SINGULARITY STRUCTURE IN MEAN CURVATURE FLOW OF MEAN CONVEX SETS

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ABSTRACT. In this note we announce results on the mean curvature flow of mean convex sets in 3-dimensions. Loosely speaking, our results justify the naive picture of mean curvature flow where the only singularities are neck pinches, and components which collapse to asymptotically round spheres.

In this note we announce results on the mean curvature flow of mean convex sets; all the statements below have natural generalizations to the setting of Riemannian 3-manifolds, but for the sake of simplicity we will primarily discuss subsets of $\mathbb{R}^3$ here. Loosely speaking, our results justify the naive picture of mean curvature flow where the only singularities are neck pinches, and components which collapse to asymptotically round spheres. Recall that a one-parameter family of smooth hypersurfaces $\{M_t\} \subset \mathbb{R}^{n+1}$ *flows by mean curvature* if

$$z_t = \mathbf{H}(z) = \Delta_{M_t} z \,, \tag{1}$$

where $z = (z_1, \ldots, z_{n+1})$ are coordinates on $\mathbb{R}^{n+1}$ and $\mathbf{H} = -H\mathbf{n}$ is the mean curvature vector. The papers [ES91] and [CGG91] defined a level set flow for any closed subset K of $\mathbb{R}^n$. This is a 1-parameter family of closed sets $K_t \subset \mathbb{R}^n$ with $K_0 = K$ (when K is a domain bounded by a smooth compact hypersurface then the evolution of ∂K for a short time interval coincides with the classical mean curvature evolution). Following [Whi00], we say that a compact subset $K \subset \mathbb{R}^n$ is *mean convex* if $K_t \subset \mathrm{Int}(K)$ for all $t > 0$. In this case there is also an associated Brakke flow $\mathcal{M} : t \mapsto M_t$ of rectifiable varifolds [Bra78, Ilm94, Whi00], and the pair $(\mathcal{M}, \mathcal{K})$, where

$$\mathcal{K} := \bigcup_{t \geq 0} K_t \times \{t\} \subset \mathbb{R}^n \times \mathbb{R}$$

is called a *mean-convex flow*, [Whi03]. The fundamental papers [Whi00, Whi03] developed a far-reaching partial regularity theory for mean curvature flow of mean convex subsets of $\mathbb{R}^n$. Our results build on [Whi00, Whi03], giving finer understanding of the singularities in the 3-dimensional case. Recall that the main result of [Whi00] asserts that the space time singular set of the region swept out by a mean–convex set in $\mathbb{R}^{n+1}$ has parabolic Hausdorff dimension at most $(n-1)$, and [Whi03] proved

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a structure theorem for blow–ups of mean–convex flows; cf. also [HS99b, HS99a]. We expect that the more refined description of singularities given here will open the way for applications of mean convex flow to geometric and/or topological problems involving mean convex surfaces.

When $(\mathcal{M}, \mathcal{K})$ is a mean convex flow in $\mathbb{R}^3$, then for almost every time t the time slice K_t is a domain with smooth boundary, [Whi00, Corollary to Theorem 1.1]. Our first result shows that the high curvature portion of such smooth time slices has standard local geometry:

Theorem 2. *For all $\epsilon > 0$ there is a number $h_0 = h_0(\epsilon)$ with the following property. If $(\mathcal{M}, \mathcal{K})$ is a mean convex flow in $\mathbb{R}^3$ and K_t is a regular time slice of $\mathcal{K}$ for some $t > 0$, then there is a decomposition $K_t = G_t \cup B_t$, such that*

- *For all $x \in G_t$, and after rescaling by the factor $\frac{h_0}{d(x,\partial K)}$ the pointed subset (K_t, x) is ϵ-close to some pointed half-space (P, p) in the pointed $C^{\frac{1}{\epsilon}}$-topology.*
- *Each component of B_t is diffeomorphic to the 3-ball or a solid torus, and for all $x \in \partial K_t \cap B_t$, the pointed subset (K_t, x) becomes, after rescaling by the factor $H(x)$, ϵ-close to a pointed convex model subset (V, v) in the pointed $C^{\frac{1}{\epsilon}}$-topology. Here $V \subset \mathbb{R}^3$ is a convex set whose tangent cone at infinity is either a point, a line, or a ray, and V looks like a round cylinder near infinity, in the following sense: for every $\delta > 0$ there is a compact set $K \subset V$, such that for every $v' \in V$ lying outside K, if we rescale V by $H(v')$, the resulting pointed subset (V, v') is δ-close to a round cylinder in the pointed $C^{\frac{1}{\delta}}$-topology.*

