lim_{k \to \infty} \epsilon_k = 0$.

Definition 2.7 *A* **pseudo-metric, or pseudo-distance function** *on a set* X *is a function* $d : X \times X \to [0, \infty)$ *which is symmetric and which satisfies the triangle inequality.*

If (X, d) is a pseudo-metric space, then we get a metric space $(\bar{X}, \bar{d})$ by letting $\bar{X}$ be the set of maximal zero diameter subsets and setting $\bar{d}(S_1, S_2) := d(s_1, s_2)$ for any $s_i \in S_i$. We define the Gromov-Hausdorff pseudo-distance between pseudo-metric spaces to be the Gromov-Hausdorff distance between the associated (quotient) metric spaces.

[5] Geodesic rays will be parametrized at unit speed except in section 10.

3 Geometric dimension bounds topological and homological dimension

In this section we show that the geometric dimension gives a lower bound for the topological and homological dimensions of a CBA space.

Lemma 3.1 *If X is a CBA space, $p \in X$, and $\mathrm{GeomDim}(\Sigma_p X) \geq n-1$, then there are sequences $R_j \to 0$, $S_j \subset X$ so that $d(S_j, p) \to 0$ and $\frac{1}{R_j} S_j$ converges to the unit ball $B^n \subset \mathbb{E}^n$ in the Gromov-Hausdorff topology.*

Proof. The statement is immediate when $n = 0$, so we assume that $n > 0$. By induction there is a sequence of sets $T_k \subset \Sigma_p X$ so that $\frac{1}{R_k} T_k \to B^{n-1} \subset \mathbb{E}^{n-1}$ in the Gromov-Hausdorff topology. If $CT_k \subseteq C_p X$ is the Euclidean cone over T_k, and $\star_k \in CT_k \cap \Sigma_p X$, then the pointed spaces $(\frac{1}{R_k} CT_k, \star_k)$ converge to $B^{n-1} \times \mathbb{R}$ in the pointed Gromov-Hausdorff topology. Since B^n embeds isometrically in $B^{n-1} \times \mathbb{R}$ we may find $\bar{T}_k \subset CT_k$ so that $(\frac{1}{R_k} \bar{T}_k, \star_k)$ converges to B^n. But by Lemma 2.1 each $\bar{T}_k$ is a Gromov-Hausdorff limit of a sequence $\frac{1}{R'_j} S'_j$ where $S'_j \subset X$, $d(S'_j, p) \to 0$. Hence the lemma. □

Proposition 3.2 *If X is a CBA space, $p \in X$, $R_k \to 0$, $S_k \subseteq X$, $d(S_k, p) \to 0$, and $\frac{1}{R_k} S_k \to B^n \subset \mathbb{E}^n$ in the Gromov-Hausdorff topology, then $H_n(X, V) \neq 0$ for some open set $V \subset X$ and $\mathrm{TopDim}(K) \geq n$ for some compact set $K \subseteq X$.*

Proof. Let e_i be the i^{th} unit coordinate vector in $\mathbb{E}^n$, and let $\alpha : \mathbb{E}^n \longrightarrow \mathbb{E}^n$ be the map with i^{th} coordinate function $d(e_i, \cdot) - 1$. Note that if $\delta \in (0,1)$ is sufficiently small, then $\alpha(\partial B(0,\delta)) \subset \mathbb{E}^n \setminus \{0\}$ and $\alpha\big|_{B(0,\delta)}$ induces an isomorphism $H_n(B(0,\delta), \partial B(0,\delta)) \longrightarrow H_n(\mathbb{E}^n, \mathbb{E}^n \setminus \{0\})$.

Let $\phi_k : B(0, R_k) \longrightarrow S_k \subseteq X$ be a sequence of $\epsilon_k R_k$-Hausdorff approximations (see definition 2.6), where $\epsilon_k \to 0$. Set $x_k^i = \phi_k(R_k e_i)$, and define $d_k : X \longrightarrow \mathbb{E}^n$ by $d_k(\cdot) = (\frac{1}{R_k}(d(x_k^1, \cdot) - R_k), \ldots, \frac{1}{R_k}(d(x_k^n, \cdot) - R_k))$. Triangulate $B(0, R_k) \subset \mathbb{E}^n$ with smooth simplices of diameter $< \epsilon_k R_k$. For sufficiently large k define a continuous map $\psi_k : B(0, R_k) \longrightarrow X$ by starting with the restriction of ϕ_k to the 0-skeleton of the triangulation, and extending it to $B(0, R_k)$ using barycentric simplices (see section 4). Define $\alpha_k : B(0,\delta) \longrightarrow \mathbb{E}^n$ by $\alpha_k(x) = (d_k \circ \psi_k)(R_k x)$. Then α_k converges uniformly to $\alpha\big|_{B(0,\delta)}$ and for sufficiently large k we have $\alpha_k(\partial B(0,\delta)) \subset \mathbb{E}^n - \setminus\{0\}$ and $H_n(\alpha_k) : H_n(B(0,\delta), \partial B(0,\delta)) \to H_n(\mathbb{E}^n, \mathbb{E}^n \setminus \{0\})$ is an isomorphism. Hence if $V_k = d_k^{-1}(\mathbb{E}^n \setminus \{0\})$, we find that for k sufficiently large $H_n(\psi_k) : H_n(B(0, \delta R_k), \partial B(0, \delta R_k)) \longrightarrow H_n(X, V_k)$ is well-defined and nontrivial. So for sufficiently large k, $d_k\big|_{\psi_k(B(0,\delta R_k))}$ cannot be uniformly approximated by continuous maps which miss the origin $0 \in \mathbb{E}^n$, so $\mathrm{TopDim}(\psi_k(B(0, \delta R_k))) \geq n$. □

Remark. The proof shows that there are $\epsilon, R > 0$ depending on $\kappa \in \mathbb{R}$ and $n \in \mathbb{N}$ so that if $B^n \subset \mathbb{E}^n$ denotes the unit ball, and a $CAT(\kappa)$-space X contains a subset S with $d_{GH}(\frac{1}{R'} S, B^n) < \epsilon$, $R' < R$, then $\mathrm{TopDim}(X) \geq n$. When $\kappa \leq 0$ one may take $R = \infty$. In fact, it suffices to have a (sufficiently small) set which is sufficiently close in the Gromov-Hausdorff topology to either a) the 0-skeleton of the first barycentric subdivision of a regular n-simplex or b) the set $\{0, \pm R' e_1, \ldots, \pm R' e_n\} \subset \mathbb{E}^n$.

4 Barycentric simplices

In this section we define a notion of simplex which provides a direct connection between the geometric dimension of a CBA space and the topological dimension of its separable subsets. The key result is Proposition 4.8, which implies that if $\mathrm{GeomDim}(X) \leq n$, then every barycentric k-simplex with $k > n$ is degenerate (see definition 4.7).

Let X be a space with curvature bounded above.

Definition 4.1 *We say that $z = (z_0, \ldots, z_n) \in X^{n+1}$ is* (κ, r)**-admissible** *if $\kappa \in \mathbb{R}$, $r < \frac{D(\kappa)}{2}$, the closed ball $\overline{B(z_i, r)}$ is a $CAT(\kappa)$ space, and $d(z_i, z_j) \leq r$ for $0 \leq i, j \leq n$. z is* **admissible** *if it is (κ, r)-admissible for some (κ, r).*

Lemma 4.2 *Let $z = (z_0, \ldots, z_n) \in X^{n+1}$ be (κ, r)-admissible, and let Y_r be the $CAT(\kappa)$ space $\cap_i \overline{B(z_i, r)}$. Then for every α in the standard n-simplex $\Delta_n = \{(\alpha_0, \ldots, \alpha_n) \in \mathbb{R}^{n+1} \mid \alpha_i \geq 0, \sum \alpha_i = 1\}$, the function $\phi_\alpha := \sum \alpha_i d^2_{z_i}$ has a unique minimum point in Y_r. This minimum point y is characterized by the property that $\sum \alpha_i d_{z_i}(y) D(d_{z_i})$ is a nonnegative function on C_yX, where $D(d_{z_i}) : C_yX \longrightarrow \mathbb{R}$ denotes the directional derivative of d_{z_i}, see Lemma 2.4. Denoting this minimum point by $\sigma_z(\alpha)$, we obtain a Lipschitz map $\sigma_z : \Delta_n \longrightarrow X$. The map σ_z is independent of the choice of (κ, r), and we call it the* **barycentric simplex** *determined by z. If $\tau : \Delta_k \longrightarrow \Delta_n$ is a k-face of Δ_n, then $\sigma_z \circ \tau$ coincides with the barycentric simplex determined by the sub-$(k+1)$-tuple of z selected by τ.*

Proof. Existence of minimum of ϕ_α. Since z is (κ, r)-admissible, the squared distance functions $d^2_{z_i}$ are all C-convex on Y_r, where $C > 0$ depends on κ, r (see section 2). Therefore ϕ_α is C-convex on Y_r for every $\alpha \in \Delta_n$, and so we may apply Lemma 2.3 to see that ϕ_α has a unique minimum in Y_r, and that this minimum does not depend on the choice of (κ, r).

σ_z is Lipschitz. Pick $y \in Y_r$, and let $\gamma : [0, l] \longrightarrow Y_r$ be the unit speed geodesic from $\sigma_z(\alpha)$ to y. The function $\phi_\alpha \circ \gamma : [0, l] \longrightarrow \mathbb{R}$ is C-convex with a minimum at 0, so the left derivate of $\phi_\alpha \circ \gamma$ at l is at least Cl; in particular, if $y \neq \sigma_z(\alpha)$ then the derivative of ϕ_α at y, $D(\phi_\alpha) = \sum 2\alpha_i d_{z_i}(y) D(d_{z_i})$, attains negative values. On the other hand, if $y = \sigma_z(\alpha')$ for some $\alpha' \in \Delta_n$, then the left derivate of $\phi_{\alpha'} \circ \gamma$ at l is ≤ 0 since $\sigma_z(\alpha')$ minimizes $\phi_{\alpha'}$. Therefore the left derivate of $(\phi_\alpha - \phi_{\alpha'}) \circ \gamma$ at l is $\geq Cl$. $d^2_{z_i}$ is $2r$-Lipschitz on Y_r, so $\phi_\alpha - \phi_{\alpha'}$ is $2r\|\alpha - \alpha'\|_{l^1}$-Lipschitz on Y_r. This gives $Cl \leq 2r\|\alpha - \alpha'\|_{l^1}$ or $d(\sigma_z(\alpha), \sigma_z(\alpha')) = l \leq \frac{2r}{C}\|\alpha - \alpha'\|_{l^1}$, and so σ_z is Lipschitz. The final statement of the lemma is immediate from the definition of σ_z. □

We will need to define maps by using the minima of more general families of functions. The following lemma provides an appropriate setup.

Lemma 4.3 *Let X be a complete metric space. Let $f_0, \ldots, f_n$ be bounded Lipschitz functions on X, let $\Omega \subset \mathbb{R}^{n+1}$, and for every $\alpha \in \mathbb{R}^{n+1}$ define $\phi_\alpha : X \longrightarrow \mathbb{R}$ by $\phi_\alpha := \sum \alpha_i f_i$. Suppose $\nu : \Omega \longrightarrow \mathbb{R}$ is a continuous positive function such that for every $\alpha \in \Omega$*

1. $\phi_\alpha^{-1}((-\infty, \nu(\alpha) + \inf \phi_\alpha)) \subset X$ *is a geodesically convex subset*[6]*.*

2. ϕ_α *is* $\nu(\alpha)$*-convex on* $\phi_\alpha^{-1}((-\infty, \nu(\alpha) + \inf \phi_\alpha))$*.*

Then

1. For each $\alpha \in \Omega$, ϕ_α *attains a minimum at a unique point in* X*.*

2. If $\sigma_f : \Omega \longrightarrow X$ *is the map taking each* $\alpha \in \Omega$ *to the minimum of* ϕ_α*, then* σ_f *is locally Lipschitz.*

If in addition a) X *has curvature bounded above, b) the* f_i*'s are all* C*-convex for some* $C \in \mathbb{R}$*, c)* $Df_i : C_xX \longrightarrow \mathbb{R}$ *denotes the derivative of* f_i *at* x *(see Lemma 2.4), and d)* $\alpha \in \Omega$*; then*

3. $x \in X$ *is the minimum of* ϕ_α *iff* $\sum \alpha_i Df_i$ *is a nonnegative function on* C_xX*.*

We omit the proof as it is similar to the proof of Lemma 4.2.

Lemma 4.4 *Let* X *be a* $CAT(1)$ *space, let* CX *be the Euclidean cone over* X*, and identify* X *with the unit sphere in* CX*. Let* $f : X \longrightarrow \mathbb{R}$ *be a Lipschitz function whose homogeneous extension* $\hat{f} : CX \longrightarrow \mathbb{R}$ *is convex. Then*

1. *If* $\gamma : [0, \theta] \longrightarrow X$ *is a unit speed geodesic, then* $(f \circ \gamma)'' + f \circ \gamma \geq 0$ *in the sense of distributions. Equivalently, for every* $t \in (0, \theta)$ *and every* $\epsilon > 0$ *there is a* $\delta > 0$ *so that* $f \circ \gamma$ *is* $(-(f \circ \gamma)(t) - \epsilon)$*-convex in the interval* $[t - \delta, t + \delta] \subset [0, \theta]$*.*

2. *For every* $\mu > 0$*,* $f^{-1}((-\infty, -\mu))$ *is a geodesically convex subset of* X*, and* f *is* μ*-convex on* $f^{-1}((-\infty, -\mu))$*.*

Proof. 1. We may take $X = \gamma([0, \theta])$ and $\theta \in (0, \pi)$. Then we may identify CX with a sector in $\mathbb{R}^2$. The first statement is obvious for smooth functions and follows for general functions by a smoothing argument.

2. If x_1, $x_2 \in X$ and $f(x_1), f(x_2) < 0$ then the segment joining x_1 to x_2 in the Hadamard space CX cannot pass through the vertex o of CX, since convexity of $\hat{f}$ would force $\hat{f}(0) < 0$, which is absurd. Therefore $d_X(x_1, x_2) < \pi$, and $f^{-1}((-\infty, 0))$ is geodesically convex. The second statement now follows from the first. □

Lemma 4.5 *If* $f_0, \dots, f_n$ *are functions on a* $CAT(1)$ *space* X *which satisfy the conditions of Lemma 4.4, then together they satisfy the conditions of Lemma 4.3 for* $\Omega := \{\alpha \in \mathbb{R}^{n+1} \mid \alpha_i \geq 0, \inf \phi_\alpha < 0\}$ *and a suitable function* ν*.*

Proof. First note that the $|f_i|$'s are bounded by some B since X has diameter $\leq \pi$ and f_i is Lipschitz; by 1 of Lemma 4.4 this tells us that the f_i are $(-B)$-convex. For $\alpha \in \Omega$, the nonnegative linear combination $\sum \alpha_i f_i$ satisfies the hypotheses of Lemma 4.4, so for every $\mu > 0$, $\phi_\alpha^{-1}((-\infty, -\mu))$ is geodesically convex, and ϕ_α is μ-convex on $\phi_\alpha^{-1}((-\infty, -\mu))$. So if we set $\nu(\alpha) = -\frac{1}{2} \inf \phi_\alpha$ then conditions 1 and 2 of Lemma 4.3 will be satisfied. □

[6]Every two points are joined by path in the subset whose length equals the distance between them.

Definition 4.6 *If $f_0, \dots, f_n$ are functions on a CBA space X, then $(f_0, \dots, f_n)$ is an* **admissible** *$(n+1)$-tuple if*

1. *Each f_i is C-convex for some $C \in \mathbb{R}$.*

2. *$f_0, \dots, f_n$ satisfy the conditions of Lemma 4.3 with with $\Omega = \Delta_n$ and a suitable function ν.*

We call the map $\sigma_f : \Delta_n \longrightarrow X$ constructed in Lemma 4.3 the **simplex** *determined by $(f_0, \dots, f_n)$. Note that a (κ, r)-admissible $(n+1)$-tuple $z = (z_0, \dots, z_n) \in X^{n+1}$ defines an admissible $(n+1)$-tuple $(d^2_{z_0}, \dots, d^2_{z_n})$ on $Y_r := \cap_i \overline{B(z_i, r)}$ whose simplex is just the barycentric simplex of z.*

Definition 4.7 *Let $(f_0, \dots, f_n)$ be admissible. Then $\sigma_f : \Delta_n \longrightarrow X$ is* **degenerate** *if $\sigma_f(\Delta_n) = \sigma_f(Bdy(\Delta_n))$. Otherwise σ_f is* **nondegenerate**, *and any $x \in \sigma_f(\Delta_n) - \sigma_f(Bdy(\Delta_n))$ is a* **nondegenerate point**. *When $z \in X^{n+1}$ is admissible then we will apply the same terminology to the barycentric simplex $\sigma_z : \Delta_n \to X$. Note that by part 3 of Lemma 4.3 it follows that $x \in X$ is a nondegenerate point of σ_f iff*

1. *There is an $\bar{\alpha} \in \Delta_n$ so that $D\phi_\alpha : \Sigma_x X \to \mathbb{R}$ is nonnegative.*

2. *For every $\alpha \in Bdy(\Delta_n)$ we have $\inf_{\Sigma_x X} D\phi_\alpha < 0$.*

Proposition 4.8 *Let X be a CBA space. If $(f_0, \dots, f_n)$ is admissible and $x \in \sigma_f(\Delta_n)$ is a nondegenerate point then* $\mathrm{GeomDim}(\Sigma_x X) \geq n - 1$. *In particular if $z = (z_0, \dots, z_n) \in X^{n+1}$ is admissible and $x \in \sigma_z(\Delta_n)$ is nondegenerate then* $\mathrm{GeomDim}(\Sigma_x X) \geq n - 1$.

