PI SPACES WITH ANALYTIC DIMENSION 1 AND ARBITRARY TOPOLOGICAL DIMENSION

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Abstract. For every n, we construct a metric measure space that is doubling, satisfies a Poincare inequality in the sense of Heinonen-Koskela, has topological dimension n, and has a measurable tangent bundle of dimension 1.

Contents

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1. Introduction

Since they were introduced in [HK98], PI spaces (metric measure spaces that are doubling and satisfy a Poincaré inequality) have been investigated extensively, leading to progress in many directions. In spite of this, there remains a gap between the structural constraints imposed by the existing theory and the properties exhibited by known examples. On the one hand existing examples come from a variety of different sources:

(1) Sub-Riemannian manifolds.
(2) Limits of sequences of Riemannian manifolds with a lower bound on the Ricci curvature [CC00].
(3) Metric measure spaces satisfying synthetic Ricci curvature conditions [Raj12].
(4) Certain Ahlfors-regular topological manifolds [Sem].
(5) Examples with “small” singular sets [Laa02, Sem96].
(6) Boundaries of hyperbolic groups [BP99].
(7) Quotient constructions [Laa00].
(8) Inverse limit constructions [CK15].
(9) Spaces obtained from the above by taking products, gluing [HK98, Thm 6.15], or passing to nice subsets [MTW13].

On the other hand, in some respects this list is somewhat limited. For instance, the examples in (2), (4), (5) are rectifiable, and many in (3) are known to be rectifiable [MN, Gig, GMR], while those in (6), (7) and (8) admit a common description as limits of inverse systems and have very similar properties. Moreover, if (X, μ) denotes one of the above examples, then the infinitesimal structure of (X, μ) has a special form, in the sense that when one blows-up (X, μ) at μ-a.e. point, one gets a metric measure space that is bilipschitz equivalent to the product of a Carnot group with an example as in (8).

Our purpose in this paper is to construct PI spaces that have different characteristics from previously known examples. Before stating our theorem, we recall that any PI space (X, μ) has a measurable (co)tangent bundle [Che99]; we will refer to its dimension as the *analytic dimension* of (X, μ).

Theorem 1.1. *For every n, there is a complete self-similar PI space (X_∞, μ_∞) with analytic dimension 1 and topological dimension n. Furthermore, for some $\alpha \in (0, 1)$:*

(1) *There is a surjective David-Semmes regular map* $\hat{\pi}^\infty : (\mathbb{R}^n, d_\alpha) \to X_\infty$*, where* d_α *is the partial snowflake metric on* $\mathbb{R}^n$ *given by*

$$d_\alpha(p, p') = |p_1 - p'_1| + \sum_{i=2}^{n} |p_i - p'_i|^\alpha .$$

In particular, letting $\mathcal{L}^n$ *denote Lebesgue measure, for* $Q = 1 + (n-1)\alpha^{-1}$*, the pushforward measure* $\mu_\infty = \hat{\pi}^\infty_\#(\mathcal{L}^n)$ *is comparable to Q-dimensional Hausdorff measure* $\mathcal{H}^Q$ *on* X_∞*, and* X_∞ *is Ahlfors Q-regular.*

(2) X_∞ *has topological and Assouad-Nagata dimension* n *(see Section 6).*

(3) $(X_\infty, \mathcal{H}^Q)$ *satisfies a* $(1,1)$*-Poincaré inequality (see Section 7).*

(4) $(X_\infty, \mathcal{H}^Q)$ *has analytic dimension* 1*: there is a single Lipschitz function* $x_\infty : X_\infty \to \mathbb{R}$ *that is a chart on all of* X_∞*, i.e. it defines the measurable differentiable structure for* $(X_\infty, \mathcal{H}^Q)$ *(see Section 8).*

(5) *Let* $\hat{\Gamma}$ *be the family of horizontal lines in* $\mathbb{R}^n$*, equipped with the obvious measure. Then the pushforward of* $\hat{\Gamma}$ *under the map* $\hat{\pi}^\infty : \mathbb{R}^n \to X_\infty$ *gives a universal Alberti representation in the sense of* [Bat15] *for* (X_∞, μ_∞) *(see Section 8).*

(6) *If* $\{p_k\} \subset X_\infty$*,* $\lambda_k \subset (0, \infty)$ *are arbitrary sequences, and* (Z, z) *is a pointed Gromov-Hausdorff limit of the sequence* $\{(\lambda_k X_\infty, p_k)\}$ *of pointed rescalings of* X_∞*, then (2)-(4) hold for* (Z, z)*. Moreover, there is a collection of at most* $N = N(n)$ *David-Semmes regular maps* $(\mathbb{R}^n, d_\alpha) \to X_\infty$ *whose images cover* X_∞*.*

For comparison, we note that all the previously known examples with analytic dimension 1 have topological dimension 1 (see [BP99, Laa00, CK15]).

We refer the reader to Section 2 for an overview of the proof of Theorem 1.1, and to Section 11 for some generalizations.

We now pose some questions concerning the relation between the topological and the analytical structure of PI spaces.

The examples in Theorem 1.1 have small analytic dimension and arbitrarily large topological dimension. One may ask if the topological dimension can be small while the analytic dimension is large. This is not an interesting question, though: there are compact subsets $X \subset [0,1]^n$ with positive Lebesgue measure such that the metric measure space $(X, \mathcal{L}^n)$ is a PI space with analytic dimension n and topological dimension 1 (see [MTW13] for the $n = 2$ case). Nonetheless, such examples are rectifiable and look on the small scale like $\mathbb{R}^n$ itself, in the sense that typical blow-ups are copies of $\mathbb{R}^n$. This motivates the following revised version of the above question:

Question 1.2. Pick $n \geq 2$. Is there a PI space of analytic dimension n and Assouad-Nagata dimension 1?

The Assouad-Nagata dimension is a notion that is metric based, scale invariant, well behaved with respect to Gromov-Hausdorff limits, and is bounded below by the topological dimension (see Section 6); in particular, the rectifiable examples mentioned above have Assouad-Nagata dimension n. Rather than using Assouad-Nagata dimension in Question 1.2, one could require instead that every blow-up (i.e. weak tangent) of X has topological dimension 1.

The spaces in Theorem 1.1 have complicated local topology. One may wonder if there are examples with similar properties that are topological manifolds:

Question 1.3. Pick $n \geq 2$. Suppose (X, μ) is a PI space homeomorphic to $\mathbb{R}^n$, such that any pointed Gromov-Hausdorff limit of any sequence of pointed rescalings of X is also homeomorphic to $\mathbb{R}^n$. What are the possibilities for the analytic dimension of (X, μ)? The Hausdorff dimension? For instance, is there such a PI space homeomorphic to $\mathbb{R}^3$ (or even $\mathbb{R}^2$) with analytic dimension 1? Or one which is Ahlfors Q-regular for large Q?

Organization of the paper. In Section 2 we define the metric space X_∞ in the $n = 2$ case and then discuss some of the key points in the proof of Theorem 1.1. The $n = 2$ case of Theorem 1.1 is proven in Sections 3-9, and the case of general n is treated in Section 10. In Section 11 we consider a more general class of examples of direct systems that have many of the same features.

2. Overview of the proof of Theorem 1.1

In this section we define X_∞ and related objects in the $n = 2$ case. We also have an informal discussion of the proof of Theorem 1.1. The proof itself appears in Sections 3-10.

Standing assumptions: The objects and notation introduced in this section will be retained up through Section 9.

A combinatorial description of partial snowflake metrics on $\mathbb{R}^2$. Pick integers m, m_v with $2 \leq m < m_v$.

For $j \in \mathbb{Z}$, let Y_j be the cell complex associated with the tiling of $\mathbb{R}^2$ by the translates of the rectangle $[0, m^{-j}] \times [0, m_v^{-j}]$. Thus the translation group $(m^{-j}\mathbb{Z}) \times (m_v^{-j}\mathbb{Z})$ acts by cellular isomorphisms on Y_j. Given $k \geq 0$, we may view Y_{j+k} as a k-fold iterated subdivision of Y_j, where at each iteration the 2-cells are subdivided m times in the horizontal direction and m_v times in the vertical direction. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\Phi((x, y)) = (m^{-1}x, m_v^{-1}y)$. Then $\Phi^k : \mathbb{R}^2 \to \mathbb{R}^2$ induces a cellular isomorphism $Y_j \to Y_{j+k}$ for all $j, k \in \mathbb{Z}$.

We now define a metric on $\mathbb{R}^2$ based on the combinatorial structure of the Y_j's. Let $\hat{d}^Y_\infty$ be the largest pseudodistance on $\mathbb{R}^2$ with the property that for every j, each cell of Y_j has $\hat{d}^Y_\infty$-diameter at most m^{-j}. One readily checks (see Lemma 5.1) that $\hat{d}^Y_\infty$ is comparable to the partial snowflake metric

$$d_\alpha((p_1, p_2), (p_1', p_2')) = |p_1 - p_1'| + |p_2 - p_2'|^\alpha ,$$

where $\alpha = \frac{\log m}{\log m_v}$.

The definition of X_∞. We will define the space X_∞ as a quotient of $(\mathbb{R}^2, \hat{d}^Y_\infty)$, where the quotient is generated by certain identifications that respect the x-coordinate. One may compare this with Laakso's construction of PI spaces as quotients of the product $[0, 1] \times C$, where C is a Cantor set [Laa00]. Henceforth we will call a 1-cell of Y_j **horizontal** (respectively **vertical**) if it is a translate of $[0, m^{-j}] \times \{0\}$ (respectively $\{0\} \times [0, m_v^{-j}]$).

Choose a large integer L (e.g. $L = 100$), and set $m = 4$ and $m_v = 3L$.

For all $k, \ell \in \mathbb{Z}$, $i \in \{1, 2, 3\}$, we define the following pair of vertical 1-cells of Y_1 (see Figure 1):

$$a_{k,l,i} = \left\{ \frac{i}{4} + k \right\} \times \left[(3\ell + i - 1)m_v^{-1}, (3\ell + i)m_v^{-1} \right] ,$$

$$a'_{k,l,i} = \left\{ \frac{i}{4} + k \right\} \times \left[(3\ell + i)m_v^{-1}, (3\ell + i + 1)m_v^{-1} \right] ,$$

Note that $a'_{k,\ell,i}$ is the image of $a_{k,\ell,i}$ under the vertical translation $(x, y) \mapsto (x, y + m_v^{-1})$. The collections $\{a_{k,\ell,i} \mid k, \ell \in \mathbb{Z},\ 1 \leq i \leq 3\}$, $\{a'_{k,\ell,i} \mid k, \ell \in \mathbb{Z},\ 1 \leq i \leq 3\}$ are invariant under translation by $\mathbb{Z}^2$ and are contained in the union of vertical lines $\{(x, y) \in \mathbb{R}^2 \mid x \in \frac{1}{4}\mathbb{Z} \setminus \mathbb{Z}\}$.

