

INFINITESIMAL STRUCTURE OF DIFFERENTIABILITY SPACES, AND METRIC DIFFERENTIATION

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ABSTRACT. We prove metric differentiation for differentiability spaces in the sense of Cheeger [Che99, Kei04a, Bat12]. As corollaries we give a new proof of one of the main results of [Che99], a proof that the Lip-lip constant of any Lip-lip space in the sense of Keith [Kei04a] is equal to 1, and new nonembeddability results.

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1. INTRODUCTION

In this paper we study the metric geometry of differentiability spaces in the sense of Cheeger [Che99, Kei04a, Bat12]. We develop the infinitesimal geometry of Lipschitz curves and Lipschitz functions, generalizing and refining earlier work on spaces satisfying Poincaré inequalities and differentiability spaces; using this we formulate and establish metric differentiation for differentiability spaces. We then give several applications of these results. They include a new proof that the minimal generalized upper gradient of a Lipschitz function is its pointwise upper

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Lipschitz constant, which is one of the main results of [Che99], an alternate proof that the Lip-lip constant of any differentiability space is equal to 1 [Sch13], and new nonembeddability results.

In order to motivate the theory and place it in context, we begin with some background. We will make some additional historical comments at the conclusion of the introduction, after stating our results.

Metric differentiation for $\mathbb{R}^n$. The first instance of metric differentiation was for Lipschitz maps $F : \mathbb{R}^n \rightarrow Z$, where Z is an arbitrary metric space; this is due to Ambrosio in the $n = 1$ case and Kirchheim for general n [Amb90, Kir94]. Although Rademacher's differentiability theorem for Lipschitz maps $\mathbb{R}^n \rightarrow \mathbb{R}^m$ does not apply in this situation, and in fact the usual notion of differentiability does not even make sense since Z has no linear structure, Ambrosio and Kirchheim introduced a new kind of differentiation — metric differentiation — and proved that it always holds. Metric differentiation associates to the map F a measurable Finsler metric, i.e. a measurable assignment $x_0 \mapsto \|\cdot\|_F(x_0)$ of a seminorm (here we identify the tangent space $T_{x_0}\mathbb{R}^n$ with $\mathbb{R}^n$ itself), which captures the geometry of the pullback distance function

$$(1.1) \quad \varrho_F(x_1, x_2) = d_Z(F(x_1), F(x_2))$$

in the sense that for almost every $x_0 \in \mathbb{R}^n$, the pseudodistance ϱ_F satisfies

$$(1.2) \quad \varrho_F(x, x_0) = \|x - x_0\|_F(x_0) + o(\|x - x_0\|_{\mathbb{R}^n}).$$

A slightly different (and stronger) way to express metric differentiation is in terms of the family of pseudodistances $\{\varrho_F^\lambda(x_0) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, \infty)\}_{\lambda \in (0, \infty)}$ obtained by rescaling ϱ_F centered at x_0 :

$$(1.3) \quad \varrho_F^\lambda(x_0)(x_1, x_2) = \lambda \cdot \varrho_F(x_0 + \lambda^{-1}x_1, x_0 + \lambda^{-1}x_2).$$

For almost every x_0 , as $\lambda \rightarrow \infty$ the pseudodistance $\varrho_F^\lambda(x_0)$ converges uniformly on compact subsets of $\mathbb{R}^n \times \mathbb{R}^n$ to the pseudodistance associated with the seminorm $\|\cdot\|_F(x_0)$. An additional aspect of metric differentiation is that for a Lipschitz curve $\gamma : I \rightarrow \mathbb{R}^n$, the length of the path $F \circ \gamma : I \rightarrow Z$ is given by integrating the speed of γ with respect to the Finsler metric $\|\cdot\|_F$,

$$(1.4) \quad \text{length}(F \circ \gamma) = \int_I \|\gamma'(t)\|_F(\gamma(t)) dt,$$

provided that for a.e. $t \in I$, the norm $\|\cdot\|_F$ is defined at $\gamma(t)$, and (1.2) holds with $x_0 = \gamma(t)$. Such curves γ exist in abundance by Fubini's theorem.

Like Rademacher's theorem for Lipschitz maps $\mathbb{R}^n \rightarrow \mathbb{R}^m$, metric differentiation for maps $\mathbb{R}^n \rightarrow Z$ as above can be proved by reducing to the $n = 1$ case. Likewise, one ingredient in our approach to metric differentiation for differentiability spaces is a specific form of the 1-dimensional case of metric differentiation due to Ambrosio-Kirchheim, [AK00b].

The $\mathbb{R}^n$ version of metric differentiation has been applied to the theory of rectifiable sets and currents in metric spaces [Kir94, AK00b, AK00a], to the theory of Sobolev spaces with metric space targets [KS93], and in geometric group theory [Kle99, Wen08, Wen06]. As an historical note, we mention that metric differentiation was discovered independently in conversations between Korevaar-Schoen and

the second author in 92-93, who were unaware of Kirchheim's work at the time [KS93].

Metric differentiation for Carnot groups. A generalization of metric differentiation to Carnot groups was established by Pauls [Pau01]. If $F : \mathbb{G} \rightarrow Z$ is a Lipschitz map from a Carnot group $\mathbb{G}$ equipped with a Carnot-Caratheodory metric to a metric space Z , then for any $x_0 \in \mathbb{G}$ one can apply the canonical rescaling of $\mathbb{G}$ to the pseudodistance ϱ_F to produce a family of rescaled pseudodistances

$$\{\varrho_F^\lambda(x_0) : \mathbb{G} \times \mathbb{G} \rightarrow [0, \infty)\}_{\lambda \in (0, \infty)}$$

analogous to (1.3). Pauls showed that there is a measurable assignment $x_0 \mapsto \|\cdot\|(x_0)$ of seminorms to the horizontal subbundle of $\mathbb{G}$, such that for almost every $x_0 \in \mathbb{G}$ with respect to Haar measure, as $\lambda \rightarrow \infty$, the rescalings $\varrho_F^\lambda(x_0)$ converge on compact subsets of $\mathbb{G} \times \mathbb{G}$ to the Carnot-Caratheodory pseudodistance associated with $\|\cdot\|_F(x_0)$; however, this convergence is only asserted to hold on the subset of pairs $(x_1, x_2) \in \mathbb{G} \times \mathbb{G}$ lying on horizontal geodesics. This restriction to special pairs is necessary even in the case of the Heisenberg group, as was shown in [KM03]. Pauls used his metric differentiation theorem to prove that nonabelian Carnot groups cannot be bilipschitz embedded in Alexandrov spaces, generalizing an earlier result of Semmes [DS97] (which was based on Pansu's version of Rademacher's theorem for mappings between Carnot groups). Another application was a second proof [CK10b] of the fact that the Heisenberg group cannot be biLipschitz embedded in L_1 (originally proved in [CK10a]).

Differentiability spaces. The main goal in this paper is to generalize metric differentiation to a large class of metric measure spaces, namely differentiability spaces. These were first introduced and studied in [Che99] without being given a name; see in particular, Theorem 4.38, Definition 4.42 and the surrounding discussion. There it was shown that PI spaces — metric measure spaces that are doubling and satisfy a Poincaré inequality in the sense of Heinonen-Koskela [HK98] — are differentiability spaces. Differentiability spaces were further studied in [Kei04a, Bat12] (under slightly different hypotheses), where they were called spaces with a strong measurable differentiable structure, and Lipschitz differentiability spaces, respectively. Examples of differentiability spaces include PI spaces such as Carnot groups with Carnot-Caratheodory metrics, and more generally Borel subsets of PI spaces, with the restricted measures. We recall (see Section 2) that a differentiability space (X, μ) has a countable collection $\{(U_i, \phi_i)\}$ of charts, where $\cup_i U_i$ has full measure in X . Also, there are canonically defined measurable tangent and cotangent bundles TX , T^*X , and for any Lipschitz function $u : X \rightarrow \mathbb{R}$, there is a well-defined differential du , which is a measurable section of T^*X .

Remark 1.5. We emphasize that the cotangent and tangent bundles are not on the same footing: the existence of the cotangent bundle follows quite directly from definition of differentiability space, whereas the tangent bundle is defined as the dual of the cotangent bundle i.e. $TX = (T^*X)^*$. It was observed in [CK09] that for PI spaces, given a Lipschitz curve γ , for certain parameter values, one can define a velocity vector $\gamma'(t) \in T_{\gamma(t)}X$ and that such velocity vectors span the tangent space almost everywhere; in [CK] “span” was upgraded to “are dense”. As will be seen below, this new geometric characterization of tangent vectors was crucial to

subsequent developments including the papers [CK], [Bat14] and the main results of the present paper, a first example being Theorem 1.6.

For a Carnot group $\mathbb{G}$ with a Carnot-Caratheodory metric, the horizontal bundle can be canonically identified with the tangent bundle $T\mathbb{G}$ of $\mathbb{G}$ viewed as a PI space. This example indicates that in order to formulate a version of metric differentiation for a differentiability space (X, μ) , one needs to identify a measurable seminorm on the tangent bundle TX and a family of geodesics that will play the role of the family of horizontal geodesics. We first discuss these in the case of the identity map $X \rightarrow X$, initially focussing on the measurable seminorm on TX ; the treatment in this special case may be viewed as part of the intrinsic structure theory of X itself.

For the remainder of the introduction (X, μ) will denote a differentiability space.

The canonical norm on TX . We now consider several ways of defining a seminorm on the tangent bundle TX ; as indicated above, these will be used in the formulation of metric differentiation in the special case of the identity map $X \rightarrow X$. In the first, we choose a countable dense set $\{x_i\} \subset X$, and let $u_i : X \rightarrow \mathbb{R}$ be distance function $u_i(x) = d(x, x_i)$. For every i , since the differential du_i is a measurable section of the cotangent bundle, by duality it defines a measurable family of linear functions on the tangent spaces, and therefore $|du_i(\cdot)|$ defines a measurable family of seminorms on TX ; taking supremum we may define

$$\|v\|_1 = \sup_i |du_i(v)|.$$

As a variations on this, we may define $\|\cdot\|_2$ and $\|\cdot\|_3$ by replacing the collection of distance functions $\{u_i\}$ with the collections of all distance functions and all 1-Lipschitz functions, respectively; note that this requires a little care since these collections are uncountable, see Lemma 2.33. Finally, it was observed in [Che99] that the pointwise upper Lipschitz constant induces a canonical measurable norm on the cotangent bundle T^*X , and by duality this yields a norm $\|\cdot\|_4$ on TX .

Theorem 1.6 (See Section 6). *The seminorms described above agree almost everywhere. In particular, they are all norms, and $\|\cdot\|_1$ is independent of the choice of the countable dense subset.*

We will henceforth use $\|\cdot\|$ denote the norms $\|\cdot\|_i$, $1 \leq i \leq 4$ on the full measure set where they are well-defined and agree.

Generic curves and pairs. We now discuss the role of curves in differentiability spaces. For this we fix a particular choice of charts $\{(U_i, \phi_i)\}$ as above. If $\gamma : I \rightarrow X$ is a Lipschitz curve, then one would like to make sense, for almost every $t \in I$, of the velocity $\gamma'(t)$ and its norm $\|\gamma'(t)\|$, where $\|\cdot\|$ is the norm from Theorem 1.6 (compare (1.4)). Clearly this is impossible for an arbitrary curve γ , since it could lie entirely in the complement of the set where the tangent bundle TX and the norm are well-defined. To address this, we work with generic curves, and generic pairs. Roughly speaking (see Section 3 for the precise definition) if $\gamma : I \rightarrow X$ is Lipschitz curve and $t \in I$, then the pair (γ, t) is *generic* if for some chart (U_i, ϕ_i) of the differentiable structure, the time t is:

- A Lebesgue density point of the inverse image $\gamma^{-1}(U_i)$.

- An approximate continuity point of the measurable function $(\phi_i \circ \gamma)' : I \rightarrow \mathbb{R}^{n_i}$.
- A density point of $\gamma^{-1}(Y)$, where $Y \subset X$ is a full measure subset of $\cup_i U_i$ where the norm $\|\cdot\|$ is well-defined.

The curve γ is *generic* if the pair (γ, t) is generic for almost every $t \in I$. It follows readily from the definitions that for any generic pair (γ, t) , both the velocity vector $\gamma'(t) \in TX$ and its norm $\|\gamma'(t)\|$ are well-defined. More generally, we may use essentially the same notions when γ is a *curve fragment* rather than a curve, i.e. a Lipschitz map $\gamma : C \rightarrow X$, where $C \subset \mathbb{R}$ is closed subset; this additional generality is essential because a differentiability space might have no nonconstant Lipschitz curves. Also, if $\mathcal{F}$ and $\mathcal{C}$ are countable collections of Lipschitz functions and bounded Borel functions respectively, we may impose the additional requirement that t is an approximate continuity point of $(f \circ \gamma)'$ and $u \circ \gamma$ for all $f \in \mathcal{F}$, $u \in \mathcal{C}$.

Metric differentiation along curves. Using the notions of genericity above, we can formulate one aspect of metric differentiation, which is a statement about curve fragments. This uses the concept of the length of a curve fragment, which is straightforward extension of the length of a curve.

Theorem 1.7. *Suppose $\gamma : C \rightarrow X$ is a curve fragment.*

- (1) *If (γ, t) is a generic pair, then t is a point of metric differentiability of γ in the sense that (1.3) holds with $F = \gamma$, $x_0 = t$, and for pairs of points x_1, x_2 where the right-hand side is defined, and moreover $\|\gamma'(t)\| = \|\frac{\partial}{\partial t}\|_\gamma(t)$.*
- (2) *If γ is generic, then the length of γ is given by*

$$\text{length}(\gamma) = \int_C \|\gamma'(t)\| d\mathcal{L},$$

where $\|\cdot\|$ is the norm of Theorem 1.6.

Theorem 1.7 is essentially just an application of Theorem 1.6, and the method of proof of the 1-dimensional version of metric differentiation given in [AK00b], which exploits a countable collection of distance functions as in the definition of $\|\cdot\|_1$; See (2.5) and Theorem 4.3.

Remark 1.8. We point out that unlike in the Carnot group case (and in particular $\mathbb{R}^n$), in a differentiability space (for instance the Laakso spaces [Laa00]) one can have, for a full measure set of points $x \in X$, two generic pairs $(\gamma_1, t_1), (\gamma_2, t_2)$ such that $\gamma_i(t_i) = x$, the velocity vectors $\gamma_1'(t_1), \gamma_2'(t_2)$ coincide, but the curves are not tangent to first order in the sense that $\limsup_{s \rightarrow 0} \frac{d(\gamma_1(t_1+s), \gamma_2(t_2+s))}{s} > 0$. Thus it is somewhat surprising that the tangent vector alone controls the speed of the curve.

The density of generic velocities in TX , and consequences. While the definition of genericity is convenient for stating results about individual curve fragments, in order to use it in statements about (X, μ) that hold at almost every point, such as Theorem 1.7, it is crucial to know that generic curve fragments exist in abundance. This is not at all obvious because the definition of a differentiability

space is based on the behavior of Lipschitz functions and does not involve curves explicitly; in particular it is not even clear why X should contain any curve fragments with positive length. To deduce the needed abundance, we invoke Bate's fundamental work on Alberti representations and differentiability spaces. Bate's work shows that one can characterize differentiability spaces by means of differentiability of Lipschitz functions along curve fragments. The main consequence that we will use here is that for μ -a.e. $p \in X$, the set of generic velocity vectors is dense in $T_p X$ (see Theorem 5.3). Here a *generic velocity vector* is the velocity vector $\gamma'(t) \in T_{\gamma(t)} X$ of a generic pair (γ, t) .

Theorem 1.6 and the density of velocity vectors leads directly to the following:

Corollary 1.9. *If $u : X \rightarrow \mathbb{R}$ is a Lipschitz function, then for μ -a.e. $p \in X$, the pointwise upper Lipschitz constant $\text{Lip } u(p)$ is the supremal normalized directional derivative of u over generic pairs (γ, t) with $\gamma(t) = p$:*

$$\text{Lip } u(p) = \sup \left\{ \frac{(u \circ \gamma)'(t)}{\|\gamma'(t)\|} = \frac{(u \circ \gamma)'(t)}{\|\frac{\partial}{\partial t}\|_{\gamma}(t)} \mid (\gamma, t) \text{ generic, } \gamma(t) = p, \gamma'(t) \neq 0 \right\}.$$

Corollary 1.9 has two further consequences. The first is a new proof of the characterization of the minimal generalized upper gradient in PI spaces as the pointwise Lipschitz constant (see Section 6.3); this was one of the main results in [Che99]. The second is a new proof of the following recent result of the third author (see Section 6.2).

Theorem 1.10 ([Sch13]). *If (X, μ) is a differentiability space, and $u : X \rightarrow \mathbb{R}$ is a Lipschitz function, then for μ -a.e. $p \in X$ we have $\text{Lip } u(p) = \text{lip } u(p)$. Here $\text{lip } u(p)$ is the pointwise lower Lipschitz constant (Definition 2.25).*

We recall that [Kei04a] introduced the Lip-lip-condition for a metric measure space, which says that for some $C \in \mathbb{R}$, and every Lipschitz function $u : X \rightarrow \mathbb{R}$, the upper and lower pointwise Lipschitz constants satisfy $\text{Lip } u \leq C \text{lip } u$ almost everywhere. Keith showed that under mild assumptions on the measure, a metric measure space satisfying a Lip-lip-condition is a differentiability space. Combining this with Theorem 1.10, it follows that one may always take $C = 1$. We note that when (X, μ) is PI space, or more generally a Borel subset of a PI space with the restricted measure, it followed from the earlier work [Che99] that $\text{Lip } u = \text{lip } u$ almost everywhere. These results indicate a strong similarity between PI spaces and differentiability spaces.

For more discussion of these results we refer the reader to the corresponding Sections.

The structure of blow-ups. For a general differentiability space, there is no natural rescaling as in the Carnot group case, so to formulate an analog of the convergence of the rescaled pseudodistances (1.3), we consider sequences of rescalings of X with the measure μ suitably renormalized, and take pointed Gromov-Hausdorff limits of the metric measure spaces, as well as the chart functions and Alberti representations. We give a brief and informal account of this here, and refer the reader to Section 7 for more discussion. For simplicity, in the following statement we assume in addition that X is a doubling metric space.

Theorem 1.11. *For μ -a.e. $x \in X$, if $\{\lambda_j\}$ is any sequence of scale factors with $\lambda_j \rightarrow \infty$, and $x \in U_i$, then there is a sequence $\{\lambda'_j\}$ such that the sequence*

$$\{(\lambda_j X, \lambda'_j \mu, x) \xrightarrow{\phi_i} (\lambda_j \mathbb{R}^{n_i}, \phi_i(x))\}$$

of pointed rescalings of the chart $\phi_i : X \rightarrow \mathbb{R}^{n_i}$ subconverges in the pointed measured Gromov-Hausdorff sense to a pointed blow-up map

$$\hat{\phi}_i : (\hat{X}, \hat{\mu}, \star) \rightarrow (T_x X, 0),$$

where $(\hat{X}, \hat{\mu})$ is a doubling metric measure space. Moreover:

- (1) *When $T_x X$ is equipped with the norm $\|\cdot\|$ of Theorem 1.6, then the map $\hat{\phi}_i : \hat{X} \rightarrow T_x X$ becomes a metric submersion (see Definition 1.12 below).*
- (2) *For every unit vector v in the normed space $(T_x X, \|\cdot\|)$, there is an Alberti representation of $\hat{\mu}$ whose support is contained in the collection of unit speed geodesics $\gamma : \mathbb{R} \rightarrow \hat{X}$ with the property that $\hat{\phi}_i \circ \gamma : \mathbb{R} \rightarrow T_x X$ has constant velocity v ; furthermore, the measure associated to each such γ is just arclength. This Alberti representation is obtained by blowing-up suitable Alberti representations in X .*

Definition 1.12. A map $f : Y \rightarrow Z$ between metric spaces is a metric submersion if it is a 1-Lipschitz surjection, and for every $y_1 \in Y$, $z_2 \in Z$, there is a $y_2 \in f^{-1}(z_2)$ such that $d(y_1, y_2) = d(f(y_1), z_2)$. Equivalently, given any two fibers $f^{-1}(z_1), f^{-1}(z_2) \subset Y$, the distance function from the fiber $f^{-1}(z_1)$ is constant and equal to $d(z_1, z_2)$ on the other fiber $f^{-1}(z_2)$.

To aid the reader's intuition, it might be helpful to look at the example $(\mathbb{R}^2, \mathcal{L}^2)$, where on $\mathbb{R}^2$ we consider the l^1 -norm; as this norm is not strictly convex, one can obtain an Alberti representation of $\mathcal{L}^2$ by using unit-speed geodesics in $\mathcal{L}^2$ with corners, i.e. geodesics which do not lie in straight lines. Blowing-up such representations at a generic point, one obtains an Alberti representation of $\mathcal{L}^2$ whose transverse measure is concentrated on the set of straight lines in $\mathbb{R}^2$.

There are precursors to Theorem 1.11 in [Che99] in the case of PI spaces. In that case the blow-ups (tangent cones) are also PI spaces, the coordinate functions blow-up to generalized linear functions, and [Che99] proved the surjectivity of the canonical map $Y \rightarrow T_x X$. Distinguished geodesics of a different sort were discussed in [Che99], namely the gradient lines of generalized linear functions; however, unlike the curves in the support of the Alberti representations of Theorem 1.11 (2), these need not be affine with respect to the blow-up chart $\hat{\phi}_i$.

Remark 1.13. The third author [Sch13] and David [Dav14] also have results related to Theorem 1.11 (1). They show that certain blow-up maps are Lipschitz quotient maps, which is a weaker version of the metric submersion property. The paper [Sch13] is concerned with the relationship between Weaver derivations [Wea00] and Alberti representations without the assumption that one has a differentiability space, so the setup there is much more general than the one considered here. We point out that our results in Section 7 have natural counterparts in that general context, under the assumption that μ is asymptotically doubling. We note that one of the main ingredients in Theorem 1.11 is a procedure for blowing-up Alberti representations, which has other applications. In particular, it allows one to blow-up

Weaver derivations under the assumption that the background measure is asymptotically doubling. We point out that, as the metric measure space X does not need to possess a group of dilations, it is not trivial to find a correct way to rescale derivations and pass to a limit; however, by taking advantage of the representation of Weaver derivations in terms of Alberti representations proven in [Sch13], one can use Theorem 7.11 to blow-up a derivation at a generic point. Moreover, as the blown-up Alberti representation is concentrated on the set of geodesic lines, the blown-up derivation corresponds to a 1-normal current (in the sense of Lang) without boundary. We refer the reader to Section 7 (in particular Theorem 7.15 and Remark 7.2) for more details.

Theorem 1.11 implies that the blow-up of any Lipschitz function at a generic point is harmonic, in the following sense.

Definition 1.14. Suppose (W, ζ) is proper metric measure space, where ζ is a locally finite Borel measure. Then a Lipschitz function $u : W \rightarrow \mathbb{R}$ is p -Lip-harmonic if for every ball $B(x, r) \subset W$, and every Lipschitz function $v : W \rightarrow \mathbb{R}$ that agrees with u outside $B(x, r)$, we have

$$\int_{B(x, r)} (\text{Lip } v)^p d\hat{\mu} \geq \int_{B(x, r)} (\text{Lip } u)^p d\hat{\mu}.$$

Theorem 1.11 yields:

Corollary 1.15. *Suppose $u : X \rightarrow \mathbb{R}$ is a Lipschitz function. Then for μ a.e. $x \in X$, for any blow-up sequence as in Theorem 1.11 there is a blow-up limit $\hat{u} : Y \rightarrow \mathbb{R}$ such that:*

- (1) $\hat{u}$ is p -Lip-harmonic for all $p \geq 1$.
- (2) For any $y \in Y$, $r \in [0, \infty)$ we have $\text{var}(u, y, r) = r \cdot \text{Lip}(u)(x)$, where $\text{var}(u, y, r)$ is the variation of $\hat{u}$ over $B(y, r)$:

$$\text{var}(u, y, r) = \sup\{|u(z) - u(y)| \mid z \in B(y, r)\}.$$

In particular, $\text{Lip}(\hat{u})(y) = \text{lip}(\hat{u})(y) = \text{Lip}(u)(x)$ for all $y \in Y$, and $\text{Lip}(u)(x)$ is also the global Lipschitz constant of $\hat{u}$.

We remark that in the terminology of [Kei04a, Sec. 6], part (2) of the corollary says that blow-ups are 1-quasilinear; this refines [Kei04a, Sec. 6], where it was shown that blow-ups are K -quasilinear for some K .

It is an open question whether a blow-up of a differentiability space must be a PI space, or even a differentiability space. Corollary 1.15 may be compared with the result from [Che99], which asserts that blow-ups of Lipschitz functions are generalized linear functions — p -harmonic functions with constant norm gradient. The proof in [Che99] is quite different however — it is based on asymptotic harmonicity and breaks down in differentiability spaces.

The results above all speak to the broader topic of the infinitesimal structure of differentiability spaces. There are a number of open questions here. The present state of knowledge makes it difficult to formulate compelling conjectures or questions in a precise form, but one may ask the following:

Question 1.16. If (X, μ) is a differentiability space, is there a countable collection $\{U_i\}$ of Borel subsets of X , such that $\mu(X \setminus \cup_i U_i) = 0$ and every U_i admits a measure-preserving isometric embedding in a PI space?

If the answer is yes, then blow-ups of differentiability spaces at generic points will also be PI spaces, so one may approach this question by trying to verify that blow-ups have various properties of PI spaces, such as quasiconvexity, a differentiable structure, etc. It is of independent interest to gain a better understanding of the structure of blow-ups in the PI space case. Known examples suggest that the blown-up Alberti representations may have accessibility properties similar to the accessibility one has in Carnot groups.

The infinitesimal geometry of Lipschitz maps. We now return to the general case of metric differentiation. Consider a Lipschitz map $F : X \rightarrow Z$, where Z is any metric space, and let $\varrho = \varrho_F$ be the pullback distance function (1.1). Our results in this case parallel what has been discussed above for the special case of the identity map $\text{id}_X : X \rightarrow X$, so we will be brief and focus on the novel features; see Section 8 for the details.

The map F gives rise to a distinguished subset of the Lipschitz functions on X , namely the set of pullbacks $u \circ F$, where $u : Z \rightarrow \mathbb{R}$ is Lipschitz, or equivalently, the set of functions $v : X \rightarrow \mathbb{R}$ that are Lipschitz with respect to the pseudodistance ϱ .

Theorem 1.17 (Theorem 8.6). *There is a canonical subbundle $\mathcal{W}_\varrho \subset T^*X$ such that the differential of any ϱ -Lipschitz function $v : X \rightarrow \mathbb{R}$ belongs to $\mathcal{W}_\varrho$ μ -almost everywhere. Moreover, for any countable dense subset $D_X \subset X$, the set of differentials of the corresponding ϱ -distance functions span $\mathcal{W}_\varrho$.*

One may construct several seminorms on TX analogous to the seminorms $\|\cdot\|_j$, $1 \leq j \leq 4$, of Theorem 1.6. For instance, given a countable dense subset $D_X \subset X$, we may define a seminorm by

$$\|\cdot\|_{1,\varrho} = \sup\{|\varrho_x| \mid x \in D_X\},$$

where ϱ_x is the ϱ -distance from x ; analogs of the other three seminorms are defined similarly, using the pseudodistance ϱ instead of the distance function d_X .

Theorem 1.18 (Theorem 8.24). *The seminorms agree almost everywhere, giving rise to a canonical seminorm $\|\cdot\|_\varrho$ on TX .*

Unlike in the case of the identity map, when $\varrho = d_X$, the canonical seminorm need not be a norm. Instead it induces a norm on the quotient bundle $TX/\mathcal{W}_\varrho^\perp$ and a dual norm $\|\cdot\|_\varrho^*$ on the canonical subbundle $\mathcal{W}_\varrho \subset T^*X$; here $\mathcal{W}_\varrho^\perp \subset TX$ is the annihilator of the $\mathcal{W}_\varrho \subset T^*X$.

There are two different ways to formulate metric differentiation in terms of blow-ups. In the first, we refine Theorem 1.11 by bringing in the sequence of rescaled pseudodistances $\{\lambda_j \varrho\}$ as well. After passing to a subsequence, these will Gromov-Hausdorff converge (in a natural sense) to a limiting pseudodistance $\hat{\varrho}$ on Y . Then in addition to conclusions (1) and (2) of Theorem 1.11, we have:

- (3) When Y and $T_p X$ are equipped with the pseudodistance $\hat{\varrho}$ and the seminorm $\|\cdot\|_\varrho$ of Theorem 1.18 respectively, the map $\hat{\phi}_i : Y \rightarrow T_p X$ is a metric submersion.

- (4) For every unit vector v in the normed space $(T_p X, \|\cdot\|)$, there is an Alberti representation of $\hat{\mu}$ whose support is contained in the collection of curves $\gamma : \mathbb{R} \rightarrow Y$ with the property that $\hat{\phi}_i \circ \gamma : \mathbb{R} \rightarrow T_p X$ has constant velocity v , γ is a unit speed d_Y -geodesic, and a constant $\|v\|_e$ -speed $\hat{\rho}$ -geodesic.

A second way to formulate the blow-up assertion is to take an ultralimit of the map $F : X \rightarrow Z$. We refer the reader to Section 8.3 for the statements.

One consequence of (4) is that the blown-up Alberti representations appearing in Theorem 1.11 (2) may be viewed as invariants of the differentiability space structure, in the following way. The definitions readily imply that if (X, μ) is a differentiability space, (Z, ν) is a metric measure space, and $F : (X, \mu) \rightarrow (Z, \nu)$ is a bilipschitz homeomorphism that is also measure class preserving in the sense that pushforward measure $F_* \mu$ and ν are mutually absolutely continuous, then (Z, ν) is also a differentiability space. When X is doubling, for almost every $p \in X$, we can then take a Gromov-Hausdorff limit of the the sequence of rescalings of F as in Theorem 1.11, to obtain a bilipschitz homeomorphism

$$\hat{F} : (\hat{X}, \hat{\rho}) \longrightarrow (\hat{Z}, \hat{F}(\hat{\rho})).$$

This blow-up map $\hat{F}$ will preserve the blow-up measures up to scale, and will preserve the blow-up Alberti representations from Theorem 1.11(2) up to a change of speed that depends only on the choice of tangent vector v .

Applications to embedding. In Section 9 we apply metric differentiation to Lipschitz maps between Carnot groups, Alexandrov spaces with curvature bounded above or below, and the inverse limit spaces in [CK13], showing that such maps are strongly constrained on an infinitesimal level.

Further discussion. We now make some remarks about the evolution of some of the main ideas in this paper — generic velocities, the proof of abundance, the structure of blow-ups, and their distinguished geodesics.

While [Che99] clarified many points at the foundation of PI spaces, the role of curves remained somewhat mysterious, and in particular velocity vectors to curves were not considered there. In fact, although Lipschitz curves were used in the original definition of a PI space by Heinonen-Koskela (which is based on upper gradients) there is an equivalent definition in which curves do not appear at all [Kei03].

The first appearance of tangent vectors to curves in the context of PI spaces was in [CK09]. There a notion similar to generic velocity vectors was introduced, and it was shown that they span the tangent space at a typical point; in addition, there was a new characterization of the minimal generalized upper gradient, which may be viewed as a precursor to Corollary 1.9. Metric differentiation for PI spaces was announced in [CK09, p.1020]. This was work of the first two authors, which led to an unpublished account of metric differentiation [CK] that was similar in several respects to the present paper. For instance, it used a notion of generic velocity vectors, and contained a blow-up statement like Theorem 1.11 involving a distinguished family of geodesics with constant velocity in the blow-up chart; however, it did not use Alberti representations. We mention that is easy to see that the collection of nongeneric Lipschitz curves $\gamma : I \rightarrow X$ has zero p -modulus,

for every p . This yields a weak form of abundance of generic curves in the PI space case. A key ingredient in [CK] was a proof of the density of the directions of generic velocity vectors based on a much deeper argument that borrowed ideas — a renorming argument and the equality $\text{Lip } u$ and the minimal generalized upper gradient — from [Che99].