Proof. The idea of the proof is to construct a Lipschitz map $\sigma_g : Bdy(\Delta_n) \to \Sigma_x X$ which indicates, for each $\alpha \in Bdy(\Delta_n)$, the direction of fastest decrease for the function $\phi_\alpha = \sum \alpha_i f_i$. We then construct a Lipschitz map $g : \Sigma_x X \to \mathbb{R}^{n+1}$ with the property that $g \circ \sigma_g$ maps the fundamental cycle of $Bdy(\Delta_n)$ to a cycle in $\mathbb{R}^{n+1}$ which is nontrivial in $H_{n-1}(W)$ where $W \subset \mathbb{R}^{n+1}$ is suitably chosen. It then follows that $g \circ \sigma_g(Bdy(\Delta_n))$ has positive $(n-1)$-dimensional Hausdorff measure, which means that at least one of the top-dimensional faces of σ_g is nondegenerate. We then argue by induction, concluding that $\mathrm{GeomDim}(\Sigma_x X) \geq n - 1$.

Pick $\bar{\alpha} = (\bar{\alpha}_0, \dots, \bar{\alpha}_n) \in \Delta_n$ so that $x = \sigma_f(\bar{\alpha})$. Let $g_i := Df_i : \Sigma_x X \longrightarrow \mathbb{R}$ be the directional derivative of f_i at x, and let $g := (g_0, \dots, g_n) : \Sigma_x X \longrightarrow \mathbb{R}^{n+1}$. By 3 of Lemma 4.3 we know that $\sum \bar{\alpha}_i g_i$ is nonnegative on $\Sigma_x X$; in particular, for every $v \in \Sigma_x X$ we have $g_i(v) \geq 0$ for some i. As x is a nondegenerate point of σ_f, we have $\inf \sum \alpha_i g_i < 0$ for every $\alpha \in Bdy(\Delta_n)$. In view of Lemma 4.5, we may apply Lemma 4.3 to $g_0, \dots, g_n$ with $\Omega = Bdy(\Delta_n)$, to get a Lipschitz map $\sigma_g : Bdy(\Delta_n) \longrightarrow \Sigma_x X$. The composition $\rho = g \circ \sigma_g : Bdy(\Delta_n) \longrightarrow \mathbb{R}^{n+1}$ has the following properties:

- For every $\alpha \in Bdy(\Delta_n)$, $\rho(\alpha)$ has at least one nonnegative coordinate and at least one negative coordinate.

- If $I \subset \{0, \dots, n\}$, and $\alpha \in Bdy(\Delta_n)$ lies in the face corresponding to I (i.e. $\alpha_j = 0$ for $j \notin I$), then the i^{th} coordinate of $\rho(\alpha)$ is negative for some $i \in I$.

Set

$$W = \{t \in \mathbb{R}^{n+1} | \text{There exists } i \text{ so that } t_i < 0 \text{ and } j \text{ so that } t_j \geq 0\},$$

and for each subset $I \subset \{0, \dots, n\}$, set $W_I = \{t \in W | \; t_i < 0 \text{ for some } i \in I\}$.

Lemma 4.9 *1. The map* $-id\big|_{Bdy(\Delta_n)} : Bdy(\Delta_n) \longrightarrow W$ *is a homotopy equivalence.*

2. W_I *is contractible when* I *is a proper subset of* $\{0, \dots, n\}$*.*

3. If $u : Bdy(\Delta_n) \to W$ *is a continuous map with* $H_{n-1}(u) : H_{n-1}(Bdy(\Delta_n)) \to H_{n-1}(W)$ *nonzero then* $u(Bdy(\Delta_n)) \subset W$ *has positive* $(n-1)$*-dimensional Hausdorff measure.*

Proof. Let $\mathbb{R}^{n+1}_{\geq} := \{t \in \mathbb{R}^{n+1} \mid t_i \geq 0 \text{ for all } i\}$, $\mathbb{R}^{n+1}_{>} := \{t \in \mathbb{R}^{n+1} \mid t_i > 0 \text{ for all } i\}$ be the nonpositive and positive orthants respectively; and define $\mathbb{R}^{n+1}_{\leq}$ and $\mathbb{R}^{n+1}_{<}$ similarly. We have $W = \mathbb{R}^{n+1} \setminus (\mathbb{R}^{n+1}_{\geq} \cup \mathbb{R}^{n+1}_{<})$. Note that W_I deformation retracts to its intersection with $Bdy(\mathbb{R}^{n+1}_{\leq}) = \mathbb{R}^{n+1}_{\leq} \setminus \mathbb{R}^{n+1}_{<}$ by moving points in the $(-1, \dots, -1)$ direction until they hit $Bdy(\mathbb{R}^{n+1}_{\leq})$; moreover this retraction $r_I : W_I \to W_I \cap Bdy(\mathbb{R}^{n+1}_{\leq})$ is Lipschitz. Also, $W_I \cap Bdy(\mathbb{R}^{n+1}_{\leq})$ deformation retracts to its intersection with $-Bdy(\Delta_n)$ via the (locally Lipschitz) radial retraction $r'_I : W_I \cap Bdy(\mathbb{R}^{n+1}_{\leq}) \to W_I \cap (-Bdy(\Delta_n))$. But $W_I \cap (-Bdy(\Delta_n))$ is the image of the I-face of $Bdy(\Delta_n)$ under $-id_{\mathbb{R}^{n+1}}$ when I is a proper subset of $\{0, \dots, n\}$ and $-Bdy(\Delta_n)$ otherwise, so 1 and 2 follow.

If $u : Bdy(\Delta_n) \to W$ is continuous and $H_{n-1}(u) \neq 0$ then $r'_{\{0,\dots,n\}} \circ r_{\{0,\dots,n\}} \circ u : Bdy(\Delta_n) \to -Bdy(\Delta_n)$ is nontrivial in H_{n-1}, so it is surjective; since $r'_{\{0,\dots,n\}}$ and $r'_{\{0,\dots,n\}}$ are both locally Lipschitz this implies that the $(n-1)$-dimensional Hausdorff measure of $u(Bdy(\Delta_n))$ is positive. This completes the proof of Lemma 4.9. □

We have shown that the Lipschitz map $\rho : Bdy(\Delta_n) \longrightarrow \mathbb{R}^{n+1}$ has image contained in W, and the image of the I-face of $Bdy(\Delta_n)$ under ρ is contained in the contractible subset W_I. Since $\rho : Bdy(\Delta_n) \longrightarrow W$ is homotopic to any other continuous map satisfying these conditions, ρ is homotopic to $-id_{\mathbb{R}^{n+1}}\big|_{Bdy\Delta_n}$; in particular ρ maps the fundamental class of $Bdy(\Delta_n)$ to a nontrivial element of $H_{n-1}(W)$. By 3 of Lemma 4.9 we conclude that $\rho(Bdy(\Delta_n))$ – and hence also $\sigma_g(Bdy(\Delta_n))$ – has positive $(n-1)$-dimensional Hausdorff measure. This means that for some $i \in \{0, \dots, n\}$, the image of the $(n-1)$-face $\{t \in Bdy(\Delta_n) \mid t_i = 0\}$ of $Bdy(\Delta_n)$ under σ_g has positive $(n-1)$-dimensional Hausdorff measure. Then $(g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_n)$ is an admissible n-tuple whose simplex $\Delta_{n-1} \to \Sigma_x X$ is nondegenerate, for its image has positive $(n-1)$-dimensional Hausdorff measure. By induction we conclude that $\mathrm{GeomDim}(\Sigma_x X) \geq n-1$. □

Corollary 4.10 *If* X *is a CBA space with* $\mathrm{GeomDim}(X) = n < \infty$*, then for every admissible* $(k+1)$*-tuple* $(f_0, \dots, f_k)$*, the image of* σ_f *has finite* n*-dimensional Hausdorff measure.*

Proof. By Proposition 4.8, each face of σ_f of dimension $> n$ is degenerate, so $\sigma_f(\Delta_k)$ is the image of the n-skeleton of Δ_k. As σ_f is Lipschitz Corollary 4.10 follows. □

Lemma 4.11 *If* X *is a CBA space with* $\mathrm{GeomDim}(X) \geq n$*, then there is an admissible* $z \in X^{n+1}$ *with* σ_z *nondegenerate.*

Proof. Pick $p \in X$ so that $\mathrm{GeomDim}(\Sigma_p X) \geq n$. By Lemma 3.1 we may find sequences $R_k \to 0$ and $S_k \subset X$ satisfying the hypotheses of Proposition 3.2. Following the proof of Proposition 3.2, the map $\Psi_k : B(0, R_k) \to X$ is constructed using barycentric simplices and $\mathrm{TopDim}(\Psi_k(B(0, R_k))) \geq n$. But this forces the image of one of the barycentric simplices σ to have topological dimension at least n; therefore [HW69, Chapter VII] $Image(\sigma)$ has Hausdorff dimension $\geq n$, and so σ is nondegenerate.

Alternatively one can show directly that if the $(n+1)$-tuple $z^k := (z_0^k, \ldots, z_n^k) \in [B(0, R_k)]^{n+1}$ consists of the vertices of a regular n-simplex inscribed in $B(0, R_k)$, then $\phi_k(z^k) = (\phi_k(z_0^k), \ldots, \phi_k(z_n^k)) \in X^{n+1}$ is admissible and nondegenerate when k is sufficiently large.

□

5 Topological dimension

In this section, we use the fact that barycentric simplices degenerate in dimensions greater than $\mathrm{GeomDim}(X)$ to show that the homological dimension of X and the topological dimension of compact subsets of X are both $\leq \mathrm{GeomDim}(X)$.

Lemma 5.1 *Let X be a CBA space.*

1. Any continuous map ϕ from a finite polyhedron P to X may be uniformly approximated by a Lipschitz map ϕ_1; if $\mathrm{GeomDim}(X) \leq n$ *then we may arrange that $\phi_1(P)$ has finite n-dimensional Hausdorff measure.*

2. Suppose $K \subseteq X$ is compact. For every $\epsilon > 0$ the inclusion $i : K \to X$ can be ϵ-approximated by a Lipschitz map $i_1 : K \to X$ which factors as $K \xrightarrow{f} P \xrightarrow{g} X$ where P is a finite polyhedron, f, g are Lipschitz, and the f-inverse image of each simplex of P has diameter $< \epsilon$; in particular the image of i_1 has finite Hausdorff dimension. If K has zero k-dimensional Hausdorff measure then we may arrange that $Dim(P) < k$; if $\mathrm{GeomDim}(X) = n < \infty$ *then we may arrange that $Dim(P) \leq n$.*

Proof. 1. Pick $\epsilon > 0$. After barycentrically subdividing P enough times, we may assume that ϕ maps the vertices of each simplex τ of P to an admissible $(j+1)$-tuple in X (definition 4.1), and $\phi(\tau) \subset X$ has diameter $< \frac{\epsilon}{2}$. Then we may define $\phi_1 : P \to X$ to be the unique Lipschitz map whose restriction to each simplex of P is just the barycentric simplex determined by its vertices. Then $d(\phi, \phi_1) < \epsilon$, and if $\mathrm{GeomDim}(X) \leq n$ then $\phi_1(P)$ has finite n-dimensional Hausdorff measure since it coincides with the image of the n-skeleton of P by Corollary 4.10.

2. Choose $\epsilon > 0$ so that any tuple of points in K with pairwise distance $< \epsilon$ is admissible. Let $\mathcal{U} = \{B_{x_i}(\frac{\epsilon}{2})\}_{i \in I}$ be a finite open cover of K by open $\frac{\epsilon}{2}$-balls (in X) centered at points in K, let $Y := \cup_{i \in I} B_{x_i}(\frac{\epsilon}{2}) \subseteq X$, let $Nerve(\mathcal{U})$ be the simplicial complex whose simplices correspond to the subsets of $\mathcal{U}$ with nonempty intersection, and let $|Nerve(\mathcal{U})|$ be the polyhedron of $Nerve(\mathcal{U})$. Using barycentric simplices as in 1 above, we get a Lipschitz map $\sigma_{\mathcal{U}}$ from $|Nerve(\mathcal{U})|$ to X. By choosing a locally Lipschitz partition of unity $\{\rho_i\}_{i \in I}$ subordinate to $\mathcal{U}$, we get a locally Lipschitz map $\rho_{\mathcal{U}} : Y \to |Nerve(\mathcal{U})|$. The composition $i_1 := (\sigma_{\mathcal{U}} \circ \rho_{\mathcal{U}})\Big|_K$ satisfies the conditions in 2. If K has zero k-dimensional Hausdorff measure, then so does $\rho_{\mathcal{U}}(K) \subseteq |Nerve(\mathcal{U})|$, since $\rho_{\mathcal{U}}$ is Lipschitz on K. So we can define a Lipschitz map π from $\rho_{\mathcal{U}}(K)$ to the $(k-1)$-skeleton of $|Nerve(\mathcal{U})|$ with the property that $\pi(t)$ lies in the closed simplex

determined by t for every $t \in \rho_{\mathcal{U}}(K)$ (starting with a top dimensional simplex τ of P, project away from a missed interior point to $Bdy\,\tau$, etc). Now $\sigma_{\mathcal{U}} \circ \pi \circ \rho_{\mathcal{U}}\Big|_K : K \to K$ factors as $K \overset{\pi\circ\rho_{\mathcal{U}}}{\longrightarrow} |(Nerve(\mathcal{U}))_{k-1}| \overset{\sigma_{\mathcal{U}}}{\longrightarrow} X$ as desired.

If $\mathrm{GeomDim}(X) \leq n$, then we already know that we can approximate $i_K : K \to K$ with a Lipschitz map j_1 whose image has finite n-dimensional Hausdorff measure, in particular zero $(n+1)$-dimensional Hausdorff measure. So we may apply the previous paragraph to get a map j_2 from $j_1(K)$ to X which approximates the inclusion $j_1(K) \to X$, and which factors through a polyhedron of dimension $\leq n$; the composition $j_2 \circ j_1$ is the desired approximation of i_k. □

Proposition 5.2 *Let X be a CBA space. Then*

1. *If $k > \mathrm{GeomDim}(X)$, then $H_k(U,V) = \{0\}$ for all open pairs (U,V) in X.*
2. *$\sup\{\mathrm{TopDim}(Y) \mid Y \subseteq X$ is compact$\} \leq \mathrm{GeomDim}(X)$.*

Proof. 1. Suppose $k > n = \mathrm{GeomDim}(X)$ and let (U,V) be an open pair in X. Given $[\alpha] \in H_k(U,V)$, there is a compact pair (K,L) and a map $\phi : (K,L) \to (U,V)$ so that $[\alpha] \in \mathrm{Im}\, H_k(\phi)$. Pick $\epsilon > 0$. According to Lemma 5.1, we may approximate the inclusion $\phi(K) \to X$ with maps $i_1 : \phi(K) \to X$ which factor through finite polyhedra of dimension $\leq n$. If $d(\phi_1, \phi)$ is sufficiently small then ϕ_1 will be homotopic to ϕ (as a map of pairs) by the geodesic homotopy. In this case we have $\mathrm{Im}\, H_k(\phi) = \mathrm{Im}\, H_k(\phi_1) = \{0\}$.