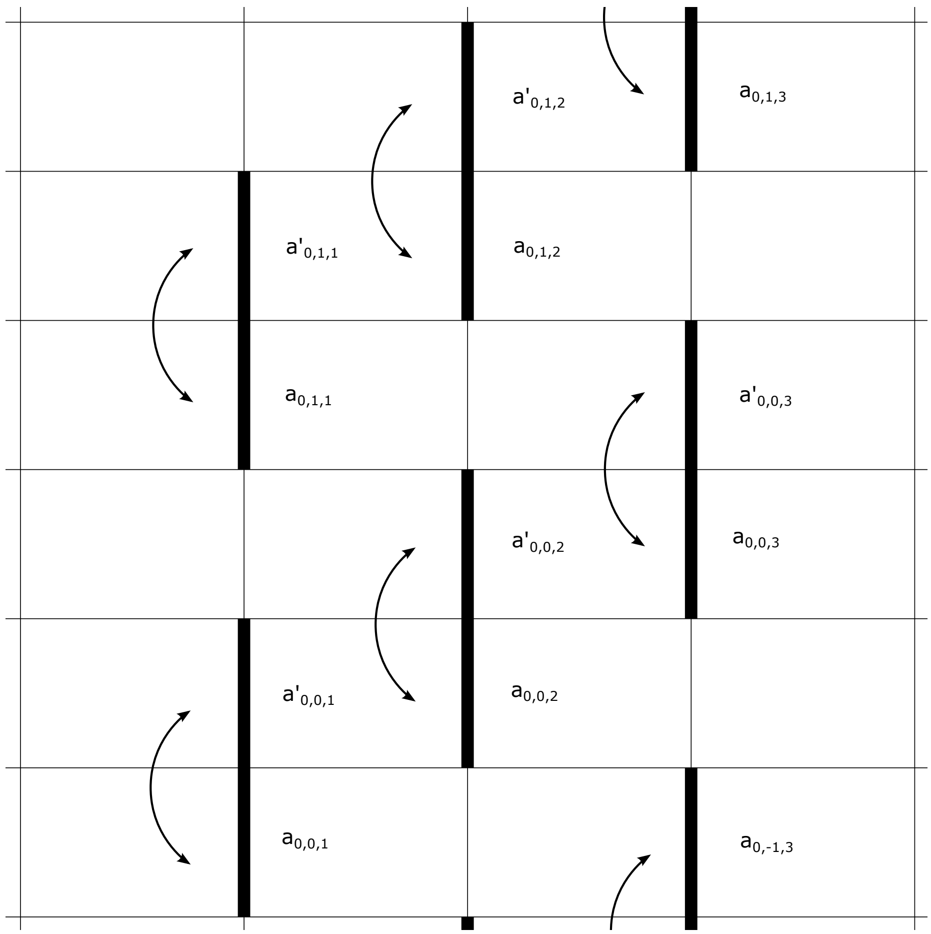

Figure 1: $L = 2$ case

Next, we define an equivalence relation $\mathcal{R}$ on $\mathbb{R}^2$ by identifying $a_{k,\ell,i}$ with $a'_{k,\ell,i}$ by the vertical translation $(x, y) \mapsto (x, y + m_v^{-1})$ for all $k, \ell \in \mathbb{Z}$, $1 \le i \le 3$. Note that $\mathcal{R}$ is invariant under translation by $\mathbb{Z}^2$.

Let $\mathcal{R}_\infty$ be the equivalence relation on $\mathbb{R}^2$ generated by the collection of pushforwards $\Phi_*^j \mathcal{R}$ for all $j \in \mathbb{Z}$. We define X_∞ to be the quotient $\mathbb{R}^2/\mathcal{R}_\infty$, and let $\hat{\pi}^\infty : \mathbb{R}^2 \to X_\infty$ be the quotient map. We metrize X_∞ using the largest pseudodistance $\hat{d}_\infty^X$ on X_∞ such that for every $j \in \mathbb{Z}$ and every 2-cell $\hat{\sigma}$ of Y_j, the projection $\hat{\pi}^\infty(\hat{\sigma}) \subset X_\infty$ has $\hat{d}_\infty^X$-diameter at most m^{-j}. It is not hard to see that $\hat{d}_\infty^X$ is the largest pseudodistance on X_∞ such that $\hat{\pi}^\infty : (\mathbb{R}^2, \hat{d}_\infty^Y) \to (X_\infty, \hat{d}_\infty^X)$ is 1-Lipschitz. Henceforth we use $\hat{d}_\infty$ instead of $\hat{d}_\infty^X$ when there is no risk of confusion.

X_∞ as a direct limit of cell complexes. While the definition of X_∞ as a quotient $\mathbb{R}^2/\mathcal{R}_\infty$ is transparent, it does not provide a convenient framework for understanding the structure of X_∞. Instead, we will analyze X_∞ by representing it as a direct limit.

For every $j \in \mathbb{Z}$, let $\mathcal{R}_j$ be the equivalence relation on $\mathbb{R}^2$ generated by $\Phi^i_*\mathcal{R}$ for all $i \in \mathbb{Z}$ with $i < j$, and let X_j be the quotient $\mathbb{R}^2/\mathcal{R}_j$ equipped with quotient topology. Since $\mathcal{R}_{j-1} \subset \mathcal{R}_j$, the quotient maps induce a direct system of topological spaces

$$\ldots \xrightarrow{\pi_{-1}} X_0 \xrightarrow{\pi_0} X_1 \xrightarrow{\pi_1} \ldots \xrightarrow{\pi_{j-1}} X_j \xrightarrow{\pi_j} \ldots$$

For all $i \in \mathbb{Z}$, $j \in \mathbb{Z} \cup \{\infty\}$ with $i \le j$, we denote the projection map $X_i \to X_j$ by π_i^j, the quotient map $\mathbb{R}^2 \to X_j$ by $\hat{\pi}^j$, and the composition $\pi_i^j \circ \hat{\pi}^i : Y_i \to X_j$ by $\hat{\pi}_i^j$.

We metrize X_j using the largest pseudodistance $\hat{d}_j$ on X_j such that for every $i \le j$ and every cell $\hat{\sigma}$ of Y_i, the image $\hat{\pi}^j(\hat{\sigma}) \subset X_j$ has $\hat{d}_j$-diameter at most m^{-i}.

Remark 2.1. Our examples were partly inspired by [CK15, Sec. 11], which gives a construction of PI spaces of topological and analytic dimension n. In fact, X_∞ arose when we attempted to find an "anisotropic" variant of the cube complex examples of [CK15, Sec. 11]. However, this leads to a situation where the projection map $\pi_0^\infty : X_\infty \to [0,1]^n$ is Lipschitz with respect to the partial snowflake metric on $[0,1]^n$, and is moreover a light map. This is incompatible with the existence of a Poincaré inequality in X_∞.

Discussion of the proof. We now give an indication of some of the key points in the proof of Theorem 1.1.

The first part of the proof, which appears in Sections 3-4, develops the combinatorial and metric structure of the direct system $\{X_j\}$. Because the equivalence relation $\mathcal{R}_j$ may be generated by identifying pairs of vertical 1-cells of the cell complex Y_j by vertical translations, the cell structure of Y_j descends to a cell structure on X_j. This cell complex has controlled combinatorics; because the distance $\hat{d}_j$ is defined combinatorially, this implies that $(X_j, \hat{d}_j)$ is doubling at the scale m^{-j}. A key estimate (Proposition 4.12) compares $(X_j, \hat{d}_j)$ with $(X_\infty, \hat{d}_\infty)$ is that for every $p, p' \in X_j$ one has

$$\hat{d}_j(p,p') - 2m^{-j} \le \hat{d}_\infty(\hat{\pi}_j^\infty(p), \hat{\pi}_j^\infty(p')) \le \hat{d}_j(p,p')\,. \tag{2.2}$$

In particular, this implies that the sequence $\{X_j\}$ Gromov-Hausdorff converges to X_∞.

With the foundation laid in Sections 3-4, several parts of Theorem 1.1 follow fairly easily:

- By a short argument (Lemma 5.2), one deduces the David-Semmes regularity of the projection $\hat{\pi}^\infty : (\mathbb{R}^2, \hat{d}^Y_\infty) \to X_\infty$, which gives part (1) of Theorem 1.1.
- The restriction of the projection $\hat{\pi}^\infty : \mathbb{R}^2 \to X_\infty$ to the boundary of the unit square $\partial[0,1]^2$ is injective. By an elementary topological argument this implies that the topological dimension of X_∞ is at least 2 (see Lemma 5.2).
- Using (2.2), one shows that there is a "good cover" of X_∞ whose inverse image under $\pi_j^\infty : X_j \to X_\infty$ approximates the decomposition of X_j into open cells. This proves that the Assouad-Nagata dimension of X_∞ is at most 2 (see Theorem 6.2).

The remaining assertions of Theorem 1.1 have to do with the analytical structure of X_∞, and are largely based on the existence of good families of curves, whose construction we now describe.

The starting point is the observation that the first coordinate $x : \mathbb{R}^2 \to \mathbb{R}$ descends to a 1-Lipschitz function $x_\infty : X_\infty \to \mathbb{R}$, and hence any horizontal geodesic segment in $\mathbb{R}^2$ projects to a geodesic in X_∞. Every 2-cell $\hat{\sigma}$ of Y_j is a rectangle of width m^{-j} and height m_v^{-j} so it yields a family $\Gamma_{\hat{\sigma}}$ of geodesic segments in X_∞ of length m^{-j}; since $\Gamma_{\hat{\sigma}}$ has a natural parametrization by an interval of length m_v^{-j}, it carries a natural measure $\nu_{\hat{\sigma}}$.

If $\hat{\sigma}$, $\hat{\sigma}'$ are 2-cells of Y_j such that the projections $\hat{\pi}^j(\sigma)$, $\hat{\pi}^j(\sigma')$ share a vertical 1-cell of X_j, then the corresponding measured families of curves $(\Gamma_{\hat{\sigma}}, \nu_{\hat{\sigma}})$, $(\Gamma_{\hat{\sigma}'}, \nu_{\hat{\sigma}'})$ may be concatenated to form a new measured family of curves. More generally, if $\sigma_1, \dots, \sigma_\ell$ is a sequence of 2-cells of X_j that form a horizontal gallery[1] in X_j (i.e. σ_{i-1} shares a vertical 1-cell with σ_i for all $1 < i \leq \ell$) then one may concatenate the corresponding curve families. To produce an abundance of such curve families, a crucial property is the "horizontal gallery accessibility" (Lemma 7.1), which is an analog of the gallery diameter bound of [CK15, Sec. 11, condition (3)]. This says that if σ, σ' are 2-cells of X_j with controlled combinatorial distance in X_j, then they may be joined by a horizontal gallery of controlled length. This accessibility property is due to the choice of the equivalence relation $\mathcal{R}$: notice that if $\hat{\sigma}$, $\hat{\sigma}'$ are 2-cells of Y_1, then one may form a horizontal gallery between their projections

[1]We have borrowed the term "gallery" from the theory of Coxeter complexes and Tits buildings.