Bate's beautiful work on Alberti representations [Bat12, Bat14] greatly strengthened the connection between curves and differentiability, providing several different alternate characterizations of differentiability spaces in terms of Alberti representations. His approach was partly motivated by the work of Alberti-Csornyei-Preiss on differentiability for subsets of $\mathbb{R}^n$, and an observation of Preiss that the characterization of the minimal generalized upper gradient in [CK09] implied the existence of Alberti representations for PI spaces [Bat14, Sec. 10].

When [Bat12] appeared, the third author used it to give a proof of $\text{Lip} = \text{lip}$ based on a renorming construction, without being aware of the contents of [CK]. Independently, the first two authors recognized that [Bat12] could be used to give a stronger and more general treatment of metric differentiation, and proposed writing the present paper.

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2. PRELIMINARIES

2.1. Standing assumptions and review of differentiability spaces. Throughout this paper, the pair (X, μ) will denote a **differentiability space**; this means that (X, d_X) is a complete, separable metric space, μ is a Radon measure, and the pair (X, μ) admits a measurable differentiable structure as recalled below, cf. [Che99, Kei04a].

We briefly highlight the main features of a differentiability space, see below for more discussion:

- (1) There is a countable collection of charts $\{(U_\alpha, \phi_\alpha)\}_\alpha$, where $U_\alpha \subset X$ is measurable and ϕ_α is Lipschitz, such that $X \setminus (\cup_\alpha U_\alpha)$ is μ -null, and each real-valued Lipschitz function f admits a first order Taylor expansion with respect to the components of $\phi_\alpha : X \rightarrow \mathbb{R}^{N_\alpha}$ at generic points of U_α , i.e. there exist a.e. unique measurable functions $\frac{\partial f}{\partial \phi_\alpha^i}$ on U_α such that:

$$(2.1) \quad f(x) = f(x_0) + \sum_{i=1}^{N_\alpha} \frac{\partial f}{\partial \phi_\alpha^i}(x_0) (\phi_\alpha^i(x) - \phi_\alpha^i(x_0)) + o(d_X(x, x_0)) \quad (\text{for } \mu\text{-a.e. } x_0 \in U_\alpha).$$

- (2) There are measurable cotangent and tangent bundles T^*X and TX (see also subsection 2.5). The fibres of T^*X are generated by the differentials of Lipschitz functions, and the tangent bundle of TX is defined *formally* by duality: part of the motivation of the present work is to give a concrete description of TX by using velocity vectors of Lipschitz curves.

- (3) Natural dual norms $\|\cdot\|_{\text{Lip}}$ and $\|\cdot\|_{\text{Lip}}^*$ on T^*X and TX respectively. The norm $\|\cdot\|_{\text{Lip}}$ is induced by the pointwise upper Lipschitz constant, i.e. for any real-valued Lipschitz functions f we have $\|df\|_{\text{Lip}} = \text{Lip } f(x)$ for μ -a.e. $x \in X$.

We recall that $\text{Lip } f(x)$ denotes the **(upper) pointwise Lipschitz constant of f at x** , that is:

$$(2.2) \quad \text{Lip } f(x) = \limsup_{r \searrow 0} \sup \left\{ \frac{|f(y) - f(x)|}{r} : d_X(x, y) \leq r \right\}.$$

We now give a brief review of some definitions from [Che99, Kei04a]; an exposition can be found in [KM11]. Let (Z, ν) be a metric measure space. Let $\mathcal{U}$ be a (countable) collection of Lipschitz functions on Z . Then $\mathcal{U}$ is **dependent at $x \in Z$** if some finite nontrivial linear combination v of elements of $\mathcal{U}$ is constant to first order at x , i.e. $|v(y) - v(x)| = o(d_X(x, y))$. Alternatively, one can say that the pointwise upper Lipschitz constant $\text{Lip } v$ of v vanishes at x . The **dimension of $\mathcal{U}$ at x** is the supremal cardinality of a subset that is linearly independent at x ; the dimension function $\dim_{\mathcal{U}} : Z \rightarrow \mathbb{N} \cup \{\infty\}$ is Borel whenever $\mathcal{U}$ is a countable collection. Suppose that $U \subset Z$ is a Borel set with positive ν -measure, and that $\phi : U \rightarrow \mathbb{R}^n$ is Lipschitz. The pair (U, ϕ) is a **chart** if the component functions $\phi_1, \dots, \phi_n$ of ϕ are independent at ν -a.e. $x \in U$, and if for each real-valued Lipschitz function f , the $(n+1)$ -tuple $(\phi_1, \dots, \phi_n, f)$ is dependent at ν -a.e. $x \in U$. In particular, there are, unique up to ν -null sets, Borel functions $\frac{\partial f}{\partial \phi_i} : U \rightarrow \mathbb{R}$ such that the Taylor expansion (2.1) holds for ν -a.e. $x_0 \in U$; in this case we also say that f is **differentiable at x_0 with respect to the $\{\phi_i\}_{i=1}^n$** .

A metric measure space (Z, ν) **admits a measurable differentiable structure** if there exists an countable collection of charts $\{(U_\alpha, \phi_\alpha)\}_\alpha$ such that $Z \setminus (\cup_\alpha U_\alpha)$ is ν -null. Without loss of generality, we will always assume that for each pair (α, β) , at each point of $U_\alpha \cap U_\beta$ the functions ϕ_α are differentiable with respect to the functions ϕ_β .

One says that a metric measure space (Z, ν) is **(almost everywhere) finite dimensional** if for any countable collection $\mathcal{U}$ of Lipschitz functions, the dimension $\dim_{\mathcal{U}}$ is finite almost everywhere. It follows from a selection argument [Che99, Kei04a] that when ν is σ -finite, then (Z, ν) admits a measurable differentiable structure if and only if it is finite dimensional. Thus, apart from being a standard condition on a measure, σ -finiteness is a natural assumption in the present topic. As the measure ν only enters through its sets of measure zero, one really only cares about the *measure class* of ν ; hence if ν is σ -finite, then without loss of generality one may take ν to be finite.

We finally give a brief justification of why we assume X to be complete in the definition of a differentiability space, which was also a working assumption in [BS11, Bat12]. Suppose (Z, ν) is a metric measure space, where Z is not necessarily complete. Denote by $\bar{Z}$ its completion, and let $\bar{\nu}$ be the pushforward of ν under the inclusion $Z \rightarrow \bar{Z}$. Then any Lipschitz function $u \in \text{Lip}(Z)$ extends uniquely to $\bar{Z}$, and since Z is dense in $\bar{Z}$, the notions of dependence and dimension for a collection $\mathcal{U} \subset \text{Lip}(Z)$ at any $x \in Z$ agree with the notions for the corresponding collection

$\bar{U} \subset \text{Lip}(\bar{Z})$. Hence (Z, ν) has a differentiable structure if and only if $(\bar{Z}, \bar{\nu})$ has a measurable differentiable structure.

2.2. The metric derivative for 1-rectifiable sets. Let Z be a separable metric space and denote by d_Z the metric on Z . We say that a pseudometric ϱ on Z is **Lipschitz compatible** if there is a nonnegative constant C such that:

$$(2.3) \quad \varrho \leq C d_Z;$$

we say that a function $f : Z \rightarrow W$ is ϱ -Lipschitz if there is a nonnegative C such that:

$$(2.4) \quad d_W(f(z_1), f(z_2)) \leq C \varrho(z_1, z_2) \quad (\forall z_1, z_2 \in Z).$$

Note that ϱ -Lipschitz functions are necessarily d_Z -Lipschitz; when referring to the background metric d_Z we will simply use the term Lipschitz. We denote by $\mathcal{H}^1$ the 1-dimensional Hausdorff measure on Z and by $\mathcal{H}_\varrho^1$ the 1-dimensional Hausdorff measure associated to the pseudometric ϱ .

We now recall metric differentiation results of [Kir94, AK00b, AT04] in the case of 1-rectifiable sets.

Let Y be a Lebesgue measurable subset of $\mathbb{R}$ and let $\gamma : Y \rightarrow Z$ be a Lipschitz map. We fix a countable dense subset $\{z_i\}$ of Z , and let u_i be the pullback of the pseudodistance function $\varrho_{z_i}(\cdot) = \varrho(\cdot, z_i)$ by the map γ . Then γ has a **ϱ -metric differential** $\varrho\text{-md}\gamma : Y \rightarrow [0, \infty)$, which is uniquely determined for $\mathcal{L}^1$ a.e. $t \in Y$, and which has the following properties:

(MD1): Rescalings of the pullback pseudometric $\gamma^* \varrho$ at t converge uniformly on compact sets to $\varrho\text{-md}\gamma(t) d_{\mathbb{R}}$, that is, the Euclidean distance scaled by the factor $\varrho\text{-md}\gamma(t)$.

(MD2): Consider a point $t \in Y$ such that:

- (1) The point t is a Lebesgue density point of Y ;
- (2) The derivatives of the functions $\{u_i\}_i$ exist at t ;
- (3) The derivatives $\{u'_i\}_i$ are approximately continuous at t ;
- (4) The function $\sup_i |u'_i|$ is approximately continuous at t .

Then the ϱ -metric differential exists at t and is given by:

$$(2.5) \quad \varrho\text{-md}\gamma(t) = \sup_i |u'_i(t)|.$$

(MD3): One has an area formula [Kir94, Thm. 7]:

$$(2.6) \quad \int_Z \# \{t \in Y : \gamma(t) = z\} d\mathcal{H}_\varrho^1(z) = \int_Y \varrho\text{-md}\gamma(t) d\mathcal{L}^1(t).$$

In the case in which the metric differential refers to the metric d_Z we will use the symbol $\text{md}\gamma$ instead of $d_Z\text{-md}\gamma$.

2.3. Alberti representations. Alberti representations were introduced in [Alb93] to prove the so-called rank-one property for BV functions; they were later applied to study the differentiability properties of Lipschitz functions $f : \mathbb{R}^N \rightarrow \mathbb{R}$

[ACP05, ACP10] and have recently been used to obtain a description of measures in differentiability spaces [Bat12]. We first give an informal definition.

An **Alberti representation** of a Radon measure μ is a generalized Lebesgue decomposition of μ in terms of 1-rectifiable measures: i.e. one writes μ as an integral:

$$(2.7) \quad \mu = \int \nu_\gamma dP(\gamma),$$

where $\{\nu_\gamma\}$ is a family of 1-rectifiable measures. The standard example is offered by Fubini's Theorem; given $x \in \mathbb{R}^{N-1}$, denote by $\gamma(x)$ the parametrized line in $\mathbb{R}^N$ given by $\gamma(x)(t) = x + te_{n+1}$; then an Alberti representation of the Lebesgue measure $\mathcal{L}^N$ is given by:

$$(2.8) \quad \mathcal{L}^N = \int_{\mathbb{R}^{N-1}} \mathcal{H}_{\gamma(x)}^1 d\mathcal{L}^{N-1}(x).$$

To make the previous account more precise we introduce more terminology. For more details we refer the reader to [Bat12] and [Sch13, Sec. 2.1]; note however, that we slightly diverge from the treatments in [Bat12, Sch13] because we discuss also unbounded 1-rectifiable sets: the need to do so becomes apparent in Section 7.

Definition 2.9. A **fragment** in X is a Lipschitz map $\gamma : C \rightarrow X$, where $C \subset \mathbb{R}$ is closed. The set of fragments in X will be denoted by $\text{Frag}(X)$.

We need to topologize $\text{Frag}(X)$; let $F(\mathbb{R} \times X)$ denote the set of closed subsets of $\mathbb{R} \times X$ with the Fell topology [Kec95, (12.7)]; we recall that a basis of the Fell topology consists those sets of the form:

$$(2.10) \quad \{F \in F(\mathbb{R} \times X) : F \cap K = \emptyset, F \cap U_i \neq \emptyset \text{ for } i = 1, \dots, n\},$$

where K is a compact subset of $\mathbb{R} \times X$, and $\{U_i\}_{i=1}^n$ is a finite collection of open subsets of $\mathbb{R} \times X$. Note that the empty set $\emptyset$ is included in $F(\mathbb{R} \times X)$ and that, if X is locally compact, the topological space $F(\mathbb{R} \times X)$ is compact. We now consider the set $F_c(\mathbb{R} \times X) = F(\mathbb{R} \times X) \setminus \{\emptyset\}$ which is, if X is locally compact, a K_σ , i.e. a countable union of compact sets. Each fragment γ can be identified with an element of $F_c(\mathbb{R} \times X)$ and thus $\text{Frag}(X)$ will be topologized as a subset of $F_c(\mathbb{R} \times X)$. We will use fragments to parametrize 1-rectifiable subsets of X .

We now briefly discuss the topology on Radon measures that allows to make sense of an integral like (2.7). Let $C_c(X)$ denote the set of continuous function defined on X with compact support; recall that the set $C_c(X)$ is a Fréchet space. We denote by $\text{Rad}(X)$ the set of (nonnegative) Radon measures on X ; as $\text{Rad}(X)$ can be identified with a subset of the dual of $C_c(X)$, we will topologize it with the restriction of the weak* topology. In particular, when we assert that a map $\psi : Z \rightarrow \text{Rad}(X)$ is Borel, we mean that for each $g \in C_c(X)$, the map:

$$(2.11) \quad z \mapsto \int_X g(x) d(\psi(z))(x)$$

is Borel.

Definition 2.12. An Alberti representation of the measure μ is a pair (P, ν) such that:

(Alb1): P is a Radon measure on $\text{Frag}(X)$;

(Alb2): The map $\nu : \text{Frag} \rightarrow \text{Rad}(X)$ is Borel and, for each $\gamma \in \text{Frag}(X)$, we have $\nu_\gamma \ll \mathcal{H}_\gamma^1$, where $\mathcal{H}_\gamma^1$ denotes the 1-dimensional Hausdorff measure on the image of γ ;

(Alb3): The measure μ can be represented as $\mu = \int_{\text{Frag}(X)} \nu_\gamma dP(\gamma)$;

(Alb4): For each Borel set $A \subset X$ and all real numbers $b \geq a$, the map $\gamma \mapsto \nu_\gamma(A \cap \gamma(\text{Dom } \gamma \cap [a, b]))$ is Borel.

We now recall some definitions regarding additional properties of Alberti representations.

Definition 2.13. An Alberti representation $\mathcal{A} = (P, \nu)$ is said to be **C -Lipschitz (resp. (C, D) -biLipschitz)** if P -a.e. γ is C -Lipschitz (resp. (C, D) -biLipschitz).

Definition 2.14. Let $\sigma : X \rightarrow [0, \infty)$ be Borel and $f : X \rightarrow \mathbb{R}$ be Lipschitz. An Alberti representation $\mathcal{A} = (P, \nu)$ is said to be **have f -speed $\geq \sigma$ (resp. $> \sigma$)** if for P -a.e. $\gamma \in \text{Frag}(X)$ and $\mathcal{L}^1 \llcorner \text{Dom } \gamma$ -a.e. t one has $(f \circ \gamma)'(t) \geq \sigma(\gamma(t)) \text{md } \gamma(t)$ (resp. $(f \circ \gamma)'(t) > \sigma(\gamma(t)) \text{md } \gamma(t)$).

Another property regards the *direction*, with respect to a finite tuple of Lipschitz functions, of the fragments used in an Alberti representation. To measure the direction one can use the notion of Euclidean cone:

Definition 2.15. Let $\theta \in (0, \pi/2)$, $v \in \mathbb{S}^{n-1}$; the **open cone** $\text{Cone}(v, \theta) \subset \mathbb{R}^n$ with axis v and opening angle θ is:

$$(2.16) \quad \text{Cone}(v, \theta) = \{u \in \mathbb{R}^q : \tan \theta \langle v, u \rangle > \|\pi_v^\perp u\|_2\},$$

where $\pi_v^\perp$ denotes the orthogonal projection on the orthogonal complement of the line $\mathbb{R}v$.

Definition 2.17. Given a Lipschitz function $f : X \rightarrow \mathbb{R}^n$, an Alberti representation $\mathcal{A} = (P, \nu)$ is said to be **in the f -direction of the open cone $\text{Cone}(v, \theta)$** if for P -a.e. $\gamma \in \text{Frag}(X)$ and $\mathcal{L}^1 \llcorner \text{Dom } \gamma$ -a.e. t one has $(f \circ \gamma)'(t) \in \text{Cone}(v, \theta)$.

For the purpose of this paper it will be convenient to obtain Alberti representations with biLipschitz constants close to 1. We will thus use the following result [Sch13, Thm. 2.64]:

Theorem 2.18. *Let X be a complete separable metric space and μ a Radon measure on X . Then the following are equivalent:*

- (1) *The measure μ admits an Alberti representation $\mathcal{A}$ in the f -direction of $\text{Cone}(v, \theta)$ with g -speed $> \sigma$;*
- (2) *For each $\varepsilon > 0$ the measure μ admits a $(1, 1 + \varepsilon)$ -biLipschitz Alberti representation $\mathcal{A}$ in the f -direction of $\text{Cone}(v, \theta)$ with g -speed $> \sigma$.*

Moreover, one can always assume that the Alberti representation is of the form $\mathcal{A} = (P, \nu)$, where P is a finite Radon measure concentrated on the set of fragments with compact domain. Additionally, one can assume that $\nu = h\Psi$ where h is a nonnegative Borel function of X and:

$$(2.19) \quad \Psi_\gamma = \gamma_\# (\mathcal{L}^1 \llcorner \text{Dom } \gamma),$$

i.e. the push-forward of the restriction of the Lebesgue measure to the domain of γ .

Sometimes we will find it useful to **restrict an Alberti representation** $\mathcal{A} = (P, \nu)$ to a Borel set $U \subset X$ by letting $\mathcal{A} \llcorner U = (P, \nu \llcorner U)$. Other times one knows the existence of Alberti representations on subsets $\{U_\alpha\}_\alpha$ and would like to glue them together. This is accomplished by the following *gluing principle* [Sch13, Thm. 2.46]:

Theorem 2.20. *Let $\{U_\alpha\}_\alpha$ be Borel subsets and suppose that for each α the measure $\mu \llcorner U_\alpha$ admits a (C, D) -biLipschitz Alberti representation in the f -direction of $\text{Cone}(v, \theta)$ with f -speed $\geq \sigma$ (or $> \sigma$); then the measure $\mu \llcorner \bigcup_\alpha U_\alpha$ also admits a (C, D) -biLipschitz Alberti representation in the f -direction of $\text{Cone}(v, \theta)$ with f -speed $\geq \sigma$ (or $> \sigma$).*

2.4. Results from Bate and Speight. We now recall some results [BS11, Bat12] on the structure of measures in differentiability spaces. The original Theorems [Che99, Kei04a] on the existence of differentiable structures required the measure μ to be doubling. Bate and Speight [BS11] found a partial converse of this:

Theorem 2.21. *If (X, μ) is a differentiability space, then:*

- The measure μ is **asymptotically doubling**, i.e. for μ -a.e. x there are $(C_x, r_x) \in (0, \infty)^2$ such that:

$$(2.22) \quad \mu(B(x, 2r)) \leq C_x \mu(B(x, r)) \quad (\forall r \leq r_x).$$

As a consequence, (X, μ) is a Vitali space, i.e. the Vitali Covering Theorem holds in (X, μ) , and thus also Lebesgue's Differentiation Theorem holds for μ .

- Every porous subset is μ -null.

It was shown in [Bat14, Lemma 8.3] that if (X, μ) is asymptotically doubling, there are countably many Borel sets $\{U_\alpha\}_\alpha$ such that $\mu(X \setminus \bigcup_\alpha U_\alpha) = 0$ and such that each U_α is doubling as a metric space. Moreover, the sets $\{U_\alpha\}_\alpha$ might be assumed to be closed or compact.

Remark 2.23. In particular, at generic points of each U_α , one can obtain blow-ups/tangent cones of (X, μ) by using Gromov's Compactness Theorem (see Section 7). In fact, as porous sets are μ -null, blowing-up $(U_\alpha, \mu \llcorner U_\alpha)$ at a point p of U_α which is a Lebesgue density point for μ , and is also a point at which U_α is not porous in the ambient space X , will yield the same metric measure spaces as blowing-up (X, μ) .

Recently Bate [Bat12] made a deep study of the structure of measures in differentiability spaces by using Alberti representations; in particular, he was able to obtain several characterizations of these spaces. For the sake of brevity we just summarize one characterization as follows:

Theorem 2.24. *The metric measure space (X, μ) is a differentiability space if and only if:*

- (1) *The measure μ is asymptotically doubling and porous sets are μ -null;*
- (2) *There is a Borel function $\tau : X \rightarrow (0, \infty)$ such that, for each real-valued Lipschitz function f , the measure μ admits an Alberti representation with f -speed $\geq \tau \text{Lip } f$.*

In [Sch13] it was shown that one may take $\tau = 1$: in subsection 6.2 we provide a proof of this fact which is independent of the results in [Sch13]. To put this in perspective we recall the following definition:

Definition 2.25. Let $f : X \rightarrow \mathbb{R}$ be Lipschitz. The **lower pointwise Lipschitz constant of f at x** is:

$$(2.26) \quad \text{lip } f(x) = \liminf_{r \searrow 0} \sup \left\{ \frac{|f(y) - f(x)|}{r} : d_X(x, y) \leq r \right\}.$$

In [Kei04a] it was shown that the existence of a measurable differentiable structure follows under the assumption that (X, μ) satisfies a **Lip-lip inequality**: this means that there is a $K \geq 1$ such that, for each real-valued Lipschitz function f , one has:

$$(2.27) \quad \text{Lip } f(x) \leq K \text{ lip } f(x) \quad (\text{for } \mu\text{-a.e. } x).$$

In particular, Theorem 2.24 implies that in a differentiability space the Lip-lip inequality holds by replacing the constant K with the function τ ; thus, showing that one can take $\tau = 1$ implies that the Lip-lip inequality self-improves to an equality. For the case of PI-spaces, the Lip-lip equality was a main result of [Che99], which followed from the more general result that, for $p > 1$, $\text{Lip } f$ is a representative of the *minimal generalized upper gradient* of f .

The result of [Bat12] that we will mainly use is the existence of Alberti representations in the directions of arbitrary cones:

Theorem 2.28. *Let (U, ψ) be an N -dimensional differentiability chart for the differentiability space (X, μ) ; then for each $v \in \mathbb{S}^{N-1}$ and each $\theta \in (0, \pi/2)$, the measure $\mu \llcorner U$ admits an Alberti representation in the ψ -direction of $\text{Cone}(v, \theta)$.*

2.5. Measurable Vector Bundles. In this paper we will work with measurable subbundles of the tangent and cotangent bundles associated to a differentiability space. Since we deal with different (measurable) seminorms on these subbundles, we need to introduce a bit of terminology to make the treatment precise. Let (Ω, Σ) be a measurable space; a **Σ -measurable vector bundle over Ω** is a quadruple $(I_{\mathcal{V}}, \{N_{\alpha}\}_{\alpha \in I_{\mathcal{V}}}, \{U_{\alpha}\}_{\alpha \in I_{\mathcal{V}}}, \{g_{\alpha, \beta}\}_{(\alpha, \beta) \in I_{\mathcal{V}, \cap}})$ such that:

- (1) The index set $I_{\mathcal{V}}$ is countable and $\{U_{\alpha}\}_{\alpha \in I_{\mathcal{V}}}$ is a cover of Ω consisting of Σ -measurable sets;
- (2) Each N_{α} is a nonnegative integer and if $U_{\alpha} \cap U_{\beta} \neq \emptyset$, then $N_{\alpha} = N_{\beta}$;
- (3) The (possibly empty set) $I_{\mathcal{V}, \cap}$ consists of those pairs $(\alpha, \beta) \in I_{\mathcal{V}} \times I_{\mathcal{V}}$ such that $U_{\alpha} \cap U_{\beta} \neq \emptyset$;
- (4) Each $g_{\alpha, \beta}$ is a Σ -measurable map $g_{\alpha, \beta} : U_{\alpha} \cap U_{\beta} \rightarrow \text{GL}(\mathbb{R}^{N_{\alpha}})$.

If $N = \sup_{\alpha} N_{\alpha} < \infty$ the bundle $\mathcal{V}$ is said **to have finite dimension N** .

A **section σ of $\mathcal{V}$** is a collection $\{\sigma_{\alpha}\}_{\alpha \in I_{\mathcal{V}}}$ of Σ -measurable maps $\sigma_{\alpha} : U_{\alpha} \rightarrow \mathbb{R}^{N_{\alpha}}$ such that:

$$(2.29) \quad g_{\alpha, \beta} \circ \sigma_{\alpha} = \sigma_{\beta}.$$

A measurable subbundle of $\mathcal{V}$ is a measurable choice of a hyperplane in each fibre. More precisely, let $\text{Gr}(\mathbb{R}^N, k)$ denote the Grassmanian of unoriented k -dimensional planes in $\mathbb{R}^N$; then a **subbundle** $\mathcal{W}$ of $\mathcal{V}$ is a pair $(\{M_\alpha\}_{I_\mathcal{V}}, \{\phi_\alpha\}_{I_\mathcal{V}})$ such that:

- (1) Each nonnegative integer M_α satisfies $M_\alpha \leq N_\alpha$ and if $(\alpha, \beta) \in I_{\mathcal{V}, \cap}$, then $M_\alpha = M_\beta$;
 - (2) Each ϕ_α is a Σ -measurable map $\phi_\alpha : U_\alpha \rightarrow \text{Gr}(\mathbb{R}^{N_\alpha}, M_\alpha)$;
 - (3) For each pair $(\alpha, \beta) \in I_{\mathcal{V}, \cap}$ the following compatibility condition holds:
- $$(2.30) \quad g_{\alpha, \beta}(\phi_\alpha(x)) = \phi_\beta(x) \quad (\forall x \in U_\alpha \cap U_\beta).$$

We now turn to the construction of seminorms on $\mathcal{V}$ (or on a subbundle). Let $\text{Sem}(\mathbb{R}^N)$ denote the set of seminorms on $\mathbb{R}^N$, and let $\text{Sem}_{+\infty}(\mathbb{R}^{N_\alpha}) = \text{Sem}(\mathbb{R}^{N_\alpha}) \cup \{+\infty\}$, which is viewed as the one-point compactification of $\text{Sem}(\mathbb{R}^{N_\alpha})$. The element $+\infty$ is interpreted as the real-valued function on $\mathbb{R}^N$ which assigns value $+\infty$ to any non-zero vector and value 0 to the null vector. A seminorm (resp. a generalized seminorm) $\|\cdot\|$ on $\mathcal{V}$ is a collection $\{\|\cdot\|_\alpha\}_{\alpha \in I_\mathcal{V}}$ of Σ -measurable maps $\|\cdot\|_\alpha : U_\alpha \rightarrow \text{Sem}(\mathbb{R}^{N_\alpha})$ (resp. $\text{Sem}_{+\infty}(\mathbb{R}^{N_\alpha})$) which satisfy, for each $(\alpha, \beta) \in I_{\mathcal{V}, \cap}$ and each $v \in \mathbb{R}^{N_\alpha}$, the following compatibility condition:

$$(2.31) \quad \|v\|_\alpha(x) = \|g_{\alpha, \beta}(v)\|_\beta(x) \quad (\forall x \in U_\alpha \cap U_\beta).$$

We will essentially work with measurable bundles where $\Omega = X$, a complete separable metric space, and where Σ is the Borel σ -algebra. However, in the case of a metric measure space (X, μ) , we implicitly identify vector bundles, sections and seminorms which agree μ -a.e. For example, consider two Borel vector bundles $\mathcal{V} = (I_\mathcal{V}, \{N_\alpha\}_{\alpha \in I_\mathcal{V}}, \{U_\alpha\}_{\alpha \in I_\mathcal{V}}, \{g_{\alpha, \beta}\}_{(\alpha, \beta) \in I_{\mathcal{V}, \cap}})$ and $\mathcal{V}' = (I'_\mathcal{V}, \{N'_{\alpha'}\}_{\alpha' \in I'_\mathcal{V}}, \{U'_{\alpha'}\}_{\alpha' \in I'_\mathcal{V}}, \{g'_{\alpha', \beta'}\}_{(\alpha', \beta') \in I'_{\mathcal{V}, \cap}})$ over X ; we identify them if:

- (1) Whenever $\mu(U_\alpha \cap U'_{\alpha'}) > 0$ one has $N_\alpha = N'_{\alpha'}$;
- (2) Whenever $\mu(U_\alpha \cap U'_{\alpha'}) > 0$ there are a μ -full measure subset $V_{\alpha, \alpha'} \subset U_\alpha \cap U'_{\alpha'}$ and a Borel map $G_{\alpha, \alpha'} : V_{\alpha, \alpha'} \rightarrow \text{GL}(\mathbb{R}^{N_\alpha})$, such that, if $\mu(U_\beta \cap U'_{\beta'}) > 0$, one has:

$$(2.32) \quad G_{\beta, \beta'} \circ g_{\alpha, \beta}(x) = g_{\alpha', \beta'} \circ G_{\alpha, \alpha'}(x) \quad (\text{for } \mu\text{-a.e. } x \in V_{\alpha, \alpha'} \cap V_{\beta, \beta'}).$$

To construct seminorms on measurable vector bundles we will use often the following lemma.

Lemma 2.33. *Let $\mathcal{V}$ be a measurable vector bundle over X and let $\{\|\cdot\|_\tau\}_{\tau \in T}$ be a countable collection of seminorms on $\mathcal{V}$. Then for $x \in U_\alpha$ and $v \in \mathbb{R}^{N_\alpha}$ let*

$$(2.34) \quad \|v\|_{T, \alpha}(x) = \sup_{\tau \in T} \|v\|_{\tau, \alpha}(x);$$

*then $\{\|\cdot\|_{T, \alpha}\}_{\alpha \in I_\mathcal{V}}$ defines a seminorm $\|\cdot\|_T$ on $\mathcal{V}$, which we call the **supremum of the seminorms** $\{\|\cdot\|_\tau\}_{\tau \in T}$. Moreover, suppose that there are a seminorm $\|\cdot\|$ on $\mathcal{V}$ and a $C \geq 0$ such that:*

$$(2.35) \quad \|\cdot\|_\tau \leq C \|\cdot\|$$

holds uniformly in τ . Then $\|\cdot\|_T$ is a seminorm on $\mathcal{V}$ and one has:

$$(2.36) \quad \|\cdot\|_T \leq C \|\cdot\|.$$

Suppose now that μ is a σ -finite Borel measure on X and let $\{\|\cdot\|_\omega\}_{\omega \in \Omega}$ be a collection of seminorms on $\mathcal{V}$ which is allowed to be uncountable. Then there is a μ -a.e. unique generalized seminorm $\|\cdot\|_\Omega$, called the **essential supremum** of the collection $\{\|\cdot\|_\omega\}_{\omega \in \Omega}$, which satisfies the following properties:

(Ess-sup1): For each section σ of $\mathcal{V}$ and each $\omega \in \Omega$ one has:

$$(2.37) \quad \|\sigma\|_\Omega \geq \|\sigma\|_\omega \quad \mu\text{-a.e.};$$

(Ess-sup2): If $\|\cdot\|'_\Omega$ is another generalized seminorm satisfying (2.37), then one has:

$$(2.38) \quad \|\cdot\|'_\Omega \geq \|\cdot\|_\Omega \quad \mu\text{-a.e.}$$

Moreover, if there are a seminorm $\|\cdot\|$ on $\mathcal{V}$ and a $C \geq 0$ such that:

$$(2.39) \quad \|\cdot\|_\omega \leq C \|\cdot\|$$

holds μ -a.e. and uniformly in ω , then $\|\cdot\|_\Omega$ can be taken to be a seminorm satisfying:

$$(2.40) \quad \|\cdot\|_\Omega \leq C \|\cdot\| \quad \mu\text{-a.e.}$$

Proof. The proof that $\|\cdot\|_\Omega$ defines a generalized seminorm, which is also a norm under the additional assumption (2.35), follows by unwinding the definition of a measurable vector bundle. To prove the second part of Lemma 2.33 we use the approach of [EK05, Prop. 5.4.7].