2. If $K \subseteq X$ is compact, we may apply 2 of Lemma 5.1 to get a map f from K to a polyhedron P of dimension $\leq n$ so that $Diam(f^{-1}(\tau)) < \epsilon$ for every simplex τ of P. This shows that open covers of K with order $\leq n+1$ are cofinal, so $\mathrm{TopDim}(K) \leq n$. □

Proposition 5.3 *Let X be a CBA space with $\mathrm{GeomDim}(X) = n < \infty$. Let I be a finite set, let $U_i \subseteq X$ be open sets such that all intersections $U_{i_1} \cap \ldots \cap U_{i_k}$ are either empty or contractible, and suppose $\cup_{i \in I} U_i \subset Y$ where Y is a contractible open set. Then if $\cap_{i\in I} U_i = \emptyset$, there is a subset $J \subseteq I$ with $|J| \leq n+1$ so that $\cap_{i \in J} U_i = \emptyset$.*

Proof. Let k be the largest integer so that every subset $I' \subseteq I$ with $|I'| \leq k$ gives $\cap_{i \in I'} U_i \neq \emptyset$. Pick $J \subset I$ with $|J| = k+1$ so that $\cap_{i\in J} U_i = \emptyset$, let $\mathcal{V} = \{U_i\}_{i \in J}$, and $V = \cup_{i\in J} U_i$. The simplicial complex $Nerve(\mathcal{V})$ is isomorphic to $Bdy(\Delta_k)$. As nonempty intersections are contractible, Mayer-Vietoris sequences show that $H_{k-1}(V) \simeq H_{k-1}(Bdy(\Delta_k)) \neq \{0\}$. From the exact sequence of the pair (Y,V) we get $H_k(Y,V) \neq \{0\}$. By Corollary 5.2 we have $k \leq n$. □

6 Minimum sets and subsets homeomorphic to $\mathbb{R}^n$

In this section we study the images of barycentric simplices. The main results are Theorem 6.3, which produces biHölder images of open sets $U \subset \mathbb{E}^n$ in spaces with nondegenerate n-simplices, and Theorem 6.8, which gives biLipschitz images of open sets $U \subset \mathbb{E}^n$ in CBA spaces with geometric dimension n.

6.1 General minimum sets

If X is a geodesic metric space and $f = (f_0, \dots, f_n) : X \to \mathbb{R}^{n+1}$ is a map with convex component functions, then $f(X) \subset \mathbb{R}^{n+1}$ is not necessarily convex, but the "bottom" of $f(X)$ behaves like a convex set in many respects. We will use this idea to produce nice subsets of X.

Notation. If $t, t' \in \mathbb{R}^n$, we say that $t \prec t'$ (resp. $t \preceq t'$) if $t_i < t'_i$ (resp $t_i \le t'_i$) for all $1 \le i \le n$. **CH(Y)** denotes the convex hull of $Y \subset \mathbb{R}^n$.

Definition 6.1 *If $S \subset \mathbb{R}^n$, then an element $s \in S \subset \mathbb{R}^n$ is* **minimal** *if there is no $s' \in S$ with $s' \prec s$.* **Minset(S)** *denotes the set of minimal elements of S; $Minset(S)$ is a closed subset of S. If $J \subset \{1, \dots, n\}$, let $\pi_J : \mathbb{R}^n \longrightarrow \mathbb{R}^{|J|}$ be the corresponding projection. The* **J-face** *of S is $\pi_J^{-1}(Minset(\pi_J(S)))$. $s \in Minset(S)$ is* **nondegenerate** *if it does not lie in the J-face of S for any nonempty proper subset $J \subset \{1, \dots, n\}$;* **Nondeg(S)** *denotes the set of nondegenerate points of S.*

Let X be a bounded complete length space with unique geodesic segments joining pairs of points. Given $x_0, x_1 \in X$ and $\lambda \in [0,1]$, we let $(1-\lambda)x_0 + \lambda x_1$ denote $\gamma(\lambda)$ where $\gamma : [0,1] \to X$ is the unique constant speed geodesic joining x_0 to x_1.

Definition 6.2 *Suppose $C > 0$, and let $f_0, \dots, f_n$ be L-Lipschitz C-convex non-negative functions on X. Set $f := (f_0, \dots, f_n) : X \longrightarrow \mathbb{R}^{n+1}$, $S := f(X) \subset \mathbb{R}^{n+1}_{\geq} := \{t \in \mathbb{R}^{n+1} \mid t_i \ge 0 \text{ for all } i\}$, $Minset(f) := f^{-1}(Minset(S))$, $Nondeg(f) := f^{-1}(Nondeg(S))$.*

Theorem 6.3
1. *If for $1 < k \le N$, $s_k \in S$ are distinct, $\lambda_k > 0$, $\sum_{k=1}^N \lambda_k = 1$, then there is an $s' \in S$ with $s' \prec \sum \lambda_k s_k$.*
2. *$Minset(\bar{S}) = Minset(S)$.*
3. *If $x \in Minset(f)$ then there is a system of weights $\alpha \in \Delta_n := \{t \in \mathbb{R}^{n+1} \mid t_i \ge 0 \text{ for all } i \text{ and } \sum t_i = 1\}$ so that x is a strict minimum for $\sum \alpha_i f_i$. Moreover, setting $\phi(t) := \sum \alpha_i t_i$, $s = f(x)$, we have*
$$\phi(s - s') \ge \frac{C}{2L^2} \parallel s - s' \parallel^2_{l^\infty} \tag{6.4}$$
for every $s' \in S$, where $\|\cdot\|_{l^\infty}$ is the l^∞ norm on $\mathbb{R}^{n+1}$. Consequently $Minset(f)$ is precisely the image of the barycentric simplex $\sigma_f : \Delta_n \longrightarrow X$, and $Nondeg(f)$ is nonempty iff σ_f is nondegenerate.
4. *If $CH(S)$ is the convex hull of S and $p \in Bdy(CH(S))$ admits a supporting linear functional $\sum_{i=0}^n \alpha_i t_i$ with $\alpha_i > 0$ for all $0 \le i \le n$, then $p \in Minset(S)$.*
5. *$f\big|_{Minset(f)} : Minset(f) \longrightarrow \mathbb{R}^{n+1}$ is a Lipschitz homeomorphism onto $Minset(S)$ with $\frac{1}{2}$-Hölder inverse. The image $f(Minset(f)) = Minset(S)$ is a Lipschitz graph over a closed subset of the hyperplane $H = \{t \in \mathbb{R}^{n+1} \big| \sum t_i = 0\}$. $Nondeg(S)$ projects to an open subset of H, so $Nondeg(f)$ is biHölder to an open subset of $\mathbb{R}^n$.*

Corollary 6.5 *If X is a CBA space with* $\mathrm{GeomDim}(X) \geq n$*, then there is an open set $U \subset \mathbb{E}^n$, and a $\frac{1}{2}$-Hölder embedding $U \to X$ with Lipschitz inverse.*

Proof of Corollary 6.5. By Lemma 4.11 there is an admissible $z \in X^{n+1}$ so that the barycentric simplex σ_z is nondegenerate. Setting $f = (f_0, \ldots, f_n) = (d^2_{z_0}, \ldots, d^2_{z_n})$ we see by 3 of Theorem 6.3 that $Nondeg(f)$ is nonempty; hence 5 of Theorem 6.3 applies. □

Proof. 1. Since $\sum_{k=1}^N \lambda_k s_k = \lambda_1 s_1 + (1-\lambda_1)(\sum_{k=2}^N \frac{\lambda_k}{(1-\lambda_1)} s_k)$, 1 follows from the strict convexity of the f_i's and induction.

2. Let $s \in \bar{S}$ be minimal in $\bar{S}$, and pick a sequence $x_k \in X$ so that $f(x_k) \to s$. By the argument of Lemma 2.3, x_k is Cauchy and so $f(\lim_{k\to\infty} x_k) = s$, which proves 2.

3. Suppose $x \in Minset(f)$, $s = f(x)$. s is minimal in $CH(S)$, for otherwise by 1, s would not be minimal in S. Therefore the convex set $CH(S)$ is disjoint from the open convex set $U = \{t \in \mathbb{R}^{n+1} \big| t \prec s\}$, and hence there is a $\phi \in (\mathbb{R}^{n+1})^* \setminus \{0\}$, $\phi(t) = \sum \alpha_i t_i$, so that $\inf \phi(CH(S)) \geq \sup \phi(U)$. So we have $\alpha_i \geq 0$ and without loss of generality we may assume $\sum \alpha_i = 1$. ϕ is the supporting linear functional sought in 3. The C-convex function $\phi \circ f$ attains a minimum at x, so for every $x' \in X$ we have

$$(\phi \circ f)(x') \geq (\phi \circ f)(x) + \frac{C}{2} d^2(x, x').$$

If $s' = f(x') \in S$, then $\| s - s' \|_{l^\infty} \leq L d(x, x')$, so the inequality (6.4) follows.

4. Suppose $p \in Bdy(CH(S))$ is supported by a linear functional $\sum_{i=0}^n \alpha_i t_i$ with $\alpha_i > 0$. Since S is a bounded subset of $\mathbb{R}^{n+1}$, every $p \in \overline{CH(S)}$ is a convex combination $\sum_{k=1}^N \mu_k \bar{s}_k$ of at most $n+2$ elements $\bar{s}_k \in \bar{S}$, $\mu_k > 0$, $\sum \mu_k = 1$. If $p \notin \bar{S}$ then $N \geq 2$, and uniform convexity of the f_i gives us $s' \in S$ with $s' \prec p$, contradicting the assumption that $CH(S)$ is supported by $\sum \alpha_i t_i$ at p. Hence $p \in \bar{S}$ and clearly $p \in Minset(\bar{S})$ so by 2 we have $p \in S$, proving 4.

5. Pick $x_1, x_2 \in Minset(f)$, and set $d = d(x_1, x_2)$, $d' = d(f(x_1), f(x_2))$. For each i we have

$$f_i(\frac{1}{2}x_1 + \frac{1}{2}x_2) \leq \frac{1}{2} f_i(x_1) + \frac{1}{2} f_i(x_2) - \frac{C}{2}(\frac{d}{2})^2$$

by the C-convexity of f_i, and

$$\frac{1}{2} f_i(x_1) + \frac{1}{2} f_i(x_2) - f_i(x_1) \leq \frac{d'}{2}$$

by the triangle inequality. Hence

$$f_i(\frac{1}{2}x_1 + \frac{1}{2}x_2) + \frac{C}{2}(\frac{d}{2})^2 - f_i(x_1) \leq \frac{d'}{2}.$$

Since $x_1 \in Minset(f)$, we have $f_i(\frac{1}{2}x_1 + \frac{1}{2}x_2) \geq f_i(x_1)$ for some $0 \leq i \leq n$, so $d' \geq \frac{Cd^2}{4}$. This shows that $f\big|_{Minset(f)}$ satisfies a reverse Hölder inequality, and so f has a Hölder inverse. This proves the first claim of 5.

To see that $Minset(S) \subset \mathbb{R}^{n+1}$ is a Lipschitz graph over a closed subset of the hyperplane $H := \{t \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1\}$, note that if $p \in Minset(S)$ then $Minset(S)$ is disjoint from the union of the orthants $\{t \in \mathbb{R}^{n+1} \mid t \prec p\}$ and $\{t \in \mathbb{R}^{n+1} \mid p \prec t\}$.

Now suppose $p \in Nondeg(S)$. By 3, p lies on the boundary of $CH(S)$ so $CH(S)$ can be supported at p by a linear functional $\phi(t) := \sum \alpha_i t_i$, $\alpha_i \geq 0$. If $\phi'(t) = \sum \alpha'_i t_i$ is any supporting functional for $CH(S)$ at p, then $\alpha'_i > 0$ for all i; otherwise a convex combination of ϕ and ϕ' would give another supporting linear functional $\phi''(t) = \sum \alpha''_i t_i$ where $\alpha''_i \geq 0$ for all i and $\alpha''_i = 0$ for some i, contradicting the nondegeneracy of p. By an obvious limiting argument, any point $p' \in Bdy(CH(S))$ sufficiently close to p also has the property that every supporting linear functional has positive coefficients; consequently, a neighborhood U of p in $Bdy(CH(S))$ is a convex hypersurface contained in $Nondeg(S)$ by 4, and this neighborhood projects to an open set in H. □

6.2 Top dimensional minimum sets

We now specialize the setup of section 6.1 to the case when X is a $CAT(\kappa)$ space with the property that $d(x_1, x_2)$ is less than $D(\kappa)$ (the diameter of the model space M^2_κ with constant curvature κ) for all $x_1, x_2 \in X$, the C-convex functions $f_0, \dots, f_n$ are the squares of distance functions d_{z_i}, $z_i \in X$, and $\mathrm{GeomDim}(X) = n$.

Theorem 6.6 *If K is a compact subset of $Nondeg(f)$, then there is a constant $L_1 > 0$ so that for every $x \in K$ and every $x' \in X$ we have*

$$d(f(x), f(x')) \geq L_1 d(x, x') \tag{6.7}$$

In particular, $f\Big|_K : K \longrightarrow \mathbb{R}^{n+1}$ is a biLipschitz homeomorphism onto its image. By part 5 of Theorem 6.3, $Nondeg(S)$ is the graph (over the hyperplane H) of a Lipschitz function defined on an open subset of H. Therefore $Nondeg(f)$ is locally biLipschitz to an open subset of $\mathbb{E}^n$.

Proof. If $x \in Nondeg(f)$ and $0 \leq i \leq n$, then there is a $y \in X$ so that $\angle_x(y, z_j) < \frac{\pi}{2}$ for every $j \neq i$. Upper semicontinuity of the angle function $x \mapsto \angle_x(y, z_i)$ (see section 2.1) implies that there is an $r_x < \frac{\pi}{2}$ so that $\angle_{x'}(y, z_j) \leq r_x$ for every $j \neq i$ when x' is sufficiently close to x. Hence by the compactness of K, we may conclude that the points in K are "uniformly nondegenerate" in the sense that there is an $r < \frac{\pi}{2}$ so that for every $x \in K$ and every $0 \leq i \leq n$ there is a $v \in \Sigma_x X$ with $\angle_x(v, \overrightarrow{xz_j}) \leq r$ for $j \neq i$.

We now show that the failure of a reverse Lipschitz condition (6.7) implies that $\mathrm{GeomDim}(\Sigma_x X) \geq n$ for some $x \in K$, which contradicts our assumption that $n = \mathrm{GeomDim}(X) \geq 1 + \mathrm{GeomDim}(\Sigma_x X)$. Choose sequences $x_k \in K$, $x'_k \in X - \{x_k\}$ so that $\lim_{k\to\infty} \frac{d(f(x_k), f(x'_k))}{d(x_k, x'_k)} = 0$. Note that $\lim_{k\to\infty} d(x_k, x'_k) = 0$, for otherwise the C-convexity of the f_i's implies that the midpoint m_k of the segment $\overline{x_k x'_k}$ will satisfy $f(m_k) \prec f(x_k)$ for sufficiently large k, contradicting $x_k \in Minset(f)$. Hence the comparison angles $\tilde{\angle}_{x_k}(x'_k, z_i)$ satisfy $\lim_{k\to\infty} \tilde{\angle}_{x_k}(x'_k, z_i) = \frac{\pi}{2}$, so $\limsup_{k\to\infty} \angle_{x_k}(x'_k, z_i) \leq \frac{\pi}{2}$. Applying the preceding paragraph, for each $0 \leq i \leq n$ and every k choose $v^i_k \in \Sigma_{x_k} X$ such that $\angle_{x_k}(v^i_k, \overrightarrow{x_k z_j}) \leq r < \frac{\pi}{2}$ for $j \neq i$. Hence for sufficiently large k, we have that $\cap_{0\leq i\leq n} B_{\overrightarrow{x_k z_i}}(r) \subseteq \cap_{0\leq i\leq n} B_{\overrightarrow{x_k z_i}}(\frac{\pi}{2}) = \emptyset$, $\cap_{j\neq i} B_{\overrightarrow{x_k z_j}}(r) \neq \emptyset$ for every i, and $\cup_{0\leq i\leq n} B_{\overrightarrow{x_k z_i}}(r) \subseteq B_{\overrightarrow{x_k x_k'}}(\pi)$. By Proposition 5.3 we have $\mathrm{GeomDim}(\Sigma_{x_k} X) \geq n$, which is a contradiction. $\Box$

Let X, z_i, f_i be as in the previous theorem.

Theorem 6.8 *Suppose $x_0 \in Nondeg(f)$, and let $s_0 = f(x_0)$ be a point where the convex hypersurface $f(Nondeg(f)) = Nondeg(S)$ is differentiable. Let $\phi = \sum \alpha_i t_i$ be the supporting linear functional for $Minset(S)$ at s_0, $\alpha_i > 0$, $\sum \alpha_i = 1$, let $P = \phi^{-1}(\phi(x_0))$ be the supporting hyperplane, $Q := C_{s_0} P \subset C_{s_0} \mathbb{R}^{n+1}$, and $g = \left(f\Big|_{Minset(f)} \right)^{-1} : Minset(S) \longrightarrow Minset(f)$. Then*

1. *g has a well-defined tangent cone $C_{s_0} g : C_{s_0} P = C_{s_0} Minset(S) = Q \longrightarrow C_{x_0} X$, in the sense that if $x_k \in Minset(f)$, $s_k = f(x_k) \in Minset(S) \setminus \{s_0\}$, $\lim_{k\to\infty} s_k = s_0$, and $\frac{1}{d(s_k,s_0)} \log_{s_0} s_k \in C_{s_0} \mathbb{R}^{k+1}$ converges to $v \in Q$, then $\frac{1}{d(s_k,s_0)} \log_{x_0}(x_k)$ converges to $(C_{s_0} g)(v)$. Hence $Minset(f)$ has a well-defined tangent cone $\hat{Q} = Im(C_{s_0} g)$ at x_0 which is isometric to $\mathbb{E}^n$. $C_{s_0} g$ is an affine map.*

2. *With respect to a suitable Euclidean metric d_1 on $\mathbb{R}^{n+1}$, we find that*
$$\lim_{s_1,s_2\to s_0} \frac{d(g(s_1), g(s_2))}{d_1(s_1, s_2)} = 1.$$
In particular, for every $\epsilon > 0$ we have an open subset $U \subset \mathbb{E}^n$ and a $(1+\epsilon)$-biLipschitz embedding $U \longrightarrow X$.