$\sigma, \sigma' \in X_1$ by using the vertical identifications $a_{k,\ell,i} \leftrightarrow a'_{k,\ell,i}$ that define $\mathcal{R}$.

The $(1,1)$-Poincaré inequality is proved using the standard approach via "pencils" of curves [Sem]. The pencils are built by combining the measured families of curves described above, for horizontal galleries of different scales, see Section 7.

To prove that the analytic dimension of X_∞ is 1, we define a "horizontal derivative" $D_\infty u$ for any Lipschitz function $u : X_\infty \to \mathbb{R}$. By using horizontal galleries again, we show that at an approximate continuity point p of $D_\infty u$, we have

$$u(q) - u(p) = (D_\infty u(p))(x_\infty(q) - x_\infty(p)) + o(d(q,p)),$$

see Section 8.

Remark 2.3. Rather than metrizing X_∞ using $\hat{d}_\infty$, an alternate approach is to define a metric on X_∞ using a distinguished set of paths. For instance, following [Laa00] one could define, for all $p, p' \in X_\infty$, the distance $d(p,p')$ to be the infimal 1-dimensional Hausdorff measure of $(\hat{\pi}^\infty)^{-1}(\gamma)$, where γ is a path from p to p'. This leads to an essentially equivalent analysis, with the details organized somewhat differently. We found the approach using $\hat{d}_\infty$ more transparent.

Notational conventions. In the following $a \approx b$ will indicate that a and b are comparable up to a uniformly bounded multiplicative factor C, and sometimes we will also write $a \approx_C b$ to highlight C. We will similarly use expressions like $a \lesssim b$ and $a \gtrsim b$.

3. The cellular structure of the direct system $\{X_j\}$

In this section we examine different aspects of the combinatorial structure of the X_j's. We remind the reader that we will retain the notation from Section 2 through Section 9. The following lemma lists the main properties of $\mathcal{R}$ and $\mathcal{R}_j$ that will be used later.

Lemma 3.1 (Properties of $\mathcal{R}$)**.**

(1) *Nontrivial cosets of $\Phi_*^{j-1}\mathcal{R}$ belong to the union of vertical lines $\{(x,y) \in \mathbb{R}^2 \mid x \in m^{-j}\mathbb{Z} \setminus m^{-(j-1)}\mathbb{Z}\}$, and any two points in the same coset lie in a vertical edge path of Y_j of combinatorial length at most* 2.
(2) *Cosets of $\mathcal{R}_j$ are contained in orbits of the action $m^{-j}\mathbb{Z} \times m_v^{-j}\mathbb{Z} \curvearrowright \mathbb{R}^2$ by translations.*

(3) *For all $j \in \mathbb{Z}$, let $\bar{\mathcal{R}}_{j+1}$ be the equivalence relation on X_j generated by the pushforward $(\hat{\pi}_j \circ \Phi^j)_*(\mathcal{R})$. Then $\bar{\mathcal{R}}_{j+1}$ is an equivalence relation on X_j, and the cosets of $\bar{\mathcal{R}}_{j+1}$ are the fibers of $\pi_j : X_j \to X_{j+1}$.*

(4) *For $i = 0, 1$, let $\hat{\sigma}_i$ be a cell of Y_j, and $\hat{p}_i$ be an interior point of $\hat{\sigma}_i$. If $\hat{p}_0 \sim_{\mathcal{R}_j} \hat{p}_1$, then there is a translation $t \in m^{-j}\mathbb{Z} \times m_v^{-j}\mathbb{Z}$ such that $t(\hat{\sigma}_0) = \hat{\sigma}_1$, and $\hat{q}_0 \sim_{\mathcal{R}_j} t(\hat{q}_0)$ for all $\hat{q}_0 \in \hat{\sigma}_0$.*

Proof. (1) and (2) follow immediately from the definition of $\mathcal{R}$ and Φ.

(3). By (1) the nontrivial cosets of $\Phi^{j-1}\mathcal{R}$ intersect only trivial cosets of $\mathcal{R}_{j-1}$. This implies that $\bar{\mathcal{R}}_{j+1}$ is an equivalence relation on X_j, and that its cosets are the fibers of $\pi_j : X_j \to X_{j+1}$.

(4). Since $\mathcal{R}_j$ is generated by $\Phi^i_*\mathcal{R}$ for $i < j$, it is generated by pairwise identifications of cells of Y_j by translations. This implies (4).

□

Lemma 3.2 (Cell structure of X_j)**.**

(1) *The collection of images of the open cells of Y_j under the projection map $\hat{\pi}^j : Y_j \to X_j$ defines a CW complex structure on X_j.*

(2) *For every cell $\hat{\sigma}$ of Y_j, the restriction of $\hat{\pi}^j$ to $\hat{\sigma}$ is a characteristic map for the open cell $\hat{\pi}^j(\mathrm{Int}(\hat{\sigma})) \subset X_j$.*

(3) *If $\hat{\sigma}_0$, $\hat{\sigma}_1$ are cells of Y_j then $\hat{\pi}^j(\mathrm{Int}(\hat{\sigma}_0)) \cap \hat{\pi}^j(\mathrm{Int}(\hat{\sigma}_1)) \neq \emptyset$ if and only if $\mathcal{R}_j$ identifies $\hat{\sigma}_0$ with $\hat{\sigma}_1$ by translation. In particular, If $\hat{\sigma}$ is a cell of Y_j, then $\hat{\pi}^j\big|_\sigma : \sigma \to X_j$ identifies certain pairs of faces of $\hat{\sigma}$ by translation.*

(4) *The cells of Y_{j+k} project under $\hat{\pi}^j : \mathbb{R}^2 \to X_j$ to define a cell complex $X_j^{(k)}$, which is the k-fold iterated subdivision of X_j.*

Proof. (1)-(3). We first show that X_j is Hausdorff.

Pick $p \in X_j$. By Lemma 3.1(2) and the fact that Y_j is invariant under $m^{-j}\mathbb{Z} \times m_v^{-j}\mathbb{Z}$, there is an $r > 0$ such that any cell $\hat{\sigma}$ of Y_j that intersects $N_r((\hat{\pi}^j)^{-1}(p))$ must intersect $(\hat{\pi}^j)^{-1}(p)$. Here $N_r(S) = \{q \in \mathbb{R}^2 \mid d_{\mathbb{R}^2}(q, S) \leq r\}$ denotes the Euclidean metric r-neighborhood.

Claim. $N_r((\hat{\pi}^j)^{-1}(p))$ *is a union of cosets of* $\mathcal{R}_j$.

Suppose $\hat{q}' \in \mathbb{R}^2$ and $\hat{q}' \sim_{\mathcal{R}_j} \hat{q}$ for some $\hat{q} \in N_r((\hat{\pi}^j)^{-1}(p))$. By the choice of r, if $\mathrm{Int}(\hat{\sigma})$ is the open cell of Y_j containing $\hat{q}$, then $\hat{\sigma}$ contains some point $\hat{p} \in (\hat{\pi}^j)^{-1}(p)$. By Lemma 3.1(4) there is a translation

$t \in m^{-j}\mathbb{Z} \times m_v^{-j}\mathbb{Z}$ such that $t(\hat{q}) = \hat{q}'$ and $\hat{p} \sim_{\mathcal{R}_j} t(\hat{p})$. But then $\hat{q}' \in B(t(\hat{p}), r) \subset N_r((\hat{\pi}^j)^{-1}(p))$, proving the claim.

For every pair of distinct points $p_0, p_1 \in X_j$, the sets $(\hat{\pi}^j)^{-1}(p_0)$, $(\hat{\pi}^j)^{-1}(p_1)$ are disjoint and lie in orbits of $m^{-j}\mathbb{Z} \times m_v^{-j}\mathbb{Z}$, and hence have positive distance from each other. By the claim, if $r > 0$ is sufficiently small, then $\hat{\pi}^j(N_r((\hat{\pi}^j)^{-1}(p_0)))$, and $\hat{\pi}^j(N_r((\hat{\pi}^j)^{-1}(p_1)))$ are disjoint open subsets of X_j. This proves that X_j is Hausdorff.

Let T_j^2 denote the quotient space $\mathbb{R}^2/(m^{-j}\mathbb{Z} \times m_v^{-j}\mathbb{Z})$, and $\pi_{T_j^2} : \mathbb{R}^2 \to T_j^2$ be the quotient map. The open cells of Y_j project under $\pi_{T_j^2}$ to open cells of T_j^2, inducing the standard cell structure on T_j^2. Note that by Lemma 3.1(2) there is a well-defined continuous map $X_j \to T_j^2$.

Now consider a cell $\hat{\sigma}$ of Y_j. The composition $Y_j \stackrel{\hat{\pi}^j}{\to} X_j \to T_j^2$ maps the interior of $\hat{\sigma}$ homeomorphically onto an open cell of T_j^2; it follows that $\hat{\pi}^j : Y_j \to X_j$ maps the interior of $\hat{\sigma}$ homeomorphically onto its image, which is therefore an open cell. Thus the restriction of $\hat{\pi}^j$ to $\hat{\sigma}$ of Y_j is a characteristic map for the open cell $\hat{\pi}^j(\mathrm{Int}(\hat{\sigma}))$.

If $\hat{\sigma}_0$, $\hat{\sigma}_1$ are cells of Y_j such that $\hat{\pi}^j(\mathrm{Int}(\hat{\sigma}_0)) \cap \hat{\pi}^j(\mathrm{Int}(\hat{\sigma}_1)) \neq \emptyset$, then by Lemma 3.1(4) we have $\hat{\pi}^j(\mathrm{Int}(\hat{\sigma}_0)) = \hat{\pi}^j(\mathrm{Int}(\hat{\sigma}_1))$. This shows that the collection of images of open cells of Y_j is a decomposition of X_j into disjoint open cells.

For any cell $\hat{\sigma}$ of Y_j, the closure of $\hat{\pi}^j(\mathrm{Int}(\hat{\sigma}))$ is just $\hat{\pi}^j(\hat{\sigma})$, and is therefore contained in the union of the images of the open cells contained in $\hat{\sigma}$. The closure of any open cell of X_j intersects only finitely many open cells.