We first observe that **(Ess-sup1)** and **(Ess-sup2)** are properties that hold up to μ -null sets, and thus we can construct $\|\cdot\|_\Omega$ independently on each $\mathcal{V}|U_\alpha$, where $\mathcal{V}|U_\alpha$ denotes the union of the fibres of $\mathcal{V}$ over the points $x \in U_\alpha$. We can therefore assume that the cardinality of $I_\mathcal{V}$ is one and identify $\mathcal{V}$ with the product $U \times \mathbb{R}^N$. Without loss of generality we can also assume that μ is a probability measure on U . We take a norm $\|\cdot\|$ on $\mathbb{R}^N$, and denote by S^{N-1} and $\mathcal{H}^{N-1}$ the corresponding unit ball and $(N-1)$ -dimensional Hausdorff measure. We finally let π be the probability measure

$$(2.41) \quad \mu \otimes \mathcal{H}^{N-1} \llcorner S^{N-1} / \mathcal{H}^{N-1}(S^{N-1}).$$

We also observe that, by possibly increasing Ω , we can assume that the collection $\{\|\cdot\|_\omega\}_{\omega \in \Omega}$ is *upward-filtering*, i.e. for all pairs $(\omega, \omega') \in \Omega^2$ there is an $\omega'' \in \Omega$ satisfying:

$$(2.42) \quad \|\cdot\|_{\omega''} = \max\{\|\cdot\|_\omega, \|\cdot\|_{\omega'}\}.$$

We now consider the increasing homeomorphism:

$$(2.43) \quad \begin{aligned} \varphi : \mathbb{R} &\rightarrow (0, 1) \\ t &\mapsto \frac{e^t}{e^t + 1}, \end{aligned}$$

and observe that the random variables $\{\varphi(\|\cdot\|_\omega)\}_{\omega \in \Omega}$ are all nonnegative and have π -expectations satisfying:

$$(2.44) \quad E[\varphi(\|\cdot\|_\omega)] \leq 1.$$

Thus the supremum:

$$(2.45) \quad q = \sup\{E[\varphi(\|\cdot\|_\omega)] : \|\cdot\|_\omega \in \Omega\}$$

is finite, and we let $T = \{\|\cdot\|_{\omega_n}\}$ denote a maximizing sequence:

$$(2.46) \quad \lim_{n \rightarrow \infty} E[\varphi(\|\cdot\|_{\omega_n})] = q.$$

The proof is completed by showing that $\|\cdot\|_T$ satisfies **(Ess-sup1)** and **(Ess-sup2)**.

We first address **(Ess-sup1)**: suppose that one has $\|\cdot\|_{\omega} > \|\cdot\|_T$ on a set of positive measure. Then, considering the sequence of norms $\{\max\{\|\cdot\|_{\omega_n}, \|\cdot\|_{\omega}\}\} \subset \Omega$ one contradicts (2.46).

We now address **(Ess-sup2)** and take a norm $\|\cdot\|'_{\Omega}$ satisfying (2.37). Let $A \subset U$ be a set of positive μ -measure: we claim that one has:

$$(2.47) \quad \lim_{n \rightarrow \infty} E[\chi_{A \times \mathbb{R}^N} \varphi(\|\cdot\|_{\omega_n})] = \sup\{E[\chi_{A \times \mathbb{R}^N} \varphi(\|\cdot\|_{\omega})] : \omega \in \Omega\} = E[\chi_{A \times \mathbb{R}^N} \varphi(\|\cdot\|_T)].$$

In fact, if any of the equalities in (2.47) failed, using that φ is positive and that the collection $\{\|\cdot\|_{\omega} : \omega \in \Omega\}$ is upward-filtering, one would contradict (2.46). As φ is increasing, we have

$$(2.48) \quad E[\chi_{A \times \mathbb{R}^N} \varphi(\|\cdot\|'_{\Omega})] \geq E[\chi_{A \times \mathbb{R}^N} \varphi(\|\cdot\|_{\omega_n})],$$

and from (2.47) it follows that:

$$(2.49) \quad E[\chi_{A \times \mathbb{R}^N} \varphi(\|\cdot\|'_{\Omega})] \geq E[\chi_{A \times \mathbb{R}^N} \varphi(\|\cdot\|_T)],$$

from which (2.38) follows. $\square$

3. GENERIC POINTS AND GENERIC VELOCITIES

In this Section we fix a complete separable metric space X and introduce a notion of genericity for pairs $(\gamma, t) \in \text{Frag}(X) \times \mathbb{R}$; this notion of genericity will be specified in terms of a quadruple $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ such that: $\mathcal{F}$ is a countable collection of real-valued Lipschitz functions defined on X , $\mathcal{C}$ is a countable collection of real-valued bounded Borel functions defined on X , D_X is a countable dense subset of X , and $\mathcal{S}$ is a countable collection of Lipschitz compatible pseudometrics on X which will always include the metric d_X .

Definition 3.1. We say that the pair (γ, t) is $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ -**generic** if:

- (Gen1):** The point t is a Lebesgue density point of $\text{Dom}(\gamma)$;
- (Gen2):** For each $f \in \mathcal{F}$ the derivative $(f \circ \gamma)'$ exists and is approximately continuous at t ;
- (Gen3):** For each $u \in \mathcal{C}$ the function $u \circ \gamma$ is approximately continuous at t ;
- (Gen4):** For each $x \in D_X$ and each $\varrho \in \mathcal{S}$ the derivative $(\varrho_x \circ \gamma)'$ exists and is approximately continuous at t ;
- (Gen5):** For each $\varrho \in \mathcal{S}$ the function $\sup_{x \in D_X} |(\varrho_x \circ \gamma)'(t)|$ is approximately continuous at t .

In the case in which $\mathcal{S}$ consists only of d_X we will just write $(\mathcal{F}, \mathcal{C}, D_X)$. Whenever a default choice of the set D_X is assumed, we will omit D_X from the notation.

Remark 3.2. We remark that the proof of [AT04, Thm. 4.1.6] shows that at a point t where **(Gen1)**, **(Gen4)** and **(Gen5)** hold, the ϱ -metric derivative $\varrho\text{-md}\gamma(t)$ exists

and equals $\sup_{x \in D_X} |(\varrho_x \circ \gamma)'(t)|$. Thus, if (γ, t) is $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ -generic, for each $\varrho \in \mathcal{S}$ the ϱ -metric derivative exists and is approximately continuous at t .

We point out that, in the case of a differentiability space (X, μ) , Definition 3.1 has a natural interpretation in terms of the μ -tangent bundle TX . Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for (X, μ) and suppose that $\mathcal{F}$ contains the components of all the coordinate functions $\{\phi_\alpha\}$, and that $\mathcal{C}$ contains all the characteristic functions $\{\chi_{U_\alpha}\}$. Suppose now that (γ, t) is $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ -generic and that $\gamma(t) \in \bigcup_\alpha U_\alpha$; then $\gamma'(t)$ is a well-defined element of TX . We are thus led to the following definition.

Definition 3.3. A $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ -generic velocity vector is an element of TX of the form $\gamma'(t)$, where (γ, t) is $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ -generic and $\gamma(t) \in \bigcup_\alpha U_\alpha$. As above, in the case in which $\mathcal{S}$ consists only of d_X , we will just write $(\mathcal{F}, \mathcal{C}, D_X)$, and we will omit D_X from the notation if a default choice of the set D_X is assumed.

We now establish measurability for generic pairs.

Lemma 3.4. *The set*

$$(3.5) \quad G(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X) = \{(\gamma, t) : (\gamma, t) \text{ is } (\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)\text{-generic}\}$$

is a Borel subset of $\text{Frag}(X) \times \mathbb{R}$.

Proof. We prove that $G(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ is Borel by showing that certain sets are Borel. Let DOM denote the set of pairs (γ, t) such that $t \in \text{Dom } \gamma$:

$$(3.6) \quad \text{DOM} = \{(\gamma, t) \in \text{Frag}(X) \times \mathbb{R} : t \in \text{Dom } \gamma\};$$

then DOM is closed. Fix $\delta > 0$ and consider the set of pairs (γ, t) where t becomes isolated below scale δ :

$$(3.7) \quad \text{ISOL}(\delta) = \{(\gamma, t) \in \text{DOM} : \text{Dom } \gamma \cap (t - \delta, t + \delta) \text{ contains only one point}\};$$

then $\text{ISOL}(\delta)$ is closed and $\text{ISOL} = \bigcup_{\delta \in \mathbb{Q}_{>0}} \text{ISOL}(\delta)$ is Borel and consists of the pairs (γ, t) where t is an isolated point of $\text{Dom } \gamma$. We can thus attempt to define, for a Lipschitz compatible pseudometric ϱ , the ϱ -metric derivative and, for f Lipschitz, the derivative of f at pairs in $\text{DOM} \setminus \text{ISOL}$. Consider the set:

$$(3.8) \quad \begin{aligned} \text{MDIFF}(\varrho) &= \{(\gamma, t) \in \text{DOM} \setminus \text{ISOL} : \varrho\text{-md}\gamma(t) \text{ exists}\} \\ &= \bigcap_{\varepsilon \in \mathbb{Q}_{>0}} \bigcup_{(\delta, \theta) \in \mathbb{Q}_{>0} \times \mathbb{Q}_{\geq 0}} \left\{ (\gamma, t) \in \text{DOM} \setminus \text{ISOL} : \forall s_1, s_2 \in (t - \delta, t + \delta) \cap \text{Dom } \gamma, \right. \\ &\quad \left. |\varrho(\gamma(s_1), \gamma(s_2)) - \theta|s_1 - s_2|| \leq \varepsilon|s_1 - s_2| \right\}; \end{aligned}$$

this set is Borel as all the sets in the curly brackets are closed in $\text{DOM} \setminus \text{ISOL}$. Modifying the definition of MDIFF by constraining θ to lie in a specified interval we also conclude that the map:

$$(3.9) \quad \begin{aligned} \text{MDer}(\varrho) : \text{MDIFF} &\rightarrow [0, \infty) \\ (\gamma, t) &\mapsto \varrho\text{-md}\gamma(t) \end{aligned}$$

is Borel. Consider now a real-valued Lipschitz function f defined on X ; the set $\text{DIFF}(f)$ where $(f \circ \gamma)'(t)$ exists is Borel because we can write it as:

$$\begin{aligned}
 (3.10) \quad \text{DIFF}(f) &= \{(\gamma, t) \in \text{DOM} \setminus \text{ISOL} : f \circ \gamma \text{ is differentiable at } t\} \\
 &= \bigcap_{\varepsilon \in \mathbb{Q}_{>0}} \bigcup_{(\delta, \theta) \in \mathbb{Q}_{>0} \times \mathbb{Q}} \left\{ (\gamma, t) \in \text{DOM} \setminus \text{ISOL} : \forall s_1, s_2 \in (t - \delta, t + \delta) \cap \text{Dom } \gamma, \right. \\
 &\quad \left. |f \circ \gamma(s_1) - f \circ \gamma(s_2) - \theta(s_1 - s_2)| \leq \varepsilon |s_1 - s_2| \right\}
 \end{aligned}$$

where the sets in curly brackets are closed in $\text{DOM} \setminus \text{ISOL}$. Constraining the θ appearing in the definition of $\text{DIFF}(f)$ to lie in a given interval we conclude that the map:

$$\begin{aligned}
 (3.11) \quad \text{Der}(f) : \text{DIFF}(f) &\rightarrow \mathbb{R} \\
 (\gamma, t) &\mapsto (f \circ \gamma)'(t)
 \end{aligned}$$

is Borel. Regarding condition **(Gen5)** we need also to take a sup of derivatives when they exist; so let Ω be a countable set of Lipschitz functions; then the set:

$$(3.12) \quad \text{DIFF}(\Omega) = \bigcap_{f \in \Omega} \text{DIFF}(f)$$

and the map:

$$\begin{aligned}
 (3.13) \quad |\text{Der}(\Omega)| : \text{DIFF}(\Omega) &\rightarrow \mathbb{R} \\
 (\gamma, t) &\mapsto \sup_{f \in \Omega} |(f \circ \gamma)'(t)|
 \end{aligned}$$

are Borel.

We now turn to questions pertaining to the approximate continuity of a function at a point in the domain of a fragment. For $L \geq 0$ we will denote by $\text{SUB}(L)$ the closed set of those fragments whose domain lies in $[-L, L]$. Suppose now that we are given a Borel set $B \subset \text{DOM}$ and a Borel map $\psi : B \rightarrow \mathbb{R}$. For $(\varepsilon, \delta, L) \in (\mathbb{Q}_{>0})^3$ let:

$$\begin{aligned}
 (3.14) \quad \tilde{\Psi}(\varepsilon, \delta, L, B, \psi) &= \left\{ (\gamma, t, s) \in \text{Frag}(X) \times \mathbb{R}^2 : \gamma \in \text{SUB}(L), \right. \\
 &\quad \left. (\gamma, t), (\gamma, s) \in B, |t - s| \leq \delta \text{ and } |\psi(\gamma, t) - \psi(\gamma, s)| \leq \varepsilon \right\};
 \end{aligned}$$

the set $\tilde{\Psi}(\varepsilon, \delta, L, B, \psi)$ is Borel and [Kec95, Thm. 17.25] shows that the map:

$$\begin{aligned}
 (3.15) \quad \text{Leb}(\varepsilon, \delta, L, B, \psi) : B &\rightarrow \mathbb{R} \\
 (\gamma, t) &\mapsto \mathcal{L}^1 \left(\left(\tilde{\Psi}(\varepsilon, \delta, L, B, \psi) \right)_{(\gamma, t)} \right)
 \end{aligned}$$

is Borel. It is then easy to prove that the sets of pairs (γ, t) where some map is approximately continuous at t is Borel; in fact, first define:

$$(3.16) \quad \text{ACONT}(\psi) = \bigcup_{L \in \mathbb{Q}_{>0}} \bigcap_{\varepsilon \in \mathbb{Q}_{>0}} \bigcup_{\delta \in \mathbb{Q}_{>0}} \bigcap_{r \in \mathbb{Q}_{>0}} \left\{ (\gamma, t) \in B : \gamma \in \text{SUB}(L), \right. \\ \left. \text{and for each } r \leq \delta \text{ one has } \text{Leb}(\varepsilon, r, L, B, \psi)(\gamma, t) \geq 2(1 - \varepsilon)r \right\},$$

which is a Borel set; then, for example, $\text{ACONT}(\text{MDer}(d_X))$ consists of the pairs (γ, t) where $\text{md } \gamma$ exists and is approximately continuous at t . In order to handle the approximate continuity for a Borel map $u : X \rightarrow \mathbb{R}$ we introduce the notation $\text{Ev}(u)$ to denote the Borel map which evaluates u at $\gamma(t)$:

$$(3.17) \quad \begin{aligned} \text{Ev}(u) : \text{DOM} &\rightarrow \mathbb{R} \\ (\gamma, t) &\mapsto u \circ \gamma(t). \end{aligned}$$

Letting $\psi : \text{DOM} \rightarrow \mathbb{R}$ to be the function which trivially maps each pair (γ, t) to 0, we see that $\text{ACONT}(\psi) = \text{LEBDENS}$ is the set of pairs (γ, t) where t is a Lebesgue density point of $\text{Dom } \gamma$.

We finally conclude that $G(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ is Borel by observing that:

$$(3.18) \quad \begin{aligned} G(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X) &= \bigcap_{\varrho \in \mathcal{S}} \text{ACONT}(\text{MDer}(\varrho)) \cap \bigcap_{f \in \mathcal{F}} \text{ACONT}(\text{Der}(f)) \\ &\cap \bigcap_{u \in \mathcal{C}} \text{ACONT}(\text{Ev}(u)) \cap \bigcap_{x \in D_X, \varrho \in \mathcal{S}} \text{ACONT}(\text{Der}(\varrho_x)) \\ &\cap \bigcap_{\varrho \in \mathcal{S}} \text{ACONT}(|\text{Der}|(\{\varrho_x\}_{x \in D_X})) \cap \text{LEBDENS}. \end{aligned}$$

□

4. METRIC DIFFERENTIALS AND SEMINORMS ON TX

In this Section we discuss the first instance of metric differentiation, Theorem 4.3. The point is that in the presence of a differentiable structure, the $\mathcal{H}_\varrho^1$ -measure of a fragment γ can be recovered using a seminorm (canonically associated to ϱ) on the tangent bundle TX associated to the differentiable structure. Let (X, μ) be a differentiability space with atlas $\{(U_\alpha, \phi_\alpha)\}$, and fix a countable dense set $D_X \subset X$.

Definition 4.1. Let Φ be a countable collection of Lipschitz functions on X ; we say that a Borel subset $V \subset \bigcup_\alpha U_\alpha$ is a **Φ -differentiability** set if:

(Diff1): The set V has full μ -measure: $\mu(\bigcup_\alpha U_\alpha \setminus V) = 0$;

(Diff2): For each $(x, f) \in V \times \Phi$, if $x \in U_\alpha$, then f is differentiable at x with respect to the coordinate functions ϕ_α .

Let $\Phi_{D_X, \varrho} = \{\varrho_x : x \in D_X\}$ and let V be a $\Phi_{D_X, \varrho}$ -differentiability set. Using Lemma 2.33, we obtain a seminorm $\|\cdot\|_{D_X, \varrho}$ on TX by defining, for $y \in V$ and $v \in T_y V$:

$$(4.2) \quad \|v\|_{D_X, \varrho} = \sup_{x \in D_X} |d\varrho_x|_y(v)|.$$

Theorem 4.3. *Let $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ be as in Section 3 and let V be a $\Phi_{D_X, \varrho}$ -differentiability set. Assume that $\mathcal{F}$ contains all the components of the coordinate functions ϕ_α , that $\mathcal{C}$ contains the characteristic functions $\{\chi_{U_\alpha}\}_\alpha \cup \{\chi_V\}$, and that $\varrho \in \mathcal{S}$. If $\gamma'(t)$ is an $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ -generic velocity vector and if $\gamma(t) \in V$, then the metric differential $\varrho\text{-md}\gamma(t)$ exists and equals $\|\gamma'(t)\|_{D_X, \varrho}$. In particular, if a fragment γ lies in V , we have:*

$$(4.4) \quad \mathcal{H}_\varrho^1(\text{Im } \gamma) = \int_{\text{Dom } \gamma} \|\gamma'(t)\|_{D_X, \varrho} dt.$$

Proof. To fix the ideas suppose that $\gamma(t) \in U_\alpha$. Because of conditions **(Gen4)**–**(Gen5)**, the argument in [AT04, Thm. 4.1.6] implies that

$$(4.5) \quad \varrho\text{-md}\gamma(t) = \sup_{x \in D_X} |(\varrho_x \circ \gamma)'(t)|;$$

as $\gamma(t) \in V$, for each $x \in D_X$ the pseudodistance function ϱ_x is differentiable at $\gamma(t)$ with respect to the coordinate functions ϕ_α ; note also that $\phi_\alpha \circ \gamma$ is differentiable at t by condition **(Gen2)**. Thus,

$$(4.6) \quad (\varrho_x \circ \gamma)'(t) = \sum_i \frac{\partial \varrho_x}{\partial \phi_\alpha^i}(\gamma(t)) (\phi_\alpha^i \circ \gamma)'(t) = d_{\varrho_x}|_{\gamma(t)}(\gamma'(t)),$$

which implies

$$(4.7) \quad \sup_{x \in D_X} |(\varrho_x \circ \gamma)'(t)| = \|\gamma'(t)\|_{D_X, \varrho}.$$

Formula (4.4) follows from the area formula (2.6) for the pseudometric ϱ by observing that for a fragment γ which lies in V , for $\mathcal{L}^1$ -a.e. $t \in \text{Dom } \gamma$, the velocity vector $\gamma'(t)$ is $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ -generic. $\square$

In Section 8 (Theorem 8.24) we will show that for different choices D_X and $\tilde{D}_X$ of the countable dense set, the seminorms $\|\cdot\|_{D_X, \varrho}$ and $\|\cdot\|_{\tilde{D}_X, \varrho}$ are the same. The proof uses the density of directions at generic points which is discussed in the next Section. For the case in which $\varrho = d_X$ this follows from Theorem 6.1.

5. DENSITY OF GENERIC DIRECTIONS AT GENERIC POINTS

In this Section we show that for μ -a.e. $x \in X$ the set of vectors in $T_x X$ which can be represented by $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ -generic velocity vectors contains a dense set of “directions” in $T_x X$. We make this idea precise with the following definition:

Definition 5.1. If V is a finite-dimensional vector space, we say that a subset $W \subset V$ **contains a dense set of directions** if:

$$(5.2) \quad \overline{[0, \infty)W} = \overline{\{tw \mid t \in [0, \infty), w \in W\}} = V.$$

We now fix an atlas $\{(U_\alpha, \phi_\alpha)\}_\alpha$ for the differentiability space (X, μ) and let N_α denote the dimension of the chart (U_α, ϕ_α) . For each α let $\{\text{Cone}(v_{\alpha,k}, \theta_{\alpha,k})\}_{k \in \mathbb{N}}$ denote a collection of open cones with $\{v_{\alpha,k}\} \subset \mathbb{S}^{N_\alpha-1}$ dense in the unit sphere and $\lim_{k \rightarrow \infty} \theta_{\alpha,k} = 0$. Using Theorem 2.28, we find Alberti representation $\mathcal{A}_k = (P_k, \nu_k)$ of μ such that, for each α , the restriction $\mathcal{A}_k \llcorner U_\alpha$ is in the ϕ_α -direction of $\text{Cone}(v_{\alpha,k}, \theta_{\alpha,k})$.

Theorem 5.3. *Let $\Gamma_0 \subset \text{Frag}(X)$ be a Borel set such that, for each k one has $P_k(\text{Frag}(X) \setminus \Gamma_0) = 0$; and let $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ be as in Definition 3.3. Then there is μ -measurable subset $Y \subset X$ with full μ -measure such that, for each $x \in Y$, the set of velocity vectors*

$$(5.4) \quad G_x = \left\{ v \in T_x X \mid v = \gamma'(t) \text{ for } \gamma \in \Gamma_0 \right. \\ \left. \text{such that } \gamma'(t) \text{ is } (\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)\text{-generic} \right\},$$

contains a dense set of directions in $T_x X$.

Proof. Let $Z_k \subset X \times \text{Frag}(X) \times \mathbb{R}$ consist of those triples (x, γ, t) satisfying:

- (1) $\gamma'(t)$ is an $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ -generic velocity vector;
- (2) $\gamma(t) = x$ and $\gamma \in \Gamma_0$;
- (3) If $\gamma'(t) \in U_\alpha$, then $(\phi_\alpha \circ \gamma)'(t) \in \text{Cone}(v_{\alpha,k}, \theta_{\alpha,k})$.

Using Lemma 3.4 we conclude that Z_k is Borel, and therefore its projection $Y_k \subset U$ on X is Suslin [Kec95], and hence μ -measurable. Note that for each $\gamma \in \Gamma_0$, as ν_k is absolutely continuous with respect to $\mathcal{H}_\gamma^1$, one has $\nu_k(\gamma)(X \setminus Y_k) = 0$, and therefore $\mu(X \setminus Y_k) = 0$. We conclude that $Y = \bigcap_k Y_k$ is a μ -full measure μ -measurable subset of X . Let $x \in Y \cap U_\alpha$, and let $v \in T_x X$; then for each $\varepsilon > 0$ we can find a k such that, for each $w \in \text{Cone}(v_{\alpha,k}, \theta_{\alpha,k})$, there is a $t_w \in [0, \infty)$ with:

$$(5.5) \quad \|v - t_w w\|_{l^2} \leq \varepsilon \|v\|_{l^2};$$

but as $x \in Y \cap U_\alpha$, there are a fragment $\gamma_k \in \Gamma_0$ and a $t_k \in \mathbb{R}$ such that the vector $\gamma'_k(t_k) \in T_x X$ is $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ -generic and $(\phi_\alpha \circ \gamma_k)'(t_k) \in \text{Cone}(v_{\alpha,k}, \theta_{\alpha,k})$; thus there is an $s_k \in [0, \infty)$ with

$$(5.6) \quad \|v - s_k (\phi_\alpha \circ \gamma_k)'(t_k)\|_{l^2} \leq \varepsilon \|v\|_{l^2},$$

which implies $\overline{[0, \infty)G_x} = T_x X$. □

6. CONSEQUENCES OF DENSITY OF GENERIC DIRECTIONS

In this section we prove the equality of various seminorms on TX (Theorem 1.6), the equality $\text{Lip } u = \text{lip } u$ a.e. (Theorem 1.10), and give a new proof that in PI spaces the minimal generalized upper gradient agrees with the pointwise Lipschitz constant.

6.1. Equality of natural seminorms on TX . The main result in this subsection is the proof of Theorem 1.6, which is based on the following result.

Theorem 6.1. *Let (X, μ) be a differentiability space and $D_X \subset X$ a countable dense set. Then the seminorm $\|\cdot\|_{D_X, d_X}$ on TX provided by (4.2) (taking $\varrho = d_X$) coincides with the norm $\|\cdot\|_{\text{Lip}}^*$ (see Section 2.1); in particular, the norm $\|\cdot\|_{D_X, d_X}$ does not depend on the choice of D_X .*

Notation: After proving Theorem 1.6, we will change to the notation $\|\cdot\|_{TX}$, $\|\cdot\|_{T^*X}$ or simply $\|\cdot\|$ to denote the canonical norms on TX and T^*X .

Theorem 6.1 can be regarded as an infinitesimal version of metric differentiation for the identity map $\text{id} : X \rightarrow X$; its proof uses the following lemma:

Lemma 6.2. *Suppose that $(V, \|\cdot\|)$ is a finite dimensional normed vector space, with dual space $(V^*, \|\cdot\|^*)$. Let W be a subset of the closed unit ball $\overline{B(\|\cdot\|)} \subset V$, such that:*

(H1): *For every $w \in W$, there is a linear functional $\alpha_w \in V^*$ with $\|\alpha_w\|^* \leq 1$, such that $\alpha_w(w) = 1$;*

(H2): *The set W contains a dense set of directions.*

Then:

- (1) *For all $w \in W$ one has $\|\alpha_w\|^* = 1$;*
- (2) *The set W is a dense subset of the unit sphere $S(\|\cdot\|)$;*
- (3) *The seminorm on V defined by $\sup_{w \in W} |\alpha_w(\cdot)|$ agrees with $\|\cdot\|$.*

Proof. Note that by **(H1)** each α_w has unit norm (which implies (1)) and that each vector $w \in W$ has unit norm, which implies that $W \subset S(\|\cdot\|)$. Let $v \in S(\|\cdot\|)$; by **(H2)**, for each $\varepsilon > 0$ there are a $w_\varepsilon \in W$ and a $t_\varepsilon \in [0, \infty)$:

$$(6.3) \quad \|v - t_\varepsilon w_\varepsilon\| \leq \varepsilon;$$

let β_v a unit norm functional on V assuming the norm at v . Then (6.3) implies:

$$(6.4) \quad |1 - t_\varepsilon \beta_v(w_\varepsilon)| \leq \varepsilon;$$

as $|\beta_v(w_\varepsilon)| \leq 1$, the previous equation implies $t_\varepsilon \geq 1 - \varepsilon$. On the other hand, evaluating with α_{w_ε} , (6.3) gives

$$(6.5) \quad |\alpha_{w_\varepsilon}(v) - t_\varepsilon| \leq \varepsilon;$$

as the functional α_{w_ε} has unit norm, $t_\varepsilon \leq 1 + \varepsilon$. We thus conclude that

$$(6.6) \quad \|v - w_\varepsilon\| \leq \|v - t_\varepsilon w_\varepsilon\| + \|(1 - t_\varepsilon)w_\varepsilon\| \leq 2\varepsilon,$$

implying (2). Note that, as the functionals α_w have unit norm,

$$(6.7) \quad \sup_{w \in W} \|\alpha_w(\cdot)\| \leq \|\cdot\|.$$

On the other hand, for each $v \in V \setminus \{0\}$ and each $\varepsilon > 0$, choose $w_\varepsilon \in W$ with

$$(6.8) \quad \left\| \frac{v}{\|v\|} - w_\varepsilon \right\| \leq \varepsilon;$$

then

$$(6.9) \quad \alpha_{w_\varepsilon} \left(\frac{v}{\|v\|} \right) \geq 1 - \varepsilon,$$

implying that:

$$(6.10) \quad \sup_{w \in W} \|\alpha_w(\cdot)\| \geq (1 - \varepsilon) \|\cdot\|,$$

from which (3) follows. □

Proof of Theorem 6.1. Let V be a differentiability set for the countable collection of Lipschitz functions $\{d_X(\cdot, x) : x \in D_X\}$. We let $\mathcal{F}$ contain the components of the coordinate functions, $\mathcal{C}$ contain the characteristic functions of the charts, and $\mathcal{S} = \{d_X\}$. Let Y be the set provided by Theorem 5.3; we will show that for each $p \in Y \cap V$ the norm $\|\cdot\|_{TX}$ and the seminorm $\|\cdot\|_{D_X, d_X}$ coincide on the fibre $T_p X$. Let $\gamma'(t) \in G_p(\mathcal{F}, \mathcal{C}, D_X)$ with $\gamma'(t) \neq 0$; then $\text{md } \gamma(t) \neq 0$. Without loss of generality we assume that p belongs to the chart U_α and we consider a functional $\sum_{i=1}^{N_\alpha} a_i d\phi_\alpha^i |_p \in T_p^* X$; then:

$$(6.11) \quad \left| \left\langle \sum_{i=1}^{N_\alpha} a_i d\phi_\alpha^i |_p, \frac{\gamma'(t)}{\text{md } \gamma(t)} \right\rangle \right| = \frac{\left| \sum_{i=1}^{N_\alpha} a_i (\phi_\alpha^i \circ \gamma)'(t) \right|}{\text{md } \gamma(t)};$$

choose $s_n \searrow 0$ such that $t + s_n \in \text{Dom } \gamma$ and note that

$$(6.12) \quad \begin{aligned} \left| \sum_{i=1}^{N_\alpha} a_i (\phi_\alpha^i \circ \gamma)'(t) \right| &= \lim_{n \rightarrow \infty} \frac{\left| \sum_{i=1}^{N_\alpha} a_i ((\phi_\alpha^i \circ \gamma)(t + s_n) - (\phi_\alpha^i \circ \gamma)(t)) \right|}{s_n} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\left| \sum_{i=1}^{N_\alpha} a_i ((\phi_\alpha^i \circ \gamma)(t + s_n) - (\phi_\alpha^i \circ \gamma)(t)) \right|}{d_X(\gamma(t + s_n), \gamma(t))} \\ &\quad \times \limsup_{n \rightarrow \infty} \frac{d_X(\gamma(t + s_n), \gamma(t))}{s_n} \\ &\leq \left\| \sum_{i=1}^{N_\alpha} a_i d\phi_\alpha^i |_p \right\|_{T^* X} \text{md } \gamma(t); \end{aligned}$$

we thus conclude that:

$$(6.13) \quad \left| \left\langle \sum_{i=1}^{N_\alpha} a_i d\phi_\alpha^i |_p, \frac{\gamma'(t)}{\text{md } \gamma(t)} \right\rangle \right| \leq \left\| \sum_{i=1}^{N_\alpha} a_i d\phi_\alpha^i |_p \right\|_{T^* X},$$

which implies $\frac{\gamma'(t)}{\text{md } \gamma(t)} \in \overline{B(\|\cdot\|_{TX}(x))}$.