3. *$\overrightarrow{x_0 z_i} \in \hat{Q}$.*

We remark that the top dimensionality (the number of functions is $1+\mathrm{GeomDim}(X)$) of the nondegenerate minimum set is essential in Theorem 6.8, see example 6.14.

Proof. The idea of the proof is that near s_0, $Nondeg(S)$ can be viewed as a graph over P of a function with small Lipschitz constant; this together with Theorem 6.6 forces $f\Big|_{Minset(f)}$ to be "approximately affine" and $Minset(f)$ to be "approximately convex" near x_0.

Lemma 6.9 (f is approximately affine) *There is a function $\epsilon(r)$ with $\lim_{r\to 0} \epsilon(r) = 0$ with the following property. If $x_1, x_2 \in Minset(f)$, $s_i = f(x_i) \in Minset(S)$; $d(x_i, x_0) < r$; λ_1, λ_2 are weights; $s_3 \in Minset(S)$ satisfies $s_3 \preceq \lambda_1 s_1 + \lambda_2 s_2$; $f(x_3) = s_3$; then*

$$d(x_3, \lambda_1 x_1 + \lambda_2 x_2) \leq \epsilon(r) d(x_1, x_2) Min(\lambda_1, \lambda_2) \tag{6.10}$$

and

$$1 - \epsilon(r) \leq \frac{d(x_1, x_3)}{\lambda_2 d(x_1, x_2)} \leq 1 + \epsilon(r), 1 - \epsilon(r) \leq \frac{d(x_2, x_3)}{\lambda_1 d(x_1, x_2)} \leq 1 + \epsilon(r). \tag{6.11}$$

In particular, the comparison triangles for $\Delta(x_1, x_2, x_3)$ and $\Delta(s_1, s_2, s_3)$ have angles tending to $\{0, \pi\}$ as $r \to 0$.

Proof. Since $Nondeg(S)$ is (locally) a convex hypersurface and $s_0 \in Nondeg(S)$ is a differentiable point, the supporting hyperplanes at points $s \in Nondeg(S)$ tend to P as $s \to s_0$. Therefore if $\pi_P : \mathbb{R}^{n+1} \longrightarrow P$ is the orthogonal projection, we have

$$\lim_{d(s_i,s_0)\to 0} \frac{d(s_1,s_2)}{d(\pi_P(s_1),\pi_P(s_2))} = 1$$

and $Nondeg(S) \cap B_{s_0}(r)$ is the graph over P of a function with Lipschitz constant tending to zero with r. Hence if λ_1, λ_2 are weights, $s_1, s_2, s_3 \in Nondeg(S)$, $s_i \in B_{s_0}(r)$ and $s_3 \preceq \lambda_1 s_1 + \lambda_2 s_2$, we have

$$d(s_3, \lambda_1 s_1 + \lambda_2 s_2) \leq \epsilon_1(r) d(s_1, s_2) Min(\lambda_1, \lambda_2)$$

where $\lim_{r\to 0} \epsilon_1(r) = 0$. With $x_i = g(s_i)$ we get $L_1 d(x_3, \lambda_1 x_1 + \lambda_2 x_2) \leq d(f(x_3), f(\lambda_1 x_1 + \lambda_2 x_2)) = d(s_3, f(\lambda_1 x_1 + \lambda_2 x_2))$ by Theorem 6.6, but since $f(\lambda_1 x_1 + \lambda_2 x_2) \preceq \lambda_1 s_1 + \lambda_2 s_2$ we have

$$d(s_3, f(\lambda_1 x_1 + \lambda_2 x_2)) \leq 2\epsilon_1(r) d(s_1, s_2) Min(\lambda_1, \lambda_2).$$

(6.10) now follows from the fact that f is Lipschitz. (6.11) follows from (6.10) (after adjusting $\epsilon(r)$ if necessary) and the triangle inequality. □

It follows easily from (6.10) and (6.11) that if $s_k \in Minset(S) \setminus \{s_0\}$ is a sequence with $\lim_{k\to\infty} s_k = s_0$, and $\frac{1}{d(s_k,s_0)} \log_{s_0} s_k \in C_{s_0}\mathbb{R}^{n+1}$ converges to some $v \in Q$, then setting $x_k = g(s_k)$, $\lim_{k\to\infty} \frac{d(x_0,x_k)}{d(s_0,s_k)}$ and $\lim_{k\to\infty} \frac{1}{d(x_0,x_k)} \log_{x_0} x_k$ exist. We therefore have a well-defined map $C_{s_0} g : Q \longrightarrow C_{s_0} X$. The estimates (6.10) and (6.11) also imply that $C_{s_0} g$ is an affine map in the sense that for every $v_1, v_2 \in Q$ we have $(C_{s_0} g)(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 (C_{s_0} g)(v_1) + \lambda_2 (C_{s_0} g)(v_2)$ for any weights λ_1, λ_2. Hence $\hat{Q} := (C_{s_0} g)(Q)$ is a convex subset. Also, the fact that g is locally biLipschitz near s_0 (Theorem 6.6) implies that $C_{s_0} g$ is biLipschitz.

If $\gamma_1, \gamma_2 : \mathbb{R} \longrightarrow Q$ are two constant speed geodesics with $d(\gamma_1(t), \gamma_2(t)) < C$, then $\eta_1 = (C_{s_0} g) \circ \gamma_1$ and $\eta_2 = (C_{s_0} g) \circ \gamma_2$ are constant speed geodesics in the Hadamard space $\hat{Q}$ which satisfy $d(\eta_1(t), \eta_2(t)) \leq L_1 C$ since g is L_1^{-1}-Lipschitz; therefore $d(\eta_1(t), \eta_2(t))$ is constant and the η_j's bound a flat strip. Hence it follows that $\hat{Q}$ is flat, and we have proved 1 of Theorem 6.8.

We now prove 2. Let $\langle \cdot, \cdot \rangle_Q$ be the inner product on Q induced from $\hat{Q}$ by $C_{s_0} g$; let $\langle \cdot, \cdot \rangle_1$ be the inner product on $C_{s_0}\mathbb{R}^{n+1}$ which extends $\langle \cdot, \cdot \rangle_Q$, which defines the same $Q^\perp$ as the standard inner product, and which agrees with the standard inner product on $Q^\perp$. Let d_1 be the distance function on $\mathbb{R}^{n+1} \simeq C_{s_0}\mathbb{R}^{n+1}$ defined by $\langle \cdot, \cdot \rangle_1$. We now have that

$$\lim_{s_1\to s_0} \frac{d(g(s_1), g(s_0))}{d_1(s_1, s_0)}$$

$$= \lim_{s_1\to s_0} \left(\frac{d(g(s_1), g(s_0))}{d(s_1, s_0)} \frac{d(s_1, s_0)}{d_1(s_1, s_0)} \right) = 1.$$

This implies the third assertion of theorem 6.8 since if $s_1, s_2 \in Minset(S)$ are sufficiently close to s_0, there will be an $s_3 \in Minset(S)$ with $d(s_3, s_0) >> \max(d(s_1, s_0), d(s_2, s_0))$ and with $\widetilde{\angle}_{s_1}(s_2, s_3) << 1$; so by (6.10) and (6.11) we get

$$\left| \frac{d(g(s_1), g(s_2))}{d_1(s_1, s_0)} - 1 \right| << 1.$$

It remains to prove 3.

Lemma 6.12 *For $0 \leq i \leq n$ let $Df_i : C_{x_0}X \longrightarrow \mathbb{R}$ denote the derivative of f_i, $Df_i(w) = -2d(x_0, z_i)\langle w, \overrightarrow{x_0 z_i}\rangle$ (cf section 2.2). Then $\sum \alpha_i Df_i$ is zero on $\hat{Q}$, where $\phi(t) = \sum \alpha_i t_i$ is the supporting linear functional for $Minset(Q)$ at s_0. Consequently Df_i is affine on $\hat{Q}$ for each i.*

Proof. If $w \in \hat{Q}$, $w = (C_{s_0}g)(v)$, then

$$\sum \alpha_i Df_i(v) = \sum \alpha_i Df_i((C_{s_0}g)(v)) = D\phi(v) = 0.$$

□

Lemma 6.13 *For $0 \leq i \leq n$, $\overrightarrow{x_0 z_i} \in \hat{Q}$.*

Proof. The affineness of the Df_i on $\hat{Q}$ implies that if $w_1, w_2 \in \hat{Q} \cap \Sigma_{x_0}X$ are opposite directions in the flat $\hat{Q}$, then $\angle(w_1, \overrightarrow{x_0 z_j}) + \angle(w_2, \overrightarrow{x_0 z_j}) = \pi$. Therefore if $\overrightarrow{x_0 z_j} \notin \hat{Q}$, we have $\max\{\angle(w, \overrightarrow{x_0 z_j}) \mid w \in \hat{Q} \cap \Sigma_{x_0}X\} < \pi$, so the $(n-1)$ sphere $\hat{Q} \cap \Sigma_{x_0}X$ lies in the contractible set $B_{\overrightarrow{x_0 z_j}}(\pi)$. By Proposition 5.3 this implies that $\mathrm{GeomDim}(\Sigma_{x_0}X) \geq n$, which contradicts our assumption that $\mathrm{GeomDim}(X) = n$. Hence the lemma. □

This completes the proof of Theorem 6.8 □

Example 6.14 *This example shows that the top dimensionality assumption of Theorem 6.8 is necessary. If $z \in X^{n+1}$, $f_i := d^2_{z_i}$ and $p \in Minset(f)$ is a nondegenerate point so that $f(p)$ is a differentiable point of $f(Minset(f))$, then C_pX need not contain an n-flat. Start with the standard upper hemisphere $S^2_+ \subset S^2$, and pick equally spaced points $\xi_0, \xi_1, \xi_2 \in \partial S^2_+$. Letting $B \subset S^2_+$ be a spherical cap centered at the North pole $N \in S^2_+$, let U_i be the geodesic cone over B with vertex at ξ_i, and set $U = \cup U_i$. Modify the metric on $S^2_+ \setminus U$ so that it has curvature $K < 1$, and so that it is in polar coordinate form with radius $\frac{\pi}{2}$. Then we have a $CAT(1)$ space Y, and let X be the Euclidean cone over Y with vertex o. Now if we take $z_i \in X$ to lie on the ray $\overline{o\xi_i}$, then the vertex $o \in C_oX$ will correspond to a differentiable point of $d_z(Minset(z))$, but C_pX doesn't contain any flats.*

7 The Tits boundary, asymptotic cones, flats, and the geometric dimension

For properties of geometric boundaries, Tits boundaries, ultralimits, and asymptotic cones, see section 2 or [KL97]. In this section ω will be a fixed non-principal ultrafilter on $\mathbb{N}$.

Theorem 7.1 *Let $\{(X_i, d_i)\}_{i \in I}$ be a family of Hadamard spaces. Then the following are equivalent:*

1. *There are sequences $i_k \in I$, $\star_k \in X_{i_k}$, $\star_k \in F_k \subset X_{i_k}$, so that $(F_k, \star_k)$ converges to $(\mathbb{E}^r, 0)$ in the pointed Gromov-Hausdorff topology. Equivalently, ω-$lim(X_{i_k}, \star_k)$ contains an r-flat.*

2. *There are sequences $i_k \in I$, $R_k \in \mathbb{R}$, $S_k \subset X_{i_k}$, so that $\lim_{k\to\infty} R_k = \infty$, and $\frac{1}{R_k}S_k$ converges to $B(1) \subset \mathbb{E}^r$ in the Gromov-Hausdorff topology. Equivalently, ω-lim$(\frac{1}{R_k}X_{i_k}, \star_k)$ contains an isometric copy of $B(1)$.*

3. *There are sequences $i_k \in I$, $\star_k \in X_{i_k}$, $\lambda_k \to \infty$ so that ω-lim$(\frac{1}{\lambda_k}X_{i_k}, \star_k)$ is a $CAT(0)$ space with geometric dimension $\geq r$.*

4. *There are sequences $i_k \in I$, $\star_k \in X_{i_k}$, so that $\partial_T(\omega\text{-}lim(X_{i_k}, \star_k))$ has geometric dimension $\geq r-1$.*

Remark. In 2, we really only use the fact that the $\frac{1}{R_k}S_k$ contain subsets which converge to $\{\pm e_1, \ldots, \pm e_r, 0\} \subset \mathbb{E}^r$, where e_i is the i^{th} standard basis vector.

Corollary 7.2 *If the X_i's have uniformly bounded geometry (in the sense that $\{(X_i, \star) \mid i \in I, \star \in X_i\}$ is a Gromov-Hausdorff precompact family of pointed metric spaces), then there is an $r_0 \in \mathbb{N}$ such that for any sequences $\lambda_k \in \mathbb{R}$, $i_k \in I$, $\star_k \in X_{i_k}$ with ω-lim $\lambda_k = 0$ we have*

1. GeomDim$(\omega\text{-}lim(\frac{1}{\lambda_k}X_{i_k}, \star_k)) \leq r_0$.
2. GeomDim$(\partial_T(\omega\text{-}lim(X_{i_k}, \star_k))) \leq r_0$.

Proof of Corollary 7.2. The uniformly bounded geometry of the X_i's implies that there is an $r_0 \in \mathbb{N}$ so that for any sequence $i_k \in I$ and any sequence of basepoints $\star_k \in X_{i_k}$, ω-lim$(X_{i_k}, \star_k)$ contains no r-flats with $r > r_0$. Therefore the corollary follows from Theorem 7.1. □

Proof of theorem 7.1.

Clearly 1 implies 2 , 3 , and 4.

3 $\Longrightarrow$ 2. Let $X_\omega = \omega\text{-lim}(\frac{1}{\lambda_k}X_{i_k}, \star_k)$. Since GeomDim$(X_\omega) \geq r$ we can find $p \in X_\omega$ so that GeomDim$(\Sigma_p X) \geq r-1$. By Lemma 3.1, there are sequences $\bar{T}_j \subset X_\omega$, $R_j \in \mathbb{R}$ so that $\lim_{k\to\infty} d(p, \bar{T}_j) \to 0$, and $\frac{1}{R_j}\bar{T}_j$ converges to $B(1) \subset \mathbb{E}^r$ in the Gromov-Hausdorff topology. This means that for each j we can find sequences $i'_k \in \mathbb{N}$, $\lambda'_k \in \mathbb{R}$, $T^j_k \subset X_{i'_k}$ so that $\frac{1}{\lambda'_k}T^j_k$ converges to $\bar{T}_j$ in the Gromov-Hausdorff topology. Passing to a suitable subsequence of the double sequence T^j_k we get a sequence S_k as described in 2.

4 $\Longrightarrow$ 2. Let Y be the Euclidean cone over $\partial_T(\omega\text{-lim}(X_{i_k}, \star_k))$, with vertex o. By Lemma 3.1, there are subsets $T_k \subset Y$ and a sequence $\lambda_k \to 0$ so that $\frac{1}{\lambda_k}T_k$ converges to $B(1) \subset \mathbb{E}^r$ in the Gromov-Hausdorff topology, and $d(T_k, o) \to 0$. Since any finite metric space $Z \subset Y$ is a Hausdorff limit of rescaled finite metric spaces in X_i (this follows easily from the definition of the Tits metric), we get sequences $\bar{\lambda}_j \to \infty$, $\bar{T}_j \subseteq \omega\text{-lim}(X_{i_k}, \star_k)$ so that $\frac{1}{\bar{\lambda}_j}\bar{T}_j$ converges to $B(1) \subset \mathbb{E}^r$ in the Gromov-Hausdorff topology. Each $\bar{T}_j$ is the Gromov-Hausdorff limit of a suitable sequence of elements of $\{S_i\}_{i\in I}$ so 2 follows.

2 $\Longrightarrow$ 1. We will show that for a suitable choice of basepoints $\star_k \in X_{i_k}$ the ultralimit ω-lim$(X_{i_k}, \star_k)$ contains an r-flat. To simplify notation slightly we assume that $I = \mathbb{N}$ and that $i_k = k \in \mathbb{N}$. We will also assume that $r \geq 2$ since the implication is trivial otherwise.