Finally, if $C \subset X_j$, then C is closed if and only if $(\hat{\pi}^j)^{-1}(C)$ is closed, which is equivalent to $(\hat{\pi}^j)^{-1}(C) \cap \hat{\sigma}$ being closed for every cell $\hat{\sigma}$ of Y_j, which happens if and only if $C \cap \hat{\pi}^j(\hat{\sigma})$ is closed.

Thus we have verified (1)-(3).

The proof of (4) is similar to the proof of (1).

□

Henceforth the notation X_j and $X_j^{(k)}$ will refer to the cell complex structure established in Lemma 3.2. A 1-cell of $X_j^{(k)}$ is **horizontal** (**vertical**) if it is the image of a horizontal (vertical) cell of Y_{j+k} under the projection map $\hat{\pi}^j : \mathbb{R}^2 \to X_j$.

We now analyze the behavior of the projection maps with respect to the combinatorial structure.

Lemma 3.3. (1) *If $p, p' \in X_j$ are distinct points and $\pi_j(p) = \pi_j(p')$, then there is a vertical edge path in the* 1*-skeleton of $X_j^{(1)}$ that contains p, p' and has combinatorial length at most* 2*.*
(2) *If σ, σ' are* 2*-cells of X_j, and $\pi_j(\sigma) \cap \pi_j(\sigma') \neq \emptyset$, then $\sigma \cap \sigma' \neq \emptyset$.*
(3) *Suppose $0 \leq j \leq j' < \infty$ and $\hat{\sigma}$, $\hat{\sigma}'$ are* 2*-cells of Y_j and $Y_{j'}$ respectively. If $\hat{\pi}^\infty(\hat{\sigma}) \cap \pi^\infty(\hat{\sigma}') \neq \emptyset$, then $\hat{\pi}^{j'}(\hat{\sigma}) \cap \hat{\pi}^{j'}(\hat{\sigma}') \neq \emptyset$.*

Proof. (1). This follows from (1) and (3) of Lemma 3.1.

(2). Suppose that $p \in \sigma$, $p' \in \sigma'$, and $\pi_j(p) = \pi_j(p')$. By (1) there is a vertical edge path in $X_j^{(1)}$ of combinatorial length at most two joining p to p'. Since the subdivision of σ in $X_j^{(1)}$ has combinatorial "height" m_v, it follows that $\sigma \cap \sigma' \neq \emptyset$.

(3). By the definition of the direct limit, we have $\hat{\pi}^\ell(\hat{\sigma}) \cap \hat{\pi}^\ell(\hat{\sigma}') \neq \emptyset$ for some ℓ. Letting $k_0 + 1$ be the minimal such ℓ, suppose we have $k_0 \geq j'$. Since σ, σ' are finite unions of 2-cells of Y_{k_0}, we may assume without loss of generality that $j = j' = k_0$. Then $\hat{\pi}^{k_0}(\hat{\sigma})$, $\hat{\pi}^{k_0}(\hat{\sigma}')$ are disjoint 2-cells of X_{k_0} whose projections to X_{k_0+1} intersect. This contradicts (2). Thus $k_0 + 1 \leq j'$, proving (3). □

The next lemma shows that the X_j's and the projection maps have bounded complexity.

Lemma 3.4.

(1) *$\pi_j : X_j \to X_{j+1}$ is injective on the* 1*-skeleton of X_j.*
(2) *If $p \in X_j$ is a* 0*-cell, then $\pi_j^{-1}(\pi_j(p)) = \{p\}$, i.e.* 0*-cells experience no collapsing.*
(3) *For every $p \in X_j$, the point inverse $(\hat{\pi}^j)^{-1}(p)$ contains at most* 3 *points.*
(4) *For every j, the link of any cell in X_j contains at most* 24 *cells.*
(5) *For every j and every* 0*-cell $v \in X_j$, the link of v is connected: any two cells σ, σ' of X_j containing v may joined by a sequence $\sigma = \tau_1, \ldots, \tau_\ell = \sigma'$ of cells of X_j containing v, where τ_{i-1} and τ_i share a* 1*-cell.*
(6) *Every cell of X_j contains at most* 9 *open cells.*
(7) *There is an $N = N(n)$ such that for every $j \in \mathbb{Z}$, every combinatorial n-ball in X_j contains at most N cells.*
(8) *For every $j \in \mathbb{Z}$ and every* 2*-cell σ of X_j, the inverse image $(\hat{\pi}^j)^{-1}(\sigma)$ may be covered by at most* 27 2*-cells of Y_j.*

Proof. (1) and (2) follow from Lemma 3.1(3) and Lemma 3.3(1).

(3). By Lemma 3.1, a nontrivial coset of $\mathcal{R}_j$ is a nontrivial coset of $\Phi^i\mathcal{R}$ for some $i < j$, and therefore contains at most 3 elements.

(4). It suffices to verify this for 0-cells. If v is a 0-cell of X_j, then $(\hat{\pi}^j)^{-1}(v)$ contains at most 3 vertices of Y_j by (3). Therefore v is contained in at most 12 2-cells and 12 1-cells.

(5). If $(\hat{\pi}^j)^{-1}(v)$ contains only one point $\hat{v} \in \mathbb{R}^2$, then the link of v is isomorphic to the link of $\hat{v}$. Otherwise for some $i < j$ the set $(\hat{\pi}^j)^{-1}(v)$ is a nontrivial coset of $\Phi^i\mathcal{R}$, and has the form $\{\hat{v}, \hat{v} \pm m_v^{-(i+1)}\}$ for some 0-cell $\hat{v}$ of Y_{i+1}, and the two vertical 1-cells emanating from $\hat{v}$ are identified by $\Phi^i\mathcal{R}$ with the vertical 1-cells emanating from $\hat{v} \pm m_v^{-(i+1)}$. Thus the link of v is homeomorphic to the quotient of three disjoint circles S^1, S^1_+, S^1_- by by identifying $p_\pm \in S^1$ with $q_\pm \in S^1_\pm$; this is connected.

(6). This follows from Lemma 3.2(3).

(7). This follows by induction on n, using (4) and (6).

(8). Suppose $\tau \subset \sigma$ is an open cell of dimension d. By Lemma 3.2(3) the inverse image of τ under the projection $\hat{\pi}^j : Y_j \to X_j$ is the disjoint union of the interiors of a collection $\hat{\tau}_1, \ldots, \hat{\tau}_\ell$ of d-cells of Y_j. By (3) we have $\ell \leq 3$. Combining this with (6) we get that $(\hat{\pi}^j)^{-1}(\sigma)$ can be covered by at most 27 cells of Y_j.

$\square$

Let x, y denote the coordinate functions on $\mathbb{R}^2$. For every $j \in \mathbb{Z}$, define $\hat{y}_j : \mathbb{R}^2 \to S^1(m_v^{-j}) = \mathbb{R}/m_v^{-j}\mathbb{Z}$ to be the composition of $y : \mathbb{R}^2 \to \mathbb{R}$ with the quotient map $\mathbb{R} \longrightarrow \mathbb{R}/m_v^{-j}\mathbb{Z}$. We will metrize $S^1(r) = \mathbb{R}/r\mathbb{Z}$ with the quotient metric $d_{S^1(r)}$.

Lemma 3.5 (The functions x_j and y_j)**.**

(1) *For all $j \in \mathbb{Z} \cup \{\infty\}$ the function x descends to a function $x_j : X_j \to \mathbb{R}$ and for all $j \in \mathbb{Z}$ the function $\hat{y}_j$ descends to a function $y_j : X_j \longrightarrow S^1(m_v^{-j}) = \mathbb{R}/m_v^{-j}\mathbb{Z}$.*
(2) *If σ is a 2-cell of X_j, then the image of σ under $x_j : X_j \to \mathbb{R}$ is an interval of length m^{-j} whose endpoints are the images of the vertical 1-cells of σ.*
(3) *If $p, p' \in X_j$ and $\pi_j(p) = \pi_j(p')$, then $d(y_j(p), y_j(p')) \leq 2\, m_v^{-(j+1)}$.*
(4) *If $\bar{\sigma}$ is a 2-cell of X_{j+1}, then the inverse image $\pi_j^{-1}(\bar{\sigma}) \subset X_j$ maps under $y_j : X_j \to \mathbb{R}/m_v^{-j}\mathbb{Z}$ to a set of diameter at most $5m_v^{-(j+1)}$.*

Proof. (1). Since the identifications generating $\mathcal{R}$ are given by vertical translations by multiples of m_v^{-1}, the cosets of $\mathcal{R}$ are contained in the orbits of the translation action $\{e\} \times m_v^{-1}\mathbb{Z} \curvearrowright \mathbb{R}^2$; likewise the cosets of $\mathcal{R}_j$ are contained in the orbits of $\{e\} \times m_v^{-(j)}\mathbb{Z}$. Therefore x and $\hat{y}_j$ descend to $X_j = X_0/\mathcal{R}_j$.

(2). We have $\sigma = \hat{\pi}^j(\hat{\sigma})$ for some 2-cell $\hat{\sigma}$ of Y_j. Then $y_j(\sigma) = y(\sigma)$, and the assertion is clear.

(3). If $\bar{e}$ is a vertical 1-cell of $X_j^{(1)}$, then $\bar{e} = \hat{\pi}^j(e)$ for some vertical 1-cell e of Y_{j+1}; therefore $y_j(\bar{e})$ has diameter $m_v^{-(j+1)}$. Applying Lemma 3.3(1) we get $d(y_j(p), y_j(p')) \leq 2\, m_v^{-(j+1)}$.

(4). Pick $p, p' \in (\pi_j)^{-1}(\bar{\sigma})$. Choose σ a 2-cell of $X_j^{(1)}$ such that $\pi_j(\sigma) = \bar{\sigma}$. Then there exist $q, q' \in \sigma$ such that $\pi_j(p) = \pi_j(q)$ and $\pi_j(p') = \pi_j(q')$. Since $\operatorname{diam}(y_j(\sigma)) = m_v^{-(j+1)}$, by (3) we get that $\operatorname{diam}(\pi_j)^{-1}(\bar{\sigma}) \leq 5\, m_v^{-(j+1)}$.

□

4. Metric structure

We now analyze the metric $\hat{d}_\infty$ by relating it to the geometry of the approximating cell complexes X_j. The main result in this section is Proposition 4.12.

Let $\hat{d}_j$ be the largest pseudodistance on X_j such that for every $i \leq j$, and every 2-cell $\hat{\sigma}$ of Y_i, the diameter of $\hat{\pi}^j(\hat{\sigma})$ is $\leq m^{-i}$.

Before proceeding, we introduce some additional terminology.

A **chain** in a set is a sequence $S_1, \ldots, S_\ell$ of subsets such that $S_{i-1} \cap S_i \neq \emptyset$ for all $1 < i \leq \ell$.