Let

$$(6.14) \quad W_p = \left\{ \frac{\gamma'(t)}{\text{md } \gamma(t)} : \gamma'(t) \neq 0 \text{ and } \gamma'(t) \in G_p(\mathcal{F}, \mathcal{C}, D_X) \right\};$$

by Theorem 5.3 the set W_p contains a dense set of directions in $T_p X$. Theorem 4.3 implies then

$$(6.15) \quad \|\gamma'(t)\|_{D_X, d_X} = \text{md } \gamma(t),$$

and so we can find a sequence $\left\{ \sum_{i=1}^{N_\alpha} a_{i,k} d\phi_\alpha^i |_p \right\} \subset \overline{B(\|\cdot\|_{T^* X})}$ such that:

(1) We have:

$$(6.16) \quad \lim_{k \rightarrow \infty} \left\langle \sum_{i=1}^{N_\alpha} a_{i,k} d\phi_\alpha^i |_p, \gamma'(t) \right\rangle = \text{md } \gamma(t);$$

(2) For each k there is an $x_k \in D_X$ with:

$$(6.17) \quad d(d_X(\cdot, x_k)) |_p = \sum_{i=1}^{N_\alpha} a_{i,k} d\phi_\alpha^i |_p.$$

By compactness we can find a subsequence of $\left\{ \sum_{i=1}^{N_\alpha} a_{i,k} d\phi_\alpha^i |_p \right\}$ converging to $\omega_{\gamma'(t)} \in \overline{B(\|\cdot\|_{T^*X})}$. Now, (6.16) implies that

$$(6.18) \quad \omega_{\gamma'(t)} \left(\frac{\gamma'(t)}{\text{md } \gamma(t)} \right) = 1;$$

so for $w \in W_p$ of the form $\frac{\gamma'(t)}{\text{md } \gamma(t)}$ let $\alpha_w = \omega_{\gamma'(t)}$; applying Lemma 6.2 we conclude that:

$$(6.19) \quad \|\cdot\|_{TX}(p) = \sup_{w \in W_p} |\alpha_w(\cdot)|;$$

but Lemma 6.2 implies also that W_p is dense in $S(\|\cdot\|_{TX} \cdot)$ and by (6.18) we conclude that for $w' \in W_p$:

$$(6.20) \quad \sup_{w \in W_p} |\alpha_w(w')| = 1 = \|w'\|_{D_X, d_X},$$

from which we have $\|\cdot\|_{TX}(p) = \|\cdot\|_{D_X, d_X}(p)$. $\square$

Proof of Theorem 1.6. Let $\|\cdot\|_1, \|\cdot\|_3$ be the seminorms as in Theorem 1.6, constructed using Lemma 2.33, and let $\|\cdot\|_4$ be the dual Lip norm $\|\cdot\|_{\text{Lip}}^*$. Clearly we have $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \|\cdot\|_3$.

We claim that

$$(6.21) \quad \|\cdot\|_3 \leq \|\cdot\|_{\text{Lip}}^* \quad \mu - a.e.$$

To see this, recall that by Lemma 2.33 there is a countable collection $\{f_i\}$ of 1-Lipschitz functions such that for μ -a.e. $p \in X$, the differentials $df_i(p) \in T_p^*X$ are well-defined, and

$$\|\cdot\|_3(p) = \sup_i |df_i(p)|.$$

Recalling that for μ a.e. $p \in X$ we have $\|df_i(p)\|_{\text{Lip}} = \text{Lip } f_i(p)$, we get that for μ a.e. $p \in X$, every i , and every $v \in T_pX$,

$$|df_i(v)| \leq \|df_i\|_{\text{Lip}} \cdot \|v\|_{\text{Lip}}^* = \text{Lip } f_i(p) \cdot \|v\|_{\text{Lip}}^* \leq \|v\|_{\text{Lip}}^*$$

since f_i is 1-Lipschitz. Taking supremum gives (6.21).

By Theorem 6.1 we have $\|\cdot\|_1 = \|\cdot\|_{\text{Lip}}^*$ μ -a.e., so Theorem 1.6 follows. $\square$

6.2. A new proof of $\text{lip } f = \text{Lip } f$ in differentiability spaces. In this subsection we provide a proof, independent of the one given in [Sch13], of the following result:

Theorem 6.22. *Let (X, μ) be a differentiability space and $f : X \rightarrow \mathbb{R}$ Lipschitz. The for all $(\varepsilon, \sigma) \in (0, 1)^2$ there is a $(1, 1 + \varepsilon)$ -biLipschitz Alberti representation of μ with f -speed $\geq \sigma \text{Lip } f$. In particular,*

$$(6.23) \quad \text{Lip } f(x) = \text{lip } f(x) \quad \text{for } \mu\text{-a.e. } x.$$

The equality (6.23) generalizes one of the main results in [Che99, Thm. 6.1], which is a consequence of the fact that, in a PI-space (X, μ) , the function $\text{Lip } f$ is a representative of the minimal generalized upper gradient g_f of f . This last statement does not make sense in a general differentiability space as one might have $g_f < \text{Lip } f$ on a positive measure set, e.g. because X might not contain enough curves and one might then have $g_f = 0$. However, in a differentiability space the concept of the *maximal slope of f along fragments passing at time $t = 0$ through x* and having 0 as a density point of their domain, remains useful and can be interpreted as *the size of the gradient of f* . The result (6.23) is also proven in [Sch13] in a conceptually different way, and there it is also shown that in a differentiability space one has $\text{Lip } f = |df|$, where $|df|$ is the *local norm* of the form df , which is the Weaver differential form associated to the function f . The proof of Theorem 6.22 relies on the following lemma.

Lemma 6.24. *Let $\|\cdot\|_{l^2}$ denote the standard l^2 -norm on $\mathbb{R}^N$, and let $\|\cdot\|'$ denote another norm on $\mathbb{R}^N$ satisfying:*

$$(6.25) \quad \frac{1}{C} \|\cdot\|_{l^2} \leq \|\cdot\|' \leq C \|\cdot\|_{l^2}.$$

Then the diameter of the set $\text{Cone}(v, \theta) \cap S(\|\cdot\|')$, with respect to the norm $\|\cdot\|'$, is at most

$$(6.26) \quad 4C^2(1 - \cos \theta + \sin \theta).$$

Proof. Let $v_1, v_2 \in \text{Cone}(v, \theta) \cap S(\|\cdot\|')$; then we can find $u_1, u_2 \in S(\|\cdot\|_{l^2})$ such that: $v_i = \frac{u_i}{\|u_i\|}$; now

$$(6.27) \quad \|u_1 - u_2\|_{l^2} \leq 2(1 - \cos \theta + \sin \theta)$$

by using the definition of Euclidean cone. Observe also that (6.25) implies:

$$(6.28) \quad \left| \|u_1\|' - \|u_2\|' \right| \leq \|u_1 - u_2\|' \leq 2C(1 - \cos \theta + \sin \theta);$$

thus

$$(6.29) \quad \begin{aligned} \left\| \frac{u_1}{\|u_1\|'} - \frac{u_2}{\|u_2\|'} \right\|' &= \left\| \frac{u_1}{\|u_1\|'} - \frac{u_2}{\|u_1\|'} + \frac{u_2}{\|u_1\|'} - \frac{u_2}{\|u_2\|'} \right\|' \\ &\leq \frac{\|u_1 - u_2\|'}{\|u_1\|'} + \frac{\|u_2\|'}{\|u_1\|' \|u_2\|'} \left| \|u_1\|' - \|u_2\|' \right| \\ &\leq C \|u_1 - u_2\|' + C \left| \|u_1\|' - \|u_2\|' \right| \\ &\leq 4C^2(1 - \cos \theta + \sin \theta). \end{aligned}$$

□

Proof of Theorem 6.22. We fix an N -dimensional chart (U, ϕ) and a countable dense set $D_X \subset X$. We will show that, for each $(\varepsilon, \sigma) \in (0, 1)^2$, the measure $\mu \llcorner U$ admits a $(1, 1 + \varepsilon)$ -biLipschitz Alberti representation with f -speed $\geq \sigma \text{Lip } f$; the result about μ will then follow by applying the *gluing principle* Theorem 2.20.

We first consider the special case in which f is of the form $\langle v_0^*, \phi \rangle$ for some $v_0^* \in \mathbb{R}^N \setminus \{0\}$. For each $\eta \in (0, 1)$ we can use Egorov and Lusin Theorems to find

disjoint compact sets $C_\alpha \in U$, and dual norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\alpha^*$ on $\mathbb{R}^N$ such that:

$$(6.30) \quad \mu \left(U \setminus \bigcup_{\alpha} C_{\alpha} \right) = 0;$$

$$(6.31) \quad \begin{aligned} \frac{1}{1+\eta} \|\cdot\|_{TX} &\leq \|\cdot\|_{\alpha} \leq (1+\eta) \|\cdot\|_{TX} \quad (\text{on the fibres of } TX|_{C_{\alpha}}); \\ \frac{1}{1+\eta} \|\cdot\|_{T^*X} &\leq \|\cdot\|_{\alpha}^* \leq (1+\eta) \|\cdot\|_{T^*X} \quad (\text{on the fibres of } T^*X|_{C_{\alpha}}). \end{aligned}$$

By Theorem 6.1 we can also assume that on the fibres of each $TX|_{C_{\alpha}}$ one has:

$$(6.32) \quad \|\cdot\|_{TX} = \|\cdot\|_{D_X, d_X}.$$

Having fixed α , we will show that $\mu \llcorner C_{\alpha}$ admits a $(1, 1 + \varepsilon)$ -biLipschitz Alberti representation with $\langle v_0^*, \phi \rangle$ -speed $\geq \sigma \|v_0^*\|_{T^*X}$ for each $\sigma \in (0, 1)$; the result about $\mu \llcorner U$ will follow again by using Theorem 2.20. As we can rescale v_0^* , we can assume that $\|v_0^*\|_{\alpha}^* = 1$; we will denote by $v_0 \in S(\|\cdot\|_{\alpha})$ a vector where v_0^* assumes the norm. We let M denote a constant such that:

$$(6.33) \quad \frac{1}{M} \|\cdot\|_{l^2} \leq \|\cdot\|_{\alpha} \leq M \|\cdot\|_{l^2}.$$

We fix $\varepsilon_0 \in (0, 1)$ and $\theta \in (0, \pi/2)$ and, using 2.28 and Theorem 2.18. we find a $(1, 1 + \varepsilon_0)$ -biLipschitz Alberti representation $\mathcal{A}$ of $\mu \llcorner C_{\alpha}$ in the ϕ -direction of $\text{Cone} \left(\frac{v_0}{\|v_0\|_{l^2}}, \theta \right)$. Let $\mathcal{F}$ contain the components of ϕ and $\{f\}$, and let $\mathcal{C}$ contain χ_U . Using the Alberti representation $\mathcal{A}$ and (6.32) we conclude that for $\mu \llcorner C_{\alpha}$ -a.e. p there is an $(\mathcal{F}, \mathcal{C})$ -generic velocity vector $\gamma'(t) \in T_p X$ such that:

$$(6.34) \quad \begin{aligned} \text{md } \gamma(t) &= \|\gamma'(t)\|_{TX} \in [1, 1 + \varepsilon_0]; \\ (\phi \circ \gamma)'(t) &\in \text{Cone} \left(\frac{v_0}{\|v_0\|_{l^2}}, \theta \right). \end{aligned}$$

In particular, (6.34) and (6.31) imply that:

$$(6.35) \quad \|\gamma'(t)\|_{\alpha} \in \left[\frac{1}{1+\eta}, (1+\eta)(1+\varepsilon_0) \right].$$

We now use Lemma 6.24 to get

$$(6.36) \quad \left\| \frac{(\phi \circ \gamma)'(t)}{\|(\phi \circ \gamma)'(t)\|_{\alpha}} - v_0 \right\|_{\alpha} \leq 4M^2(1 - \cos \theta + \sin \theta);$$

as

$$(6.37) \quad |1 - \|(\phi \circ \gamma)'(t)\|_{\alpha}| \leq \max \left(1 - \frac{1}{1+\eta}, (1+\eta)(1+\varepsilon_0) - 1 \right),$$

we obtain

$$(6.38) \quad \begin{aligned} \|(\phi \circ \gamma)'(t) - v_0\|_{\alpha} &\leq 4M^2(1 - \cos \theta + \sin \theta) \\ &\quad + \max \left(1 - \frac{1}{1+\eta}, (1+\eta)(1+\varepsilon_0) - 1 \right) \\ &= a(\eta, \varepsilon_0, \theta), \end{aligned}$$

where $\lim_{\eta, \varepsilon_0, \theta \rightarrow 0} a(\eta, \varepsilon_0, \theta) = 0$. Recall that $t \in \text{Dom } \gamma$ is a Lebesgue density point, and assume that $\langle v_0^*, \phi \rangle \circ \gamma$, which is $M\mathbf{L}(\phi)(1 + \varepsilon_0)$ -Lipschitz because of (6.33), has been extended to a neighbourhood of t by using MacShane's Lemma:

$$\begin{aligned}
 \langle v_0^*, \phi \rangle \circ \gamma(t+h) - \langle v_0^*, \phi \rangle \circ \gamma(t) &= \int_t^{t+h} (\langle v_0^*, \phi \rangle \circ \gamma)'(s) ds \\
 &\geq \int_{[t, t+h] \cap \text{Dom } \gamma} (\langle v_0^*, \phi \rangle \circ \gamma)'(s) \\
 &\quad - \underbrace{M\mathbf{L}(\phi)(1 + \varepsilon_0)\mathcal{L}^1([t, t+h] \cap \text{Dom } \gamma)}_{o(h)} \\
 (6.39) \quad &= \int_{[t, t+h] \cap \text{Dom } \gamma} \langle v_0^*, v_0 \rangle ds \\
 &\quad + \int_{[t, t+h] \cap \text{Dom } \gamma} \langle v_0^*, (\phi \circ \gamma)'(s) - v_0 \rangle ds + o(h) \\
 &\geq \mathcal{L}^1([t, t+h] \cap \text{Dom } \gamma) - h a(\eta, \varepsilon_0, \theta) + o(h),
 \end{aligned}$$

where in the last step we used the approximate continuity of $(\phi \circ \gamma)'(s)$ at t . Now (6.39) implies that

$$(6.40) \quad (\langle v_0^*, \phi \rangle \circ \gamma)'(t) \geq \frac{1 - a(\eta, \varepsilon_0, \theta)}{(1 + \eta)^2(1 + \varepsilon_0)} \text{md } \gamma(t) \text{Lip } \langle v_0^*, \phi \rangle(\gamma(t)),$$

and it suffices to choose $\eta, \varepsilon_0, \theta$ small enough to guarantee

$$\begin{aligned}
 (6.41) \quad &\frac{1 - a(\eta, \varepsilon_0, \theta)}{(1 + \eta)^2(1 + \varepsilon_0)} \geq \sigma; \\
 &\varepsilon \geq \varepsilon_0.
 \end{aligned}$$

We now consider the general case in which df is not constant. We let $V \subset U$ be a full-measure Borel subset where f is differentiable with respect to the chart functions ϕ . On the set where $df = 0$ we have $\text{Lip } f = 0$, so we can assume that $df \neq 0$ on V . We fix $\eta > 0$ and use Lusin and Egorov Theorems to find disjoint compact sets $C_\alpha \subset V$ and $v_\alpha^* \in \mathbb{R}^N \setminus \{0\}$ such that $\mu(V \setminus \bigcup_\alpha C_\alpha) = 0$ and:

$$(6.42) \quad \|df(x) - v_\alpha^*\|_{T^*X} \leq \|df(x)\|_{T^*X} \quad (\forall x \in C_\alpha).$$

We fix $\sigma' \in (0, 1)$ and, using the special case $f = \langle v_\alpha^*, \phi \rangle$, we obtain a $(1, 1 + \varepsilon)$ -biLipschitz Alberti representation $\mathcal{A}_\alpha = (P_\alpha, \nu_\alpha)$ of $\mu \llcorner C_\alpha$ with $\langle v_\alpha^*, \phi \rangle$ -speed $\geq \sigma' \|v_\alpha^*\|_{T^*X}$; then for P_α -a.e. γ we have:

$$\begin{aligned}
 (6.43) \quad (f \circ \gamma)'(t) &\geq (\langle v_\alpha^*, \phi \rangle \circ \gamma)'(t) - \eta \text{md } \gamma(t) \|df\|_{T^*X} \\
 &\geq (\sigma' \|v_\alpha^*\|_{T^*X} - \eta \|df\|_{T^*X}) \text{md } \gamma(t) \\
 &\geq (\sigma' - (1 + \sigma')\eta) \text{Lip } f(\gamma(t)) \text{md } \gamma(t),
 \end{aligned}$$

and it suffices to choose η small enough and σ' close to 1 to guarantee that $\sigma' - (1 + \sigma')\eta \geq \sigma$.

The proof of (6.23) is now immediate. Let $\mathcal{F}$ contain the components of the chart functions and f , and let $\mathcal{C}$ contain the characteristic functions of the charts. Now, for each $\sigma \in (0, 1)$, we conclude that for μ -a.e. $x \in X$ there is an $(\mathcal{F}, \mathcal{C})$ -generic velocity vector $\gamma'(t) \in T_x X$ with

$$(6.44) \quad (f \circ \gamma)'(t) \geq \sigma \text{Lip } f(\gamma(t)) \text{md } \gamma(t);$$

observing that

$$(6.45) \quad |(f \circ \gamma)'(t)| \leq \text{lip } f(\gamma(t)) \text{ md } \gamma(t),$$

we conclude that the Borel set

$$(6.46) \quad \{x \in X : \text{lip } f(x) \geq \sigma \text{ Lip } f(x)\}$$

has full μ -measure, and then let $\sigma \nearrow 1$. □

6.3. A new proof of $g_f = \text{Lip } f$ in PI-spaces. In this subsection we give a new proof of the characterization of the **minimal generalized upper gradient** g_f of a Lipschitz function f in a PI-space. We will assume that the reader is familiar with the material in [Che99]; in particular, we will denote by $H^{1,p}(X, \mu)$ the Sobolev space introduced by Cheeger in [Che99, Sec. 2]. One of the main results in [Che99] states that, if $p \in (1, \infty)$ and if (X, μ) is a PI-space, then $H^{1,p}(X, \mu)$ is reflexive. Since we will use the reflexivity of $H^{1,p}(X, \mu)$, throughout this subsection the power p will be taken to lie in $(1, \infty)$. Our goal is to give a new proof of [Che99, Thm. 6.1]:

Theorem 6.47. *If (X, μ) is a PI-space and if $f \in \text{Lip}(X) \cap H^{1,p}(X, \mu)$, then $\text{Lip } f$ is a representative of the minimal generalized upper gradient of f (which is then independent of the power $p > 1$).*

We first give some remarks on how the new proof differs from the original one. The original proof contained two steps:

- (S1): Proof of Theorem 6.47 under the additional assumption that (X, μ) is a length space.
- (S2): Removing the assumption that (X, μ) is a length space.

The argument for (S1) was motivated by the observation that, whenever (X, μ) is a length space and g is a *continuous* upper gradient of f , then $g \geq \text{Lip } f$ holds at each point. Therefore the strategy in [Che99] was to prove an approximation result [Che99, Thm. 5.3] which states that for any $f \in \text{Lip}(X) \cap H^{1,p}(X, \mu)$ there is a sequence $(f_k, h_k) \subset H^{1,p}(X, \mu) \times L^p(\mu)$ such that $f_k \rightarrow f$ in $H^{1,p}(X, \mu)$, the function v_k is a continuous upper gradient of f_k , and $v_k \rightarrow g_f$ in $L^p(\mu)$. This approximation result is probably the most technical part of Cheeger's original proof.

The first simplification of the new argument is that one does not need to handle first the case in which (X, μ) is a length space. The strategy of the proof is motivated by the observation (Lemma 6.48) that if g is a *bounded* upper gradient of f , then $g \geq \text{Lip } f$ holds μ -a.e.: this is where Alberti representations are used. Had the minimal generalized upper gradient been defined by minimizing the p -energy on *bounded* upper gradients, then Theorem 6.47 would have followed directly from Lemma 6.48. However, as an upper gradient in $L^p(\mu)$ can be infinite on a null set, one needs, roughly speaking, to approximate f in $\text{Lip}(X) \cap H^{1,p}(X, \mu)$ by functions which have bounded upper gradients. Here we use an instance of the argument “modulus equals capacity” [Zie69] which appears also in the proof of [Che99, Thm. 5.3]: however, as we do not need to build approximations which use continuous upper gradients, there are fewer technical details to handle.

The following lemma relates bounded upper gradients and Alberti representations.

Lemma 6.48. *If (X, μ) is a PI-space, $u : X \rightarrow \mathbb{R}$ is Lipschitz, and g is a bounded upper gradient of u , then*

$$(6.49) \quad g \geq \text{Lip } u \quad \mu\text{-a.e.}$$

Proof. For each $\varepsilon > 0$ we can find countably many disjoint compact sets $\{K_\alpha\}$ and nonnegative real numbers $\{\lambda_\alpha\}$ such that:

- (1) For each $x \in K_\alpha$ one has $g(x) \in [\lambda_\alpha, \lambda_\alpha + \varepsilon]$;
- (2) The $\{K_\alpha\}$ cover X in measure: $\mu(X \setminus \bigcup_\alpha K_\alpha) = 0$.

By Theorem 6.22 for $\mu \llcorner K_\alpha$ -a.e. x there is a $(1, 1 + \varepsilon)$ -biLipschitz fragment γ :

- (1) The domain $\text{Dom } \gamma$ is a subset of $[-1, \infty)$;
- (2) One has $\gamma(0) = x$ and:

$$(6.50) \quad \lim_{r \searrow 0} \frac{\mathcal{L}^1(\text{Dom } \gamma \cap (-r, r))}{2r} = 1;$$

- (3) The point 0 is an approximate continuity point of $(u \circ \gamma)'$ and

$$(6.51) \quad (u \circ \gamma)'(0) \geq \frac{1}{1 + \varepsilon} \text{Lip } u(x).$$

Let $[c, d]$ be the minimal interval containing $\text{Dom } \gamma$ and let $\{(a_i, b_i)\}$ denote the set of component of $[c, d] \setminus \text{Dom } \gamma$; we extend γ on each interval (a_i, b_i) by choosing a C -quasigeodesic joining $\gamma(a_i)$ to $\gamma(b_i)$: note that this is possible because a PI-space is C -quasiconvex for some C [Che99, Sec. 17]¹. Then:

$$(6.52) \quad \left| \int_0^r (u \circ \gamma)'(s) ds \right| = |u(\gamma(r)) - u(x)| \leq \int_0^r g \circ \gamma \, \text{md } \gamma(t) dt \leq (\lambda_\alpha + \varepsilon)(1 + \varepsilon)r + o(r);$$

dividing by r and letting $r \searrow 0$ we get:

$$(6.53) \quad \text{Lip } u(x) \leq (1 + \varepsilon)^2 (g(x) + \varepsilon),$$

and the result follows letting $\varepsilon \searrow 0$. $\square$

Remark 6.54. Note that in Lemma 6.48 we had to work with bounded upper gradients to establish (6.52); in fact, to apply the Fundamental Theorem of Calculus, one needs curves, and the K_α might only contain fragments, and thus, filling-in the fragments in K_α using that a PI-space is quasiconvex might produce curves where g is unbounded or infinite on a set of positive length. Note also that in a PI-space one can use curves instead of fragments in building Alberti representations; this follows from a general observation in [Sch14] that if μ is a Radon measure on a quasiconvex metric space X , a Lipschitz Alberti representation of μ can be replaced by one which gives the same derivation and whose probability measure has support contained in the set of curves in X .

To prove Theorem 6.47 we can just consider, as in [Che99], upper gradients which are lower semicontinuous. In fact, the Vitali-Carathéodory Theorem [Rud87, Thm. 2.25] states that for any $h \in L^1(\mu)$ and any $\varepsilon > 0$ there are functions u and v such that $u \leq h \leq v$, u is upper semicontinuous and bounded from above, v is

¹This result is due to Semmes.

lower semicontinuous and bounded from below, and $\|u - v\|_{L^1(\mu)} < \varepsilon$. Thus, any upper gradient of f can be replaced, up to slightly increasing the $L^p(\mu)$ -norm, by one which is lower semicontinuous and bounded below by a small positive constant. We thus only need to prove:

Theorem 6.55. *Suppose (X, μ) is a PI-space, u is a real-valued Lipschitz function defined on X and g is a lower-semicontinuous upper gradient of u . Then:*

$$(6.56) \quad g \geq \text{Lip } u \quad \mu\text{-a.e.}$$

To prove Theorem 6.55 we recall a consequence of the Poincaré inequality, which follows from the characterization of the Poincaré inequality in terms of the maximal function associated to an upper gradient [HK98, Lem. 5.15]. Suppose that g is an upper gradient for the function u and that $g \in L^p(\mu)$; consider for $N \in (0, \infty)$ the set:

$$(6.57) \quad A(g, N) = \left\{ x \in X : \sup_{r>0} \int_{B(x,r)} g^p d\mu \leq N^p \right\};$$

then if $x, y \in A(g, N)$ are Lebesgue points of u , one has

$$(6.58) \quad |u(x) - u(y)| \leq CNd(x, y),$$

where C is a universal constant that depends only on the PI-space (X, μ) .

Proof. Let N, M be natural numbers and $S = A(g, N) \cap B(x, M)$; it suffices to show that (6.56) holds $\mu \llcorner S$ -a.e. Fix $\varepsilon > 0$ and let:

$$(6.59) \quad u_n(x) = \inf \left\{ \int_{\gamma} (g \wedge n + \varepsilon) d\mathcal{H}_{\gamma}^1 + u(y) : \gamma \text{ is a Lipschitz curve joining } x \text{ to } y \in S \right\}.$$

As (X, μ) is C -quasiconvex for some C , the function u_n is $C(n + \varepsilon)$ -Lipschitz. Note also that $h_n = g \wedge n + \varepsilon$ is an upper gradient of u_n . We let $\{x_{j,m}\}_{j \in J_m}$ be a finite $\frac{1}{m}$ -dense set in S , which exists because X is proper. Using the fact that the h_n are lower-semicontinuous and uniformly bounded away from zero (compare [Zie69, 3.3, 3.4] and [Che99, Lem. 5.18]), it follows that for each $m \in \mathbb{N}$ there is an $N_m \in \mathbb{N}$ such that, for $n \geq N_m$, one has:

$$(6.60) \quad u_n(x_{j,m}) = u(x_{j,m}) \quad (\forall j \in J_m).$$

Let v_n be obtained by truncating u_n so that

$$(6.61) \quad |v_n(x)| \leq \sup_{y \in B(x, M)} |u_n(y)|;$$

thus, for $n \geq N_m$ one has:

$$(6.62) \quad v_n(x_{j,m}) = u(x_{j,m}) \quad (\forall j \in J_m).$$

Note that h_n is an upper gradient of v_n and that (6.58) implies that the functions v_{N_m} , when restricted to S , are uniformly $C(N + \varepsilon)$ -Lipschitz; therefore, (6.61) implies that $v_{N_m} \rightarrow u$ uniformly on S . As the Banach space $H^{1,p}(\mu \llcorner B(x, M))$ is reflexive, by applying Mazur's Lemma we can find Lipschitz functions w_n and integers Q_n such that:

- (1) The sequence $\{w_n\}$ converges to the function w in $H^{1,p}(\mu \llcorner B(x, M))$ and $w = u$ on S ;

- (2) Each function w_n is a convex combination of finitely many of the functions v_{N_m} ;
- (3) The function $g \wedge Q_n + \varepsilon$ is an upper gradient for w_n .

We then recall that in a PI-space there is a constant C such that, for each Lipschitz function f , one has $Cg_f \geq \text{Lip } f$ μ -a.e. As $w_n \rightarrow w$ in $H^{1,p}(\mu \llcorner B(x, M))$, one has that the generalized minimal upper gradients $\{g_{w_n-w}\}$ converge to 0 in $L^p(\mu \llcorner B(x, M))$; by the locality property of generalized minimal upper gradients [Che99, Cor. 2.25], as $u = w$ on S , we have that $\{g_{w_n-u}\}$ converges to 0 in $L^p(\mu \llcorner S)$; we thus conclude that $\text{Lip}(w_n - u) \rightarrow 0$ in $L^p(\mu \llcorner S)$. As $|\text{Lip } w_n - \text{Lip } u| \leq \text{Lip}(w_n - u)$, we can then pass to a subsequence such that $\text{Lip } w_n \rightarrow \text{Lip } u$ $\mu \llcorner S$ -a.e. Now, by Lemma 6.48 we have that $g \wedge Q_n + \varepsilon \geq \text{Lip } w_n$ holds $\mu \llcorner S$ -a.e., and thus $g \geq \text{Lip } u$ holds $\mu \llcorner S$ -a.e. $\square$

7. THE GEOMETRY OF BLOW-UPS/TANGENT CONES

In this section we show that, if (X, μ) is a differentiability space, blowing-up the measure μ at a generic point yields measures which possess Alberti representations concentrated on distinguished geodesic lines on which the blow-ups of the chart functions have constant derivatives, and are harmonic. This generalizes the fact that in PI-spaces the blow-ups are generalized linear functions. Weaker versions of the results presented here, where the blow-up of the measure is not discussed, have been obtained in [Sch13], and [Dav14]. The result in [Sch13] is more general than [Dav14] because it applies also in the context of Weaver derivations: we point out that the results in this section, under the assumption that μ is asymptotically doubling, have natural counterparts in that context. We first recall some notions of blow-ups of metric measure spaces and Lipschitz functions. Note that we use the terminology *blow-up* to avoid a conflict with the word *tangent* which is used for different objects in this paper; often, instances of what we call *blow-ups* are called *tangent cones* / *tangent spaces* in the literature.

7.1. Blow-ups of metric measure spaces and Lipschitz maps.

Definition 7.1. A **blow-up of a metric space X at a point p** is a (complete) pointed metric space (Y, q) which is a pointed Gromov-Hausdorff limit of a sequence $(\frac{1}{r_n}X, p)$ where $r_n \searrow 0$: the notation $\frac{1}{r_n}X$ means that the metric on X is rescaled by $1/r_n$; the class of blow-ups of X at p is denoted by $\text{Bw-up}(X, p)$.

Remark 7.2. In Subsection 8.3 we discuss blow-ups of metric spaces in a more general context which requires the notion of ultralimits: under suitable assumptions on X , a sequence $(\frac{1}{r_n}X, p)$ will always be precompact and the two notions will agree. This is the case, for example, if X is a doubling metric space. However, in the context of differentiability spaces we merely know (Theorem 2.21) that μ is asymptotically doubling, and that porous sets are μ -null. This implies that, for μ -a.e. $p \in X$, there is a compact set S_p such that: S_p is metrically doubling, and for each $\varepsilon > 0$, there is an $r_0 > 0$ such that, for each $r \leq r_0$, the set $S_p \cap B(p, r)$ is εr -dense in $B(p, r)$. This allows essentially to reduce the existence of blow-ups to the case in which X is doubling.