By assumption $X_k \supset \frac{1}{R_k}S_k \longrightarrow B(1) \subset \mathbb{E}^r$ in the Gromov-Hausdorff topology, so we can find a sequence $\phi_k : \mathbb{E}^r \supset B(0, R_k) \to X_k$ of $\epsilon_k R_k$-Hausdorff approximations (definition 2.6) where $\epsilon_k \to 0$. For $1 \leq i \leq r$ let $e_i \in \mathbb{E}^r$ be the i^{th} standard basis vector, let $z_k^{\pm i} := \phi_k(\pm R_k e_i)$ and let $Z_k := \{z_k^{\pm i}\}_{1 \leq i \leq r} \subset X_k$. Define $f_k : X_k \to \mathbb{R}$ to be the average distance from $Z_k \subset X_k$, i.e. $f_k(p) := \frac{1}{2r}\sum_{z \in Z_k} d(p, z)$. f_k is 1-Lipschitz and convex. Choose $\star_k \in X_k$ so that $f_k(\star_k) \leq \epsilon_k + \inf_{X_k} f_k$. We will extract an ultralimit of the configuration $\{\overline{\star_k z_k^{\pm i}}\}_{1 \leq i \leq r}$ of geodesic segments to get a configuration of geodesic rays and then we will show that these geodesic rays span an r-flat.

Lemma 7.3 $\lim_{k\to\infty} \frac{1}{R_k} d(\star_k, \phi_k(0)) = 0.$

Proof. We have $\limsup_{k\to\infty} \frac{1}{R_k} f_k(\star_k) \leq \limsup_{k\to\infty} \frac{1}{R_k} f_k(\phi_k(0)) = 1$. On the other hand $\liminf_{k\to\infty} \frac{1}{R_k}[d(z_k^i, \star_k) + d(\star_k, z_k^{-i})] \geq \liminf_{k\to\infty} \frac{1}{R_k} d(z_k^i, z_k^{-i}) = 2$, so $\liminf_{k\to\infty} \frac{1}{R_k} f_k(\star_k) \geq 1$. Combining these inequalities we get $\lim_{k\to\infty} \frac{1}{R_k} f_k(\star_k) = 1$, $\lim_{k\to\infty} \frac{1}{R_k}[d(z_k^i, \star_k) + d(\star_k, z_k^{-i})] = 2$, $\lim_{k\to\infty} \frac{1}{R_k} d(\star_k, \overline{z_k^i z_k^{-i}}) = \lim_{k\to\infty} \frac{1}{R_k} d(\star_k, \overline{\phi_k(0) z_k^i} \cup \overline{\phi_k(0) z_k^{-i}}) = 0$. This forces $\lim_{k\to\infty} \frac{1}{R_k} d(\star_k, \phi_k(0)) = 0$ when $r \geq 2$ since $\lim_{k\to\infty} \tilde{\angle}_{\phi_k(0)}(z_k^{\pm i}, z_k^{\pm j}) = \frac{\pi}{2}$ when $i \neq j$. □

Now consider the ultralimit $(X_\omega, \star_\omega) := \omega\text{-lim}(X_k, \star_k)$. We have geodesic rays $\omega\text{-lim}\,\overline{\star_k z_k^{\pm i}}$ with ideal boundary points $\xi^{\pm i} \in \partial_T X_\omega$. If we let $f_k^{\pm i}(\cdot) := d(z_k^{\pm i}, \cdot) - d(z_k^{\pm i}, \star_k)$ be the normalized distance function, then $\omega\text{-lim}\, f_k^{\pm i}$ is the Busemann function $b_{\xi^{\pm i}}$ of the geodesic ray $\overline{\star_\omega \xi^{\pm i}}$. Therefore $\star_\omega$ is a minimum of $b := \frac{1}{2r}\sum_i (b_{\xi^i} + b_{\xi^{-i}})$ because if $x_\omega = \omega\text{-lim}\, x_k \in X_\omega$, then

$$b(\star_\omega) = 0 = \omega\text{-lim}(-\epsilon_k) \leq \omega\text{-lim}[f_k(x_k) - f_k(\star_k)] = \omega\text{-lim}\,\frac{1}{2r}\sum_i [f_k^i(x_k) + f_k^{-i}(x_k)]$$

$$= \frac{1}{2r}\sum_i [b_{\xi^i}(x_\omega) + b_{\xi^{-i}}(x_\omega)] =: b(x_\omega).$$

On the other hand, by Lemma 7.3 we have $\omega\text{-lim}\,\tilde{\angle}_{\star_k}(z_k^{\pm i}, z_k^{\pm j}) = \frac{\pi}{2}$ ($\tilde{\angle}$ denotes the comparison angle, see section 2.1) when $1 \leq i \neq j \leq r$, so we conclude that $\angle_T(\xi^{\pm i}, \xi^{\pm j}) \leq \frac{\pi}{2}$ for $1 \leq i \neq j \leq r$. Set $w^{\pm i} := \overrightarrow{\star_\omega \xi^{\pm i}} \in \Sigma_{\star_\omega} X_\omega$. We have $\angle_{\star_\omega}(w^{\pm i}, w^{\pm j}) \leq \angle_T(\xi^{\pm i}, \xi^{\pm j}) \leq \frac{\pi}{2}$. The directional derivative of b in the direction of $v \in \Sigma_{\star_\omega} X_\omega$ is (see equation 2.5)

$$-\frac{1}{2r}\sum_i [\langle w^i, v\rangle + \langle w^{-i}, v\rangle].$$

With $v = w^{\pm j}$ we find that the directional derivative is ≤ 0 since $\langle w^{\pm i}, w^{\pm j}\rangle \geq 0$ when $i \neq j$ and $\langle w^j, w^{\pm j}\rangle + \langle w^{-j}, w^{\pm j}\rangle \geq 0$. As $\star_\omega$ minimizes b we have equality everywhere, forcing $\angle_{\star_\omega}(w^i, w^{-i}) = \angle_T(\xi^i, \xi^{-i}) = \pi$ and $\angle_{\star_\omega}(w^{\pm i}, w^{\pm j}) = \angle_T(\xi^{\pm i}, \xi^{\pm j}) = \frac{\pi}{2}$ for $i \neq j$. This easily implies that the convex hull of the rays $\overline{\star_\omega \xi^{\pm i}}$ is an r-flat $F_\omega \subset X_\omega$ (to see this, assume inductively that the convex hull $CH(\{\overline{\star_\omega \xi^{\pm i}}\}_{i \leq m})$ is an m-flat F_m in X_ω, and observe that the nonnegative convex function $b_{\xi^{+(m+1)}} + b_{\xi^{-(m+1)}}$ is zero on F_m, so the convex hull $CH(F_m \cup \{\overline{\star_\omega \xi^{\pm(m+1)}}\}$ is isometric to $F_m \times \mathbb{R} \simeq \mathbb{E}^{m+1}$). □

Corollary 7.4 *If X is a locally compact $CAT(0)$ space with cocompact isometry group, then every compact subset of any asymptotic cone of X has topological dimension* $\leq \mathrm{TopDim}(X)$.

8 Proofs of Theorems A, B, and C

Proof of Theorem A. For $1 \leq i \leq 4$ let $n_i \in \mathbb{N} \cup \{\infty\}$ denote the i^{th} quantity in the statement of Theorem A. Lemma 3.1 proves $\mathrm{GeomDim}(X) \leq n_3$. Proposition 5.2 proves $n_1 \leq \mathrm{GeomDim}(X)$ and $n_2 \leq \mathrm{GeomDim}(X)$; Proposition 3.2 proves $n_3 \leq n_1$ and $n_3 \leq n_2$. Obviously $n_4 \leq n_1$. Theorem 6.3 proves $\mathrm{GeomDim}(X) \leq n_4$. □

Proof of Theorem B. Let $n = \mathrm{GeomDim}(X)$. By Lemma 4.11 there is a $z \in X^{n+1}$ which is (κ, r)-admissible for some κ, r, such that σ_z is nondegenerate. By 3 of Theorem 6.3 if we let $f_i := d^2_{z_i} : \cap B_{z_i}(r) \to \mathbb{R}$, then $Nondeg(f)$ is nonempty. By 1 of Theorem 6.8 we have $p \in X$ whose space of directions contains an isometrically embedded standard $(n-1)$-sphere $Z \subseteq \Sigma_p X$. Since Z is an absolute neighborhood retract, there is an open neighborhood $V \subset \Sigma_p X$ of Z so that $H_{n-1}(Z) \longrightarrow H_{n-1}(V)$ is a monomorphism. Then $H_{n-1}(V) \longrightarrow H_{n-1}(\Sigma_p X)$ is also a monomorphism since $H_n(\Sigma_p X, V) = 0$ by Theorem A. So $H_{n-1}(\Sigma_p X) \neq 0$. We may find a map $\phi : Z \longrightarrow X - \{p\}$ so that $\phi(Z)$ is in the domain Y of $\log_{\Sigma_p X}$ and $\log_{\Sigma_p X} \circ \phi$ is arbitrarily close to the inclusion $Z \longrightarrow \Sigma_p X$. Identifying Z with the unit sphere $S^{n-1} \subset \mathbb{E}^n$ by an isometry, we define a map $\bar{\phi} : \mathbb{E}^n \supset B^n \to Y$ by declaring that if $\lambda \in [0,1]$, $x \in S^{n-1}$, then $\bar{\phi}(\lambda x)$ is the unique point on the segment $\overline{p\phi(x)}$ at distance $\lambda d(p, \phi(x))$ from p. Hence we get a map of pairs $\bar{\phi} : (B^n, S^{n-1}) \longrightarrow (Y, Y - \{p\})$ which is nontrivial on H_n since its boundary is nontrivial. Hence $H_n(X, X - \{p\}) \neq \{0\}$ by excision. This shows that each of the quantities in the statement of Theorem B is $\geq \mathrm{GeomDim}(X)$. The remaining inequalities are contained in Theorem A. □

Proof of Theorem C. See Theorem 7.1. □

9 Questions from *Asymptotic Invariants of infinite groups.*

On pp.127-33 of [Gro93] Gromov discusses a number of issues relating to the large-scale geometry of Hadamard spaces. The main results of this paper – especially the more general version of Theorem C formulated in Theorem 7.1 – settle many of the questions raised in Gromov's discussion provided one replaces topological dimension (Gromov's "dim") with compact topological dimension:

Definition 9.1 *If Z is a topological space, then the* **compact topological dimension** *of Z is*

$$\mathrm{CTopDim}(Z) := \sup\{\mathrm{TopDim}(K) \mid K \subset Z \text{ is compact}\}.$$

We now comment on some of the questions.

DimCon$_\omega$*X* $\leq$ *DimX, [Gro93, p.129].* We reformulate this as follows: if X is an arbitrary Hadamard space, then $\mathrm{CTopDim}(X_\omega) \leq \mathrm{CTopDim}(X)$ for any asymptotic cone of X. To see this, suppose X_ω is an asymptotic cone of X and $\mathrm{CTopDim}(X_\omega) \geq k$. Then by Theorem A we have sequences $R_j \to 0$, $S_j \subset X_\omega$ so that $d(p, S_j) \to 0$ for some $p \in X_\omega$ and $\frac{1}{R_j} S_j \to B(1) \subset \mathbb{E}^k$ in the Gromov-Hausdorff topology. From the properties of ultralimits (see section 2.3 property 5) we see that there are sequences $R'_j \to 0$, $T_j \subset X$ so that $\frac{1}{R'_j} T_j \to B(1) \subset \mathbb{E}^k$ in the Gromov-Hausdorff topology. By the remark after Proposition 3.2 we get $\mathrm{CTopDim}(X) \geq k$.

Existence of regular points, [Gro93, p.132-3]. Gromov defines a point p in a CBA space to be regular if the tangent cone C_pX satisfies $DimC_pX \geq DimX$, and asks when regular points exist. If we replace Dim with $CTopDim$, then Theorem A gives

$$\mathrm{CTopDim}(X) = \mathrm{GeomDim}(X) = \sup_{p \in X} \mathrm{GeomDim}(C_pX) = \sup_{p \in X} \mathrm{CTopDim}(C_pX).$$

So regular points always exist when $\mathrm{CTopDim}(X) < \infty$. In the general case $\mathrm{CTopDim}(X)$ can be locally finite even when $\mathrm{CTopDim}(X) = \infty$; for example, take the disjoint union $\mathbb{E}^1 \cup \mathbb{E}^2 \cup \ldots$, and glue on a segment of length 1 starting at $0 \in \mathbb{E}^1$ and ending at $0 \in \mathbb{E}^k$.

Rank's and Rank$^+$*'s, [Gro93, pp.127-33].* Gromov gives seven definitions of rank for Hadamard spaces and then raises the issue of whether they coincide for Hadamard spaces with cocompact isometry groups, or, more generally, if the "plusified" ranks $Rank^+_I, \ldots, Rank^+_{VII}$ agree for arbitrary Hadamard spaces. Theorem 7.1 in this paper settles this question completely for $Rank^+_I, \ldots, Rank^+_{IV}$ provided one uses compact topological dimension instead of the usual topological dimension.

Let (X, d) be a Hadamard space. To see that the ranks $Rank^+_I, \ldots, Rank^+_{IV}$ (redefined using CTopDim instead of TopDim) are equal we will apply Theorem 7.1 to the one-element family of spaces (X, d). For $1 \leq i \leq 4$ let $r_i \in \mathbb{N} \cup \infty$ be the supremum of the r's which satisfy the i^{th} statement in Theorem 7.1. Let $\mathcal{X}^+$ be the plusification of X, i.e. the collection of ultralimits of the form $\omega\text{-lim}(X, d, \star_i)$ where $\star_i$ is a sequence in X.

Notice that r_1 is exactly the same as $Rank^+_{II}X$. Similarly, $Rank^+_{IV}X = r_4$ since by Theorem A, part 1 we have $\mathrm{GeomDim}(\partial_T Z) = \mathrm{CTopDim}(\partial_T Z)$ for any Hadamard space Z.

Suppose $Rank^+_I X \geq r$. Then there is an $X' \in \mathcal{X}^+$ and an asymptotic cone $(X')_\omega$ of X' with $\mathrm{CTopDim}((X')_\omega) \geq r$. Therefore by Theorem A, there are sequences $R_k \to 0$, $S_k \subset (X')_\omega$ so that S_k is finite, and $\frac{1}{R_k} S_k$ converges to $B(1) \subset \mathbb{E}^r$ in the Gromov-Hausdorff topology. Each S_k is a Gromov-Hausdorff limit of a sequence $\frac{1}{R_{kl}} S_{kl}$ where $S_{kl} \subset X'$ is finite and $R_{kl} \to \infty$ (see section 2.3 property 5); and each $S_{kl} \subset X'$ is a Gromov-Hausdorff limit of a sequence of finite subsets of X itself. By a diagonal construction we find sequences $R'_k \to \infty$ and $T_k \subset X$ so that $\frac{1}{R'_k} T_k \to B(1) \subset \mathbb{E}^r$ in the Gromov-Hausdorff topology. Hence 2 of Theorem 7.1 is satisfied for this r. Taking suprema we get $Rank^+_I X \leq r_2$.

Note that $Rank^+_{III}X \leq Rank^+_I X$ since an r-quasiflat in $X' \in \mathcal{X}^+$ produces a biLipschitz embedded copy of $\mathbb{E}^r$ in asymptotic cones of X'. But $r_1 = Rank^+_{II}X \leq Rank^+_{III}X \leq Rank^+_I X \leq r_2$ so the $Rank^+$'s are all equal to the r_i's.

10 Convex length spaces

A convex (length) space is a complete length space (X, d) such that if $\gamma_1 : [a_1, b_1] \to X$ and $\gamma_2 : [a_2, b_2] \to X$ are constant speed geodesics, then the function $d \circ (\gamma_1, \gamma_2) : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ is convex. Convex spaces were introduced by Busemann (see [Bus55]); the literature about them seems to be quite limited: [Rin61, Gro78, Bow95, AB90]. Hadamard spaces are convex length spaces, as are Banach spaces with a strictly convex norm. Our goal in this section is to prove Theorem 1. In section 10.1 we discuss elementary properties of convex spaces, in section 10.2 we give a proof of a special case of the differentiation theorem from [KS93], and in section 10.3 we construct flats in convex spaces whose asymptotic cone contains a flat.

10.1 Background

We recall some basic facts about convex spaces, omitting proofs whenever the standard proofs in the Hadamard space case extend without significant modification to convex length spaces.