Definition 4.1. Let $\hat{\sigma}_1, \ldots, \hat{\sigma}_\ell$ be a sequence, where $\hat{\sigma}_i$ is a 2-cell of Y_{j_i}. Then:

- For $j \in \mathbb{Z} \cup \{\infty\}$, the sequence $\{\hat{\sigma}_i\}$ is a j**-chain joining** $p, p' \in X_j$ if the projections $\hat{\pi}_0^j(\hat{\sigma}_0), \ldots, \hat{\pi}^j(\hat{\sigma}_\ell)$ form a chain in X_j, and $p \in \hat{\pi}^j(\hat{\sigma}_0)$, $p' \in \hat{\pi}^j(\hat{\sigma}_\ell)$. By convention, the empty sequence is a chain joining every point to itself.
- The **length** and **generation** of the sequence $\{\hat{\sigma}_i\}$ are $\sum_i m^{-j_i}$ and $\max_i j_i$ respectively.

By Definition 4.1, for $p, p' \in X_\infty$, we have

$$\hat{d}_\infty(p, p') = \inf\{\operatorname{length}(\{\hat{\sigma}_i\}) \mid \{\hat{\sigma}_i\} \text{ is an } \infty\text{-chain joining } p, p'\},$$

and for all $p, p' \in X_j$

$$\hat{d}_j(p,p') = \inf \left\{ \text{length}(\{\hat{\sigma}_i\}) \;\middle|\; \begin{array}{c} \{\hat{\sigma}_i\} \text{ is a } j\text{-chain of generation} \\ \leq j \text{ joining } p, p' \end{array} \right\},$$

Note that it follows that $\hat{d}_j$ restricts to a discrete metric on each 2-cell σ of X_j.

Lemma 4.2. *If $\hat{\sigma}_1, \ldots, \hat{\sigma}_\ell$ is an ∞-chain of generation g, then it is a j-chain for all $j \geq g$.*

Proof. This follows from Lemma 3.3(3). □

Lemma 4.3. *Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ be a collection of subsets of a set, and $S, S' \in \mathcal{C}_1$. Assume that $\mathcal{C}$ contains a chain from S to S', and that every element of $\mathcal{C}_2$ is contained in some element of $\mathcal{C}_1$. Then $\mathcal{C}_1$ contains a chain from S to S'.*

Proof. Starting with a chain in $\mathcal{C}$ from S to S', one may inductively reduce the number of elements from $\mathcal{C}_2$ by replacing each one with an element of $\mathcal{C}_1$ that contains it. □

Definition 4.4. A **gallery** in $X_j^{(k)}$ is a sequence of 2-cells $\sigma_1, \ldots, \sigma_\ell$ of $X_j^{(k)}$ such that $\sigma_{i-1} \cap \sigma_i$ contains a 1-cell e_i of $X_j^{(k)}$ for every $1 < i \leq \ell$; the gallery is **vertical** if for all $1 < i \leq \ell$ the 1-cell e_i is horizontal, and **horizontal** if e_i is vertical for all $1 < i \leq \ell$.

Lemma 4.5. *There exist $N_1 = N_1(C)$, $N_2 = N_2(C)$ such that for every $j \in \mathbb{Z}$, every $\hat{d}_j$-ball $B(p, Cm^{-j}) \subset X_j$ is contained in a combinatorial N_1-ball of X_j, and is contained in a union of at most N_2 2-cells of X_j.*

Proof. Fix $C < \infty$ and $p \in X_j$.

If $\hat{d}_j(p', p) < Cm^{-j}$, then there is a j-chain $\hat{\sigma}_1^0, \ldots, \hat{\sigma}_{\ell_0}^0$ from p to p' of length $< Cm^{-j}$. Letting j_i be the generation of $\hat{\sigma}_i^0$, this yields

$$m^{-j_i} < Cm^{-j}, \quad \ell_0 \cdot m^{-j} < Cm^{-j}$$

so

$$j - j_i < \frac{\log C}{\log m} \tag{4.6}$$

and $\ell_0 < C$. Now (4.6) implies that $\hat{\sigma}_i^0$ contains at most $C_1 = C_1(C)$ 2-cells of Y_j. The collection $\mathcal{C}$ of all 2-cells of Y_j contained in $\cup_i \hat{\sigma}_i^0$ contains at most $C_1 \cdot \ell_0$ 2-cells. Also $\mathcal{C}$ contains a j-chain $\hat{\sigma}_1, \ldots, \hat{\sigma}_\ell$

from p to p', where $\ell \leq C_1\ell_0 < C_1 \cdot C$. Putting $N_1 = C_1 \cdot C$, we have shown that p' lies in the combinatorial N_1-ball of the open cell containing p.

By Lemma 3.4(7) the N_1-ball contains at most $N_2 = N_2(C)$ cells.

□

The next two lemmas provide lower bounds on the combinatorial length of certain types of chains of 2-cells in X_{j+1}.

Lemma 4.7. *Let $\sigma \subset X_j$ be a 2-cell with vertical 1-cells e_1, e_2. If $\bar\sigma_1, \ldots, \bar\sigma_\ell \subset \pi_j(\sigma)$ is a chain of 2-cells of X_{j+1} that joins $\pi_j(e_1)$ to $\pi_j(e_2)$, then $\ell \geq m$.*

Proof. Since $\{\bar\sigma_i\}$ is a chain in X_{j+1}, the images under $x_{j+1} : X_{j+1} \to [0,1]$ form a chain in $[0,1]$ joining $x_{j+1}(\pi_j(e_1))$ to $x_{j+1}(\pi_j(e_2))$. But by Lemma 3.5(2) the image $x_{j+1}(\bar\sigma_i)$ is a segment of length $m^{-(j+1)}$, while $|x_{j+1}(\pi_j(e_1)) - x_{j+1}(\pi_j(e_2))| = m^{-j}$. Thus

$$m^{-j} = \operatorname{diam}(\pi_j(e_1) \cup \pi_j(e_2)) \leq \sum_i \operatorname{diam}(x_{j+1}(\bar\sigma_i)) = \ell \cdot m^{-(j+1)}$$

so $\ell \geq m$. □

Definition 4.8. The **open star (resp. closed star)** of a cell σ in a cell complex is the union of the open (respectively closed) cells whose closure contains σ.

Lemma 4.9. *Let e be a vertical 1-cell of X_j, and $\operatorname{St}(e, X_j)$ be the closed star of e in X_j. Let $\bar\sigma_1, \ldots, \bar\sigma_\ell$ be a chain of 2-cells in X_{j+1} such that:*

(1) $\bar\sigma_1, \ldots, \bar\sigma_\ell \subset \pi_j(\operatorname{St}(e, X_j))$.
(2) $\bar\sigma_1 \cap \pi_j(e) \neq \emptyset$.
(3) *There is a 2-cell τ of X_j that is not contained in $\operatorname{St}(e, X_j)$, such that $\bar\sigma_\ell$ intersects $\pi_j(\tau)$.*

Then one of the following holds:

(a) *For some $\bar\sigma_i$, there is a $p \in \pi_j^{-1}(\bar\sigma_i)$ such that $d(y_j(p), 0) \leq 2m_v^{-(j+1)}$.*
(b) *There is a 2-cell σ of X_j that is contained in $\operatorname{St}(e, X_j)$, with vertical 1-cells e, e_1 such that some subchain of $\{\bar\sigma_i\}$ is contained in $\pi_j(\sigma)$ and joins $\pi_j(e)$ to $\pi_j(e_1)$.*

Moreover in case (a) we have

$$\ell \geq \frac{1}{5} \cdot \left[m_v^{(j+1)} \operatorname{dist}(y_j(\pi_j^{-1}(\bar\sigma_1)), 0) - 2\right] . \tag{4.10}$$

Proof. Suppose (a) holds. Since the sets $\{y_j(\pi_j^{-1}(\bar\sigma_k))\}_{1\le k\le i}$ each have diameter $\le 5m_v^{-(j+1)}$ by Lemma 3.5(4), and they form a chain in $S^1(m_v^{-j})$ joining $y_j(\pi_j^{-1}(\bar\sigma_1))$ to $B(0,2m_v^{-(j+1)})\subset S^1(m_v^{-j})$, the triangle inequality gives (4.10).

Now suppose (a) does not hold, i.e. for every $\bar\sigma_i$ and every $p\in\pi_j^{-1}(\bar\sigma_i)$,

$$d(y_j(p),0) > 2m_v^{-(j+1)}\,. \tag{4.11}$$

We may assume without loss of generality that $\bar\sigma_1,\ldots,\bar\sigma_\ell$ is a minimal chain satisfying the hypotheses of the Lemma, so that:

- (i) For all $i>1$, the 2-cell $\bar\sigma_i$ does not intersect $\pi_j(e)$.
- (ii) For all $i<\ell$, the 2-cell $\bar\sigma_i$ does not intersect $\pi_j(\tau')$ for any 2-cell τ' of X_j not contained in $\mathrm{St}(e,X_j)$.

For every i, there is a 2-cell σ_i of $X_j^{(1)}$ such that $\pi_j(\sigma_i)=\bar\sigma_i$, and $\sigma_i\subset\mathrm{St}(e,X_j)$. The 2-cell σ_i is contained in a unique 2-cell τ_i of X_j that contains e.

Let τ be as in (3), and pick $\bar p\in\bar\sigma_\ell\cap\pi_j(\tau)$. Choose $p_\ell\in\sigma_\ell$, $p\in\tau$ such that $\pi_j(p_\ell)=\pi_j(p)=\bar p$. By Lemma 3.3(1) there is a vertical edge path γ in $X_j^{(1)}$ of combinatorial length at most 2 that contains p_ℓ and p. Because $\tau\not\subset\mathrm{St}(e,X_j)$, we know that $e\not\subset\tau$, and so $\tau\ne\tau_\ell$. Therefore γ must intersect $\partial\tau_\ell$ and $\partial\tau$. In view of (4.11) we conclude that p_ℓ and p both lie in the interior of a vertical 1-cell $e_1\subset\tau_\ell\cap\tau$. Since $e\not\subset\tau$, it follows that $e_1\ne e$, so e_1 is the second vertical 1-cell of $\partial\tau_\ell$.