Recall that if the sequence $(\frac{1}{r_n}X, p)$ converges to (Y, q) in the pointed Gromov-Hausdorff sense, there is a pointed metric space (Z, z) such that there are isometric embeddings $\iota_n : (\frac{1}{r_n}X, p) \rightarrow (Z, z)$ and $\iota : (Y, q) \rightarrow (Z, z)$, and, for each $R > 0$, one has:

$$(7.3) \quad \begin{aligned} \lim_{n \rightarrow \infty} \sup_{y \in B(z, R) \cap \iota(Y)} \text{dist} \left(\iota_n \left(\frac{1}{r_n}X \right), \{y\} \right) &= 0, \\ \lim_{n \rightarrow \infty} \sup_{y \in B(z, R) \cap \iota_n(\frac{1}{r_n}X)} \text{dist} (\iota(Y), \{y\}) &= 0. \end{aligned}$$

In particular, each $q' \in Y$ can be *approximated* by a sequence $p'_n \in \frac{1}{r_n}X$ such that $\iota_n(p'_n) \rightarrow \iota(q')$ in Z . This notion can be made independent of the embedding in Z and one can **represent** each point $q' \in Y$ by some sequence $(p'_n) \subset X$ of points converging to p (compare the treatment with ultralimits in subsection 8.3). Moreover, if (p'_n) represents q' , and if $(\tilde{p}'_n)$ represents $\tilde{q}'$, we have:

$$(7.4) \quad d_Y(q', \tilde{q}') = \lim_{n \rightarrow \infty} \frac{d_X(p'_n, \tilde{p}'_n)}{r_n}.$$

Definition 7.5. Let (X, μ) be a metric measure space; a **blow-up of (X, μ) at p** is a triple (Y, ν, q) such that one has $(\frac{1}{r_n}X, p) \rightarrow (Y, q) \in \text{Bw-up}(X, p)$, and, having chosen a pointed metric space (Z, z) and isometric embeddings $\iota_n : (\frac{1}{r_n}X, p) \rightarrow (Z, z)$ and $\iota : (Y, q) \rightarrow (Z, z)$ such that (7.3) holds, one has:

$$(7.6) \quad (\iota_n)_\# \frac{\mu}{\mu(B(p, r_n))} \xrightarrow{w^*} \iota_\# \nu. \quad (\text{convergence in the weak}^* \text{ topology}).$$

The set of blow-ups of (X, μ) at p will be denoted by $\text{Bw-up}(X, \mu, p)$.

Remark 7.7. Note that if μ is asymptotically doubling and if porous sets are μ -null, then for μ -a.e. $p \in X$ one has $\text{Bw-up}(X, \mu, p) \neq \emptyset$. In fact, at a generic point p , for each sequence of scaling factors $r_n \searrow 0$, there is a subsequence r_{n_k} such that $(\frac{1}{r_{n_k}}X, p) \rightarrow (Y, q) \in \text{Bw-up}(X, p)$, and there is a doubling measure ν such that (7.6) holds.

We finally discuss blow-ups of Lipschitz mappings which take values into Euclidean spaces.

Definition 7.8. Let (X, μ) be a metric measure space and $\psi : X \rightarrow \mathbb{R}^N$ a Lipschitz map; then a **blow-up of (X, μ, ψ) at p** is a tuple (Y, ν, φ, q) such that one has that $(Y, \nu, q) \in \text{Bw-up}(X, \mu, p)$, where the blow-up is realized by considering scaling factors $r_n \searrow 0$, and where $\varphi : Y \rightarrow \mathbb{R}^N$ is a Lipschitz function such that, whenever $(p'_n) \subset X$ represents q' , one has:

$$(7.9) \quad \varphi(q') = \lim_{n \rightarrow \infty} \frac{\psi(p'_n) - \psi(p)}{r_n}.$$

The set of blow-ups of (X, μ, ψ) at p will be denoted by $\text{Bw-up}(X, \mu, \psi, p)$.

Remark 7.10. If μ is asymptotically doubling and porous sets are μ -null, then for μ -a.e. $p \in X$ one has that $\text{Bw-up}(X, \mu, \psi, p) \neq \emptyset$ by an application of Ascoli-Arzelá.

7.2. Blowing up Alberti representations. We can now state the main result of this Section.

Theorem 7.11. *Let (U, ψ) be an N -dimensional differentiability chart for the differentiability space (X, μ) ; then for $\mu \llcorner U$ -a.e. p , for each blow-up $(Y, \nu, \varphi, q) \in \text{Bw-up}(X, \mu, \psi, p)$ and for each unit vector $v_0 \in T_p X$, the measure ν admits an Alberti representation $\mathcal{A} = (Q, \Phi)$ where:*

- (1) Q is concentrated on the set $\text{Lines}(\varphi, v_0)$ of unit speed geodesic lines in Y with $(\varphi \circ \gamma)' = v_0$;
- (2) For each $\gamma \in \text{Lines}(\varphi, v_0)$ the measure Φ_γ is given by:

$$(7.12) \quad \Phi_\gamma = \mathcal{H}_\gamma^1.$$

Suppose that $X' \subset X$ and that the measures μ' and $\mu \llcorner X'$ are in the same measure class. Then an application of measure differentiation shows that for $\mu \llcorner X'$ -a.e. p the sets $\text{Bw-up}(X', \mu', p)$ and $\text{Bw-up}(X, \mu, p)$ coincide. Given $(Y, \nu, \varphi, q) \in \text{Bw-up}(X, \mu, \psi, p)$ we will then obtain the Alberti representations of ν by *blowing-up* Alberti representations of measures $\mu' \ll \mu$ which admit Alberti representations of a special form.

Definition 7.13 (Simplified Alberti representations). We say that the Alberti representation $\mathcal{A} = (P, \Psi)$ of the measure μ' is **simplified** if there are $(C_0, D_0, \tau_0) \in (0, \infty)^3$ such that:

- (1) The measure P is finite and is supported on the set of (C_0, D_0) -biLipschitz fragments whose domain is a subset of $[0, \tau_0]$;
- (2) Denoting by $M(X)$ the set of finite Radon measures on X , Ψ is the Borel map:

$$(7.14) \quad \begin{aligned} \Psi : \text{Frag}(X) &\rightarrow M(X) \\ \gamma &\mapsto \gamma_\# (\mathcal{L}^1 \llcorner \text{Dom } \gamma). \end{aligned}$$

To prove Theorem 7.11 we will use the following technical result about blow-ups of a simplified Alberti representation $\mathcal{A}$.

Theorem 7.15. *Suppose that the simplified Alberti representation $\mathcal{A}$ of the finite measure $\mu' \ll \mu$ is in the ψ -direction of a cone $\mathcal{C}$ and that it has $\langle v_0, \psi \rangle$ -speed $\geq \sigma_0 \|v_0\|_{T^*X}$. Then there is a Borel set U with full μ' -measure such that for each $p \in U$, for each $(Y, \nu, \varphi, q) \in \text{Bw-up}(X, \mu, \psi, p)$ and each $R_0 > 0$ the measure $\nu \llcorner B(q, R_0)$ admits an Alberti representation $\mathcal{A}_{R_0} = (Q_{R_0}, \Phi)$ such that:*

- (1) The finite Radon measure Q_{R_0} has support contained in a compact set $\mathcal{S}_{R_0} \subset \text{Frag}(Y)$ of geodesic segments;
- (2) The total mass of Q_{R_0} is bounded by $\frac{D_0}{2R_0} (\text{As}(\mu, p))^{\log_2 R_0 + 1}$, where $\text{As}(\mu, p)$ denotes the asymptotic doubling constant of μ at p , i.e.:

$$(7.16) \quad \text{As}(\mu, p) = \limsup_{r \searrow 0} \frac{\mu(B(p, 2r))}{\mu(B(p, r))};$$

- (3) The set $\mathcal{S}_{R_0}$ consists of those geodesic segments γ which have domain contained in $[0, \frac{4R_0}{C_0}]$, image contained in $\bar{B}(q, 2R_0)$, which have both endpoints

lying outside of $B(q, \frac{3}{2}R_0)$, which intersect $\bar{B}(q, R_0)$, which have constant speed $\theta_\gamma \in [C_0, D_0]$, which satisfy:

$$(7.17) \quad \begin{aligned} \operatorname{sgn}(s_2 - s_1) \langle v_0, \varphi \circ \gamma(s_2) - \varphi \circ \gamma(s_1) \rangle \\ \geq \sigma_0 \theta_\gamma (s_2 - s_1) \operatorname{Lip}(\langle v_0, \psi \rangle)(p) \quad (\forall s_1, s_2 \in \operatorname{Dom} \gamma), \end{aligned}$$

and such that there is a $w_\gamma \in \bar{\mathcal{C}}$ for which the following holds:

$$(7.18) \quad \varphi \circ \gamma(s_2) - \varphi \circ \gamma(s_1) = (s_2 - s_1) w_\gamma \quad (\forall s_1, s_2 \in \operatorname{Dom} \gamma);$$

(4) For each $\gamma \in \mathcal{S}_{R_0}$ the measure Φ_γ is given by:

$$(7.19) \quad \Phi_\gamma = \frac{1}{\theta_\gamma} \mathcal{H}_\gamma^1 \llcorner B(q, r_0).$$

We now introduce a bit of terminology to split the measure on fragments in a *good* and a *bad* part.

Definition 7.20. Let $\varepsilon > 0$ and $S > 0$; we denote by $\widetilde{\operatorname{Reg}}(\psi, \mathcal{C}, v_0, \sigma_0, [C_0, D_0], \tau_0, \varepsilon, S)$ the set of pairs $(\gamma, p) \in \operatorname{Frag}(X) \times X$ such that:

(Reg1): The fragment γ is $[C_0, D_0]$ -biLipschitz with domain contained in $[0, \tau_0]$;

(Reg2): There is a $t \in \operatorname{Dom} \gamma$ with $p = \gamma(t)$ and for each $r_1, r_2 \leq S$ one has:

$$(7.21) \quad \mathcal{L}^1(\operatorname{Dom} \gamma \cap [t - r_1, t + r_2]) \geq (1 - \varepsilon)(r_1 + r_2);$$

(Reg3): There are a $\theta \in [C_0, D_0]$ and $w \in \mathcal{C}$ such that if $r \leq S$ and $s_1, s_2 \in [t - r, t + r] \cap \operatorname{Dom} \gamma$ one has:

$$(7.22) \quad \begin{aligned} |d(\gamma(s_1), \gamma(s_2)) - \theta|s_1 - s_2|| &\leq \varepsilon|s_1 - s_2| \\ |\psi \circ \gamma(s_1) - \psi \circ \gamma(s_2) - w(s_1 - s_2)| &\leq \varepsilon|s_1 - s_2|; \end{aligned}$$

(Reg4): If $r \leq S$ and $s_1, s_2 \in [t - r, t + r]$ with $s_1 \leq s_2$ then:

$$(7.23) \quad \langle v_0, \psi \circ \gamma(s_2) - \psi \circ \gamma(s_1) \rangle \geq (\sigma_0 - \varepsilon) \theta \operatorname{Lip}(\langle v_0, \varphi \rangle)(p)(s_2 - s_1).$$

In the following we will usually fix a choice of $(\psi, \mathcal{C}, v_0, \sigma_0, [C_0, D_0], \tau_0)$ and vary $(\varepsilon, S) \in (0, \infty)^2$; we thus introduce the shorter notation $\operatorname{PAR}(\varepsilon, S)$ for $(\psi, \mathcal{C}, v_0, \sigma_0, [C_0, D_0], \tau_0, \varepsilon, S)$. We denote by $\operatorname{Reg}(\operatorname{PAR}(\varepsilon, S))$ the subset of those $(\gamma, p) \in \widetilde{\operatorname{Reg}}(\operatorname{PAR}(\varepsilon, S))$ such that:

(Reg5): For all $r_1, r_2 \leq S$ one has:

$$(7.24) \quad \mathcal{L}^1 \left(\gamma^{-1} \left(\widetilde{\operatorname{Reg}}(\operatorname{PAR}(\varepsilon, S)) \right)_\gamma \cap [t - r_1, t + r_2] \right) \geq (1 - \varepsilon)(r_1 + r_2),$$

where then notation $\left(\widetilde{\operatorname{Reg}}(\operatorname{PAR}(\varepsilon, S)) \right)_\gamma$ denotes the γ -section of the set $\widetilde{\operatorname{Reg}}(\operatorname{PAR}(\varepsilon, S))$.

Lemma 7.25. *The set $\operatorname{Reg}(\operatorname{PAR}(\varepsilon, S))$ is a Borel subset of $\operatorname{Frag}(X) \times X$.*

Proof. We first show that the set $\widetilde{\operatorname{Reg}}(\operatorname{PAR}(\varepsilon, S))$ is Borel. The set of fragments satisfying **(Reg1)** is closed in $\operatorname{Frag}(X)$. Now consider the set

$$(7.26) \quad \operatorname{IMG} = \{(\gamma, p) \in \operatorname{Frag}(X) \times X : p \in \gamma(\operatorname{Dom} \gamma)\},$$

which is closed in $\text{Frag}(X) \times X$; let $\text{IMG}(C_0, D_0, \tau_0)$ denote the closed subset of those $(\gamma, p) \in \text{IMG}$ such that γ satisfies **(Reg1)**; then the map:

$$(7.27) \quad \begin{aligned} \text{Inv} : \text{IMG}(C_0, D_0, \tau_0) &\rightarrow \mathbb{R} \\ (\gamma, p) &\mapsto \gamma^{-1}(p) \end{aligned}$$

is continuous. Using an argument similar to that used to prove that the map defined at (3.15) is Borel, we see that, for fixed $r_1, r_2 > 0$, the map:

$$(7.28) \quad \begin{aligned} \psi_{r_1, r_2} : \text{Frag}(X) \times \mathbb{R} &\rightarrow \mathbb{R} \\ (\gamma, t) &\mapsto \mathcal{L}^1(\text{Dom } \gamma \cap [t - r_1, t + r_2]) \end{aligned}$$

is Borel; then the set of pairs (γ, p) satisfying **(Reg1)**–**(Reg2)** is Borel since it can be written as:

$$(7.29) \quad \bigcap_{r_1, r_2 \in [0, S] \cap \mathbb{Q}} \left\{ (\gamma, p) \in \text{IMG}(C_0, D_0, \tau_0) : \psi_{r_1, r_2}(\gamma, \gamma^{-1}(p)) \geq (1 - \varepsilon)(r_1 + r_2) \right\}.$$

That the set of pairs satisfying **(Reg3)**–**(Reg4)** is Borel follows by arguments similar to those used in the proof of Lemma 3.4, compare (3.8), (3.10).

Consider the set:

$$(7.30) \quad \text{TRIP} = \{(\gamma, p, t) \in \text{Frag}(X) \times X \times \mathbb{R} : t \in \text{Dom } \gamma, \gamma(t) = p\},$$

which is closed in $\text{Frag}(X) \times X \times \mathbb{R}$. We now fix $r_1, r_2 \geq 0$ and define the Borel set:

$$(7.31) \quad A_{r_1, r_2} = \left\{ (\gamma, p, t) \in \widetilde{\text{Reg}}(\text{PAR}(\varepsilon, S)) \times \mathbb{R} \cap \text{TRIP} : \gamma^{-1}(p) \in [t - r_1, t + r_2] \right\};$$

using [Kec95, Thm. 7.25] we get that the map:

$$(7.32) \quad \begin{aligned} \Omega_{r_1, r_2} : \text{Frag}(X) \times \mathbb{R} \times M(X) &\rightarrow \mathbb{R} \\ (\gamma, t, \mu) &\mapsto \mu\left((A_{r_1, r_2})_{(\gamma, t)}\right) \end{aligned}$$

is Borel. The proof that $\text{Reg}(\text{PAR}(\varepsilon, S))$ is Borel is completed by observing that **(Reg5)** can be expressed as:

$$(7.33) \quad \Omega_{r_1, r_2}(\gamma, \gamma^{-1}(p), \Psi(\gamma)) \geq (1 - \varepsilon)(r_1 + r_2) \quad (\forall r_1, r_2 \in [0, S] \cap \mathbb{Q}).$$

□

Consider the map Ψ defined in (7.14); we can decompose the measures $\Psi(\gamma)$ as follows:

$$(7.34) \quad \begin{aligned} \Psi_{\text{PAR}(\varepsilon, S)}(\gamma) &= \Psi(\gamma) \llcorner (\text{Reg}(\text{PAR}(\varepsilon, S)))_\gamma; \\ \Psi_{\text{PAR}(\varepsilon, S)}^c(\gamma) &= \Psi(\gamma) \llcorner (\text{Reg}(\text{PAR}(\varepsilon, S)))_\gamma^c. \end{aligned}$$

Lemma 7.35. *The maps $\Psi_{\text{PAR}(\varepsilon, S)}$ and $\Psi_{\text{PAR}(\varepsilon, S)}^c$ are Borel. Thus, given an Alberti representation $\mathcal{A}$ of the finite measure μ' satisfying the assumptions of Theorem 7.15, we can define the finite Radon measures:*

$$(7.36) \quad \begin{aligned} \mu'_{\text{PAR}(\varepsilon, S)} &= \int_{\text{Frag}(X)} \Psi_{\text{PAR}(\varepsilon, S)}(\gamma) dP(\gamma) \\ \mu'^c_{\text{PAR}(\varepsilon, S)} &= \int_{\text{Frag}(X)} \Psi_{\text{PAR}(\varepsilon, S)}^c(\gamma) dP(\gamma), \end{aligned}$$

which satisfy:

$$(7.37) \quad \mu'_{\text{PAR}(\varepsilon, S)} + \mu'^c_{\text{PAR}(\varepsilon, S)} = \mu',$$

and:

$$(7.38) \quad \lim_{S \searrow 0} \left\| \mu'^c_{\text{PAR}(\varepsilon, S)} \right\|_{M(X)} = 0.$$

Proof. By [Kec95, Thm. 7.25] the map:

$$(7.39) \quad \begin{aligned} \Omega : \text{Frag}(X) \times M(X) &\rightarrow \mathbb{R} \\ (\gamma, \mu) &\mapsto \mu \llcorner (\text{Reg}(\text{PAR}(\varepsilon, S)))_\gamma \end{aligned}$$

is Borel, and thus $\Psi_{\text{PAR}(\varepsilon, S)}$ is Borel as $\Psi_{\text{PAR}(\varepsilon, S)}(\gamma)$ can be written as $\Omega(\gamma, \Psi(\gamma))$; the proof for $\Psi^c_{\text{PAR}(\varepsilon, S)}$ is similar.

Now note that $\left\| \Psi^c_{\text{PAR}(\varepsilon, S)}(\gamma) \right\| \leq \tau_0$ and that, for each γ , one has

$$(7.40) \quad \lim_{S \searrow 0} \left\| \Psi^c_{\text{PAR}(\varepsilon, S)}(\gamma) \right\| = 0,$$

as for $\mathcal{L}^1$ -a.e. $t \in \text{Dom } \gamma$ there is an $S(t)$ such that, for $s \leq S(t)$, one has $(\gamma, \gamma(t)) \in \text{Reg}(\text{PAR}(\varepsilon, S))$. Then (7.38) follows by the Dominated Convergence Theorem. $\square$

The following lemma follows from a standard argument in measure differentiation.

Lemma 7.41. *Let $\{\varepsilon_m\} \subset (0, \infty)$ be a sequence with $\sum_m \varepsilon_m < \infty$; then there are a Borel $U \subset X$ and a sequence of pairs $\{(s_m, S_m)\}_m \subset (0, \infty)^2$ such that:*

- (1) *One has $\mu(X \setminus U) \leq \sum_m \varepsilon_m$ and, for each m , one also has $s_m \leq S_m$;*
- (2) *For each $x \in U$ and for each $r \leq s_m$, one has:*

$$(7.42) \quad \mu'^c_{\text{PAR}(\varepsilon_m, S_m)}(B(x, r)) \leq \varepsilon_m \mu'(B(x, r)).$$

Proof of Theorem 7.15. We fix a sequence ε_m such that $\sum_m \varepsilon_m < \infty$: the set U is the intersection of the set provided by Lemma 7.41 and the set of points p where the limit:

$$(7.43) \quad \lim_{r \searrow 0} \frac{\mu'(B(x, r))}{\mu(B(x, r))}$$

exists and is finite. Having fixed a point $p \in U$, we let r_n be a sequence converging to 0 such that the rescalings

$$(7.44) \quad \left(\frac{1}{r_n} X, \frac{\mu'}{\mu(B(p, r_n))}, \frac{\psi - \psi(p)}{r_n}, p \right)$$

converge to (Y, ν, φ, q) in the measured Gromov-Hausdorff sense. We let $X_n = \frac{1}{r_n} X$. As in the following we consider simultaneously different metric spaces, we will use subscripts to denote objects which “live” in a given metric space, e.g. $B_{X_n}(p, R_0)$ denotes the ball of radius R_0 and center p in the metric space X_n .

By the theory of measured Gromov-Hausdorff convergence we can find a compact metric space Z , which is a convex compact subset of some Banach space (e.g. ℓ^∞), which satisfies the following properties:

(Z1): There are isometric embeddings:

$$(7.45) \quad \begin{aligned} J_n &: (\bar{B}_{X_n}(p, 4R_0), p) \rightarrow (Z, q_Z) \\ J_\infty &: (\bar{B}_Y(q, 4R_0), q) \rightarrow (Z, q_Z); \end{aligned}$$

in the following we will often implicitly identify balls like $B_{X_n}(p, r)$ and $B_Y(q, r)$ with their images in Z ;

(Z2): There are compact sets $K_n, \tilde{K}_n \subset Z$ and a sequence $\eta_n \searrow 0$ such that:

$$(7.46) \quad \begin{aligned} \bar{B}_{X_n}(p, R_0) &\subset K_n \subset \bar{B}_{X_n}(p, R_0 + \eta_n) \\ \bar{B}_{X_n}(p, 2R_0) &\subset \tilde{K}_n \subset \bar{B}_{X_n}(p, 2R_0 + \eta_n) \\ d_{Z,H}(K_n, \bar{B}_Y(q, R_0)) &\leq \eta_n \\ d_{Z,H}(\tilde{K}_n, \bar{B}_Y(q, 2R_0)) &\leq \eta_n, \end{aligned}$$

where $d_{Z,H}(\cdot, \cdot)$ denotes the Hausdorff distance between subsets of Z ;

(Z3): There is an $\mathbf{L}(\psi)$ -Lipschitz function $\psi_Z : Z \rightarrow \mathbb{R}^N$ such that, denoting by ψ_{X_n} the restriction $\psi_Z|_{\bar{B}_{X_n}(p, 2R_0)}$ and by ψ_Y the restriction $\psi_Z|_{\bar{B}_Y(q, 2R_0)}$, one has:

$$(7.47) \quad \begin{aligned} \psi_{X_n} \circ J_n &= \frac{\psi - \psi(p)}{r_n} \\ \psi_Y \circ J_\infty &= \varphi; \end{aligned}$$

(Z4): Letting μ_n and μ_∞ denote, respectively, the measures

$$(7.48) \quad \begin{aligned} J_{n\#} \frac{\mu' \llcorner B_{X_n}(p, R_0)}{\mu'(B_X(p, r_n))} \\ J_{\infty\#} \nu \llcorner B_Y(q, R_0), \end{aligned}$$

one has $\mu_n \xrightarrow{w^*} \mu_\infty$.

We chose Z convex to “fill-in” fragments to Lipschitz curves; specifically, let $\text{Curves}(Z)$ denote the set of Lipschitz maps $\gamma : K \rightarrow Z$, where $K \subset \mathbb{R}$ is a (possibly degenerate) compact interval; we topologize $\text{Curves}(Z)$ with the Vietoris topology. Let

$$(7.49) \quad \text{Fill} : \text{Frag}(Z) \rightarrow \text{Curves}(Z)$$

be the map which extends a fragment γ to a Lipschitz curve, with domain the minimal compact interval $I(\gamma)$ containing $\text{Dom } \gamma$, by extending γ linearly on each component of $I(\gamma) \setminus \text{Dom } \gamma$. The map Fill is continuous.

Let $\Gamma_{X_n} \subset \text{Frag}(X)$ denote the set of those $[C_0, D_0]$ -biLipschitz fragments which intersect $\bar{B}_X(p, 2R_0 r_n)$; note that Γ_{X_n} is closed. We define maps:

$$(7.50) \quad \text{Rep}_n : \Gamma_{X_n} \rightarrow \text{Frag}(Z)$$

by composing $J_n \circ (\gamma|_{\gamma^{-1}(\bar{B}_X(p, 2R_0))})$, where we naturally identify γ with a fragment in X_n , with the unique affine map $A_\gamma : \mathbb{R} \rightarrow \mathbb{R}$ which has dilating factor $\frac{1}{r_n}$ and which maps the point:

$$(7.51) \quad \min \{t : t \in \gamma^{-1}(\bar{B}_X(p, 2R_0))\}$$

to 0. Note that Rep_n is continuous.

We will now refer back to the map Ψ defined in (7.14), adding subscripts regarding the metric space. From the definition of Rep_n we see that:

$$(7.52) \quad r_n \Psi_Z(\text{Rep}_n(\gamma)) = J_{n\#} \Psi_{X_n}(\gamma) \lrcorner \bar{B}_{X_n}(p, 2R_0)$$

Let $g \in C_c(Z)$ so that:

$$(7.53) \quad \lim_{n \rightarrow \infty} \int g d\mu_n = \int g d\mu_\infty;$$

then

$$(7.54) \quad \begin{aligned} \int g d\mu_n &= \frac{1}{\mu'(B_X(p, r_n))} \int_{B_{X_n}(p, R_0)} g \circ J_n d\mu' \\ &= \frac{1}{\mu'(B_X(p, r_n))} \int_{B_{X_n}(p, R_0)} g \circ J_n d(\mu'_{\text{PAR}(\varepsilon_m, S_m)} + \mu'^c_{\text{PAR}(\varepsilon_m, S_m)}); \end{aligned}$$

Note that

$$(7.55) \quad \left| \frac{1}{\mu'(B_X(p, r_n))} \int_{B_{X_n}(p, R_0)} g \circ J_n d\mu'^c_{\text{PAR}(\varepsilon_m, S_m)} \right| \leq \|g\|_\infty \frac{\mu'^c_{\text{PAR}(\varepsilon_m, S_m)}(B_X(p, r_n R_0))}{\mu'(B_X(p, r_n))};$$

for n sufficiently large $r_n R_0 \leq s_m$ so that by (7.42) and using that μ' is doubling we conclude that:

$$(7.56) \quad \lim_{n \rightarrow \infty} \frac{1}{\mu'(B_X(p, r_n))} \int_{B_{X_n}(p, R_0)} g \circ J_n d\mu'^c_{\text{PAR}(\varepsilon_m, S_m)} = 0.$$

We also introduce some notation to deal with regularity in Z and X ; so we let:

$$(7.57) \quad \begin{aligned} \text{PAR}_X(\varepsilon, S) &= (\psi, \mathcal{C}, v_0, \sigma_0, [C_0, D_0], \tau_0, \varepsilon, S) \\ \text{PAR}_Z(\varepsilon, S) &= \left(\psi, \mathcal{C}, v_0, \sigma_0, [C_0, D_0], \frac{4R_0}{C_0}, \varepsilon, S \right); \end{aligned}$$

in particular, inspection of conditions **(Reg1)**–**(Reg5)** shows that:

$$(7.58) \quad \begin{aligned} &\frac{1}{\mu'(B_X(p, r_n))} \int_{B_{X_n}(p, R_0)} g \circ J_n d\mu'_{\text{PAR}_X(\varepsilon_m, S_m)} \\ &= \frac{1}{\mu'(B_X(p, r_n))} \int_{\Gamma_{X_n}} dP(\gamma) \int_Z g r_n \chi_{B_{X_n}(p, R_0)} d\Psi_{\text{PAR}_Z(\varepsilon_m, S_m/r_n)}(\text{Rep}_n(\gamma)). \end{aligned}$$

Let $\tilde{\Gamma}_{X_n}$ be the Borel subset of those $\gamma \in \Gamma_{X_n}$ such that:

$$(7.59) \quad \chi_{B_{X_n}(p, R_0)} \Psi_{\text{PAR}_Z(\varepsilon_m, S_m/r_n)}(\text{Rep}_n(\gamma)) \neq 0;$$

then (7.59) implies that there is a $p_\gamma = \gamma(t) \in (\text{Reg}(\text{PAR}_X(\varepsilon_m, S_m)))_\gamma \cap B_X(p, r_n R_0)$. Note that the set $\mathcal{B}_{\gamma, n} = \gamma^{-1}(\bar{B}_X(p, 2r_n R_0))$ has diameter at most $\frac{4R_0 r_n}{C_0}$; let a_γ, b_γ be minimal such that the interval $[t_\gamma - a_\gamma, t_\gamma + b_\gamma]$ contains $\gamma^{-1}(\mathcal{B}_{\gamma, n})$. For n -sufficiently large one has $a_\gamma, b_\gamma \leq S_m$ so that by **(Reg2)** the $\varepsilon_m(a_\gamma + b_\gamma)$ -neighbourhood of $\mathcal{B}_{\gamma, n}$ contains $[t_\gamma - a_\gamma, t_\gamma + b_\gamma]$. A similar conclusion holds for

the smallest interval containing $\gamma^{-1}(B_{X_n}(p, R_0))$ from which we get:

$$(7.60) \quad r_n \left\| \Psi_{\text{PAR}_Z(\varepsilon_m, S_m/r_n)}(\text{Rep}_n(\gamma)) \lrcorner B_{X_n}(p, R_0) - \Psi_Z(\text{Fill} \circ \text{Rep}_n(\gamma)) \lrcorner B_Z(q_z, R_0) \right\| \leq \varepsilon_m r_n \frac{2R_0}{C_0}.$$

Note also that:

$$(7.61) \quad \gamma \left(\text{Dom } \gamma \cap [t_\gamma - \frac{r_n R_0}{D_0}, t_\gamma + \frac{r_n R_0}{D_0}] \right) \subset B_X(p, 2r_n R_0);$$

as for n sufficiently large one has $\frac{r_n R_0}{D_0} \leq S_m$, we have:

$$(7.62) \quad r_n \Psi_{\text{PAR}_Z(\varepsilon_m, S_m/r_n)}(\text{Fill} \circ \text{Rep}_n(\gamma))(B_{X_n}(p, 2r_n R_0)) \geq 2(1 - \varepsilon_m) \frac{r_n R_0}{D_0}.$$

For n sufficiently large we also have $\frac{3r_n R_0}{C_0} \leq S_m$ which implies:

$$(7.63) \quad \begin{aligned} \mathcal{L}^1 \left(\text{Dom } \gamma \cap \left[t_\gamma, t_\gamma + \frac{3r_n R_0}{C_0} \right] \right) &\geq (1 - \varepsilon_m) \frac{3R_0}{C_0} r_n \\ \mathcal{L}^1 \left(\text{Dom } \gamma \cap \left[t_\gamma - \frac{3r_n R_0}{C_0}, t_\gamma \right] \right) &\geq (1 - \varepsilon_m) \frac{3R_0}{C_0} r_n; \end{aligned}$$

so we can find $s_{1,\gamma} \leq t_\gamma \leq s_{2,\gamma}$ with:

$$(7.64) \quad \begin{aligned} |t_\gamma - s_{i,\gamma}| &\geq (1 - \varepsilon_m) \frac{3r_n R_0}{C_0} \quad (\text{for } i = 1, 2) \\ d_X(p, \gamma(s_{i,\gamma})) &\geq \left((1 - \varepsilon_m) \frac{3R_0}{C_0} - R_0 \right) r_n \quad (\text{for } i = 1, 2); \end{aligned}$$

in particular, for m sufficiently large (7.64) implies that the maximum and minimum point in $\mathcal{B}_{\gamma,n}$ are mapped by γ outside of $B_X(p, \frac{3}{2}R_0)$. Thus, the endpoints of $\text{Fill} \circ \text{Rep}_n(\gamma)$ lie out of $B_Z(q_z, \frac{3}{2}R_0)$.