Let X be a convex space. Two geodesic rays[7] $\gamma_1 : [0, \infty) \to X$, $\gamma_2 : [0, \infty) \to X$ are **asymptotic** if $d(\gamma_1(t), \gamma_2(t))$ is bounded. Given any $p \in X$ and any geodesic ray $\gamma_1 : [0, \infty) \to X$ there is a unique geodesic ray γ_2 with $\gamma_2(0) = p$ which is asymptotic to γ_1. Asymptoticity is an equivalence relation on geodesic rays, and we use $C_\infty X$ to denote the set of equivalence classes. For any $p \in X$ we may view $C_\infty X$ as a subset of $\{\gamma \in C([0, \infty), X) \mid \gamma(0) = p\}$; the compact-open topology on $C([0, \infty), X)$ induces a subspace topology on $C_\infty X$, which is independent of p. The **geometric boundary**, $\partial_\infty X$, is the subset of $C_\infty X$ determined by the unit speed geodesic rays. The union $\bar{X} := X \cup \partial_\infty X$ inherits a natural topology, which is compact when X is locally compact. The Tits distance between two geodesic rays γ_1, γ_2 is

$$d_{Tits}(\gamma_1, \gamma_2) := \lim_{t \to \infty} \frac{d(\gamma_1(t), \gamma_2(t))}{t}.$$

d_{Tits} defines a metric on $C_\infty X$; we call the resulting metric space the **Tits cone** and denote it by $C_T X$. d_{Tits} is lower semicontinuous with respect to the product topology on $C_\infty X \times C_\infty X$. The **Tits boundary** is the subset $\partial_T X \subset C_T X$ determined by the unit speed geodesic rays. If $p \in X$, and $\xi \in \partial_T X$, then $\overline{p\xi}$ denotes (the image of) the geodesic ray with initial point p. We define Busemann functions for unit speed geodesic rays as in the Hadamard case; unlike the Hadamard case, the Busemann functions of asymptotic geodesic rays need not differ by a constant. However, the following properties still hold:

Lemma 10.1 *1. If p_k, $x_k \in X$, and the geodesic segments $\overline{p_k x_k}$ converge to a geodesic ray $\overline{p_\infty \xi} \subset X$, then the Busemann function b_ξ of the ray $\overline{p_\infty \xi}$ satisfies*

$$b_\xi \geq \limsup_{k \to \infty} \left[d(x_k, \cdot) - d(x_k, p_k) \right] \tag{10.2}$$

Recall that in the Hadamard case b_ξ is the limit of the normalized distance functions $[d(x_k, \cdot) - d(x_k, p_k)]$.

[7] In this section all geodesics rays will be parametrized at constant (not necessarily unit) speed.

2. If $p \in X$, $\xi \in \partial_T X$, b_ξ is the Busemann function for the ray $\overline{p\xi}$, and the unit speed geodesic ray $\gamma : [0, \infty) \to X$ is a "gradient line" for b_ξ:

$$b_\xi(\gamma(t)) = b_\xi(\gamma(0)) - t,$$

then γ is asymptotic to $\overline{p\xi}$.

3. If $p \in X$, ξ_1, $\xi_2 \in \partial_T X$, b_{ξ_i} denotes the Busemann function of $\overline{p\xi_i}$, and $\overline{p\xi_1} \cup \overline{p\xi_2}$ forms a geodesic, then $b_{\xi_1} + b_{\xi_2} \geq 0$ on X. In this case if $x \in X$ and $(b_{\xi_1} + b_{\xi_2})(x) = 0$, then $\overline{x\xi_1} \cup \overline{x\xi_2}$ is a geodesic parallel to $\overline{p\xi_1} \cup \overline{p\xi_2}$.

We will use the following result:

Theorem 10.3 *([Rin61, p.432, par. 7 and p. 463, par.20], [Bow95, Lemma 1.1 and remark after its proof]) If X is a convex length space, and γ_1, $\gamma_2 : \mathbb{R} \to X$ are constant speed geodesics with $d(\gamma_1(t), \gamma_2(t)) < C$, then γ_1 and γ_2 bound a flat Minkowskian strip (a convex subset isometric to the region in a normed plane bounded by two parallel lines.).*

A weaker convexity condition. Convexity is not preserved by limit operations: a sequence of strictly convex norms on $\mathbb{R}^n$ may converge to a non-strictly convex norm. To remedy this defect we introduce a weaker convexity condition below: we only insist on the convexity of the distance function when it is restricted to a distinguished collection of geodesic segments.

Definition 10.4 *A family $\mathcal{G}$ of constant speed geodesics (or geodesic segments) in a length space X is* **adequate** *if*

a) Each pair of points in X is joined by a geodesic segment in $\mathcal{G}$;

and

b) $\mathcal{G}$ is closed under precomposition with affine maps: if $\gamma \in \mathcal{G}$, $\gamma : [a, b] \to X$, and $\alpha : [c, d] \to [a, b]$ is affine, then $\gamma \circ \alpha : [c, d] \to X$ is in $\mathcal{G}$.

Definition 10.5 *A length space X with distance function d is* **convex with respect to an adequate family of geodesics $\mathcal{G}$ in** *X if $d \circ (\gamma_1, \gamma_2)$ is convex for all pairs γ_1, $\gamma_2 \in \mathcal{G}$. X is* **often convex** *if it is convex with respect to some adequate family of geodesics.*

If for each $i \in \mathbb{N}$, X_i is a length space convex with respect to $\mathcal{G}_i$, then for any choice of basepoints $\star_i \in X_i$, the ultralimit $(X_i, \star_i)$ is convex with respect to the adequate family

$$\mathcal{G}_\omega := \{\omega\text{-}\lim \gamma_i \mid \exists C > 0 \text{ such that } \forall i \in \mathbb{N},\ \gamma_i : [a, b] \to X_i,\ d(\gamma_i(a), \star_i),\ d(\gamma_i(b), \star_i) < C\}.$$

In particular, any asymptotic cone of an often convex length space is an often convex length space.

Lemma 10.6 *(B. Leeb) Let (X, d) be a locally compact convex length space. If $\lambda_i \to 0$, and $(X_\omega, d_\omega, \star_\omega) := \omega\text{-}\lim(X, \lambda_i d, \star)$ (i.e. X_ω is an asymptotic cone with fixed basepoints), then there is a canonical isometric embedding $i : C_T X \to X_\omega$ and a 1-Lipschitz retraction $\rho : X_\omega \to i(C_T X)$.*

Proof. Given a geodesic ray $\gamma : [0, \infty) \to X$, $(\gamma(\frac{1}{\lambda_i}))$ defines a point $i(\gamma)$ in X_ω. This clearly defines an isometric embedding $i : C_T X \to X_\omega$. To obtain the retraction ρ, we use the compactification $\bar{X} := X \cup \partial_\infty X$ and let $\rho((x_i)) \in C_T X$ be the ultralimit of $x_i \in \bar{X}$ normalized by $\omega\text{-}\lim \lambda_i d(x_i, \star)$. $\square$

10.2 Differentiating maps into metric spaces

The next result is a slight reformulation of a special case of the differentiation theory of [KS93, section 1.9]. Since this case is somewhat simpler than the general L^p version in [KS93], we give a proof here.

We will use d_0, $\|\cdot\|_0$, and $B_0(x, r)$ to denote the Euclidean metric, the Euclidean norm, and a Euclidean ball, respectively.

Theorem 10.7 (Korevaar-Schoen) *Suppose $d : U \times U \to \mathbb{R}$ is a pseudo-distance (definition 2.7) on an open subset $U \subseteq \mathbb{R}^n$ which is L-Lipschitz with respect to d_0, i.e. $d \leq L d_0$. Then there is a measurable function $\rho : U \times \mathbb{R}^n \to \mathbb{R}$ so that for a.e. $x \in U$,*

i) $\rho(x, \cdot)$ is a semi-norm on $\mathbb{R}^n$,

and

ii) for every $v \in \mathbb{R}^n - \{0\}$ we have

$$\limsup_{r \to 0} \left\{ \frac{d(y, y + rv)}{r} - \rho(x, v) \mid y \in B_0(x, r) \right\} = 0. \tag{10.8}$$

In other words, the pseudo-distance d behaves infinitesimally like a measurable Finsler pseudo-metric.

Corollary 10.9 *Let U, d, and $\rho : U \times \mathbb{R}^n \to [0, \infty)$ be as in Theorem 10.7, and for every $x \in \mathbb{R}^n$ define a pseudo-metric $d_{x,r}$ on $\mathbb{R}^n$ by $d_{x,r}(y, z) = d(x + ry, x + rz)$. Then for a.e. $x \in U$ the family of pseudo-metrics $\frac{1}{r} d_{x,r}$ converges uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^n$ as $r \to 0$ to the pseudo-metric defined by the semi-norm $\rho(x, \cdot)$. In particular, if $K \subset \mathbb{R}^n$ is a compact subset, then the pseudo-metric spaces $(K, \frac{1}{r} d_{x,r}, 0)$ converge in the Gromov-Hausdorff topology to K with the pseudo-metric determined by $\rho(x, \cdot)$.*

Proof of Corollary 10.9. Pick $x \in U$ so that (10.8) holds, and let $\bar{d}_x : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ be the pseudo-metric defined by $\rho(x, \cdot)$. The pseudo-metrics $\frac{1}{r} d_{x,r}$ are L-Lipschitz, and by (10.8) they converge pointwise on $\mathbb{R}^n \times \mathbb{R}^n$ to $\bar{d}_x$; therefore they converge uniformly on compact sets.

If $K \subset \mathbb{R}^n$ is compact, then $id_K : (K, \frac{1}{r} d_{x,r}) \to (K, \bar{d}_x)$ is an $\epsilon(r)$-Hausdorff approximation where $\lim_{r \to 0} \epsilon(r) = 0$. $\square$

Proof of Theorem 10.7. The proof is based on a covering argument, and is analogous to the proof of the Rademacher-Stepanoff theorem on the differentiability of Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}$.

Since (truncated) cylinders will be used repeatedly below, we will use the notation $Cyl(a,b,t)$, $a,b \in \mathbb{R}^n$, $t \in (0,\infty)$ to denote the **cylinder with core segment** $\overline{ab}$ **and thickness** t:

$$Cyl(a,b,t) := \{s+n \mid s \in \overline{ab},\ n \perp \overline{ab},\ \|n\|_0 < t\}.$$

The **caps** of the cylinder $Cyl(a,b,t)$ are the subsets

$$\{a+n \mid n \perp \overline{ab},\ \|n\|_0 \leq t\}$$

and

$$\{b+n \mid n \perp \overline{ab},\ \|n\|_0 \leq t\}.$$

We now introduce functions that compare the pseudo-distance function d with the Euclidean distance function d_0; these functions are analogous to directional derivatives. Define measurable functions $\underline{\rho} : U \times \mathbb{R}^n \to \mathbb{R}$, $\overline{\rho} : U \times \mathbb{R}^n \to \mathbb{R}$ by

$$\underline{\rho}(x,v) = \liminf_{r \to 0} \left\{ \frac{d(y, y+rv)}{r} \mid y \in B_0(x,r) \right\}$$

and

$$\overline{\rho}(x,v) = \limsup_{r \to 0} \left\{ \frac{d(y, y+rv)}{r} \mid y \in B_0(x,r) \right\}.$$

Observe that $\underline{\rho}(x,\cdot)$ and $\overline{\rho}(x,\cdot)$ are L-Lipschitz with respect to d_0 for each $x \in U$:

$$|\underline{\rho}(x,v_1) - \underline{\rho}(x,v_2)| \leq L\|v_1 - v_2\|_0 \tag{10.10}$$

and

$$|\overline{\rho}(x,v_1) - \underline{\rho}(x,v_2)| \leq L\|v_1 - v_2\|_0 \tag{10.11}$$

for every $v_1, v_2 \in \mathbb{R}^n$.

Given $x \in U$, $v \in \mathbb{R}^n - \{0\}$, $\mu, \nu \in (0,\infty)$, we let $\mathcal{C}(x,v,\mu,\nu)$ be the collection of cylinders $Cyl(y, y+rv, r\nu) \subset U$ where $r > 0$, $y \in B_0(x,r)$, and

$$\frac{d(y, y+rv)}{r} < \underline{\rho}(x,v) + \mu. \tag{10.12}$$

Note that since $d \leq Ld_0$, if we take a cylinder $Cyl(y, y+rv, r\nu) \in \mathcal{C}(x,v,\mu,\nu)$, and a pair of points $u, v \in Cyl(y, y+rv, r\nu)$ – one from each cap of $Cyl(y, y+rv, r\nu)$ – then by the triangle inequality for d we have

$$\frac{d(u,v)}{r} \leq \underline{\rho}(x,v) + \mu + 2L\nu. \tag{10.13}$$

Elements of $\mathcal{C}(x,v,\mu,\nu)$ are somewhat off-centered cylinders with direction v where the d-distance between caps is approximately infimal (among such cylinders).

For given $v \in \mathbb{R}^n - \{0\}$, $\nu \in (0,\infty)$, cylinders of the form $Cyl(y, y+rv, r\nu)$ with $y \in B_0(x,r)$ are contained in the closed ball $\overline{B_0(x, r(1+\|v\|_0+\nu))}$ and have uniform density there:

$$\frac{\mathcal{L}^n(Cyl(y, y+rv, r\nu))}{\mathcal{L}^n(B_0(x, r(1+\|v\|_0+\nu)))} = \frac{(r\|v\|_0)(\omega_{n-1}(r\nu)^{n-1})}{\omega_n\,[r(1+\|v\|_0+\nu)]^n} = \frac{c(n)\|v\|_0\nu^{n-1}}{(1+\|v\|_0+\nu)^n} \tag{10.14}$$

Lemma 10.15 *Suppose $v \in \mathbb{R}^n - \{0\}$ and $x \in U$ is an approximate continuity point of $\underline{\rho}(\cdot, v) : U \to \mathbb{R}$. Then $\overline{\rho}(x, \alpha v) \leq |\alpha| \underline{\rho}(x, v)$ for every $\alpha \in \mathbb{R}$.*

Proof of lemma. Our goal is to show that if $r \in (0, \infty)$ is sufficiently small and $y \in B_0(x, r)$, then $\frac{d(y, y+r\alpha v)}{r} \lesssim |\alpha| \underline{\rho}(x, v)$. To prove this we thicken the segment $\overline{y(y + r\alpha v)}$ into the cylinder $Cyl(y, y + r\alpha v, r\nu)$, and estimate the d-distance between its caps.

The lemma is obvious when $v = 0$ or $\alpha = 0$, so we assume henceforth that $v \neq 0$, $\alpha \neq 0$.

Pick $\epsilon_1 \in (0, \infty)$, $\nu_1 \in (0, \infty)$. Since $x \in U$ is an approximate continuity point of $\underline{\rho}(\cdot, v)$, the density of the set $\{z \mid |\underline{\rho}(z, v) - \underline{\rho}(x, v)| < \epsilon_1\}$ in $B_0(x, r)$ tends to 1 as r tends to zero. In view of the density estimate (10.14), the density of $\{z \mid |\underline{\rho}(z, v) - \underline{\rho}(x, v)| < \epsilon_1\}$ in cylinders of form $Cyl(y, y + r\alpha v, r\nu_1)$, $y \in B_0(x, r)$, also tends to 1 as $r \to 0$. Choose $r_1 \in (0, \infty)$ so that the latter density is $\geq 1 - \epsilon_1$ when $r \leq r_1$.