Let i_0 be the minimal i such that the subchain $\bar\sigma_i,\ldots,\bar\sigma_\ell$ is contained in τ_ℓ, i.e. such that $\tau_i=\ldots=\tau_\ell$. Suppose $i_0>1$. Choose $p_{i_0-1}\in\sigma_{i_0-1}$, $p_{i_0}\in\sigma_{i_0}$ such that $\pi_j(p_{i_0-1})=\pi_j(p_{i_0})\in\bar\sigma_{i_0-1}\cap\bar\sigma_{i_0}\subset\pi_j(\tau_{i_0-1})\cap\pi_j(\tau_\ell)$. Reasoning as above, we get that p_{i_0-1} and p_{i_0} belong to the interior of a vertical 1-cell e' of X_j, where $e'\subset\partial\tau_{i_0-1}\cap\partial\tau_\ell$. We cannot have $e'=e$ or $e'=e_1$ by the minimality of the chain $\bar\sigma_1,\ldots,\bar\sigma_\ell$. This is a contradiction. Hence $i_0=1$ and $\bar\sigma_1,\ldots,\bar\sigma_\ell\subset\tau_\ell$. Putting $\sigma=\tau_\ell$ we have shown that (b) holds. □

Proposition 4.12. *For all $k\in\mathbb{Z}$ and $p,p'\in X_k$,*

$$\hat d_\infty(\pi_k^\infty(p),\pi_k^\infty(p'))\ge\hat d_k(p,p')-2m^{-k}\,. \tag{4.13}$$

Proof. Fix $k\in\mathbb{Z}$, $p,p'\in X_k$, and let $\bar p=\pi_k^\infty(p)$, $\bar p'=\pi_k^\infty(p')$. Choose 2-cells $\hat\sigma,\hat\sigma'$ of Y_k such that $p\in\hat\pi^k(\sigma)$, $p'\in\hat\pi^k(\sigma')$.

Let $\hat\sigma_1^0,\ldots,\hat\sigma_{\ell_0}^0$ be an ∞-chain joining $\bar p$ to $\bar p'$. We enlarge this to a new ∞-chain $\hat\sigma,\hat\sigma_1^0,\ldots,\hat\sigma_{\ell_0}^0,\hat\sigma'$, which we relabel as $\hat\sigma_1,\ldots,\hat\sigma_\ell$, where

$\ell = \ell_0 + 2$. Then $\{\hat\sigma_i\}$ is an ∞-chain joining $\bar p$ to $\bar p'$ of length

$$\leq \mathrm{length}(\{\hat\sigma_i^0\}) + 2m^{-k}\,.$$

We will show that we can reduce the generation of the chain $\{\hat\sigma_i\}$ to k without increasing its length, so as to obtain a k-chain from $\bar p$ to $\bar p'$ of length at most $\mathrm{length}(\{\hat\sigma_i\})$. This implies that $\hat d_k(p,p') \leq \mathrm{length}(\{\hat\sigma_i^0\}) + 2m^{-k}$. Taking the infimum over all ∞-chains from p to p' yields (4.13).

Let $j+1$ be the generation of $\{\hat\sigma_i\}$. If $j+1 \leq k$ we are done, so we assume that $j \geq k$. Moreover, we may assume that there is no other ∞-chain $\hat\sigma'_0, \ldots, \hat\sigma'_{\ell'}$ such that:

- $\hat\sigma'_0 = \hat\sigma_0$, $\hat\sigma'_{\ell'} = \hat\sigma_\ell$.
- $\mathrm{length}(\{\hat\sigma'_i\}) \leq \mathrm{length}(\{\hat\sigma_i\})$.
- The generation of $\{\hat\sigma'_i\}$ is at most $j+1$.
- $\{\hat\sigma'_i\}$ has fewer cells of generation $j+1$ than $\{\hat\sigma_i\}$.

Since $\{\hat\sigma_i\}$ is an ∞-chain, by Lemma 3.3 the sequence $\hat\sigma_1, \ldots, \hat\sigma_\ell$ is a $(j+1)$-chain.

Before proceeding further, we first indicate the rough idea of the argument. Although it takes values in $S^1(m_v^{-j})$, we view $y_j : X_j \to S^1(m_v^{-j})$ as a "height function". Since m_v is much larger than m, a string of cells in $\hat\sigma_1, \ldots, \hat\sigma_\ell$ of maximal generation $j+1 > k$ cannot move efficiently in the "vertical" direction, i.e. it cannot change y_j efficiently. Thus any such string is forced to move roughly horizontally, and it may then be replaced by cells of lower generation without increasing $\mathrm{length}(\{\hat\sigma_i\})$. This contradicts the choice of $\{\hat\sigma_i\}$.

We now resume the proof of the proposition.

Let $\hat\sigma_{i_1}, \ldots, \hat\sigma_{i_2}$ be a maximal string of consecutive cells from $\{\hat\sigma_i\}$ of generation $j+1$, so that both $\hat\sigma_{i_1-1}$ and $\hat\sigma_{i_2+1}$ have generation $\leq j$. For $1 \leq i \leq \ell$, let $\sigma_i = \hat\pi^j(\hat\sigma_i)$, $\bar\sigma_i = \hat\pi^{j+1}(\hat\sigma_i)$. For $i_1 \leq i \leq i_2$ let $\hat\tau_i$ be the unique 2-cell of Y_j containing $\hat\sigma_i$, and put $\tau_i = \hat\pi^j(\hat\tau_i)$.

Step 1. If e is a vertical 1-cell of X_j, and $\bar\sigma_{i_3} \cap \pi_j(e) \neq \emptyset$ for some $i_1 \leq i_3 \leq i_2$, then $\mathrm{dist}(y_j((\pi_j)^{-1}(\bar\sigma_{i_3})), 0) < 11m \cdot m_v^{-(j+1)}$.

Let $\bar\sigma_{i_4}, \ldots, \bar\sigma_{i_5}$ be a maximal subsequence of $\bar\sigma_{i_1}, \ldots, \bar\sigma_{i_2}$ such that:

- $i_4 \leq i_3 \leq i_5$, i.e. $\bar\sigma_{i_3}$ belongs to the subsequence.
- The cells $\bar\sigma_{i_4}, \ldots, \bar\sigma_{i_5}$ are all contained in $\pi_j(\mathrm{St}(e, X_j))$.

Claim. At least one of the two cells $\bar{\sigma}_{i_4}, \bar{\sigma}_{i_5}$ must intersect $\pi_j(\tau)$, for some 2-cell τ of X_j that is not contained in $\mathrm{St}(e, X_j)$.

Proof of claim. Suppose the claim were false.

Assume first that σ_{i_4-1} has generation $j+1$. By the maximality of $\bar{\sigma}_{i_3}, \ldots, \bar{\sigma}_{i_5}$, we have $\bar{\sigma}_{i_4-1} \not\subset \pi_j(\mathrm{St}(e, X_j))$, so $\tau_{i_4-1} \not\subset \mathrm{St}(e, X_j)$. Then $\emptyset \neq \bar{\sigma}_{i_4-1} \cap \bar{\sigma}_{i_4} \subset \pi_j(\tau_{i_4-1}) \cap \bar{\sigma}_{i_4}$, proving the claim. Thus we may assume that σ_{i_4-1} has generation $\leq j$, and likewise for σ_{i_5+1}.

Since σ_{i_4-1} is a union of 2-cells of X_j, for some 2-cell τ of X_j, we have $\tau \subset \sigma_{i_4-1}$ and $\bar{\sigma}_{i_4} \cap \pi_j(\tau) \neq \emptyset$. If $\tau \not\subset \mathrm{St}(e, X_j)$, then the claim follows, so we assume that $\tau \subset \mathrm{St}(e, X_j)$, and in particular $e \subset \tau$. Likewise we may assume that σ_{i_5+1} contains a 2-cell τ' of X_j that contains e. Now $\sigma_{i_4-1} \cap \sigma_{i_5+1} \neq \emptyset$, so we may shorten the $j+1$-chain $\{\hat{\sigma}_i\}$ by deleting $\hat{\sigma}_{i_4}, \ldots, \hat{\sigma}_{i_5}$, which is a contradiction. Thus the claim holds. $\square$

We now assume without loss of generality that the claim holds for $\bar{\sigma}_{i_5}$. Hence $\bar{\sigma}_{i_3}, \ldots, \bar{\sigma}_{i_5}$ satisfies the hypotheses of Lemma 4.9.

First suppose that conclusion (a) of Lemma 4.9 holds. Then (4.10) gives

$$i_5 - i_3 + 1 \geq \frac{1}{5} \cdot \left[m_v^{(j+1)} \operatorname{dist}(y_j(\pi_j^{-1}(\bar{\sigma}_{i_3})), 0) - 2 \right] . \tag{4.14}$$

Because $\bar{\sigma}_{i_3}, \bar{\sigma}_{i_5} \subset \mathrm{St}(e, X_j)$, we have $\tau_{i_3}, \tau_{i_5} \subset \mathrm{St}(e, X_j)$, so $e \subset \tau_{i_3} \cap \tau_{i_5}$. It follows that we obtain a $(j+1)$ chain with fewer cells of generation $j+1$ by replacing $\hat{\sigma}_{i_3}, \ldots, \hat{\sigma}_{i_5}$ with the two cells τ_{i_3}, τ_{i_5}. This new chain has length

$$\begin{aligned} &\mathrm{length}(\{\hat{\sigma}_i\}_{i=1}^{\ell}) - \mathrm{length}(\hat{\sigma}_{i_3}, \ldots, \hat{\sigma}_{i_5}) + \mathrm{length}(\hat{\tau}_{i_3}, \hat{\tau}_{i_5}) \\ &\quad = \mathrm{length}(\{\hat{\sigma}_i\}_{i=1}^{\ell}) - m^{-(j+1)}(i_5 - i_3 + 1) - 2m^{-j} \\ &\quad = \mathrm{length}(\{\hat{\sigma}_i\}_{i=1}^{\ell}) - m^{-(j+1)}((i_5 - i_3 + 1) - 2m) . \end{aligned}$$

By the minimality of $\{\hat{\sigma}_i\}_{i=1}^{\ell}$ we obtain

$$(i_5 - i_3 + 1) - 2m \leq 0$$

Using (4.14) we get

$$\operatorname{dist}(y_j(\pi_j^{-1}(\bar{\sigma}_{i_3})), 0) \leq (10m + 2) m_v^{-(j+1)} < 11m \cdot m_v^{-(j+1)} .$$

Now suppose Case (b) of Lemma 4.9 holds, and let σ be as in (b). Then the subchain given by (b) satisfies the hypotheses of Lemma 4.7, and so it contains at least m cells; we may replace these with σ and obtain a new $j+1$-chain with length at most length$(\{\hat{\sigma}_i\}$, and fewer cells of generation $j+1$. This contradicts the choice of $\{\hat{\sigma}_i\}$.

This completes Step 1.

Step 2. We have

$$\operatorname{dist}(y_j(\pi_j^{-1}(\bar{\sigma}_{i_3})),0) < 16m \cdot m_v^{-(j+1)} \tag{4.15}$$

for all $i_1 \le i_3 \le i_2$.