We now obtain an upper estimate for $P(\tilde{\Gamma}_{X_n})$ (note that we assume that n is sufficiently large depending on m):

$$(7.65) \quad \begin{aligned} 2(1 - \varepsilon_m) \frac{r_n R_0}{D_0} P(\tilde{\Gamma}_{X_n}) &\leq r_n \int_{\text{Frag}(X)} \Psi_{\text{PAR}_Z(\varepsilon_m, S_m/r_n)}(\text{Rep}_n(\gamma))(B_X(p, 2r_n R_0)) \, dP(\gamma) \\ &\leq \mu'(B_X(p, 2R_0 r_n)); \end{aligned}$$

in particular, using (7.60),

(7.66)

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1}{\mu'(B_X(p, r_n))} \left| \int_{\tilde{\Gamma}_{X_n}} dP(\gamma) \int g r_n \chi_{B_{X_n}(p, R_0)} d\Psi_{\text{PAR}_Z(\varepsilon_m, S_m/r_n)}(\text{Rep}_n(\gamma)) \right. \\
& \quad \left. - \int_{\tilde{\Gamma}_{X_n}} dP(\gamma) \int g r_n \chi_{B_Z(q_Z, R_0)} d\Psi_{\text{PAR}_Z(\varepsilon_m, S_m/r_n)}(\text{Fill} \circ \text{Rep}_n(\gamma)) \right| \\
& \leq \limsup_{n \rightarrow \infty} \|g\|_\infty P(\tilde{\Gamma}_{X_n}) r_n \varepsilon_m \frac{2R_0}{C_0} \\
& \leq \limsup_{n \rightarrow \infty} \|g\|_\infty \frac{\varepsilon_m}{1 - \varepsilon_m} \frac{D_0}{C_0} \frac{\mu'(B_X(p, 2r_n R_0))}{\mu'(B_X(p, r_n))} \\
& = O(\varepsilon_m),
\end{aligned}$$

where in the last step we used that μ' is doubling. As $n \rightarrow \infty$ we can send $m \rightarrow \infty$ so that the left hand side of (7.66) converges to 0.

Let $\Omega_m \subset \text{Curves}(Z)$ denote the set of D_0 -Lipschitz curves so that there are a $\theta_\gamma \in [C_0, D_0]$ and a $w \in \bar{\mathcal{C}}$ such that (note the constant C_2 will be specified later):

($\Omega 1$): For all $s_1, s_2 \in \text{Dom } \gamma$ one has:

$$(7.67) \quad |d_Z(\gamma(s_1), \gamma(s_2)) - \theta_\gamma |s_1 - s_2|| \leq C_2 \varepsilon_m;$$

($\Omega 2$): The domain of γ is a subset $[0, \frac{4R_0}{C_0}]$;

($\Omega 3$): The image of γ is contained in the $C_2 \varepsilon_m$ -neighbourhood of $\bar{B}_{X_n}(p, 2R_0)$;

($\Omega 4$): For all $s_1, s_2 \in \text{Dom } \gamma$ one has:

$$(7.68) \quad |\psi_Z \circ \gamma(s_1) - \psi_Z \circ \gamma(s_2) - w |s_1 - s_2|| \leq C_2 \varepsilon_m;$$

($\Omega 5$): For all $s_1, s_2 \in \text{Dom } \gamma$ with $s_2 \geq s_1$ one has:

$$(7.69) \quad \langle v_0, \psi_Z \circ \gamma(s_2) - \psi_Z \circ \gamma(s_1) \rangle \geq (\sigma_0 - \varepsilon_m) \theta_\gamma \text{Lip}(\langle v_0, \varphi \rangle)(p)(s_2 - s_1) - C_2 \varepsilon_m.$$

Note that the set Ω_m is compact. We also define Ω_∞ by requiring in ($\Omega 3$) that γ lies in $\bar{B}_Y(p, 2R_0)$ and that the error term ε_m is replaced by 0. In view of (**Reg1**)–(**Reg5**), for an appropriate choice of C_2 one has $\text{Fill} \circ \text{Rep}_n(\tilde{\Gamma}_{X_n}) \subset \Omega_m$ for $n \geq N(m)$. If we let P_n denote the Radon measure on $\text{Curves}(Z)$:

$$(7.70) \quad P_n = \frac{1}{\mu'(B_X(p, r_n))} r_n \text{Fill} \circ \text{Rep}_n(\gamma)_\# P \llcorner \tilde{\Gamma}_{X_n};$$

we have that P_n has support contained in Ω_m for $n \geq N(m)$. By (7.65) the total mass of P_n is bounded by:

$$(7.71) \quad \frac{D_0}{2(1 - \varepsilon_m)R_0} \frac{\mu'(B_X(p, 2R_0 r_n))}{\mu'(B_X(p, r_n))}.$$

Moreover, an application of Ascoli-Arzelà shows that the set $\Omega = \bigcup_m \Omega_m \cup \Omega_\infty$ is compact; we can thus find a subsequence $n_m \geq N(m)$ such that $P_{n_m} \xrightarrow{w^*} Q_{R_0}$. The previous discussion on the properties of the fragments in $\tilde{\Gamma}_{X_n}$ implies that the

support $\text{spt } Q_{R_0}$ of Q_{R_0} is a subset of $\mathcal{S}_{R_0} \subset \Omega_\infty$ and that point (2) in the statement of this Theorem follows from (7.71). We now observe that:

$$(7.72) \quad \frac{1}{\mu'(B_X(p, r_n))} \int_{\tilde{\Gamma}_{X_{n_m}}} dP(\gamma) \int g r_n \chi_{B_Z(q_Z, R_0)} d\Psi_Z (\text{Fill} \circ \text{Rep}_{n_m}(\gamma)) \\ = \int_{\Omega} dP_{n_m}(\gamma) \int g \chi_{B_Z(q_Z, R_0)} d\Psi_Z(\gamma);$$

fix a $\xi \in (0, 1)$, and let ψ_ξ be a continuous function, which takes values in $[0, 1]$ and which equals 1 on $\bar{B}_Z(q_Z, R_0 + \xi)$ and which vanishes out of $B_Z(q_Z, R_0 + 2\xi)$; then:

$$(7.73) \quad \left| \int_{\Omega} dP_{n_m}(\gamma) \int g (\chi_{B_Z(q_Z, R_0)} - \psi_\xi) d\Psi_Z(\gamma) \right| \leq P_{n_m}(\Omega) \|g\|_\infty \frac{2\xi}{C_0};$$

as the map $\gamma \mapsto \int g \psi_\xi d\Psi_Z(\gamma)$ is continuous:

$$(7.74) \quad \lim_{m \rightarrow \infty} \int_{\Omega} dP_{n_m}(\gamma) \int g \psi_\xi d\Psi_Z(\gamma) = \int_{\Omega} dQ_{R_0}(\gamma) \int g \psi_\xi d\Psi_Z(\gamma).$$

Also,

$$(7.75) \quad \left| \int_{\Omega} dQ_{R_0}(\gamma) \int g \psi_\xi d\Psi_Z(\gamma) - \int_{\Omega} dQ_{R_0}(\gamma) \int g d\Psi_Z(\gamma) \lrcorner B_Y(q, R_0) \right| \leq Q_{R_0}(\Omega) \|g\|_\infty \frac{\xi}{C_0};$$

so we conclude that:

$$(7.76) \quad \lim_{m \rightarrow \infty} \int g d\mu_{n_m} = \int_{\mathcal{S}_{R_0}} dQ_{R_0} \int g d\Psi_Z(\gamma) \lrcorner B_Y(q, R_0);$$

in particular, if we let:

$$(7.77) \quad \Phi_\gamma = \Psi_Z(\gamma) \lrcorner B_Y(q, R_0) = \frac{1}{\theta_\gamma} \mathcal{H}_\gamma^1,$$

we get that (Q_{R_0}, Φ) gives an Alberti representation of $\nu \lrcorner B_Y(q, R_0)$. It might be worth noting that in (7.77) we used that γ is a geodesic with constant speed θ_γ and that the function $\gamma \mapsto \theta_\gamma$ is continuous. $\square$

Lemma 7.78. *There is a Borel $U \subset X$ with full μ -measure such that, for each $p \in U$, for each $(Y, \nu, \varphi, q) \in \text{Bw-up}(X, \mu, \psi, p)$, for each $R_0 > 0$ and each $v_0 \in S(\|\cdot\|_{p, \text{Lip}^*})$, the measure $\nu \lrcorner B(q, R_0)$ admits an Alberti representation (Q_{R_0}, Φ) which satisfies the following conditions:*

- (1) *The measure Q_{R_0} is a finite Radon measure with total mass at most*

$$\frac{1}{2R_0} (\text{As}(\mu, p))^{\log_2 R_0 + 1};$$

- (2) *The support of Q_{R_0} is contained in a compact set $\mathcal{S}_{R_0} \subset \text{Frag}(Y)$ which consists of the unit-speed geodesic segments γ whose domain lies in $[0, 4R_0]$, whose image lies in $\bar{B}(q, 2R_0)$, which have both endpoints lying outside of $B(q, \frac{3}{2}R_0)$, which intersect $\bar{B}(q, R_0)$, and which satisfy:*

$$(7.79) \quad \varphi \circ \gamma(s_2) - \varphi \circ \gamma(s_1) = (s_2 - s_1)v_0 \quad (\forall s_1, s_2 \in \text{Dom } \gamma);$$

- (3) *For each $\gamma \in \mathcal{S}_{R_0}$ the measure Φ_γ is given by:*

$$(7.80) \quad \Phi_\gamma = \mathcal{H}_\gamma^1 \lrcorner B(q, R_0).$$

Proof. By Theorem 6.22 we can choose Borel maps $v_n : X \rightarrow TX$ ² with $1 \leq \|v_n\|_{TX} \leq 1 + \frac{1}{n}$ and such that:

- (1) For each $x \in X$ the closure of the set $\{v_n(x)\}_n$ contains $S(\|\cdot\|_{p,\text{Lip}^*})$;
- (2) For each n there is a measure μ'_n in the same measure class of μ and there are countably many disjoint compact sets $\{K_{n,\alpha}\}$ whose union has μ -negligible complement and such that the function v_n is constant on each $K_{n,\alpha}$;
- (3) The measure $\mu'_n \llcorner K_{n,\alpha}$ admits a simplified and $(1, 1 + \frac{1}{n})$ -biLipschitz Alberti representation $\mathcal{A}_{n,\alpha}$ in the ψ -direction of the cone $\mathcal{C}(v_n \llcorner K_{n,\alpha} / \|v_n \llcorner K_{n,\alpha}\|_2, \pi/2n)$ with $\langle v_n \llcorner K_{n,\alpha}, \psi \rangle$ -speed $\geq (1 - 1/n)$.

Let U_n be a Borel subset of $\bigcup_\alpha K_{n,\alpha}$ with full μ -measure and such that, for each $p \in U_n \cap K_{n,\alpha}$, the conclusion of Theorem 7.15 holds taking $\mathcal{A} = \mathcal{A}_{n,\alpha}$. Let $U = \bigcap U_n$ and fix $p \in U$ and $(Y, \nu, \varphi, q) \in \text{Bw-up}(X, \mu, \psi, p)$. Choose a sequence n_m such that $v_{n_m}(p) \rightarrow v_0$ and let Q_{R_0, n_m} , Φ_{n_m} and $\mathcal{S}_{R_0, n_m}$ be the measures and sets of geodesics provided by Theorem 7.15. By Ascoli-Arzelá the set $\Omega = \mathcal{S}_{R_0} \bigcup_m \mathcal{S}_{R_0, n_m}$ is a compact subset of $\text{Frag}(Y)$; as the measures Q_{R_0, n_m} are uniformly bounded and supported in Ω , we can pass to a subsequence such that $Q_{R_0, n_m} \xrightarrow{w^*} Q_{R_0}$. Note also that Q_{R_0} is supported in $\mathcal{S}_{R_0}$. For $g \in C_b(Y)$ one proves that:

$$(7.81) \quad \lim_{m \rightarrow \infty} \int_{\Omega} dQ_{R_0, n_m}(\gamma) \int g d(\Phi_{n_m})_\gamma = \int_{\mathcal{S}_{R_0}} dQ_{R_0}(\gamma) \int g d\Phi_\gamma$$

by using an argument similar to the one used to derive the estimates (7.73) and (7.75). Thus the pair (Q_{R_0}, Φ) provides the desired Alberti representation. $\square$

To prove Theorem 7.11 we need to introduce a bit more of terminology. We can regard parametrized Lipschitz curves in Y , whose domain is a possibly infinite interval of $\mathbb{R}$, as elements of $F_c(\mathbb{R} \times X)$ by identifying them with their graph. We denote by $\text{Geo}(Y)$ the set of unit speed geodesic segments, half-lines or lines in Y ; note that $\text{Geo}(Y)$ is a K_σ . Moreover, if we let:

$$(7.82) \quad \begin{aligned} \Phi : \text{Geo}(Y) &\rightarrow \text{Rad} \\ \gamma &\mapsto \mathcal{H}_\gamma^1, \end{aligned}$$

then, for each $g \in C_c(Y)$, the map:

$$(7.83) \quad \begin{aligned} \Phi_g : \text{Geo}(Y) &\rightarrow \mathbb{R} \\ \gamma &\rightarrow \int g d\Phi_\gamma = \int_{\mathbb{R}} g \circ \gamma(t) dt \end{aligned}$$

is continuous.

Proof of Theorem 7.11. Let U be the μ -full measure subset provided by Lemma 7.78 and consider $p \in U$ and $(Y, \nu, \varphi, q) \in \text{Bw-up}(X, \mu, \psi, p)$. Fix a diverging sequence of radii $\{R_n\}$ with $R_n > 2R_{n-1}$ and let Q_{R_n} and $\mathcal{S}_{R_n}$ be the corresponding measures and sets provided by Lemma 7.78. Note that $\mathcal{S}_{R_n}$ can also be regarded as a compact subset of F_c ; in particular, for $i \leq n$ we define the sets:

$$(7.84) \quad \mathcal{S}_{n,i} = \{\gamma \in \mathcal{S}_{R_n} : \text{dist}(\gamma, q) \in (R_{i-1}, R_i]\},$$

²A choice of the representative of TX is implied.

where we take $R_0 = 0$, and observe that the sets $\mathcal{S}_{n,i}$ are Borel. We also consider the following Borel subsets of $F_c \times \mathbb{R}$:

$$(7.85) \quad \tilde{\mathcal{S}}_{i,n} = \{(\gamma, t) : t \in \text{Dom } \gamma, \gamma \in \mathcal{S}_{i,n}, d(\gamma(t), q) \in (R_{i-1}, R_i]\};$$

note that the sets $\tilde{\mathcal{S}}_{i,n}$ have compact sections, i.e. for $\gamma \in F_c$, each section $(\tilde{\mathcal{S}}_{i,n})_\gamma$ is compact. By the Lusin-Novikov Uniformization Theorem [Kec95, Thm. 18.10] we can find Borel maps $\tau_{i,n} : \mathcal{S}_{i,n} \rightarrow \mathbb{R}$ such that $(\gamma, \tau_{i,n}(\gamma)) \in \tilde{\mathcal{S}}_{i,n}$. In particular, we can define a Borel map $\text{Tran}_n : \text{Geo}(Y) \rightarrow \text{Geo}(Y)$ by requiring that for $\gamma \in \tilde{\mathcal{S}}_{i,n}$ the geodesic segment $\text{Tran}_n(\gamma)$ is the composition of γ with the translation by $\tau_{n,i}(\gamma)$. Note that if $\gamma \in \tilde{\mathcal{S}}_{i,n}$ the extremes of $\text{Dom } \gamma$ are at distance at least $\frac{3}{2}R_n - R_i$ from $\tau_{n,i}(\gamma)$ and so

$$(7.86) \quad \left[-\frac{R_n}{2}, \frac{R_n}{2}\right] \subset \text{Dom } \text{Tran}_n(\gamma) \subset [-3R_n, 3R_n].$$

Let $Q_n = \text{Tran}_n \# Q_{R_n}$ and denote by $K(m, i)$ the set of geodesic segments γ whose domain is contained in $[-3R_m, 3R_m]$, and which intersect $\bar{B}(q, R_i)$ in a point $p_\gamma = \gamma(t_\gamma)$ where t_γ is at distance at most $2R_i$ from 0. The set $K(\infty, i)$ is defined similarly by requiring γ to be a geodesic line. Note that the sets:

$$(7.87) \quad K(i) = K(\infty, i) \cup \bigcup_m K(m, i)$$

are compact and that Q_n is concentrated on the set $\bigcup_{i \leq n} K(n, i)$. We now obtain an upper bound on $Q_n(K(i))$:

$$(7.88) \quad \nu(B(q, 2R_i)) = \int_{\text{Geo}(Y)} \mathcal{H}_\gamma^1(B(q, 2R_i)) dQ_n(\gamma) \geq \int_{K(i)} \mathcal{H}_\gamma^1(B(q, 2R_i)) dQ_n(\gamma);$$

if $\gamma \in \text{Tran}_n(\mathcal{S}_{n,l})$ and if $l \leq i$ and $n \geq i$, one has $\mathcal{H}_\gamma^1(B(q, 2R_i)) \geq \frac{R_i}{2}$ so from (7.88) we obtain:

$$(7.89) \quad Q_n(K(i)) \leq 2 \frac{\nu(B(q, 2R_i))}{R_i}.$$

In particular, we can pass to a subsequence and find a Radon measure Q on $\text{Geo}(Y)$ such that for each i one has $Q_n \llcorner K(i) \xrightarrow{w^*} Q \llcorner K(i)$; in particular $Q_n \xrightarrow{w^*} Q$. Moreover, as Q_n is concentrated on $\bigcup_i K(n, i)$, the measure Q has support contained in $\text{Lines}(\varphi, v_0)$. To show that (Q, Φ) gives an Alberti representation of ν we take $g \in C_c(Y)$ and choose i sufficiently large so that $\text{spt } g \subset B(q, R_i)$:

$$(7.90) \quad \int_Y g d\nu = \int_{K(i)} dQ_n(\gamma) \int g d\mathcal{H}_\gamma^1 = \int_{K(i)} \Phi_g(\gamma) dQ_n(\gamma),$$

and

$$(7.91) \quad \lim_{n \rightarrow \infty} \int_{K(i)} \Phi_g(\gamma) dQ_n(\gamma) = \int_{K(i)} \Phi_g(\gamma) dQ(\gamma) = \int_{\text{Geo}(Y)} dQ(\gamma) \int g d\mathcal{H}_\gamma^1.$$

□

We now state an immediate consequence of Theorem 7.11 in terms of the canonical maps from blow-ups of X to the fibres of TX .

Definition 7.92. Let $(Y, \nu, \varphi, q) \in (X, \mu, \psi, p)$ be realized by choosing scales $r_n \searrow 0$; suppose that the Lipschitz function $f : X \rightarrow \mathbb{R}$ is differentiable at p with respect to the $\{\psi^i\}_{i=1}^N$. Then the maps:

$$(7.93) \quad \frac{f - f(p)}{r_n} : \frac{1}{r_n} X \rightarrow \mathbb{R}$$

converge to the map $g : Y \rightarrow \mathbb{R}$ given by:

$$(7.94) \quad g(y) = \sum_{i=1}^N \frac{\partial f}{\partial \psi^i}(p) \varphi(y).$$

In particular, we obtain a canonical map $E : Y \rightarrow T_p X$ by letting:

$$(7.95) \quad \left\langle \sum_{i=1}^N a_i d\psi^i|_p, E(y) \right\rangle = \sum_{i=1}^N a_i \varphi(y).$$

Corollary 7.96. *Let $p \in U$ be a point where the conclusion of Theorem 7.11 holds; then the canonical map $E : Y \rightarrow T_p X$ is surjective. Moreover, for each $\tilde{q} \in Y$ there is a line $\gamma \in \text{Lines}(\varphi, v_0)$ passing through it, and there is a $c_\gamma \in \mathbb{R}$ such that:*

$$(7.97) \quad E(\gamma(t)) = v_0(t - c_\gamma) \quad (\forall t \in \mathbb{R}).$$

Corollary 7.96 generalizes [Che99, Sec. 13] where the surjectivity of the map E was proven for the case in which (X, μ) is a PI-space. The surjectivity of the map E in the case in which (X, μ) is a differentiability space has already been proven in [Sch13, Dav14].

7.3. Harmonicity of blow-up functions. In this subsection we prove Corollary 1.15.

Proof of Corollary 1.15. Let $u : X \rightarrow \mathbb{R}$ be a Lipschitz function, and suppose that $x \in X$ a point of differentiability of u where x is as in the statement of Theorem 1.11. Choose a unit vector $\xi \in (T_x X, \|\cdot\|_{TX}(x))$ supporting $du \in T_x^* X$, i.e.

$$(7.98) \quad du(\xi) = \|du(x)\|_{T^*X} = \|du(x)\|_{T^*X} \cdot \|\xi\|_{TX}.$$

Since x is a point of differentiability, the blow-up $\hat{u}$ of u will be of the form $\hat{u} = \alpha \circ \hat{\phi}_i$ for some $\alpha \in T_x^* X$. Now consider an Alberti representation for $\hat{\mu}$ as in Theorem 1.11 (2), which is supported on unit speed geodesics γ with $(\hat{u}_i \circ \gamma)' \equiv \xi$. Fix such a unit speed geodesic $\gamma : \mathbb{R} \rightarrow \hat{X}$. Note that for all $t \in \mathbb{R}$

$$(7.99) \quad (\hat{u} \circ \gamma)'(t) = \alpha \left((\hat{\phi}_i)'(t) \right) = \alpha(\xi) = \|\hat{u}(x)\|_{T^*X} \quad \text{and} \quad \text{Lip}(\hat{u})(\gamma(t)) = (\text{Lip}(\hat{u} \circ \gamma))(t).$$

If $v : \hat{X} \rightarrow \mathbb{R}$ is Lipschitz and agrees with $\hat{u}$ outside a compact subset $K \subset \hat{X}$, then for all $t \in \mathbb{R}$ we have $\text{Lip}(v \circ \gamma)(t) \leq \text{Lip}(v)(\gamma(t))$, and for $\mathcal{L}$ -a.e. $t \in \mathbb{R} \setminus K$ we have

$$(7.100) \quad \text{Lip}(v \circ \gamma) = |(v \circ \gamma)'(t)| = |(\hat{u} \circ \gamma)'(t)| = \text{Lip}(\hat{u} \circ \gamma)(t) = \text{Lip}(\hat{u})(\gamma(t)).$$

Therefore if $\gamma^{-1}(K) \subset [a, b]$, then

$$\begin{aligned}
 (7.101) \quad & \int_{\gamma^{-1}(K)} [\text{Lip}(v)(\gamma(t))]^p dt - \int_{\gamma^{-1}(K)} [\text{Lip}(\hat{u})(\gamma(t))]^p dt \\
 & \geq \int_{\gamma^{-1}(K)} [\text{Lip}(v \circ \gamma)(t)]^p dt \\
 & \quad - \int_{\gamma^{-1}(K)} [\text{Lip}(\hat{u} \circ \gamma)(t)]^p dt \\
 & = \int_{[a,b]} [\text{Lip}(v \circ \gamma)(t)]^p dt \\
 & \quad - \int_{[a,b]} [\text{Lip}(\hat{u} \circ \gamma)(t)]^p dt \\
 & \geq \int_{[a,b]} |(v \circ \gamma)'(t)|^p dt \\
 & \quad - \int_{[a,b]} |(\hat{u} \circ \gamma)'(t)|^p dt \\
 & \geq 0
 \end{aligned}$$

by Jensen's inequality. Integrating this with respect to the measure on curves coming from the Alberti representation, we get that

$$(7.102) \quad \int_K [\text{Lip}(v)]^p d\hat{\mu} \geq \int_K [\text{Lip}(\hat{u})]^p d\hat{\mu}.$$

□

8. LIPSCHITZ MAPPINGS $f : X \rightarrow Z$ AND METRIC DIFFERENTIATION

8.1. The canonical subbundle determined by a pseudodistance. In this subsection we associate a canonical subbundle $\mathcal{W}_\varrho$ of T^*X to a Lipschitz compatible pseudometric ϱ ; we denote by C_ϱ the Lipschitz constant of ϱ , that is, $\varrho \leq C_\varrho d_X$.

Definition 8.1. Let Φ be a countable set of ϱ -Lipschitz functions and let V be a Φ -differentiability set; we define a subbundle $\mathcal{W}_\Phi$ of T^*X by letting, for $x \in V$, the fibre $\mathcal{W}_\Phi(x)$ equal the linear span of $\{df(x) : f \in \Phi\}$.

The collection $\text{Sub}(\varrho)$ of subbundles associated to countable sets of ϱ -Lipschitz functions has a partial order $\preceq$: we say that $\mathcal{W}_\Phi \preceq \mathcal{W}_{\Phi'}$ if for μ -a.e. $x \in X$ one has $\mathcal{W}_\Phi(x) \subseteq \mathcal{W}_{\Phi'}(x)$.

Lemma 8.2. *The poset $(\text{Sub}(\varrho), \preceq)$ contains a maximal element $\mathcal{W}_\varrho$ which we call the canonical subbundle associated to ϱ .*

Proof. As the constructions depend only on the measure class of μ , we can assume that μ is a probability measure. We basically follow the argument used in the proof of Lemma 2.33: to each $\mathcal{W}_\Phi \in \text{Sub}(\varrho)$ we associate a “size”, which is the expectation of the random variable $\dim \mathcal{W}_\Phi$:

$$(8.3) \quad \|\mathcal{W}_\Phi\| = \int \dim \mathcal{W}_\Phi(x) d\mu(x);$$

note that the finite dimensionality of T^*X implies that

$$(8.4) \quad S = \sup_{\mathcal{W}_\Phi \in \text{Sub}(\varrho)} \|\mathcal{W}_\Phi\| < \infty.$$

Let $\mathcal{W}_{\Phi_n}$ be a maximizing sequence and let $\Phi_\infty = \bigcup_n \Phi_n$; then $\|\mathcal{W}_{\Phi_\infty}\| = S$. Suppose, by contradiction, that for some $\mathcal{W}_\Phi \in \text{Sub}(\varrho)$ one has $\mathcal{W}_\Phi \not\leq \mathcal{W}_{\Phi_\infty}$; then there is a positive measure set V such that, if $x \in V$, one has

$$(8.5) \quad \mathcal{W}_{\Phi_\infty}(x) \subsetneq \text{span}(\mathcal{W}_\Phi(x) \cup \mathcal{W}_{\Phi_\infty}(x));$$

but then we obtain the contradiction $\|\mathcal{W}_{\Phi \cup \Phi_\infty}\| > S$. $\square$

Let $D_X \subset X$ be a countable dense set and $\Phi_{D_X, \varrho} = \{\varrho_x : x \in D_X\}$; we let $\mathcal{W}_{D_X, \varrho} = \mathcal{W}_{\Phi_{D_X, \varrho}}$. We now show that $\mathcal{W}_{D_X, \varrho}$ equals $\mathcal{W}_\varrho$: this is a stronger result than [Kei04b, Thm. 2.7] because it applies to subbundles associated to Lipschitz compatible pseudometrics.

Theorem 8.6. *For any countable dense set $D_X \subset X$ we have $\mathcal{W}_{D_X, \varrho} = \mathcal{W}_\varrho$.*

We offer two conceptually different proofs of Theorem 8.6.

Proof of Theorem 8.6 via a measurable Hahn-Banach. As $\mathcal{W}_{D_X, \varrho} \leq \mathcal{W}_\varrho$, assume by contradiction that there is a positive measure Borel set U such that, for each $x \in U$ one has:

$$(8.7) \quad \mathcal{W}_{D_X, \varrho}(x) \subsetneq \mathcal{W}_\varrho(x).$$

Without loss of generality we can assume that there are 1-Lipschitz functions $\{\phi_i\}_{i=1}^N$ such that $(U, \{\phi_i\}_{i=1}^N)$ is a differentiability chart. Let

$$(8.8) \quad \tilde{U} = \left\{ (x, a) \in U \times \mathbb{R}^N : \sum_{i=1}^N a_i d\phi_i(x) \in \mathcal{W}_\varrho(x) \cap S(\|\cdot\|_{T^*X}(x)), \text{ and } \right. \\ \left. \text{dist}_{\|\cdot\|_{T^*X}(x)} \left(\sum_{i=1}^N a_i d\phi_i(x), \mathcal{W}_{D_X, \varrho}(x) \right) \geq \frac{1}{2} \right\}.$$

Note that the distances in the fibre T_x^*X are computed with respect to the norm $\|\cdot\|_{T^*X}(x)$. The set $\tilde{U}$ is Borel and by (8.7) for each $x \in U$ the section $\tilde{U}_x$ is nonempty (compare [Rud91, Lem. 4.22]) and compact. By the Lusin-Novikov Uniformization Theorem [Kec95, Thm. 18.10] we obtain a unit-norm Borel section ω of $\mathcal{W}_\varrho|_U$ satisfying:

$$(8.9) \quad \text{dist}_{\|\cdot\|_{T^*X}(x)}(\omega(x), \mathcal{W}_{D_X, \varrho}(x)) \geq \frac{1}{2} \quad (\forall x \in U).$$

Using Hahn-Banach in each fibre T_x^*X and an argument similar to the one above, we obtain a Borel section ξ of $TX|_U$ such that:

$$(8.10) \quad \begin{aligned} \|\xi\|_{TX} &\leq 2; \\ \langle \omega(x), \xi(x) \rangle &= 1 \quad (\forall x \in U), \end{aligned}$$

and such that $\xi(x)$ is annihilated by the functionals in $\mathcal{W}_{D_X, \varrho}(x)$. Up to shrinking U we can assume that there are $\tilde{N} \leq N$, $(1, \varrho)$ -Lipschitz functions $\{\psi_i\}_{i=1}^{\tilde{N}}$ and

bounded Borel maps $s_i : U \rightarrow \mathbb{R}$ satisfying:

$$(8.11) \quad \begin{aligned} & \|s_i\| \leq C; \\ & \omega = \sum_{i=1}^{\tilde{N}} s_i d\psi_i. \end{aligned}$$

Let $\mathcal{F}$ contain the ϕ_i , the ψ_i and the components of the chart functions; let $\mathcal{C}$ contain χ_U , the s_i and the characteristic functions of the charts; let $\mathcal{S}$ contain d_X and ϱ ; by Theorem 5.3 we obtain an μ -measurable subset $V \subset U$ of full μ -measure with $G_x(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ containing a dense set of directions in $T_x X$ for each $x \in V$. In particular, fix $\varepsilon > 0$ and let $\gamma'(t) \in T_x X$ be an $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ -generic velocity vector such that:

$$(8.12) \quad \|\xi(x) - \gamma'(t)\|_{TX} \leq \varepsilon;$$

then

$$(8.13) \quad |\langle d\varrho_x, \gamma'(t) \rangle| \leq C_\varrho \varepsilon + |\langle d\varrho_x, \xi(x) \rangle| = C_\varrho \varepsilon;$$

by Theorem 4.3 we conclude that:

$$(8.14) \quad \varrho\text{-md}\gamma(t) \leq C_\varrho \varepsilon.$$

However, $\|\gamma'(t)\|_{TX} \leq 2 + \varepsilon$ and so

$$(8.15) \quad \langle \omega, \gamma'(t) \rangle \geq \frac{1}{2} - \varepsilon(2 + \varepsilon);$$

note also that

$$(8.16) \quad |\langle \omega, \gamma'(t) \rangle| = \left| \sum_{i=1}^{\tilde{N}} s_i(\gamma(t)) (\psi_i \circ \gamma)'(t) \right| \leq \tilde{N} C \max_i |(\psi_i \circ \gamma)'(t)|;$$

now choose $s_n \searrow 0$ with $t + s_n \in \text{Dom } \gamma$; we have:

$$(8.17) \quad |\psi_i \circ \gamma(t + s_n) - \psi_i \circ \gamma(t)| \leq \varrho(\gamma(t + s_n), \gamma(t)) \leq o(s_n) + \int_{[t, t+s_n] \cap \text{Dom } \gamma} \varrho\text{-md}\gamma(\tau) d\tau;$$

dividing by s_n and letting $n \nearrow \infty$ we get:

$$(8.18) \quad |(\psi_i \circ \gamma)'(t)| \leq \varrho\text{-md}\gamma(t).$$

Combining (8.14), (8.15) and (8.18) we conclude that:

$$(8.19) \quad \frac{1}{2} - \varepsilon(2 + \varepsilon) \leq \tilde{N} C C_\varrho \varepsilon$$

which yields a contradiction if ε is sufficiently small. $\square$

Proof of Theorem 8.6 via Weaver derivations. We show that if $K \subset X$ is compact and if f is ϱ -Lipschitz, for μ -a.e. $x \in K$ one has $df(x) \in \mathcal{W}_{D_X, \varrho}(x)$. Fix $n \in \mathbb{N}$ and choose a finite subset $\{x_k\}_{k \in I_n} \subset D_X$ such that each $x \in K$ lies within d_X -distance at most $\frac{1}{n}$ from some x_k . To fix the ideas, suppose that f is (C, ϱ) -Lipschitz and define $f_n : K \rightarrow \mathbb{R}$ by:

$$(8.20) \quad f_n(x) = \inf \{f(x_k) + C\varrho(x, x_k) : k \in I_n\}.$$

The functions $\{f_n\}_n$ are uniformly (C, ϱ) -Lipschitz and hence uniformly $(C C_\varrho, d_X)$ -Lipschitz. By [Sch13, Thm. 4.1] the exterior derivative operator d associated to the differentiable structure is weak* continuous. In particular, let $L^2(\mu \llcorner K, T^*X)$ denote

the L^2 -space of sections of $T^*X \mid K$. Note that the dual of $L^2(\mu \llcorner K, T^*X)$ is $L^2(\mu \llcorner K, TX)$ and that these spaces are both reflexive by finite dimensionality of T^*X . Then as the $f_n \rightarrow f$ pointwise in K , we have that $df_n \rightarrow df$ weakly in $L^2(\mu \llcorner K, T^*X)$, and Mazur's Lemma and a standard argument give tail convex combinations g_n of the functions f_n with $dg_n \rightarrow df$ $\mu \llcorner K$ -a.e. So the proof is completed if we show that each dg_n is a section of $\mathcal{W}_{D_X, \varrho}$, which happens if each df_n is a section of $\mathcal{W}_{D_X, \varrho}$. But for each n there are closed subsets $\{C_i\}_{i \in I_n}$ of K , such that $f_n \mid C_i = f(x_i) + C \varrho_{x_i}$, which gives $df_n \mid C_i = C d\varrho_{x_i}$. $\square$

We now associate to $\mathcal{W}_\varrho$ two a priori different norms on TX . Roughly speaking, we maximize the seminorms induced by sections of $\mathcal{W}_\varrho$. Recall that if f is ϱ -Lipschitz we can define the “big Lip” with respect to ϱ :

$$(8.21) \quad \varrho\text{-Lip}f(x) = \limsup_{r \searrow 0} \sup \left\{ \frac{|f(x) - f(y)|}{r} : \varrho(x, y) \leq r \right\},$$

and that the map $x \mapsto \varrho\text{-Lip}f(x)$ is Borel.