Fix a cylinder $C = Cyl(y, y + r\alpha v, r\nu_1)$ where $r \leq r_1$, $y \in B_0(x, r)$. Let $T = \{z \in Interior(C) \mid |\underline{\rho}(z, v) - \underline{\rho}(x, v)| < \epsilon_1\}$. Pick $\mu_2, \nu_2 \in (0, \infty)$, and let $\mathcal{D}$ be the collection of cylinders in $\mathcal{C}(t, v, \mu_2, \nu_2)$ which are contained in C, where t ranges over T. The density estimate (10.14) implies that $\mathcal{D}$ is a Vitali cover[8] of T. So by a standard covering argument there is a disjoint subcollection $\mathcal{D}' \subset \mathcal{D}$ so that $\mathcal{L}^n(T \setminus (\cup_{D \in \mathcal{D}'} D)) = 0$. Hence $\cup_{D \in \mathcal{D}'} D$ has density $\geq 1 - \epsilon_1$ in C. By Fubini's theorem, there is a segment $\overline{y'(y' + r\alpha v)}$ with endpoints in the caps of C so that the density of $(\cup_{D \in \mathcal{D}'} D) \cap \overline{y'(y' + r\alpha v)}$ in the segment $\overline{y'(y' + r\alpha v)}$ is $\geq 1 - \epsilon_1$. Applying the cap separation estimate (10.13), the Lipschitz estimate $d \leq L d_0$, and the triangle inequality for d, we get

$$\frac{d(y', y' + r\alpha v)}{r} \leq |\alpha| (\underline{\rho}(x, v) + \epsilon_1 + \mu_2 + 2L\nu_2)$$

and so

$$\frac{d(y, y + r\alpha v)}{r} \leq |\alpha| (\underline{\rho}(x, v) + \epsilon_1 + \mu_2 + 2L\nu_2) + 2L\nu_1. \tag{10.16}$$

If $\epsilon \in (0, \infty)$ is given and ϵ_1, μ_2, ν_1, ν_2 are chosen so that the right hand side of (10.16) is $< |\alpha| \underline{\rho}(x, v) + \epsilon$, and r_1 is chosen accordingly, then we have

$$\frac{d(y, y + r\alpha v)}{r} < |\alpha| \underline{\rho}(x, v) + \epsilon$$

provided $r \leq r_1$ and $y \in B_0(x, r)$. This proves the lemma. □

Proof of theorem 10.7 continued. Define $U_0 \subset U$ to be the set of all $x \in U$ which are approximate continuity points of $\underline{\rho}(\cdot, v)$ for all $v \in \mathbb{Q}^n$; since $\mathbb{Q}^n$ is countable we have $\mathcal{L}^n(U - U_0) = 0$. If $x \in U_0$, then Lemma 10.15 and the triangle inequality for d imply

[8] A Vitali cover of a set $S \subset \mathbb{R}^n$ is a collection of measurable sets $Y_i \subset \mathbb{R}^n$ with the following property: there is a density $\delta > 0$ so that for every $s \in S$ and every $r > 0$, there is an $r' < r$ and an i so that $Y_i \subset B_0(s, r')$ and the density $\frac{\mathcal{L}^n(Y_i)}{\mathcal{L}^n(B_0(s, r'))}$ is $> \delta$.

the inequality $\underline{\rho}(x, v_1 + v_2) \leq \underline{\rho}(x, v_1) + \underline{\rho}(x, v_2)$ for $v_1, v_2 \in \mathbb{Q}^n$. If $v \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$, then by Lemma 10.15 we have

$$\underline{\rho}(x, \alpha v) \leq \overline{\rho}(x, \alpha v) \leq |\alpha| \underline{\rho}(x, v) \tag{10.17}$$

If $|\alpha| \neq 0$ then we may replace α with α^{-1} in (10.17), thereby deducing that $\underline{\rho}(x, \alpha v) = |\alpha| \underline{\rho}(x, v)$ for every $\alpha \in \mathbb{Q}$, $v \in \mathbb{Q}^n$. Since $\underline{\rho}(x, \cdot) : \mathbb{R}^n \to [0, \infty)$ is L-Lipschitz, the homogeneity and subadditivity of $\underline{\rho}(x, \cdot)\big|_{\mathbb{Q}^n}$ extends to $\mathbb{R}^n$; so $\underline{\rho}(x, \cdot)$ is a semi-norm on $\mathbb{R}^n$. Letting $\rho = \underline{\rho}$, we have proved Theorem 10.7. □

Proposition 10.18 *Let X be a metric space, let $U \subset \mathbb{E}^n$ be an open subset, and suppose $f : U \to X$ is a Lipschitz map. Then either*

a) There is a $p \in f(U)$, and sequences $R_k \in (0, \infty)$, $S_k \subset X$ so that $R_k \to 0$, $d(p, S_k) \to 0$, and $\frac{1}{R_k} S_k$ converges in the Gromov-Hausdorff topology to the unit ball in a normed space $(\mathbb{R}^n, \| \cdot \|)$,

or

b) $\mathcal{H}^n(f(U)) = 0$.

Proof. This is a consequence of Corollary 10.9 and a covering argument.

Notation. If $\mathcal{S}$ is a collection of subsets of a pseudo-metric space Z with distance function δ, then

$$\|\mathcal{S}\| := \sup_{S \in \mathcal{S}} Diam(S).$$

If $\alpha \in [0, \infty)$, then

$$\mathcal{H}^\alpha_\delta(Z, \mathcal{S}) := \sum_{S \in \mathcal{S}} \omega_\alpha \, [Diam(S)]^\alpha$$

where ω_α is a universal constant. The α-dimensional Hausdorff measure of (Z, δ) is

$$\mathcal{H}^\alpha_\delta(Z) := \lim_{\epsilon \to 0} \inf \{ \mathcal{H}^\alpha_\delta(Z, \mathcal{S}) \mid \mathcal{S} \text{ is a countable cover of } Z, \ \|\mathcal{S}\| < \epsilon \}.$$

When δ is clear from the context we will omit it from the notation.

Let d_X be the distance function on X, and let $d : U \times U \to [0, \infty)$ be the pullback of d_X by f:

$$d(u_1, u_2) := d_X(f(u_1), f(u_2)).$$

Applying Theorem 10.7 to d, let $\rho : U \times \mathbb{R}^n \to [0, \infty)$ be the measurable function which satisfies i) and ii) for a.e. $x \in U$, and let $U_0 \subset U$ be the set where i) and ii) hold.

We first show that if $x \in U_0$ and $\rho(x, \cdot) : \mathbb{R}^n \to \mathbb{R}$ is a norm on $\mathbb{R}^n$, then a) holds. Set $\| \cdot \| := \rho(x, \cdot)$, and let B be the unit ball $\| \cdot \|^{-1}([0, 1]) \subset \mathbb{R}^n$. By Corollary 10.9, $(x + rB, \frac{1}{r} d)$ converges to B in the Gromov-Hausdorff topology, so a) holds.

Let U_1 be the set of $x \in U_0$ for which $\rho(x, \cdot)$ is not a norm. We will show that if $\mathcal{L}^n(U \setminus U_1) = 0$ then $\mathcal{H}^n_d(U) = 0$; this clearly implies b).

Lemma 10.19 *Given $x \in U_1$ and $\epsilon \in (0, \infty)$ there is an $r_0 \in (0, \infty)$ so that every ball $B_0(x, r)$ with $r < r_0$ admits a cover $\mathcal{C}$ with*

$$\mathcal{H}^n_d(B_0(x, r), \mathcal{C}) < \epsilon \mathcal{L}^n(B_0(x, r)) \tag{10.20}$$

Proof of lemma. Let $\|\cdot\|_1 := \rho(x,\cdot)$, and let d_1 be the pseudo-distance function on $\mathbb{R}^n$ associated with the semi-norm $\|\cdot\|_1$. By assumption $\|\cdot\|_1$ is not a norm, so it is zero on some 1-dimensional subspace $V \subseteq \mathbb{R}^n$. Let $\|\cdot\|_2 : \mathbb{R}^n/V \to [0,\infty)$ denote the induced semi-norm on the quotient space $\mathbb{R}^n/V$, and let d_2 be the corresponding distance function. If $\lambda \in (0,1)$, the set $\pi(B_0(0,1)) \subseteq \mathbb{R}^n/V$ can be covered by $C_1(\frac{1}{\lambda})^{n-1}$ d_2-balls of radius λ, since $(\mathbb{R}^n/V, \|\cdot\|_2)$ is a normed vector space of dimension $\leq n-1$. By Corollary 10.9 the family of pseudo-metric spaces $(B_0(x,r), \frac{1}{r}d)$ converges to $(B_0(0,1), d_1)$, which is isometric to $(\pi(B_0(0,1)), d_2)$ via $\pi : \mathbb{R}^n \to \mathbb{R}^n/V$. Therefore when r is sufficiently small $(B_0(x,r), d)$ can be covered by $C_1(\frac{1}{\lambda})^{n-1}$ d-balls of radius $2r\lambda$; for such a covering $\mathcal{C}$ we have

$$\mathcal{H}^n_d(B_0(x,r),\mathcal{C}) = \sum_{C\in\mathcal{C}} \omega_n\,[diam_d(C)]^n$$

$$\leq C_2\lambda r^n.$$

If λ is sufficiently small (10.20) will be satisfied. ☐

Proof of Proposition 10.18 continued. Since U is a countable union of sets with finite Lebesgue measure and $\mathcal{H}^n_d$ is countable subadditive, it suffices to treat the case that $\mathcal{L}^n(U) < \infty$. We assume that $\mathcal{L}^n(U_1) = \mathcal{L}^n(U)$. Pick $\epsilon > 0$. By Lemma 10.19 there is a Vitali cover $\mathcal{D}$ of U_1 by Euclidean balls contained in U, such that each ball $B \in \mathcal{D}$ admits a cover $\mathcal{C}_B$ satisfying a) $\mathcal{H}^n_d(B,\mathcal{C}_B) < \epsilon\mathcal{L}^n(B)$ and b) $Diam_d(C) < \epsilon$ for every $C \in \mathcal{C}_B$. By the Vitali covering lemma there is a disjoint subcollection $\mathcal{D}' \subseteq \mathcal{D}$ so that $\mathcal{L}^n(U_1 - \cup_{B\in\mathcal{D}'}B) = 0$. Letting $\mathcal{C}' = \cup_{B\in\mathcal{D}'}\mathcal{C}_B$ we get

$$\mathcal{H}^n_d(\cup_{B\in\mathcal{D}'}B, \mathcal{C}')$$

$$\leq \sum_{B\in\mathcal{D}'} \epsilon\mathcal{L}^n(B) = \epsilon\mathcal{L}^n(U).$$

Since $d \leq Ld_0$,

$$\mathcal{H}^n_d(U - \cup_{B\in\mathcal{D}'}B) \leq (L^n)\mathcal{L}^n(U - \cup_{B\in\mathcal{D}'}B) = 0,$$

and therefore we can find a cover $\mathcal{C}''$ of $U - \cup_{B\in\mathcal{D}'}B$ with a) $Diam_dC < \epsilon$ for every $C \in \mathcal{C}'$ and b) $\mathcal{H}^n_d(U - \cup_{B\in\mathcal{D}'}B, \mathcal{C}'') < \epsilon\mathcal{L}^n(U)$. So $\mathcal{H}^n_d(U, \mathcal{C}'\cup\mathcal{C}'') \leq 2\epsilon\mathcal{L}^n(U)$, and we conclude that $\mathcal{H}^n_d(U) = 0$. ☐

Proposition 10.21 *Let X be an often convex space*[9]*, let $V \subset U \subset X$ be open subsets, and suppose $H_n(U,V) \neq \{0\}$. Then there is a $p \in X$, and sequences $R_k \in (0,\infty)$, $S_k \subset X$ so that $R_k \to 0$, $d(p,S_k) \to 0$, and $\frac{1}{R_k}S_k$ converges in the Gromov-Hausdorff topology to a unit ball in a normed space $(\mathbb{R}^n, \|\cdot\|)$.*

Proof. By Proposition 10.18 is suffices to show that $H_n(U,V) = \{0\}$ for every open pair in X provided the image of every Lipschitz map from $\mathbb{R}^k$ into X has zero k-dimensional Hausdorff measure when $k \geq n$. We prove this by modifying the proofs of Lemma 5.1 and Proposition 5.2.

[9] We don't really need X to be often convex. It's enough to be able to cone off Lipschitz maps $\sigma : \Delta_n \to X$ at a point $x \in X$ to obtain a Lipschitz $\sigma' : \Delta_{n+1} \to X$ with $Diam(\operatorname{Im}\sigma') < C\cdot Diam(\operatorname{Im}\sigma)$.

Given a finite polyhedron P and a map σ_0' of its 0-skeleton $(P)_0$ into X, we may produce a Lipschitz map $\sigma : P \to X$ as follows. Let $Sd\,P$ be the first barycentric subdivision of P. Extend σ_0' to a map $\sigma_0 : (Sd\,P)_0 \to X$ by letting $\sigma_0(v) = \sigma_0'(w)$ where $v \in (Sd\,P)_0 \setminus (P)_0$, and w is a vertex of the P-simplex determined by v. Inductively extend $\sigma_{j-1} : (Sd\,P)_{j-1} \to X$ to $\sigma_j : (Sd\,P)_j \to X$ by coning at barycenters. This gives us a Lipschitz map $\sigma : P \to X$ with the property that for each simplex τ of P, $Diam(\sigma(\tau)) \leq Diam(\sigma_0'(Vertex(\tau)))$.

We may use the extension process defined above in the proof of Lemma 5.1 instead of using barycentric simplices; in particular if $K \subseteq X$ is a compact set with zero k-dimensional Hausdorff measure, then the inclusion $i_K : K \to X$ can be approximated by maps which factor as $K \to P \to X$ where $Dim(P) < k$. Pick $\alpha \in H_n(U, V)$. Then α is in the image of $H_n(f)$ for some map of pairs $f : (M, N) \to (U, V)$ where M is a polyhedron of dimension n. Adapting the argument of Lemma 5.1 part 1 to our often convex space, we may approximate f with a Lipschitz map f_1; by assumption $\mathrm{Im}\, f_1$ has zero n-dimensional Hausdorff measure, and when $d(f_1, f)$ is sufficiently small $f_1 : (M, N) \to (U, V)$ will be homotopic (as a map of pairs) to f. The inclusion $i_{\mathrm{Im}\, f_1} : \mathrm{Im}\, f_1 \to X$ may be approximated by a map $g : \mathrm{Im}\, f_1 \to X$ which factors through a polyhedron of dimension $< n$. Hence if $d(g, i_{\mathrm{Im}\, f_1})$ is sufficiently small we get

$$H_n(f) = H_n(f_1) = H_n(g \circ f) = H_n(g) \circ H_n(f_1) = 0$$

so $[\alpha] = 0$. □

10.3 Producing flats in convex length spaces

The next result links the large-scale geometry of convex spaces with their local structure.

Proposition 10.22 *Let (X, d) be a locally compact convex length space with cocompact isometry group. Suppose there are sequences $R_k \in (0, \infty)$, $S_k \subset X$, and a normed vector space $(\mathbb{R}^n, \|\cdot\|)$, so that $R_k \to \infty$, and $\frac{1}{R_k} S_k$ converges to $B(1) \subset (\mathbb{R}^n, \|\cdot\|)$ in the Gromov-Hausdorff topology; equivalently, suppose that $(\mathbb{R}^n, \|\cdot\|)$ can be isometrically embedded in some asymptotic cone X_ω of X. Then there is an isometric embedding of some n-dimensional Banach space in X. If n is the maximum[10] dimension of Banach spaces which isometrically embed in X, then $(\mathbb{R}^n, \|\cdot\|)$ itself can be isometrically embedded in X.*

Proof. The proof is very similar to the proof of Theorem 7.1: for each k we find a (approximate) minimum p_k of the average distance from a finite set of points $F_k \subset S_k$, and then extract a convergent subsequence of the set of segments $\{\overline{p_k s} \mid s \in F_k\}$ to produce a configuration of rays that "spans" a flat subspace in X.

Let $B \subset (\mathbb{R}^n, \|\cdot\|)$ be the unit ball $\|\cdot\|^{-1}([0, 1])$, and let d_B denote the induced distance function on B. Suppose $F \subset \partial B$ is a centrally symmetric $(-F = F)$ finite collection of points where $\|\cdot\|$ is differentiable. By assumption there are sequences

[10] If the unit ball in a Banach space V can be covered by m balls of radius $\frac{1}{2}$, then $Dim(V) \leq m$. Therefore the local compactness of X and cocompactness of $Isom(X)$ implies that there is a bound on the dimension of Banach spaces which isometrically embed in X.

$R_k \to \infty$, $\epsilon_k \to 0$ and a sequence $\phi_k : (B, R_k d_B) \to (S_k, d)$ of $R_k\epsilon_k$-Hausdorff approximations (see definition 2.6). Define $f_k : X \to \mathbb{R}$ by

$$f_k(\cdot) := \frac{1}{|F|}\sum_{x\in F} d(\phi_k(x), \cdot).$$

As in the proof of Theorem 7.1 (Lemma 7.3) we would like to claim that f_k attains a minimum at a point close to $\phi_k(0)$; unfortunately the lack of uniform convexity of f_k makes it difficult to control the location of the minima of f_k. Instead we will work with an approximate minimum p_k of f_k which is close to $\phi_k(0)$ (Lemma 10.29).

Lemma 10.23

$$f_k(\phi_k(0)) \leq \inf f_k + \frac{3}{2}R_k\epsilon_k. \tag{10.24}$$

Proof of lemma. Using the fact that ϕ_k is an $\epsilon_k R_k$-Hausdorff approximation, we have

$$d(\phi_k(x), y) + d(y, \phi_k(-x)) \geq d(\phi_k(x), \phi_k(-x)) \tag{10.25}$$

$$\geq R_k d_B(x, -x) - R_k\epsilon_k = [R_k d_B(x, 0) - \frac{R_k\epsilon_k}{2}] + [R_k d_B(-x, 0) - \frac{R_k\epsilon_k}{2}]. \tag{10.26}$$

Therefore for every $y \in X$

$$f_k(y) \geq \frac{1}{|F|}\sum_{x\in F}[R_k d_B(x, 0) - \frac{R_k\epsilon_k}{2}]. \tag{10.27}$$

And since $d(\phi_k(x), \phi_k(0)) \leq R_k d_B(x, 0) + R_k\epsilon_k$ we have

$$f_k(\phi_k(0)) \leq \frac{1}{|F|}\sum_{x\in F}[R_k d(x, 0) + R_k\epsilon_k]. \tag{10.28}$$

Combining (10.27) and (10.28) we get (10.24). □

Pick a sequence $\lambda_k \in (0, \infty)$ such that $\lambda_k \to \infty$ and $\lambda_k\epsilon_k \to 0$.