Note that the bounds on the geometry deteriorate as one approaches ∂K; this is by necessity since no regularity condition has been imposed on K. If K happens to be smooth, then standard estimates for smooth mean curvature flow control the geometry of K_t when $t \lesssim \sqrt{r}$, where r is the normal injectivity radius of ∂K. Theorem 2 may be compared with the recent work of Huisken-Sinestrari [HS], where a similar geometric description was obtained for mean curvature flow of smooth hypersurfaces in $\mathbb{R}^n$ where the sum of the first two principal curvatures is positive. The results in [Per02, sections 11, 12] are also in a similar spirit. Note that their results only apply to the evolution prior to the formation of the first singularity, whereas our results, like those in [Whi00, Whi03], apply even after the formation of a singularity. (In fact, the methods yield a decomposition of arbitrary time slices, which we omit for the sake of simplicity.)

It follows from the strong maximum principle and compactness that the sets ∂K_t for $t \geq 0$ are disjoint, and define a "singular foliation" of the original set K. Our next theorem proves Hölder regularity of the singular set of the foliation ∂K_t.

Theorem 3. *The foliation defined by the sets ∂K_t is smooth on the complement of a closed subset $S \subset K$ which satisfies the following Reifenberg-type condition: for all $\epsilon > 0$ there is an $r_0 = r_0(\epsilon)$ such that if $r < r_0$ and $x \in S$, then there is a line*

$A \subset \mathbb{R}^3$ *such that* $S \cap B(x,r)$ *is contained in the tubular neighborhood* $N_{\epsilon r}(A)$*. In particular,* S *lies in a* 1*-dimensional topological submanifold* $\gamma \subset K$ *which admits a* C^α*-biHölder parametrization for all* $\alpha < 1$*. Furthermore, the mean curvature defines a proper function on* $K \setminus S$*.*

After passing through a singularity the topological type of a surface flowing by mean curvature can change. In [Whi95] White proved some results comparing the homology of the surface before and after such a singularity. Our next theorem shows that the region between two regular time slices is obtained from the earlier time slice by attaching 2 and 3–handles. Recall that attaching a k–handle to the boundary of an n-manifold N is essentially just the process of attaching a fattened-up k–disk to ∂N along the $(k-1)$–sphere, i.e. one glues $D^k \times D^{n-k}$ to ∂N along $\partial D^k \times D^{n-k}$.

Theorem 4. *If* $0 \leq t < t'$ *and* $K_t, K_{t'}$ *are regular time slices, then* $K_t \setminus \mathrm{Int}(K_{t'})$ *is a compact* 3*-manifold with boundary which may be obtained from* ∂K_t *by attaching* k*-handles for* $k = 2, 3$*.*

Our final theorem deals with mean convex flow in a general 3-manifold, where the flow may converge as time tends to infinity to a set K_∞ with nonempty interior.

Theorem 5. *Let* M *be a compact Riemannian* 3*-manifold, and* $K \subset M$ *a mean convex subset with smooth boundary. Then as* $t \to \infty$*, the intersection of the sets* K_t *converges to a (possibly empty) domain* $K_\infty \subset \mathrm{Int}(K) \subset M$*, where each boundary component of* K_∞ *is a smooth, weakly stable minimal surface, and* $\mathrm{genus}(\partial K_\infty) \leq \mathrm{genus}(\partial K)$*. Furthermore, any compact minimal surface in* $K \setminus \mathrm{Int}(K_\infty)$ *is contained in* ∂K_∞*; in particular* ∂K_∞ *is homologically minimizing in the domain* $K \setminus \mathrm{Int}(K_\infty)$*.*

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