Let $\text{Sec}_1(\varrho)$ denote the set of those sections ω of $\mathcal{W}_\varrho$ which are locally the differential of a $(1, \varrho)$ -Lipschitz function; i.e. $\omega \in \text{Sec}_1(\varrho)$ if and only if there are countably many disjoint Borel sets $\{V_\beta\}_\beta$ and countably many $(1, \varrho)$ -Lipschitz functions $\{f_\beta\}_\beta$ such that $\mu(X \setminus \bigcup_\beta V_\beta) = 0$ and $\omega \mid V_\beta = df_\beta \mid V_\beta$. To each $\omega \in \text{Sec}_1(\varrho)$ we associate a seminorm p_ω on TX by letting:

$$(8.22) \quad p_\omega(v) = |\langle \omega, v \rangle|.$$

We observe that $p_\omega \leq C_\varrho \|\cdot\|_{TX}$ and denote by $\|\cdot\|_{\varrho, \text{LIP}}$ the essential supremum (Lemma 2.33) of the collection $\{\|\cdot\|_\omega\}_{\omega \in \text{Sec}_1(\varrho)}$.

Another way of obtaining seminorms on TX is to use arbitrary sections of $\mathcal{W}_\varrho$ and rescale them by the local ϱ -Lipschitz constant; note, however, that if u, v are both ϱ -Lipschitz, one can have $du = dv$ and $\varrho\text{-Lip}u \neq \varrho\text{-Lip}v$ on a set of positive measure. We are thus led to use a slightly more complicated framework. Let $\text{Sec}_*(\varrho)$ denote the set of countable pairs $\tilde{\omega} = \{(V_\beta, f_\beta)\}$ where the V_β are disjoint Borel sets with $\mu(X \setminus \bigcup_\beta V_\beta) = 0$, and the f_β are ϱ -Lipschitz functions. To each $\tilde{\omega} \in \text{Sec}_*(\varrho)$ we associate a seminorm $p_{\tilde{\omega}}$ on TX by letting, for $x \in V_\beta$ and $v \in T_x X$:

$$(8.23) \quad p_{\tilde{\omega}}(v) = \begin{cases} 0 & \text{if } \varrho\text{-Lip}f_\beta(x) = 0 \\ \frac{|\langle df_\beta(x), v \rangle|}{\varrho\text{-Lip}f_\beta(x)} & \text{otherwise;} \end{cases}$$

we denote by $\|\cdot\|_{\varrho, \text{Lip}}$ the essential supremum (Lemma 2.33) of the collection $\{\|\cdot\|_\omega\}_{\omega \in \text{Sec}_*(\varrho)}$.

Theorem 8.24. *Let $D_X \subset X$ be a countable dense set. Then one has:*

$$(8.25) \quad \|\cdot\|_{D_X, \varrho} = \|\cdot\|_{\varrho, \text{LIP}} = \|\cdot\|_{\varrho, \text{Lip}};$$

in particular, if $D'_X \subset X$ is another countable dense set:

$$(8.26) \quad \|\cdot\|_{D_X, \varrho} = \|\cdot\|_{\tilde{D}_X, \varrho};$$

*in the sequel, we will denote the **canonical norm** (8.25) by $\|\cdot\|_\varrho$.*

Proof. Each pseudodistance function ϱ_x gives rise to an element of $\text{Sec}_1(\varrho)$ and so $\|\cdot\|_{D_X, \varrho} \leq \|\cdot\|_{\varrho, \text{Lip}}$; to each $\omega \in \text{Sec}_1(\varrho)$ one can associate $\tilde{\omega} \in \text{Sec}_*(\varrho)$ with $p_\omega \leq p_{\tilde{\omega}}$ and so $\|\cdot\|_{\varrho, \text{LIP}} \leq \|\cdot\|_{\varrho, \text{Lip}}$. We thus just prove that:

$$(8.27) \quad \|\cdot\|_{\varrho, \text{Lip}} \leq \|\cdot\|_{D_X, \varrho}.$$

It suffices to show that for any $\tilde{\omega} = \{(V_\beta, f_\beta)\} \in \text{Sec}_*(\varrho)$ one has

$$(8.28) \quad p_{\tilde{\omega}} \leq \|\cdot\|_{D_X, \varrho}.$$

Let $\mathcal{F}$ contain the components of the chart functions and the functions $\{f_\beta\}_\beta$; let $\mathcal{C}$ contain the characteristic functions of the charts and the characteristic functions $\{\chi_{V_\beta}\}_\beta$; let $\mathcal{S}$ contain d_X and ϱ . Let V be an $\{f_\beta\}_\beta$ -differentiability set and fix β ; let $V'_\beta = V \cap V_\beta$; by Theorem 5.3 there is a full μ -measure μ -measurable subset $W_\beta \subset V'_\beta$ such that, for each $x \in W_\beta$ the set of $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ -generic velocity vectors contains a dense set of directions. In particular, for each $v \in T_x X$ and $\varepsilon > 0$ we can find an $(\mathcal{F}, \mathcal{C}, \mathcal{S}, D_X)$ -generic velocity vector $\gamma'(t) \in T_x X$ with $\|v - \gamma'(t)\|_{TX} \leq \varepsilon$. Assume that $\varrho\text{-Lip} f_\beta(x) > 0$; note that the derivative $(f_\beta \circ \gamma)'(t)$ exists and is approximately continuous at t . Without loss of generality assume that $(f_\beta \circ \gamma)'(t) \neq 0$; then we can find $s_n \searrow 0$ such that $t + s_n \in \text{Dom } \gamma$ and $\varrho(\gamma(t + s_n), \gamma(t)) = r_n > 0$. We now obtain the estimate:

$$(8.29) \quad \begin{aligned} |(f_\beta \circ \gamma)(t + s_n) - (f_\beta \circ \gamma)(t)| &\leq \\ &\leq \sup \left\{ \frac{|f_\beta(y) - f_\beta(x)|}{r_n} : \varrho(y, x) \leq r_n \right\} \\ &\quad \times \varrho(\gamma(t + s_n), \gamma(t)) \\ &\leq (\varrho\text{-Lip} f_\beta(x) + O(1/n)) \left(\int_{[t, t+s_n] \cap \text{Dom } \gamma} \varrho\text{-md} \gamma(\tau) d\tau \right. \\ &\quad \left. + o(s_n) \right); \end{aligned}$$

dividing by s_n and letting $n \nearrow \infty$ we get, by approximate continuity of $\varrho\text{-md} \gamma$ at t :

$$(8.30) \quad |(f_\beta \circ \gamma)'(t)| \leq \varrho\text{-Lip} f_\beta(x) \varrho\text{-md} \gamma(t).$$

Now Theorem 4.3 implies that $\varrho\text{-md} \gamma(t) = \|\gamma'(t)\|_{D_X, \varrho}$ and so:

$$(8.31) \quad \begin{aligned} |\langle df_\beta, \gamma'(t) \rangle| &\leq \varrho\text{-Lip} f_\beta(x) \|\gamma'(t)\|_{D_X, \varrho} \\ &\leq \varrho\text{-Lip} f_\beta(x) \|v\|_{D_X, \varrho} + \varepsilon C_\varrho \varrho\text{-Lip} f_\beta(x); \end{aligned}$$

let L denote the global Lipschitz constant of f_β ; then:

$$(8.32) \quad |\langle df_\beta, v \rangle| \leq \varrho\text{-Lip} f_\beta(x) \|v\|_{D_X, \varrho} + \varepsilon C_\varrho \varrho\text{-Lip} f_\beta(x) + \varepsilon L \|v\|_{TX};$$

so (8.28) follows by letting $\varepsilon \searrow 0$. $\square$

8.2. Metric Differentiation for Lipschitz maps. We now reformulate the results of the previous subsection for a Lipschitz map $F : X \rightarrow Z$; throughout this subsection ϱ will denote the pull-back pseudometric F^*d_Z . Putting together Theorems 8.6 and 8.24 we obtain:

Theorem 8.33. *Associated to the map F there is a canonical subbundle $\mathcal{W}_F$ of T^*X such that:*

- (1) For each $g \in F^*(\text{Lip}(Z))$ (i.e. $g = h \circ F$ for some $h \in \text{Lip}(Z)$) the section dg lies in $\mathcal{W}_F$;
- (2) For each countable dense set $D_X \subset X$ the subbundle $\mathcal{W}_F$ coincides with the subbundle spanned by the sections $\{d\varrho_x : x \in D_X\}$.

Suppose now that $\mathcal{F}$ contains the components of the chart functions of (X, μ) , that $\mathcal{C}$ contains the characteristic functions of the charts, and suppose also that $\mathcal{S}$ contains the pseudometric ϱ . The subbundle $\mathcal{W}_F$ induces a canonical seminorm $\|\cdot\|_F = \|\cdot\|_\varrho$ on TX such that, for each $(\mathcal{F}, \mathcal{C}, \mathcal{S})$ -generic velocity vector $\gamma'(t)$ one has:

$$(8.34) \quad \|\gamma'(t)\|_F = \lim_{s \rightarrow 0} \frac{d_Z(F \circ \gamma(t+s), F \circ \gamma(t))}{|s|}.$$

Remark 8.35. In practice, it does not matter whether metric differentiation is formulated in terms of pseudometrics or Lipschitz maps. In fact, consider a Lipschitz compatible pseudometric ϱ on X and associate to it the Lipschitz map:

$$(8.36) \quad \begin{aligned} F : X &\rightarrow l^\infty(D_X) \\ y &\mapsto \{\varrho_x(y)\}_{x \in D_X}; \end{aligned}$$

then we get $\|\cdot\|_\varrho = \|\cdot\|_F$ and $\mathcal{W}_\varrho = \mathcal{W}_F$.

We now specialize the discussion to the case in which (Z, ν) is a differentiability space; throughout the remainder of this subsection we will fix choices of countable dense sets $D_X \subset X$ and $D_Z \subset Z$. The case of interest is when the measure $F_\# \mu$ is absolutely continuous with respect to ν . Using the Radon-Nikodym Theorem we can find a Borel subset $V_0 \subset Z$ such that $F_\# \mu \ll V_0$ and $\nu \ll V_0$ are in the same measure class. The case of interest is when $\nu(V_0) > 0$, which we will assume throughout the remainder of this subsection.

Let $U_0 = F^{-1}(V_0)$ and suppose that $g \in \text{Lip}(Z)$ is differentiable at z_0 with respect to the Lipschitz functions $\{\psi^i\}_{i=1}^M$; suppose now that $z_0 = F(x_0)$ and that the functions $\{\psi^i \circ F\}_{i=1}^M$ are differentiable at x_0 with respect to the functions $\{\phi^j\}_{j=1}^N$. We then obtain the chain rule:

$$(8.37) \quad g \circ F(x) - g \circ F(x_0) = \sum_{i=1}^M \sum_{j=1}^N \frac{\partial g}{\partial \psi^i}(z_0) \frac{\partial(\psi^i \circ F)}{\partial \phi^j}(x_0) (\phi^j(x) - \phi^j(x_0)) + o(d_X(x, x_0)).$$

The following Corollary is a consequence of the chain rule (8.37):

Corollary 8.38. *Let $\{(U_\alpha, \phi_\alpha)\}_\alpha$ be an atlas for (X, μ) and $\{(V_\beta, \psi_\beta)\}_\beta$ an atlas for (Z, ν) . Then the subbundle $\mathcal{W}_F|_{U_0}$ is spanned by the sections $\{d(\psi_\beta^i \circ F)\}_{\beta, i}$.*

Definition 8.39. As the measures $F_\# \mu \ll V_0$ and $\nu \ll V_0$ are in the same measure class, we obtain a pull-back map:

$$(8.40) \quad F^* : T^*Z|_{V_0} \rightarrow T^*X|_{U_0},$$

which maps each section dg of $T^*Z|_{V_0}$ to the section $F^*dg = d(g \circ F)$ of $T^*X|_{U_0}$. We define the push-forward map:

$$(8.41) \quad F_* : TX|_{U_0} \rightarrow TZ|_{V_0}$$

by duality; that is, for $x \in U_0$, $v \in T_x X$ and $g \in \text{Lip}(Z)$ we let:

$$(8.42) \quad \langle F_*(v), dg|_{F(x)} \rangle = \langle v, (F^* dg)_x \rangle.$$

We conclude this subsection by proving:

Theorem 8.43. *Let $\mathcal{W}_F^\perp$ denote the annihilator of $\mathcal{W}_F$: i.e. the fibre $\mathcal{W}_F^\perp(x)$ consists of those vectors in $T_x X$ which are annihilated by the functionals in $\mathcal{W}_F(x)$. The seminorm $\|\cdot\|_F$ induces a norm on the quotient bundle $TX/\mathcal{W}_F^\perp$ which we will still denote by $\|\cdot\|_F$. Then F_* induces an injective isometry:*

$$(8.44) \quad F_* : (TX/\mathcal{W}_F^\perp \mid U_0, \|\cdot\|_F) \rightarrow (TZ \mid V_0, \|\cdot\|_{TZ}).$$

The proof of Theorem 8.43 uses the following generalization of Theorem 5.3, whose proof is omitted.

Lemma 8.45. *Suppose that $\mathcal{F}$ contains the components of the $\{\phi_\alpha\}_\alpha$ and of the $\{\psi_\beta \circ F\}_\beta$; suppose that $\mathcal{C}$ contains the $\{\chi_{U_\alpha}\}_\alpha$, the $\{\chi_{V_\beta}\}_\beta$ and χ_{U_0} ; suppose that $\mathcal{S}$ contains ϱ . Suppose also that $\mathcal{F}'$ contains the components of the $\{\psi_\beta\}_\beta$ and that $\mathcal{C}'$ contains the $\{\chi_{V_\beta}\}$ and χ_{V_0} . Let*

$$(8.46) \quad G_x(\mathcal{F}, \mathcal{C}, \mathcal{S}; \mathcal{F}', \mathcal{S}') = \left\{ v \in T_x X : v = \gamma'(t), \text{ where } \gamma'(t) \text{ is } (\mathcal{F}, \mathcal{C}, \mathcal{S})\text{-generic} \right. \\ \left. \text{and } (F \circ \gamma)'(t) \text{ is } (\mathcal{F}', \mathcal{C}')\text{-generic} \right\};$$

then there is a full μ -measure μ -measurable subset $U_1 \subset U_0$ such that, for each $x \in U_1$, $G_x(\mathcal{F}, \mathcal{C}, \mathcal{S}; \mathcal{F}', \mathcal{S}')$ contains a dense set of directions.

Proof of Theorem 8.43. We apply Lemma 8.45 and show that for each $x \in U_1$ and each $v \in T_x X$ one has:

$$(8.47) \quad \|v\|_F = \|F_*(v)\|_{TZ};$$

by density of directions, we just need to show (8.47) for $v = \gamma'(t)$ where $\gamma'(t)$ is $(\mathcal{F}, \mathcal{C}, \mathcal{S})$ -generic and $(F \circ \gamma)'(t)$ is $(\mathcal{F}', \mathcal{C}')$ -generic. By Theorem 4.3 applied in X to the pseudometric ϱ we get:

$$(8.48) \quad \|\gamma'(t)\|_F = \varrho\text{-md}\gamma(t);$$

note that by the definition of the ϱ -metric differential we have:

$$(8.49) \quad \varrho\text{-md}\gamma(t) = \text{md } F \circ \gamma(t);$$

finally, applying again Theorem 4.3 in Z to the metric d_Z , we get:

$$(8.50) \quad \|F_*\gamma'(t)\|_{TZ} = \|(F \circ \gamma)'(t)\|_{TZ} = \text{md } F \circ \gamma(t).$$

□

8.3. Metric differentiation and blow-ups. In this subsection we generalize the results of Section 7 in the case in which one considers either a Lipschitz compatible pseudometric ϱ on X or a Lipschitz map $F : X \rightarrow Z$.

Definition 8.51. Let ϱ be a Lipschitz compatible pseudometric on X and (U, ψ) be an N -dimensional differentiability chart. A **blow-up of (X, μ, ψ, ϱ) at p** along the scales $r_n \searrow 0$ is a tuple $(Y, \nu, \varphi, \tilde{\varrho}, q)$ such that:

- (1) The tuple (Y, ν, φ, q) is a blow-up of (X, μ, ψ) at p , i.e. the tuples:

$$(8.52) \quad \left(\frac{1}{r_n} X, \frac{\mu}{\mu(B(p, r_n))}, \frac{\psi - \psi(p)}{r_n}, p \right)$$

converge to (Y, ν, φ, q) in the measured Gromov-Hausdorff sense;

- (2) $\tilde{\varrho}$ is a Lipschitz compatible pseudometric on Y and if the points $y, y' \in Y$ are represented, respectively, by the sequences $[x_n], [x'_n] \subset X$, then:

$$(8.53) \quad \tilde{\varrho}(y, y') = \lim_{n \rightarrow \infty} \frac{\varrho(x_n, x'_n)}{r_n}.$$

We denote by $\text{Bw-up}(X, \mu, \psi, \varrho, p)$ the set of blow-ups of (X, μ, ψ, ϱ) at p .

Theorem 8.54. Let (U, ψ) be an N -dimensional differentiability chart for the differentiability space (X, μ) , and let ϱ be a Lipschitz compatible pseudometric. Then for $\mu \llcorner U$ -a.e. p , for each blow-up $(Y, \nu, \varphi, \tilde{\varrho}, q) \in \text{Bw-up}(X, \mu, \psi, \varrho, p)$, and for each unit vector $v_0 \in T_p X$, the measure ν admits an Alberti representation $\mathcal{A} = (Q, \Phi)$ where:

- (1) Q is concentrated on the set $\text{Lines}(\varphi, v_0, \tilde{\varrho})$ of unit speed geodesic lines in Y with:

$$(8.55) \quad \begin{aligned} (\varphi \circ \gamma)' &= v_0; \\ \tilde{\varrho}(\gamma(t), \gamma(s)) &= \|v_0\|_{\varrho} |t - s|; \end{aligned}$$

- (2) For each $\gamma \in \text{Lines}(\varphi, v_0, \tilde{\varrho})$ the measure Φ_γ is given by:

$$(8.56) \quad \Phi_\gamma = \mathcal{H}_\gamma^1.$$

Proof. The proof follows the method used to prove Theorem 7.11; we just:

- (1) add in condition **(Reg3)** that:

$$(8.57) \quad \left| \varrho(\gamma(s_1), \gamma(s_2)) - \|\gamma'(t)\|_{\varrho} |s_1 - s_2| \right| \leq \varepsilon |s_1 - s_2|;$$

- (2) require in Lemma 7.41 that U consists of points at which the map $x \mapsto \|\cdot\|_{\varrho}(x)$ is approximately continuous.

□

We now discuss what happens in the case of a Lipschitz map $F : X \rightarrow Z$. When we defined blow-ups of the chart functions there was no issue with the target space because $\mathbb{R}^N$ possesses a group of dilations. For a general map $F : X \rightarrow Z$ we first need to use ultrametrics [KL97, Sec. 2.4] to blow-up Z ; we recall here the relevant constructions.

Definition 8.58. Let (Z, z_0) denote a pointed metric space and let $r_n \searrow 0$; we define a **blow-up** (W, w_0) **of** (Z, z_0) along the scales $r_n \searrow 0$ as an ultralimit of the sequence of pointed metric spaces $(\frac{1}{r_n}Z, z_0)$. Specifically, we choose a nonprincipal ultrafilter ω and consider the set $\tilde{W}$ of those sequences $[z_n] \subset Z$ such that:

$$(8.59) \quad \limsup_{n \rightarrow \infty} \frac{d_Z(z_n, z_0)}{r_n} < \infty.$$

We define a pseudometric $d_{\tilde{W}}$ on $\tilde{W}$ by:

$$(8.60) \quad d_{\tilde{W}}([z_n], [z'_n]) = \lim_{\omega} \frac{d_Z(z_n, z'_n)}{r_n}.$$

On $\tilde{W}$ we consider the equivalence relation:

$$(8.61) \quad [z_n] \sim [z'_n] \iff d_{\tilde{W}}([z_n], [z'_n]) = 0;$$

then $d_{\tilde{W}}$ induces a metric d_W on the quotient space $W = \tilde{W} / \sim$, and the base point w_0 is the equivalence class of the constant sequence $[z_0]$. We denote the set of blow-ups of Z at z_0 by $\text{Bw-up}(Z, z_0)$.

Consider now the case of a Lipschitz map $F : X \rightarrow Z$; having fixed scales $r_n \searrow 0$, we construct blow-ups $(Y, q) \in \text{Bw-up}(X, p)$ and $(W, w_0) \in \text{Bw-up}(Z, F(p))$. We then obtain a Lipschitz map $G : (Y, q) \rightarrow (W, w_0)$ by blowing up the graph of F at $(p, F(p))$. Specifically, if $[x_n] \subset X$ represents the point $y \in Y$, we let $G(y)$ be the equivalence class of the sequence $[F(x_n)]$. In general, we say **that a tuple** $(Y, \nu, \varphi, q; G, W, w_0)$ **is a blow-up of** $(X, \mu, \psi; F, Z)$ **at** p if: $(Y, \nu, \varphi, q) \in \text{Bw-up}(X, \mu, \psi, p)$, $(W, w_0) \in \text{Bw-up}(Z, F(p))$, and G is obtained by blowing up $F : X \rightarrow Z$ at p . We denote the set of blow-ups of $(X, \mu, \psi; F, Z)$ at p by $\text{Bw-up}(X, \mu, \psi, p; F, Z)$.

Applying Theorem 8.54 to the pseudometric F^*d_Z we get:

Theorem 8.62. *Let (U, ψ) be an N -dimensional differentiability chart for the differentiability space (X, μ) , and let $F : X \rightarrow Z$ be a Lipschitz map. Then for $\mu \llcorner U$ -a.e. p , for each blow-up $(Y, \nu, \varphi, q; G, W, w_0) \in \text{Bw-up}(X, \mu, \psi, p; F, Z)$, and for each unit vector $v_0 \in T_p X$, the measure ν admits an Alberti representation $\mathcal{A} = (Q, \Phi)$ where:*

- (1) Q is concentrated on the set $\text{Lines}(\varphi, v_0, G)$ of unit speed geodesic lines in Y with:

$$(8.63) \quad \begin{aligned} (\varphi \circ \gamma)' &= v_0; \\ d_W(G \circ \gamma(t), G \circ \gamma(s)) &= \|v_0\|_F |t - s|; \end{aligned}$$

- (2) For each $\gamma \in \text{Lines}(\varphi, v_0, G)$ the measure Φ_γ is given by:

$$(8.64) \quad \Phi_\gamma = \mathcal{H}_\gamma^1.$$

Remark 8.65. In [Che99, Sec. 10] it was shown that if (X, μ) is a PI-space and if f is a real-valued Lipschitz map defined on X , at μ -a.e. p , blowing-up f at p always produces a *generalized linear function* g ; in particular, the corresponding space Y contains through each point a geodesic line γ on which the blow-up F is affine, and such that γ behaves as an integral curve of the gradient of F . Applying Theorem 8.62 to the case in which $F = f$, one gets, through each point of Y , many geodesic lines on which the blow-up G is affine, and these geodesic lines can be used to

obtain a Fubini-like decomposition of the measure ν . Among these geodesic lines, those where the slope of G is maximal correspond to the vector v_0 which is the derivative of f at p with respect to the coordinate functions ψ .

9. EXAMPLES

In this Section we provide some examples of how metric differentiation can be used to constrain the infinitesimal geometry of a Lipschitz map $F : X \rightarrow Y$, where X is a differentiability space. We will use a class of examples of differentiability spaces introduced in [CK13].

Definition 9.1 (Admissible inverse systems). We consider an inverse system of metric measure graphs:

$$(9.2) \quad \cdots \xleftarrow{\pi_{i-1}} X_i \xleftarrow{\pi_i} X_{i+1} \xleftarrow{\pi_{i+1}} \cdots,$$

where the index i can range either over $\mathbb{Z}$ or over $\mathbb{N} \cup \{0\}$: in the former case we will say that the inverse system is **signed**, and in the latter case that it is **unsigned**. Having fixed an integer $m \geq 2$ and parameters $\Delta, C, \theta \in (0, \infty)$, we say that the inverse system $\{X_i, \pi_i\}$ is **admissible** if it satisfies the following axioms:

(Ad1): Each metric space (X_i, d_i) is a nonempty connected graph with vertices of valence $\leq \Delta$ and such that each edge of X_i is isometric to an interval of length m^{-i} with respect to the path metric d_i ;

(Ad2): Let X'_i denote the graph obtained by subdividing each edge of X_i into m edges of length $m^{-(i+1)}$. Then π_i induces a map $\pi_i : (X_{i+1}, d_{i+1}) \rightarrow (X'_i, d_i)$ which is open, simplicial and an isometry on every edge;

(Ad3): For each $x_i \in X'_i$ the inverse image $\pi_i^{-1}(x_i) \subset X_{i+1}$ has d_{i+1} -diameter at most $\theta m^{-(i+1)}$;

(Ad4): Each graph X_i is equipped with a measure μ_i which restricts to a multiple of arclength on each edge; if e_1, e_2 are two adjacent edges of X_i we have:

$$(9.3) \quad \frac{\mu_i(e_1)}{\mu_i(e_2)} \in [C^{-1}, C];$$

(Ad5): The measures $\{\mu_i\}$ are compatible with the projections $\{\pi_i\}$: $\pi_{i\#}\mu_{i+1} = \mu_i$;

(Ad6): Let $\text{St}(x, G)$ denote the star of a vertex x in a graph G , i.e. the union of all the edges containing x . Then, for each vertex $v'_i \in X'_i$ and each $v_{i+1} \in \pi_i^{-1}(v'_i)$, the quantity:

$$(9.4) \quad \frac{\mu_{i+1}(\pi_i^{-1}(e'_i) \cap \text{St}(v_{i+1}, X_{i+1}))}{\mu_i(e'_i)}$$

is the same for all edges $e'_i \in \text{St}(v'_i, X'_i)$;

(Ad7): If the inverse system $\{X_i, \pi_i\}$ is unsigned we will assume that $X_0 \simeq [0, 1]$, $\mu_0 = \mathcal{L}^1 \llcorner [0, 1]$ and we will denote by φ_i the map:

$$(9.5) \quad \varphi_i = \pi_1 \circ \cdots \circ \pi_{i-1}.$$

If the inverse system $\{X_i, \pi_i\}$ is signed we require the existence of open surjective maps $\varphi_i : X_i \rightarrow \mathbb{R}$ which are, regarding $\mathbb{R}$ as a graph of edges $\{[km^{-i}, (k+1)m^{-i}]\}_{k \in \mathbb{Z}}$, simplicial and restrict to isometries on every edge. Moreover, we require that the $\{\varphi_i\}$ are compatible with the $\{\pi_i\}$:

$$(9.6) \quad \varphi_i \circ \pi_i = \varphi_{i+1} \quad (\forall i).$$

An immediate consequence of the axioms **(Ad1)**–**(Ad7)** is that the metric measure spaces (X_i, d_i, μ_i) converge in the measured Gromov-Hausdorff sense³ to a metric measure space $(X_\infty, d_\infty, \mu_\infty)$ which is called **the inverse limit** of the admissible inverse system. If $\{X_i, \pi_i\}$ is unsigned, then (X_∞, d_∞) is compact geodesic and μ_∞ is a doubling probability measure; if $\{X_i, \pi_i\}$ is signed, (X_∞, d_∞) is proper geodesic and μ_∞ is a doubling measure. In both cases there are 1-Lipschitz maps $\pi_{\infty, k} : X_\infty \rightarrow X_k$ satisfying:

$$(9.7) \quad \begin{aligned} \pi_{k-1} \circ \pi_{\infty, k} &= \pi_{\infty, k-1} \\ \pi_{\infty, k} \# \mu_\infty &= \mu_k. \end{aligned}$$

For $j > k$ we will use the short-hand notation $\pi_{j, k}$ to denote the map $\pi_k \circ \dots \circ \pi_{j-1}$. Moreover, the maps $\varphi_i : X_i \rightarrow \mathbb{R}$ or $[0, 1]$ pass to the limit giving a 1-Lipschitz map $\varphi_\infty : X_\infty \rightarrow \mathbb{R}$ or $[0, 1]$ satisfying:

$$(9.8) \quad \varphi_\infty(q) = \varphi_i(\pi_{\infty, i}(q)) \quad (\forall q \in X_\infty, \forall i \in \mathbb{Z} \text{ or } \mathbb{N} \cup \{0\})$$

We now define a special class of paths in X_i or X_∞ .