Lemma 10.29 *For each k we may choose a point $p_k \in B(\phi_k(0), \frac{3}{2}\frac{R_k}{\lambda_k})$ such that for every $y \in X - \{p_k\}$ we have*

$$\frac{f_k(y) - f_k(p_k)}{d(y, p_k)} \geq -2\lambda_k\epsilon_k \tag{10.30}$$

Proof of lemma. Consider the closed set

$$Y_k := \{\phi_k(0)\} \cup \left\{ y \in X \setminus \{\phi_k(0)\} \mid \frac{f_k(y) - f_k(\phi_k(0))}{d(y, \phi_k(0))} \leq -\lambda_k\epsilon_k \right\}.$$

By (10.24) we have $Y_k \subseteq B(\phi_k(0), \frac{3}{2}\frac{R_k}{\lambda_k})$. Therefore f_k attains a minimum at some $p_k \in Y_k$. But then (10.30) holds for every $y \in X$, for otherwise we get

$$f_k(y) - f_k(\phi_k(0)) = (f_k(y) - f_k(p_k)) + (f_k(p_k) - f_k(\phi_k(0)))$$

$$< -\lambda_k \epsilon_k (d(y, p_k) + d(p_k, \phi_k(0))) \leq -\lambda_k \epsilon_k (d(y, \phi_k(0)))$$

so $y \in Y_k$ and $f_k(y) < f_k(p_k)$ which is absurd. □

After composing ϕ_k with a suitable sequence of isometries, and after passing to a suitable subsequence, we may assume that p_k converges to a point p_∞ and the geodesic segments $\overline{p_k \phi_k(x)}$ converge to geodesic rays $\overline{p_\infty \phi_\infty(x)}$, where $\phi_\infty(x) \in \partial_\infty X$. Let $b_{\phi_\infty(x)}$ denote the Busemann function of the ray $\overline{p_\infty \phi_\infty(x)}$. By Lemma 10.1 we have

$$b_{\phi_\infty(x)} \geq \limsup_{k \to \infty} [d(\phi_k(x), \cdot) - d(\phi_k(x), p_k)]$$

for each $x \in F$. Letting

$$\bar{f}_k := f_k - f_k(p_k) = \frac{1}{|F|} \sum_{x \in F} [d(\phi_k(x), \cdot) - d(\phi_k(x), p_k)]$$

we have

$$\limsup_{k \to \infty} \bar{f}_k \leq f_\infty := \frac{1}{|F|} \sum_{x \in F} b_{\phi_\infty(x)}.$$

By Lemma 10.29 we have, for every $y \in X$

$$\liminf_{k \to \infty} [\bar{f}_k(y) - \bar{f}_k(p_k)]$$

$$\geq \liminf_{k \to \infty} (-2\lambda_k \epsilon_k) d(y, p_k) = 0$$

so p_∞ minimizes f_∞.

Our next goal is to show that f_∞ is minimal along each of the rays $\overline{p_\infty \phi_\infty(x)}$, $x \in F$.

Lemma 10.31 *For each $v \in \partial B$ let b_v denote the Busemann function of the ray $t \mapsto tv$. For every $v \in \partial B$ there is a geodesic ray $\overline{p_\infty \xi}$ so that for every $y \in \overline{p_\infty \xi} - p_\infty$, and every $x \in F$ we have*

$$\frac{b_{\phi_\infty(x)}(y)}{d(p_\infty, y)} \leq \lim_{t \to 0} \frac{d_B(tv, x) - d_B(0, x)}{t} = b_x(v) \tag{10.32}$$

In other words the Busemann function $b_{\phi_\infty(x)}$ decreases at least as fast along $\overline{p_\infty \phi_\infty(x)}$ as $d_B(x, \cdot)$ (initially) decreases along $\overline{0v}$ (this rate is the same as the value of b_x at v).

Proof of lemma. Pick α, $l \in (0, \infty)$. We will first show that there is a $y_l \in X$ so that $d(p_\infty, y_l) = l$ and

$$\frac{b_{\phi_\infty(x)}(y_l)}{d(p_\infty, y_l)} \leq \frac{d}{dt} d_B(tv, 0)\Big|_{t=0} + \alpha \quad for\, every\, x \in F$$

Choose $t > 0$ small enough that $\frac{d_B(tv,x)-d_B(0,x)}{t} < \frac{d}{dt}d_B(tv,x) + \alpha$ for all $x \in F$. Since $\phi_k : (B, R_k d_B) \to (S_k, d)$ is an $\epsilon_k R_k$-Hausdorff approximation,

$$\lim_{k\to\infty} \frac{d(\phi_k(tv), \phi_k(x)) - d(\phi_k(0), \phi_k(x))}{R_k}$$

$$< t\left[\frac{d}{dt}d_B(tv,x) + \alpha\right] + 2\epsilon_k.$$

Letting (when k is sufficiently large) $y_k \in \overline{p_k\phi_k(tv)}$ be the point with $d(p_k, y_k) = l$, and letting $z_k \in \overline{p_k\phi_k(x)}$ be the point with

$$\frac{d(z_k,p_k)}{d(\phi_k(x),p_k)} = \frac{d(y_k,p_k)}{d(\phi_k(tv),p_k)}$$

the convexity of d gives

$$\frac{d(y_k,z_k)}{d(y_k,p_k)} \leq \frac{d(\phi_k(tv),\phi_k(x))}{d(\phi_k(tv),p_k)}.$$

Hence

$$\frac{[d(y_k,z_k) - d(p_k,z_k)]}{d(p_k,y_k)}$$

$$= \frac{d(y_k,z_k)}{d(p_k,y_k)} - \frac{d(p_k,z_k)}{d(p_k,y_k)} \leq \frac{d(\phi_k(tv),\phi_k(x))}{d(\phi_k(tv),p_k)} - \frac{d(p_k,\phi_k(x))}{d(p_k,\phi_k(tv))}$$

$$\to \frac{d_B(tv,x) - d_B(0,x)}{d(tv,0)} \quad as\, k \to \infty.$$

Passing to subsequences if necessary, we have $y_k \to y_\infty$, $z_k \to z_\infty \in \overline{p_\infty\phi_\infty(x)}$, $d(y_\infty, p_\infty) = l$, and

$$\frac{b_{\phi_\infty(x)}(y_\infty)}{d(y_\infty,p_\infty)} \leq \frac{[d(y_\infty,z_\infty) - d(p_\infty,z_\infty)]}{d(y_\infty,p_\infty)}$$

$$< \frac{d}{dt}d_B(tv,x)\Big|_{t=0} + \alpha \quad for\, all\, x \in F.$$

α can be made arbitrarily small, l there is a $y(l)$ with $d(y(l), p_\infty) = l$ and $\frac{b_{\phi_\infty(x)}(y(l))}{d(y(l),p_\infty)} \leq \frac{d}{dt}d_B(tv,x)\Big|_{t=0}$. Taking a limit of the segments $\overline{p_\infty y(l)}$ as $l \to \infty$ we get a ray $\overline{p_\infty\xi}$ satisfying the conditions of the lemma. □

Elements $x \in F$ were chosen so that $d_B(x,\cdot)$ is differentiable at $0 \in B$. By the triangle inequality the function $d_B(x,\cdot) + d_B(-x,\cdot)$ attains a minimum at $0 \in B$, so its derivative is zero there. Hence if $v \in \partial B$ and $\overline{p_\infty\xi}$ are as in Lemma 10.31, then for every $y \in \overline{p_\infty\xi}$ we have

$$(b_{\phi_\infty(x)} + b_{\phi_\infty(-x)})(y) \leq \frac{d}{dt}\left(d_B(x,tv) + d_B(-x,tv)\right)\Big|_{t=0} d(y,p_\infty)$$

$$= 0.$$

Summing over $x \in F$ we get

$$f_\infty(y) = \frac{1}{2|F|}\sum_{x\in F}(b_{\phi_\infty(x)} + b_{\phi_\infty(-x)})(y)$$
$$\leq 0 = f_\infty(p_\infty) = \inf f_\infty.$$

Therefore we have $f_\infty(y) = 0$ and $b_{\phi_\infty(x)}(y) = \frac{d}{dt}d_B(x,tv)\Big|_{t=0}d(y,p_\infty)$ for all $y \in \overline{p_\infty\xi}$ and every $x \in F$.

Pick $x_0 \in F$. Applying the reasoning in the previous paragraph with $v = x_0$ we conclude that $\overline{p_\infty\xi}$ is a "gradient line" for $b_{\phi_\infty(x_0)}$, forcing $\overline{p_\infty\xi} = \overline{p_\infty\phi_\infty(x_0)}$ by Lemma 10.1. Hence $b_{\phi_\infty(-x_0)}(y) = -b_{\phi_\infty(x_0)}(y) = d(p_\infty, y)$ for every $y \in \overline{p_\infty\phi_\infty(x_0)}$, which means that $\overline{p_\infty\phi_\infty(x_0)} \cup \overline{p_\infty\phi_\infty(-x_0)}$ is a geodesic. By Lemma 10.1 we get $b_{\phi_\infty(x_0)} + b_{\phi_\infty(-x_0)} \geq 0$ on X, and the convex set $C = f_\infty^{-1}(0)$ is ruled by geodesics parallel to $\overline{\phi_\infty(-x_0)p_\infty} \cup \overline{p_\infty\phi_\infty(x_0)}$. Therefore we may define an isometric $\mathbb{R}$-action $\Psi_x : \mathbb{R}\times C \to C$ on C by flowing in the direction of $\phi_\infty(x_0)$.

Lemma 10.33 *The $\mathbb{R}$-actions $\{\Psi_x\}_{x\in F}$ commute.*

Proof. Pick $x_1,\ x_2 \in F,\ p \in C$. For each $t \in \mathbb{R}$, $\Psi_{x_j}(t) : C \to C$ maps each geodesic to a parallel geodesic; in particular if we flow the geodesic $\overline{\phi_\infty(-x_2)p\phi_\infty(x_2)}$ by $\Psi_{x_1}(-T)$ and $\Psi_{x_1}(T)$ we get a pair of geodesics which bound a flat strip Y_T (Theorem 10.3) containing $\overline{\phi_\infty(-x_2)p\phi_\infty(x_2)}$. The pointed Hausdorff limit of (Y_T, p) as $T \to \infty$ is a flat (Minkowski) plane containing the geodesics $\overline{\phi_\infty(-x_1)p\phi_\infty(x_1)}$ and $\overline{\phi_\infty(-x_2)p\phi_\infty(x_2)}$. So clearly the flows $\Psi_{x_1}(t_1)$ and $\Psi_{x_2}(t_2)$ commute at p for every $t_1,\ t_2 \in \mathbb{R}$. □

Proof of Proposition 10.22 continued. Let V be the free $\mathbb{R}$-vector space on the set F. By Lemma 10.33 we get an action $\rho : V \times C \to C$ by setting $\rho(\sum_{x_i\in F} t_ix_i, c) = [\Phi_{x_1}(t_1)\circ\ldots\circ\Phi_{x_m}(t_m)](c)$. The action ρ has the property that for every $x \in F$, $c \in C$, the map $t \mapsto \rho(tx, c)$ is a unit speed geodesic, and for every $x_1,\ x_2 \in F$

$$\frac{d}{dt}b_{\phi_\infty(x_1)}(\rho(tx_2,c)) = \frac{d}{dt}b_{\phi_\infty(x_1)}(\Phi_{x_2}(t)(c)) = \frac{d}{dt}d_B(x_1,tv_2)\Big|_{t=0}. \qquad (10.34)$$

Therefore we may apply Proposition 2.3 of [Bow95] to see that each V-orbit in C is a convex subset isometric to a normed space. The Busemann functions $b_{\phi_\infty(x)}$ are affine functions on each orbit $V(c)$, and (10.34) implies that they span a space of dimension $\geq Dim(Span(F))$; therefore $Dim(V(c)) \geq Dim(Span(F))$. Since $\|\cdot\|$ is differentiable at a dense set of points in ∂B, we may pick F so that $Span(F) = \mathbb{R}^n$; this proves Proposition 10.22 except for the last claim.

Now assume that the V-orbit $V(p_\infty)$ has dimension $\leq n$ (which will be true if n is the maximal dimension of a flat in X). We have $b_{\phi_\infty(x)}(p_\infty) = 0$ for every $x \in F$. We have an affine map $\alpha := \rho(\cdot, p_\infty) : V \to V(p_\infty)$, and a corresponding map $\bar\alpha : V \to \mathbb{R}^n$ given by $\bar\alpha(\sum t_ix_i) = \sum t_ix_i$. We also have affine maps $\beta := (b_{\phi_\infty(x)})_{x\in F} : V(p_\infty) \to V$ and $\bar\beta : (b_x)_{x\in F} : \mathbb{R}^n \to V$. By 10.34 we have $\beta\circ\alpha = \bar\beta\circ\bar\alpha$. Provided $Span\{b_x\} = \mathbb{R}^{n*}$, $\bar\beta$ is an isomorphism onto its image; and since $Dim(V) \leq n$ we conclude that β is also an isomorphism. If β^{-1} is a left inverse for β, then $\beta^{-1}\circ\beta : \mathbb{R}^n \to V(p_\infty)$ is an affine isomorphism which preserves distance in each direction $x \in F$.

Now take a nested sequence $F_1 \subset F_2 \subset \ldots \subset F_k \subset \ldots \subset \partial B$ so that $\cup F_i \subset \partial B$ is dense, for every i we have $-F_i = F_i$, and $d_B(x, \cdot)$ is differentiable at 0 for every $x \in F_i$. Then by the reasoning of the preceding paragraph we will get a sequence $\Psi_i : (\mathbb{R}^n, \|\cdot\|) \to X$ of affine maps so that Ψ_i is isometric in each direction $x \in F_i$. Passing to a convergent subsequence modulo $Isom(X)$, we get our isometric embedding $\Psi_\infty : (\mathbb{R}^n, \|\cdot\|) \to X$. □

10.4 The proof of Theorem D

Proof of Theorem D. Let n_i be the i^{th} number listed in the statement of the theorem, $1 \leq i \leq 7$. Clearly $n_2 \geq n_1$.

($n_4 \geq n_1$ and $n_3 \geq n_1$). An isometric embedding $\phi : (\mathbb{R}^k, \|\cdot\|) \to X$ induces an isometric embedding $\psi : (\mathbb{R}^k, \|\cdot\|) \to C_T X$, so clearly $n_4 \geq n_1$. Since $\mathbb{R}^k$ is an absolute retract, there is a retraction $\rho : C_T X \to \psi(C_T X)$, so the map $\phi : (\mathbb{R}^k, \mathbb{R}^k - \{0\}) \to (C_T X, \rho^{-1}(\psi(\mathbb{R}^k - \{0\})))$ induces a monomorphism on homology, and hence $n_3 \geq k$. Therefore $n_3 \geq n_1$.

($n_5 \geq n_2$.) If $\phi : (\mathbb{R}^k, \|\cdot\|) \to X$ is a quasi-isometric embedding, and $\star_i \in X$ is any sequence of basepoints, then there is a sequence of isometries $g_i : X \to X$ so that $d(g_i \circ \phi(0), \star_i)$ is bounded. Hence for any sequence of scale factors $\lambda_i \to 0$ the asymptotic cone $(X_\omega, \star_\omega) := \omega\text{-lim}(\lambda_i X, \star_i)$ receives biLipschitz embeddings

$$\omega\text{-lim}(g_i \circ \phi) : \omega\text{-lim}(\mathbb{R}^k, \lambda_i \|\cdot\|) \simeq (\mathbb{R}^k, \|\cdot\|) \to X_\omega.$$

Reasoning as in the preceding paragraph we conclude that $n_5 \geq n_2$.

($n_1 \geq n_7$.) This is proposition 10.22.

($n_6 \geq n_5$.) Recall from section 10.1 that any asymptotic cone X_ω of X is an often convex space. The inequality $n_6 \geq n_5$ follows from Proposition 10.21.

($n_5 \geq n_3$, and $n_6 \geq n_4$) By Lemma 10.6 we have an isometric embedding of $C_T X$ into any asymptotic cone of X with fixed basepoints $X_\omega := \omega\text{-lim}(\lambda_i X, \star)$, and a retraction $X_\omega \to C_T X$. So clearly the two inequalities hold.

($n_7 \geq n_6$). Recall that every compact set C of an ultralimit $\omega\text{-lim}(Z_i, \star_i)$ is a Gromov-Hausdorff limit of a sequence of finite sets $W_i \subseteq Z_i$. If $n_6 \geq k$ there is an asymptotic cone $X_\omega := \omega\text{-lim}(\lambda_i X_i, \star_i)$, sequences $R_j \to 0$, $S_j \subset X_\omega$, so that $\frac{1}{R_j} S_j$ converges to the unit ball in $(\mathbb{R}^k, \|\cdot\|)$ in the Gromov-Hausdorff topology. For each j there is a sequence $T_j^l \subset X$ so that $\lambda_i T_j^l \to S_j$. Passing to a suitable subsequence of the double sequence T_j^l and picking scale factors accordingly we get $n_7 \geq k$. □

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