Suppose i_3 violates (4.15). Choose i_4 maximal such that $i_3 \le i_4 \le i_2$ and $\tau_i = \tau_{i_3}$ for all $i_3 \le i \le i_4$. Since we obtain a comparison chain by replacing $\hat{\sigma}_{i_3}, \ldots, \hat{\sigma}_{i_4}$ with $\hat{\tau}_{i_3}$, we get that

$$m^{-(j+1)}(i_4 - i_3 + 1) = \operatorname{length}(\hat{\sigma}_{i_3}, \ldots, \hat{\sigma}_{i_4}) < \operatorname{length}(\hat{\tau}_{i_3}) = m^{-j}\,,$$

i.e. $i_5 - i_3 + 1 < m$. Applying Lemma 3.5(4) and the fact that the sets $y_j(\pi_j^{-1}(\bar{\sigma}_i))$ form a chain in $S^1(m_v^{-j})$, we get

$$\operatorname{dist}(y_j(\pi_j^{-1}(\bar{\sigma}_{i_4})),0) \ge 11m \cdot m_v^{-(j+1)}\,. \tag{4.16}$$

First suppose $i_4 < i_2$, i.e. that $\operatorname{gen}(\hat{\sigma}_{i_4+1}) = j+1$. Then $\tau_{i_4+1} \ne \tau_{i_4}$ and $\pi_j(\tau_{i_4+1}) \cap \pi_j(\tau_{i_4}) \ne \emptyset$. By (4.16) it follows that $\bar{\sigma}_{i_4}$ must intersect $\pi_j(e)$ for some vertical 1-cell e of τ_{i_4}. But then (4.16) contradicts Step 1.

Now suppose $i_4 = i_2$. Then $\operatorname{gen}(\hat{\sigma}_{i_4+1}) \le j$, and so σ_{i_4+1} contains a 2-cell τ of X_j such that $\bar{\sigma}_{i_4} \cap \pi_j(\tau) \ne \emptyset$. If $\tau = \tau_{i_3} = \tau_{i_4}$, then we may discard $\hat{\sigma}_{i_3}, \ldots, \hat{\sigma}_{i_4}$, contradicting the minimality of the chain $\hat{\sigma}_1, \ldots, \hat{\sigma}_\ell$. Therefore $\tau \ne \tau_{i_3}$, and reasoning as above we find that $\bar{\sigma}_{i_4} \cap \pi_j(e) \ne \emptyset$ for some vertical 1-cell of τ_{i_3}. Using (4.16), we again get a contradiction to Step 1.

Step 3. Shifting $\bar{\sigma}_{i_1}, \ldots, \bar{\sigma}_{i_2}$ toward the 1-skeleton of X_j.

Pick $i_1 \le i \le i_2$. Since $\sigma_i \subset \pi_j^{-1}(\bar{\sigma}_i)$, by Step 2 and Lemma 3.5(4) we have

$$\begin{aligned}\operatorname{dist}(y_j(\sigma_i),0) &\le \operatorname{dist}(y_j(\pi_j^{-1}(\bar{\sigma}_i)),0) + \operatorname{diam}(y_j(\pi_j^{-1}(\bar{\sigma}_i)))\\ &< (16m+5)\cdot m_v^{-(j+1)}\,.\end{aligned}$$

Hence there is vertical gallery (see Definition 4.4) $\mu_{i1}, \ldots, \mu_{i\ell_i}$ in $X_j^{(1)}$ with $\ell_i < 16m+5$ that starts from $\sigma_i = \mu_{i1}$, such that $\mu_{i\ell_i}$ contains a horizontal 1-cell e_i' of $X_j^{(1)}$ that is contained in a horizontal 1-cell e_i of X_j, see Figure ???. Since the gallery is vertical, we have $x_j(\sigma_{i-1}) = x_j(e_{i-1}')$, $x_j(\sigma_i) = x_j(e_i') \subset \mathbb{R}$.

Claim. $e_{i-1}' \cap e_i' \ne \emptyset$ for all $i_1 < i \le i_2$.

Proof of claim. The images $x_j(\sigma_{i-1})$, $x_j(\sigma_i) \subset \mathbb{R}$ are intervals of length $m^{-(j+1)}$ that have an endpoint in common because $x_j(\sigma_{i-1}) = x_{j+1}(\bar{\sigma}_{i-1})$, $x_j(\sigma_i) = x_{j+1}(\bar{\sigma}_i)$, and $\bar{\sigma}_{i-1} \cap \bar{\sigma}_i \neq \emptyset$. We may therefore choose 0-cells v_{i-1}, v_i of $X_j^{(1)}$ such that $v_{i-1} \in e'_{i-1}$, $v_i \in e'_i$, and $x_j(v_{i-1}) = x_j(v_i)$. Choose 0-cells p_{i-1}, p_i of $X_j^{(1)}$ such that $p_{i-1} \in \sigma_{i-1}$, $p_i \in \sigma_i$, $\pi_j(p_{i-1}) = \pi_j(p_i)$, $x_j(p_{i-1}) = x_j(p_i) = x_j(v_{i-1}) = x_j(v_i)$. We may join p_{i-1} to p_i with a vertical edge path in $X_j^{(1)}$ of combinatorial length at most 2 by Lemma 3.3(1). Using the vertical cells in the vertical galleries $\{\mu_{i-1n}\}$, $\{\mu_{in}\}$, we get vertical edge paths γ_{i-1}, γ_i in $X_j^{(1)}$ joining v_{i-1} to p_{i-1} and v_i to p_i, respectively, where γ_{i-1}, γ_i have combinatorial length $< 16m + 5$. Concatenating γ_{i-1}, γ, γ_i we get a vertical edge path joining v_{i-1} to v_i of combinatorial length $< 32m + 12 = 140$. Since $m_v = 3L \geq 300$, this forces $v_{i-1} = v_i$, proving the claim. □

Claim. $\sigma_{i_1-1} \cap e'_{i_1} \neq \emptyset$ and $e'_{i_2} \cap \sigma_{i_2+1} \neq \emptyset$.

Proof of claim. There is a 2-cell $\tau \subset \sigma_{i_1-1}$ of X_j such that $\pi_j(\tau) \cap \bar{\sigma}_{i_1} \neq \emptyset$. We may find 0-cells p, p_{i_1} of $X_j^{(1)}$ such that $p \in \tau$, $p_{i_1} \in \sigma_{i_1}$, and $\pi_j(p) = \pi_j(p_{i_1})$. There is a vertical edge path γ in $X_j^{(1)}$ from p to p_{i_1} of combinatorial length at most 2, and a vertical edge path γ_{i_1} in $X_j^{(1)}$ of combinatorial length $< 16m$ joining p_{i_1} to an endpoint v_{i_1} of e'_{i_1}. Combining γ and γ_{i_1} we get a vertical edge path of length $< 16m + 2$ in $X_j^{(1)}$ starting at $p \in \tau$ and ending in the 1-skeleton of X_j. This implies that $v_{i_1} \in \partial\tau$. Hence $\sigma_{i_1-1} \cap e'_{i_1} \neq \emptyset$ and the claim holds. □

Combining the two claims, we get that

$$\sigma_1, \ldots, \sigma_{i_1-1}, e'_{i_1}, \ldots, e'_{i_2}, \sigma_{i_2+1}, \ldots, \sigma_\ell \tag{4.17}$$

forms a chain in X_j. Therefore, after modifying $\hat{\sigma}_i$ for $i_1 \leq i \leq i_2$ if necessary, we may assume without loss of generality that $\sigma_i = \mu_{i\ell_i}$, and in particular σ_i contains e'_i.

Step 4. The final contradiction.

Pick i_3 such that $i_1 \leq i_3 \leq i_2$.

Suppose some interior point of e_{i_3} is contained in $\sigma_{i_1-1} \cup \sigma_{i_2+1}$. Since $\sigma_{i_1-1} \cup \sigma_{i_2+1}$ is a subcomplex of X_j, this implies that e_{i_3} is contained in $\sigma_{i_1-1} \cup \sigma_{i_2+1}$. Therefore applying Lemma 4.3 we may delete the edge e'_{i_3} from the collection (4.17), and it will still contain a chain, and likewise we may delete $\hat{\sigma}_{i_3}$ from the collection $\hat{\sigma}_1, \ldots, \hat{\sigma}_\ell$ and it will still contain a chain; this contradicts the minimality of $\{\hat{\sigma}_i\}$.

Now suppose e_{i_3} is contained in $e'_{i_1} \cup \ldots \cup e'_{i_2}$. Taking the union of the collection (4.17) with e_{i_3} and applying Lemma 4.3, it follows that we may remove from the collection (4.17) each e'_i that is contained in e_{i_3}, and add e_{i_3}, and the resulting collection will contain a chain from σ_1 to σ_ℓ. Therefore we may remove from $\hat{\sigma}_1, \ldots, \hat{\sigma}_\ell$ each $\hat{\sigma}_i$ with $i_1 \leq i \leq i_2$ such that $e'_i \subset e_{i_3}$, and add the 2-cell $\hat{\tau}_{i_3}$, and the resulting collection will contain a $j+1$-chain that has length at most length$(\{\hat{\sigma}_i\})$ and fewer cells of generation $j+1$, contradicting the definition of $\hat{\sigma}_1, \ldots, \hat{\sigma}_\ell$.

Therefore e_{i_3} is not contained in $\sigma_{i_1-1} \cup \sigma_{i_1} \cup \ldots \cup \sigma_{i_2} \cup \sigma_{i_2+1}$. But then we may remove from the $j+1$-chain $\hat{\sigma}_1, \ldots, \hat{\sigma}_\ell$ some 2-cell $\hat{\sigma}_i$, with $i_1 \leq i \leq i_2$ and $e'_i \subset e_{i_3}$ and still have a chain. This contradicts the minimality of $\hat{\sigma}_1, \ldots, \hat{\sigma}_\ell$.

We conclude that $j+1 \leq k$, completing the proof of Proposition 4.12.

□

The following two corollaries of Proposition 4.12 relate the distance in X_∞ or X_j with the combinatorial distance.

Corollary 4.18. *For every C there is an $N = N(C)$ such that if $p, p' \in X_j$ and $\hat{d}_\infty(\pi_j^\infty(p), \pi_j^\infty(p')) < Cm^{-j}$, then there is a chain of at most N cells of X_j joining p to p'.*

Proof. This follows from Proposition 4.12 and Lemma 4.5. □

Corollary 4.19. *For $p_0, p_1 \in X_\infty$ let*

$$J(p_0,p_1) = \left\{ j \;\middle|\; \begin{array}{c} (\pi_j^\infty)^{-1}(p_0), (\pi_j^\infty)^{-1}(p_1) \textit{ do not} \\ \textit{intersect adjacent cells of } X_j \end{array} \right\}, \tag{4.20}$$

and let $\underline{J}(p_0,p_1) = \inf J(p_0,p_1)$, where as usual the infimum of the empty set is ∞. Then

$$m^{-\underline{J}(p_0,p_1)} \leq \hat{d}_\infty(p_0,p_1) \leq 2m \cdot m^{-\underline{J}(p_0,p_1)}, \tag{4.21}$$

where by convention we let $m^{-\underline{J}(p_0,p_1)} = 0$ when $\underline{J}(p_0,p_1) = \infty$.