Definition 9.9. Let $I \subseteq \mathbb{R}$ be connected and $\gamma : I \rightarrow X_i$ continuous, where we allow $i = \infty$. We say that γ is a **monotone geodesic** if $\varphi_i \circ \gamma : I \rightarrow \mathbb{R}$ or $[0, 1]$ is either a strictly increasing or decreasing affine map. In particular, the axioms **(Ad1)**–**(Ad7)** imply that γ is a constant speed geodesic in (X_i, d_i) . Moreover, by axioms **(Ad2)** and **(Ad7)**, if $j > i$ and if $\gamma_i : I \rightarrow X_i$ is a monotone geodesic, then for each $q_j \in \pi_{j, i}^{-1}(\gamma_i(I))$, one can lift γ_i to obtain a monotone geodesic $\gamma_j : I \rightarrow X_j$ passing through q_j and satisfying $\pi_{j, i} \circ \gamma_j = \gamma_i$.

We now summarize some important consequences of the axioms **(Ad1)**–**(Ad7)**.

Theorem 9.10. *Let $\{X_i, \pi_i\}$ be an admissible inverse system and let X_∞ denote the inverse limit; then:*

- (1) *The metric measure space $(X_\infty, d_\infty, \mu_\infty)$ admits a $(1, 1)$ -Poincaré inequality; in particular, it is a differentiability space with a single differentiability chart $(X_\infty, \varphi_\infty)$;*
- (2) *If $\{X_i, \pi_i\}$ is unsigned, then μ_∞ admits an Alberti representation $(P, \mathcal{H}_\gamma^1)$, where P is a probability measure supported in the set of monotone geodesics $\gamma : [0, 1] \rightarrow X_\infty$ which satisfy:*

$$(9.11) \quad \varphi_\infty \circ \gamma(t) = t \quad (\forall t \in [0, 1]).$$

If $\{X_i, \pi_i\}$ is signed, then μ_∞ admits an Alberti representation $(Q, \mathcal{H}_\gamma^1)$, where Q is a Radon measure supported in the set of monotone geodesic (lines) $\gamma : \mathbb{R} \rightarrow X_\infty$ which satisfy:

$$(9.12) \quad \varphi_\infty \circ \gamma(t) = t \quad (\forall t \in \mathbb{R}).$$

³If $\{X_i, \pi_i\}$ is signed we consider the convergence in the pointed sense by choosing basepoints $\{q_i\}_{i \in \mathbb{Z}}$ satisfying $\pi_i(q_{i+1}) = q_i$.

- (3) For each i , there is an isometric embedding $\iota_i : X_i \rightarrow X_\infty$ such that $\pi_{\infty,i} \circ \iota_i = \text{id}_{X_i}$, and $\iota_i(X_i)$ is $C(\theta, \Delta, C) m^{-i}$ -dense in X_∞ . Moreover, having fixed a basepoint $q \in Y_\infty$, we can choose the $\{\iota_i\}$ so that $\iota_{i-1}(X_{i-1}) \subset \iota_i(X_i)$ and $q \in \iota_i(Y_i)$.

The proof of Theorem 9.10 is contained in [CK13]; note, however, that in [CK13] only the case of what we call unsigned inverse systems is discussed: the modifications for the case of signed inverse systems are straightforward. Alberti representations are not explicitly mentioned in [CK13], but part (2) in Theorem 9.10 follows from the discussion in [CK13, Sec. 6].

We now describe the blow-ups of a inverse limit space: these can be described using signed inverse systems. Recall that if X is a metric space, for $\sigma > 0$ the notation σX denotes the metric space $(X, \sigma d_X)$.

Theorem 9.13. *Let X_∞ be the inverse limit of an admissible inverse system $\{X_i, \pi_i\}$ and let $\psi = \varphi_\infty$; if the system is unsigned assume also that $p \notin \psi^{-1}(\{0, 1\})$. Then each element of $\text{Bw-up}(X_\infty, \mu_\infty, \psi, p)$ is of the form $(\sigma Y_\infty, c \cdot \nu_\infty, \sigma \cdot \varphi, q)$ where:*

- (1) *The metric measure space $(Y_\infty, d_\infty, \nu_\infty)$ is the inverse limit of a signed admissible inverse system $\{Y_i, \pi_i\}$, and φ is the function φ_∞ corresponding to Y_∞ ;*
- (2) *The parameters σ and c satisfy:*

$$(9.14) \quad \begin{aligned} \sigma &\in [1, m] \\ c &= \frac{1}{\nu_\infty(B_{Y_\infty}(q, 1/\sigma))}; \end{aligned}$$

- (3) *The basepoint q satisfies $\varphi(q) = 0$.*

For $p \in X_\infty$ let $\partial_\psi|_p \in T_p X_\infty$ denote the tangent vector to a unit-speed monotone increasing geodesic segment passing through p . Note that $\partial_\psi|_p$ provides a basis of $T_p X_\infty$.

Theorem 9.15. *Let X_∞ be the inverse limit of an admissible inverse system $\{X_i, \pi_i\}$. Let $F : X_\infty \rightarrow Z$ be Lipschitz. Then there is a full μ_∞ -measure subset $S_F \subset X_\infty$ such that, for each $p \in S_F$ and each $(\sigma Y_\infty, c \cdot \nu_\infty, \sigma \cdot \varphi, q; G, W, w_0) \in \text{Bw-up}(X_\infty, \mu_\infty, \psi, p; F, Z)$, one has that G maps each unit-speed monotone geodesic line $\gamma : \mathbb{R} \rightarrow Y_\infty$ to a (possibly degenerate⁴) geodesic line in W with constant speed $\sigma^{-1} \|\partial_\psi|_p\|_F$.*

Proof. We apply Theorem 8.62 and consider a blow-up $(\sigma Y_\infty, c \cdot \nu_\infty, \sigma \cdot \varphi, q; G, W, w_0)$ at a point p where the conclusion of that Theorem holds. By reparametrizing geodesic lines we obtain an Alberti representation (Q, Φ) of $c \cdot \nu_\infty$ with Q concentrated on the set Ω of monotone increasing geodesic lines in Y_∞ which are mapped by G to geodesic lines of constant speed $\sigma^{-1} \|\partial_\psi|_p\|_F$. Let $\gamma : \mathbb{R} \rightarrow Y_\infty$ be a unit-speed monotone increasing geodesic line, and consider the monotone geodesic line:

$$(9.16) \quad \gamma_j = \pi_{\infty,j} \circ \gamma : \mathbb{R} \rightarrow Y_j.$$

⁴This happens iff $\|\partial_\psi|_p\|_F = 0$, i.e. when $G \circ \gamma$ is constant.

The map $\pi_{\infty,j}$ can be used to push-forward monotone geodesic lines from Y_∞ to Y_j ; thus, $(\pi_{\infty,j\#}Q, \Phi)$ gives an Alberti representation of $c \cdot \nu_j$. As Y_j is a graph and as ν_j restricts to a multiple of arclength on each edge, there must be a $\tilde{\gamma}_j \in \Omega$ which, up to composition with a translation by some $a_j \in \mathbb{R}$, lifts γ_j . By Axiom **(Ad3)** we conclude that:

$$(9.17) \quad d_{Y_\infty}(\gamma(t), \tilde{\gamma}_j(t+a)) \leq C(\theta, m) m^{-j} \quad (\forall t \in \mathbb{R});$$

letting $j \nearrow \infty$ we conclude that $G \circ \gamma$ is a geodesic line in W with constant speed $\sigma^{-1} \|\partial_\psi \lfloor_p\|_F$. $\square$

We now study Lipschitz maps $F : X_\infty \rightarrow Z$ where Z is a $\text{CBB}(\kappa)$ -space, i.e. a metric space with curvature bounded from below by κ . For a review of the properties of $\text{CBB}(\kappa)$ -spaces used here we refer the reader to [Pau01, Sec. 6.2]. The properties of $\text{CBB}(\kappa)$ -spaces that we will use are:

- (1) Blow-ups of $\text{CBB}(\kappa)$ -spaces are $\text{CBB}(0)$ -spaces;
- (2) In a $\text{CBB}(0)$ -space geodesics do not branch; i.e. suppose that $\gamma_i : [0, 1] \rightarrow \text{CBB}(0)$ ($i = 1, 2$) is a geodesic and that, for some $\varepsilon > 0$, one has $\gamma_1|_{[0, \varepsilon]} = \gamma_2|_{[0, \varepsilon]}$; then one of the sets $\gamma_i([0, 1])$ ($i = 1, 2$) contains the other.

Theorem 9.18. *Let $F : X_\infty \rightarrow Z$ be Lipschitz where Z is a $\text{CBB}(\kappa)$ -space. Let:*

$$(9.19) \quad \begin{aligned} S_{F,0} &= \{p \in S_F : \|\partial_\psi \lfloor_p\|_F = 0\} \\ S_{F,>} &= \{p \in S_F : \|\partial_\psi \lfloor_p\|_F > 0\}; \end{aligned}$$

then:

- (1) For μ_∞ -a.e. $p \in S_{F,0}$, for each blow-up $(\sigma Y_\infty, c \cdot \nu_\infty, \sigma \cdot \varphi, q; G, W, w_0) \in \text{Bw-up}(X_\infty, \mu_\infty, \psi, p; F, Z)$ one has $G(Y_\infty) = \{w_0\}$;
- (2) For μ_∞ -a.e. $p \in S_{F,>0}$, for each blow-up $(\sigma Y_\infty, c \cdot \nu_\infty, \sigma \cdot \varphi, q; G, W, w_0) \in \text{Bw-up}(X_\infty, \mu_\infty, \psi, p; F, Z)$ there is a geodesic $\Gamma : \mathbb{R} \rightarrow W$ with constant speed $\sigma^{-1} \|\partial_\psi \lfloor_p\|_F$ and such that the following diagram commutes:

$$(9.20) \quad \begin{array}{ccc} Y_\infty & \xrightarrow{G} & W \\ \varphi \downarrow & \nearrow \Gamma & \\ \mathbb{R} & & \end{array}$$

Proof. Let $p \in S_{F,0}$: then G maps monotone geodesic lines of Y_∞ to points; since each $\iota_i(Y_i)$ is connected and Cm^{-i} -dense in Y_∞ , we conclude that all monotone geodesics are mapped to w_0 .

Let now $p \in S_{F,>}$ and consider the family of maps

$$(9.21) \quad G_k = G \circ \iota_k : Y_k \rightarrow Y_\infty;$$

we will first show that G_{k+1} collapses the fibres of $\pi_k : Y_{k+1} \rightarrow Y'_k$, i.e. that whenever $q_{1,k+1}, q_{2,k+1} \in \pi^{-1}(q_k)$, then $G_{k+1}(q_{i,k+1}) = G_k(q_k)$ for $i = 1, 2$. By **(Ad3)** we can take a unit-speed geodesic segment γ_{k+1} joining $q_{1,k+1}$ to $q_{2,k+1}$. We can divide $\text{Dom } \gamma_{k+1}$ into maximal subintervals $I_0 \cup \dots \cup I_N$ such that each $\gamma_{k+1}|_{I_l}$ is either a monotone increasing or a monotone decreasing geodesic. We

now consider, for $l < N$, adjacent subintervals I_l and I_{l+1} and assume, without loss of generality, that $\gamma_{k+1}|_{I_l}$ is increasing and $\gamma_{k+1}|_{I_{l+1}}$ is decreasing. Let t_l be the terminal point of I_l and let e be an edge of Y'_l containing $\pi_k(\gamma_{k+1}(t_l))$ and a point q'_k satisfying:

$$(9.22) \quad \varphi_k(q'_k) > \varphi_k(\pi_k(\gamma_{k+1}(t_l))).$$

Lift e to an edge $\tilde{e}$ of Y_{k+1} containing $\gamma_{k+1}(t_l)$; then we can construct monotone geodesics $\gamma_{l,k+1}$ and $\gamma_{l+1,k+1}$ such that the image of $\gamma_{i,k+1}$ contains $\gamma_{k+1}(I_l) \cup \tilde{e}$ for $i = l, l+1$. By Theorem 9.15 the curves $\{G_{k+1} \circ \gamma_{i,k+1}\}_{i=l,l+1}$ are geodesics in a CBB(0) space and so they cannot branch. In particular, if $s_l \in I_l$ and $s_{l+1} \in I_{l+1}$ are such that $\varphi_{k+1}(\gamma_{k+1}(s_l)) = \varphi_{k+1}(\gamma_{k+1}(s_{l+1}))$, then $G_{k+1}(\gamma_{k+1}(s_l)) = G_{k+1}(\gamma_{k+1}(s_{l+1}))$. Thus, by induction on l , we conclude that the value of $G_{k+1}(\gamma_{k+1}(t))$ is determined by the value of $\varphi_{k+1}(\gamma_{k+1}(t))$; therefore, $G_{k+1}(q_{1,k+1}) = G_{k+1}(q_{2,k+1})$. As $\iota_k(q_k)$ can be identified with a point $\iota_{k+1}(q_{0,k+1})$ with $\pi_k(q_{0,k+1}) = q_k$, we conclude that $G_{k+1}(q_{1,k+1}) = G_k(q_k)$.

Let $\gamma : \mathbb{R} \rightarrow Y_\infty$ be a monotone increasing unit speed geodesic of Y_∞ which satisfies:

$$(9.23) \quad \begin{aligned} \gamma(0) &= q \\ \varphi(\gamma(t)) &= t \quad (\forall t \in \mathbb{R}). \end{aligned}$$

We take $\Gamma = G \circ \gamma$ and observe that Theorem 9.15 implies that Γ is a geodesic in W with constant speed $\sigma^{-1} \|\partial_\psi|_p\|_F$. Let $\tilde{q} \in Y_\infty$ and fix $j > 0$; we can find $\tilde{q}_j \in Y_j$ such that:

$$(9.24) \quad \begin{aligned} d_{Y_\infty}(\tilde{q}, \iota_j(\tilde{q}_j)) &\leq C(\theta, \Delta, C, m)m^{-j} \\ \varphi(\tilde{q}) &= \varphi_j(\tilde{q}_j); \end{aligned}$$

we now consider the monotone geodesic line $\gamma_j = \pi_{\infty,j} \circ \gamma$ in Y_j , and observe that, by axioms **(Ad3)**, **(Ad7)** and by Theorem 9.10(3), we have:

$$(9.25) \quad \begin{aligned} d_{Y_\infty}(\gamma(t), \iota_j(\gamma_j(t))) &\leq C(\theta, \Delta, C, m)m^{-j} \quad (\forall t \in \mathbb{R}) \\ \varphi_j(\gamma_j(t)) &= t \quad (\forall t \in \mathbb{R}). \end{aligned}$$

Choose $k \in \mathbb{Z}$ such that m^{-k} is larger than the distance between $\tilde{q}_j$ and $\gamma_j(\mathbb{R})$; then there is a vertex $v \in Y_k$ such that $\pi_{j,k}(q_j)$ lies in $\text{St}(v)$ and $\pi_{j,k} \circ \gamma_j$ passes through v . By further decreasing k we can then arrange that $\pi_{j,k}(q_j) \in \pi_{j,k}(\gamma_j(\mathbb{R}))$; by compatibility with $\{\varphi_i\}$ we conclude that $\pi_{j,k}(\tilde{q}_j) = \pi_{j,k}(\gamma_j(\varphi(\tilde{q}_j)))$. Using inductively the argument on the collapse of the fibres we get:

$$(9.26) \quad G_j(\tilde{q}_j) = G_j(\gamma_j(\varphi(\tilde{q}_j))),$$

from which we get:

$$(9.27) \quad d_W(G(\tilde{q}), G(\gamma(\varphi(\tilde{q})))) \leq 2\mathbf{L}(F) \times Cm^{-j},$$

and the result follows letting $j \nearrow \infty$. $\square$

We now study Lipschitz maps $F : X_\infty \rightarrow Z$ where Z is a $\text{CBA}(\kappa)$ -space, i.e. a metric space with curvature bounded from above by κ . For a review of the properties of $\text{CBA}(\kappa)$ -spaces used here we refer the reader to [Pau01, Sec. 6.1]. The properties of $\text{CBA}(\kappa)$ -spaces that we will use are:

- (1) Blow-ups of $\text{CBA}(\kappa)$ -spaces are $\text{CAT}(0)$ -spaces, i.e. the condition about triangle comparison holds globally;

- (2) If Z is a $CAT(0)$ space and $\gamma_i : [0, \infty) \rightarrow Z$ ($i = 1, 2$) are geodesic rays with the sets $\{\gamma_i([0, \infty))\}_{i=1,2}$ at bounded distance from each other, and such that $\gamma_1(0) = \gamma_2(0)$, then the sets $\{\gamma_i([0, \infty))\}_{i=1,2}$ coincide.

Theorem 9.28. *Let $F : X_\infty \rightarrow Z$ be Lipschitz, where Z is a $CBA(\kappa)$ -space. Using the same notation as in Theorem 9.18:*

- (1) *For μ_∞ -a.e. $p \in S_{F,0}$, for each blow-up $(\sigma Y_\infty, c \cdot \nu_\infty, \sigma \cdot \varphi, q; G, W, w_0) \in \text{Bw-up}(X_\infty, \mu_\infty, \psi, p; F, Z)$ one has $G(Y_\infty) = \{w_0\}$;*
- (2) *For μ_∞ -a.e. $p \in S_{F,>0}$, for each blow-up $(\sigma Y_\infty, c \cdot \nu_\infty, \sigma \cdot \varphi, q; G, W, w_0) \in \text{Bw-up}(X_\infty, \mu_\infty, \psi, p; F, Z)$ there is a geodesic $\Gamma : \mathbb{R} \rightarrow W$ with constant speed $\sigma^{-1} \|\partial_\psi|_p\|_F$ and such that the following diagram commutes:*

$$(9.29) \quad \begin{array}{ccc} Y_\infty & \xrightarrow{G} & W \\ \varphi \downarrow & \nearrow \Gamma & \\ \mathbb{R} & & \end{array}$$

Proof. The proof follows the same line of reasoning as that of Theorem 9.18: the only difference is that a new argument is required to show that G_{k+1} collapses the fibres of $\pi_k : Y_{k+1} \rightarrow Y'_k$. We will first show the following: suppose that there are monotone geodesic segments $\gamma_{i,k+1} : [0, T] \rightarrow Y_{k+1}$ ($i = 1, 2$) satisfying:

- (1) $\gamma_{1,k+1}(0) = \gamma_{2,k+1}(0)$;
- (2) $\pi_k(\gamma_{1,k+1}(t)) = \pi_k(\gamma_{2,k+1}(t))$ for each $t \in [0, T]$;

then for each $t \in [0, T]$ one has $G_{k+1}(\gamma_{1,k+1}(t)) = G_{k+1}(\gamma_{2,k+1}(t))$. Without loss of generality, we can assume that the geodesics $\gamma_{i,k+1}$ are increasing; we take a monotone ray:

$$(9.30) \quad \gamma_k : [0, \infty) \rightarrow Y_k$$

which coincides with $\pi_k \circ \gamma_{i,k+1}$ when restricted to $[0, T]$. Using **(Ad2)**, we lift γ_k in Y_{k+1} to extend each $\gamma_{i,k+1}$ to a ray:

$$(9.31) \quad \gamma_{i,k+1} : [0, \infty) \rightarrow Y_{k+1}.$$

By **(Ad3)** we have:

$$(9.32) \quad d_{Y_{k+1}}(\gamma_{1,k+1}(t), \gamma_{2,k+1}(t)) \leq \theta \cdot m^{-(k+1)} \quad (\forall t \in \mathbb{R});$$

by Theorem 9.15 the maps $\{G_{k+1} \circ \gamma_{i,k+1}\}_{i=1,2}$ are geodesic rays in a $CAT(0)$ -space which issue from the same point, which are at bounded distance from each other, and which have the same speed; therefore, we conclude that:

$$(9.33) \quad G_{k+1}(\gamma_{1,k+1}(t)) = G_{k+1}(\gamma_{2,k+1}(t)) \quad (\forall t \in \mathbb{R}).$$

Now take a unit-speed geodesic segment γ_{k+1} joining $q_{1,k+1}$ to $q_{2,k+1}$. We can divide $\text{Dom } \gamma_{k+1}$ into maximal subintervals $I_0 \cup \dots \cup I_N$ such that each $\gamma_{k+1}|_{I_l}$ is either a monotone increasing or a monotone decreasing geodesic. Using the previous argument on pairs $(\gamma_{k+1}|_{I_l}, \gamma_{k+1}|_{I_{l+1}})$ for $l < N$, we conclude that if $s_1, s_2 \in \text{Dom } \gamma_{k+1}$ are such that:

$$(9.34) \quad \varphi_{k+1}(\gamma_{k+1}(s_1)) = \varphi_{k+1}(\gamma_{k+1}(s_2)),$$

then one has $G_{k+1}(\gamma_{k+1}(s_1)) = G_{k+1}(\gamma_{k+1}(s_2))$, from which the claim about the collapse of the fibres follows. $\square$

We now consider Lipschitz maps $F : X_\infty \rightarrow \mathbb{G}$, where $\mathbb{G}$ is a Carnot group equipped with the Carnot-Caratheodory distance corresponding to the choice of a norm $\|\cdot\|_{\mathfrak{h}}$ on the horizontal layer $\mathfrak{h}$ of its Lie algebra $\mathfrak{g}$. We refer to [Pau01] for more background on Carnot groups. For the moment, we recall that there is a canonical 1-Lipschitz projection $\pi_{\mathbb{G}} : \mathbb{G} \rightarrow \mathfrak{h}$, that the blow-up of $\mathbb{G}$ at any point g is unique and can be canonically identified with $\mathbb{G}$, and that the basepoint can be taken to be the identity $0_{\mathbb{G}} \in \mathbb{G}$.

Theorem 9.35. *Let $F : X_\infty \rightarrow \mathbb{G}$ be Lipschitz, and let $\eta = \pi_{\mathbb{G}} \circ F$. Using the same notation as in Theorem 9.18:*

- (1) *For μ_∞ -a.e. $p \in S_{F,0}$, for each for each blow-up $(\sigma Y_\infty, c \cdot \nu_\infty, \sigma \cdot \varphi, q; G, \mathbb{G}, 0_{\mathbb{G}}) \in \text{Bw-up}(X_\infty, \mu_\infty, \psi, p; F, \mathbb{G})$ one has $G(Y_\infty) = \{0_{\mathbb{G}}\}$;*
- (2) *Let $p \in S_{F,>0}$ be an approximate continuity point for the derivatives $\partial_\psi \eta$ and let $v(p) = \partial_\psi \eta(p) \in \mathfrak{h}$. Considering a blow-up $(\sigma Y_\infty, c \cdot \nu_\infty, \sigma \cdot \varphi, q; G, \mathbb{G}, 0_{\mathbb{G}}) \in \text{Bw-up}(X_\infty, \mu_\infty, \psi, p; F, \mathbb{G})$, and denoting by $\Gamma : \mathbb{R} \rightarrow \mathfrak{h} \subset \mathbb{G}$ the horizontal line $\Gamma(t) = \exp(v(p) \cdot t)$, the following diagram commutes:*

$$(9.36) \quad \begin{array}{ccc} Y_\infty & \xrightarrow{G} & \mathbb{G} \\ \sigma \varphi \downarrow & \nearrow \Gamma & \\ \mathbb{R} & & \end{array}$$

Moreover, one has $\|v(p)\|_{\mathfrak{h}} = \|\partial_\psi|_p\|_F$.

Proof. We focus on the case $p \in S_{F,>}$. A modification of the argument of Theorem 9.15 implies that, whenever p is an approximate continuity point of the derivative $\partial_\psi \eta$, G maps unit-speed geodesic lines of σY_∞ to left translates of the line $\Gamma(t) = \exp(v(p) \cdot t)$. Moreover, these translates must have constant speed $\|\partial_\psi|_p\|_F$, which implies that $\|v(p)\|_{\mathfrak{h}} = \|\partial_\psi|_p\|_F$. One can then argue as in 9.18, as horizontal lines in $\mathbb{G}$ with the same velocity vector cannot branch. $\square$

We now focus on Lipschitz maps $F : \mathbb{G} \rightarrow Z$ and recall a few more things about Carnot groups. Let N denote the dimension of $\mathfrak{g}$ and recall that the exponential map $\exp : \mathfrak{g} \simeq \mathbb{R}^N \rightarrow \mathbb{G}$ can be used to identify (topologically) $\mathbb{G}$ with $\mathbb{R}^N$. We will use this identification to introduce coordinates on $\mathbb{G}$ and to transfer the Lebesgue measure from $\mathbb{R}^N$ to $\mathbb{G}$. Moreover, given $v \in \mathfrak{h} \setminus \{0\}$ we will denote by $v^\perp$ its orthogonal complement with respect to the standard product on $\mathbb{R}^N$. For $w \in v^\perp$ denote by $\Gamma_{v,w}$ the geodesic line:

$$(9.37) \quad \Gamma_{v,w}(t) = \exp(vt) \cdot w;$$

recall also that, given $g \in \mathbb{G}$, there are unique $(t, w) \in \mathbb{R} \times v^\perp$ such that $\exp(tv) \cdot w = g$. We will denote by $\pi_{\mathbb{G},v}$ the map $g \mapsto t$, which is $1/\|v\|_{\mathfrak{h}}$ -Lipschitz. Using the identification $\mathbb{G} \simeq \mathbb{R}^N$, we will denote by $\mu_{\mathbb{G}}$ the multiple of Lebesgue measure

which gives unit measure to the unit ball⁵ of $\mathbb{G}$. The metric measure space $(\mathbb{G}, \mu_{\mathbb{G}})$ is then a differentiability space and, for each $v \in \mathfrak{h} \setminus \{0\}$, one gets an Alberti representation of the form:

$$(9.38) \quad \mu = C \times \int_{v^\perp} \mathcal{H}_{\Gamma_{v,w}}^1 d\mathcal{L}^{N-1}(w),$$

where $\mathcal{L}^{N-1}(w)$ denotes the Lebesgue measure on $v^\perp$, and C is a constant. A global chart for the differentiable structure is $(\mathbb{G}, \pi_{\mathbb{G}})$ and, for each $g \in \mathbb{G}$, the set $\text{Bw-up}(\mathbb{G}, \mu_{\mathbb{G}}, \pi_{\mathbb{G}}, g)$ consists of the single tuple $(\mathbb{G}, \mu_{\mathbb{G}}, \pi_{\mathbb{G}}, 0_{\mathbb{G}})$. Using the Alberti representations (9.38), we can restate 8.62 for Carnot groups: this is just Pauls' metric differentiation Theorem [Pau01, Thm. 4.7].

Theorem 9.39. *If $F : \mathbb{G} \rightarrow Z$ is Lipschitz, then there is a full $\mu_{\mathbb{G}}$ -measure Borel subset S_F such that, for each $g \in S_F$ and each blowup $(\mathbb{G}, \mu_{\mathbb{G}}, \pi_{\mathbb{G}}, 0_{\mathbb{G}}; G, W, w_0) \in \text{Bw-up}(\mathbb{G}, \mu_{\mathbb{G}}, \pi_{\mathbb{G}}, g; F, Z)$, G maps each $\Gamma_{v,w}$ to a geodesic line of W with constant speed $\|v\|_F(g)$.*

We now study maps $F : \mathbb{G} \rightarrow X_\infty$.

Theorem 9.40. *Let $F : \mathbb{G} \rightarrow X_\infty$ be Lipschitz and let:*

$$(9.41) \quad \begin{aligned} S_{F,0} &= \{g \in S_F : \|\cdot\|_F(g) = 0\} \\ S_{F,>} &= \{g \in S_F : \|\cdot\|_F(g) \neq 0\}; \end{aligned}$$

then:

- (1) *For $\mu_{\mathbb{G}}$ -a.e. $g \in S_{F,0}$, for each blow-up $(\mathbb{G}, \mu_{\mathbb{G}}, \pi_{\mathbb{G}}, 0_{\mathbb{G}}; G, \sigma Y_\infty, q) \in \text{Bw-up}(\mathbb{G}, \mu_{\mathbb{G}}, \pi_{\mathbb{G}}, g; F, X_\infty)$ one has $G(Y_\infty) = \{q\}$;*
- (2) *For $\mu_{\mathbb{G}}$ -a.e. $g \in S_{F,>0}$ the set of those $v \in \mathfrak{h}$ with $\|v\|_F(g) = 0$ lies in a $(\dim \mathfrak{h} - 1)$ -dimensional hyperplane of $\mathfrak{h}$. Moreover, for each blow-up $(\mathbb{G}, \mu_{\mathbb{G}}, \pi_{\mathbb{G}}, 0_{\mathbb{G}}; G, \sigma Y_\infty, q) \in \text{Bw-up}(\mathbb{G}, \mu_{\mathbb{G}}, \pi_{\mathbb{G}}, g; F, X_\infty)$, if we denote by $\gamma : \mathbb{R} \rightarrow Y_\infty$ a unit-speed monotone increasing geodesic satisfying:*

$$(9.42) \quad \begin{aligned} \varphi(\gamma(t)) &= t \quad (\forall t \in \mathbb{R}) \\ \gamma(0) &= q, \end{aligned}$$

then, if for $v \in \mathfrak{h}$ satisfying $\|v\|_F(g) \neq 0$ we let $\gamma_v(t) = \gamma(t \times \sigma^{-1}v \cdot \partial_{\pi_{\mathbb{G}}}(\psi \circ F)(g))$, the following diagram commutes:

$$(9.43) \quad \begin{array}{ccc} \mathbb{G} & \xrightarrow{G} & Y_\infty \\ \pi_{\mathbb{G},v} \downarrow & \nearrow \gamma_v & \\ \mathbb{R} & & \end{array}$$

Proof. We focus on the case in which $g \in S_{F,>}$ is a point of approximate continuity of $\partial_{\pi_{\mathbb{G}}}(\psi \circ F)$. Then the map $G \circ \Gamma_{v,w}$ satisfies:

$$(9.44) \quad \frac{d\varphi(G \circ \Gamma_{v,w}(t))}{dt} = \sigma^{-1}v \cdot \partial_{\pi_{\mathbb{G}}}(\psi \circ F)(g),$$

⁵With respect to the Carnot-Caratheodory metric.

where we remind the reader that $\psi = \varphi_{X_\infty}$ and $\varphi = \varphi_{Y_\infty}$. In particular,

$$(9.45) \quad \|v\|_F(g) = |v \cdot \partial_{\pi_G}(\psi \circ F)(g)|,$$

which implies that the set of those $v \in \mathfrak{h}$ which are annihilated by $\|\cdot\|_F(g)$ is the hyperplane:

$$(9.46) \quad v \cdot \partial_{\pi_G}(\psi \circ F)(g) = 0.$$

Suppose now that $\|v\|_F(g) \neq 0$ and consider the monotone geodesic:

$$(9.47) \quad \gamma_{v,w,k} = \pi_{\infty,k} \circ G \circ \Gamma_{v,w};$$

if two monotone geodesics $\gamma_1, \gamma_2 : \mathbb{R} \rightarrow Y_k$ satisfying

$$(9.48) \quad d_{Y_k}(\gamma_1(0), \gamma_2(0)) \leq \frac{m^{-k}}{2}$$

are distinct, then at some point $t \in \mathbb{R}$ one must have:

$$(9.49) \quad d_{Y_k}(\gamma_1(t), \gamma_2(t)) \geq m^{-k}/2.$$

In particular, if $w_1, w_2 \in \mathbb{G}$ are at distance $< m^{-k}\sigma/(2\mathbf{L}(F))$ in $\mathbb{G}$, then the maps $\gamma_{v,w_1,k}$ and $\gamma_{v,w_2,k}$ coincide. As the set $\pi_{\mathbb{G},v}^{-1}(0)$ is connected, we conclude that:

$$(9.50) \quad \pi_{\infty,k} \circ G \circ \Gamma_{v,w}(t) = \pi_{\infty,k} \circ \gamma(t \times \sigma^{-1} \|v\|_F(g)).$$

The result follows letting $k \nearrow \infty$. $\square$

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