Proof. Note that $\underline{J}(p_0,p_1) = \inf J(p_0,p_1) = 1 + \sup(\mathbb{Z} \setminus J(p_0,p_1))$.

If $k \notin J(p_0, p_1)$, then $(\pi_k^\infty)^{-1}(p_0)$ and $(\pi_k^\infty)^{-1}(p_1)$ intersect adjacent cells, and this gives $\hat{d}_\infty(p_0, p_1) \le 2m^{-k}$ by the definition of $\hat{d}_\infty$. Thus

$$\begin{aligned}\hat{d}_\infty(p_0, p_1) &\le 2 \inf\{m^{-k} \mid k \notin J(p_0, p_1)\} \\ &\le 2m \cdot \inf\{m^{-(k+1)} \mid k \notin J(p_0, p_1)\} \\ &= 2m \cdot \sup\{m^{-k} \mid k \in J(p_0, p_1)\} \\ &= 2m \cdot m^{\underline{J}(p_0, p_1)}.\end{aligned}$$

If $k \in J(p_0, p_1)$ then then any k-chain connecting $(\pi_k^\infty)^{-1}(p_0)$ with $(\pi_k^\infty)^{-1}(p_1)$ must contain at least 3 cells, and hence

$$\hat{d}_k((\pi_k^\infty)^{-1}(p_0), (\pi_k^\infty)^{-1}(p_1)) \ge 3m^{-k}. \tag{4.22}$$

The lower bound in (4.21) then follows applying Proposition 4.12. □

We remark that the following lemma is not really essential to the discussion. Even without knowing that $\hat{d}_\infty$ is a distance, we could quotient out the sets of zero diameter and work in the resulting metric space, cf. Section 11.

Lemma 4.23. *Keeping the notation from Corollary 4.19, the set $J(p_0, p_1)$ is nonempty iff $p_0 \neq p_1$; in particular $\hat{d}_\infty$ is a distance function on X_∞.*

Proof. Clearly $J(p_0, p_1) \neq \emptyset \implies p_0 \neq p_1$. We will show that if $J(p_0, p_1)$ is empty then $p_0 = p_1$.

For all $j \in \mathbb{Z}$, since $j \notin J(p_0, p_1)$, the sets $(\pi_j^\infty)^{-1}(p_0)$, $(\pi_j^\infty)^{-1}(p_1)$ intersect adjacent cells of X_j, and therefore the values of x_j on these sets agree to within error $2m^{-j}$. Since $x_\infty(p_i) = x_j((\pi_j^\infty)^{-1}(p_i))$, we get that $|x_\infty(p_0) - x_\infty(p_1)| \le 2m^{-j}$. As this holds for all j we conclude that $x_\infty(p_0) = x_\infty(p_1)$.

Let $t = x_\infty(p_0) = x_\infty(p_1)$.

If $t \in \cup_{j\in\mathbb{Z}} (m^{-j}\mathbb{Z})$ select $j_0 \in \mathbb{Z}$ such that $t \in m^{-j_0}\mathbb{Z}$, and otherwise let $j_0 \in \mathbb{Z}$ be arbitrary. Then $t \notin m^{-(j+1)}\mathbb{Z} \setminus m^{-j}\mathbb{Z}$ for every $j \ge j_0$, and by the definition of $\mathcal{R}_j$, it follows that $(\pi_{j_0}^\infty)^{-1}(p_i)$ contains a unique element $\hat{p}_i \in X_{j_0}$ for $i = 0, 1$.

Let $j_1 \ge 0$ be arbitrary.

Claim. *$\hat{p}_0$ may be joined to $\hat{p}_1$ by a chain of at most four 2-cells of $X_{j_0}^{(j_1)}$.*

Pick $j \ge j_0$, to be determined later.

If $t \in m^{-j}\mathbb{Z}$, we define $t_\pm = t \pm m^{-j}$; if $t \notin m^{-j}\mathbb{Z}$ then for some $k_j \in \mathbb{Z}$ we have $t \in (k_j m^{-j}, (k_j+1)m^{-j})$, and then we let $t_- = k_j m^{-j}$, $t_+ = (k_j+1)m^{-j}$. Let $g_j = \min(\{\ell \mid t_- \in m^{-\ell}\mathbb{Z}\}, \{\ell \mid t_+ \in m^{-\ell}\mathbb{Z}\})$. Since $|t_\pm - t| \leq m^{-j}$, we will have

$$\min(g_j - j_0, j - j_0) > j_1 \tag{4.24}$$

for all but finitely many j, so we assume (4.24) holds.

Choose adjacent cells τ_0, τ_1 of X_j such that $\tau_i \cap (\pi_j^\infty)^{-1}(p_i) \neq \emptyset$, and cells $\hat{\tau}_0, \hat{\tau}_1$ of X_{j_0} such that $\pi^j_{j_0}(\hat{\tau}_i) = \tau_i$. Let $v \in \tau_0 \cap \tau_1$ be a 0-cell of X_j, and choose $\hat{v}_i \in \hat{\tau}_i$ such that $\pi^j_{j_0}(\hat{v}_i) = v$.

Case 1: $x_j(v) = t$. As $\pi^j_{j_0}$ is injective on $x_j^{-1}(t)$, we have $\hat{v}_0 = \hat{v}_1$, so $\hat{\tau}_0 \cap \hat{\tau}_1 \neq \emptyset$. Thus $\hat{p}_0, \hat{p}_1$ may be joined by a chain of at most two 2-cells of $X^{(j-j_0)}_{j_0}$, and hence also by a chain of at most two 2-cells of $X^{(j_1)}_{j_0}$, proving the claim.

Case 2: $x_j(v) \neq t$. Then $x_j(v) \in \{t_-, t_+\}$. Since $\pi^j_{j_0}(\hat{v}_i) = v$ it follows that $\hat{v}_0$ may be joined to $\hat{v}_0$ by a vertical edge path in $X^{(g_j - j_0)}_{j_0}$ of combinatorial length at most 2. Hence $\hat{p}_0$ may be joined to $\hat{p}_1$ may be joined by a chain of at most four 2-cells of $X^{(j_1)}_{j_0}$, and the claim holds in this case.

Since j_1 is arbitrary, the claim forces $\hat{p}_0 = \hat{p}_1$.

$\square$

5. David-Semmes regularity of the projection $\mathbb{R}^2 \to X_\infty$ and the lower bound on the topological dimension

In this section we prove part (1) of Theorem 1.1, and the lower bound in part (2), in the $n = 2$ case.

Let $\hat{d}^Y_\infty$, α, and d_α be as in Section 2. We recall that $\mathcal{L}^2$ denotes Lebesgue measure on $\mathbb{R}^2$, and for $j \in \mathbb{Z} \cup \{\infty\}$ we let $\mu_j = (\hat{\pi}^j)_\# \mathcal{L}^2$.

Lemma 5.1. *There is a constant C such that $C^{-1} d_\alpha \leq \hat{d}^Y_\infty \leq C\, d_\alpha$. Moreover, the Q-dimensional Hausdorff measure is uniformly comparable to a Lebesgue measure.*

Proof. The metric on $Y = \mathbb{R}^2$ is uniformly comparable to a product metric $d_{hor} \times d_{vert}$; d_{hor} is just the standard metric on $\mathbb{R}$; on the other hand, d_{vert} is the largest pseudodistance which makes each cell of generation $j \in \mathbb{Z}$ of the standard m_v-adic subdivision of $\mathbb{R}$ have diameter at most m^{-j}. Specifically, for any $x, x' \in \mathbb{R}$, the distance $d_{vert}(x, x')$

is the infimum of the quantities $\sum_i \text{length}(\sigma_i)$, where $\sigma_1, \ldots, \sigma_\ell$ is a chain of m_v-adic intervals joining x to x', and $\text{length}(\sigma_i) = m^{-j_i}$ when σ_i has generation j_i (cf. Section 4); since $|x - x'| \leq \sum_i m_v^{-j_i} = \sum_i (\text{length}(\sigma_i))^{\frac{1}{\alpha}}$ one readily checks that d_{vert} is uniformly comparable to the snowflake of the standard metric on $\mathbb{R}$ by the power $\alpha = \frac{\log m}{\log m_v}$.

Finally, as d^Y_∞-balls of radius r have Lebesgue measure $\approx r^Q$, we conclude that $\mathcal{H}^Q \approx \mathcal{L}^2$ $\square$

Lemma 5.2 (David-Semmes regularity of $\hat{\pi}^\infty$)**.**

(1) *The projection map* $\hat{\pi}^\infty : (\mathbb{R}^2, \hat{d}^Y_\infty) \to (X_\infty, \hat{d}_\infty)$ *is a David-Semmes regular map.*
(2) $(X_\infty, \hat{d}_\infty)$ *is Ahlfors Q-regular, where* $Q = 1 + \alpha^{-1}$.
(3) $(X_\infty, \hat{d}_\infty)$ *has topological dimension at least* 2.

Proof. (1). Suppose σ is a 2-cell of Y_j. Then by the definition of $\hat{d}_\infty$, the pullback of $(\hat{\pi}^\infty)^* \hat{d}_\infty$ is a pseudodistance on $\mathbb{R}^2$ with respect to which the diameter of σ is $\leq m^{-j}$. From the definition of $\hat{d}^Y_\infty$ we therefore have $\hat{d}^Y_\infty \leq (\hat{\pi}^\infty)^* \hat{d}_\infty$, so $\hat{\pi}^\infty : (\mathbb{R}^2, \hat{d}^Y_\infty) \to (X_\infty, \hat{d}_\infty)$ is 1-Lipschitz.

Pick $\bar{p} \in X_\infty$, $r > 0$.

We want to show that $(\hat{\pi}^\infty)^{-1}(B(\bar{p}, r))$ may be covered by a controlled number of balls of radius comparable to r in $(\mathbb{R}^2, \hat{d}^Y_\infty)$. To that end, we choose $j \in \mathbb{Z}$ such that $m^{-(j+1)} < r \leq m^{-j}$, and pick $p \in (\pi^\infty_j)^{-1}(\bar{p})$.

Since $(\pi^\infty_j)^{-1}(B(\bar{p}, m^{-j})) \subset B(p, 3m^{-j})$ by Proposition 4.12, we have

$$(\hat{\pi}^\infty)^{-1}(B(\bar{p}, r)) \subset (\hat{\pi}^\infty)^{-1}(B(\bar{p}, m^{-j}))$$
$$= (\hat{\pi}^j)^{-1}(\pi^\infty_j)^{-1}(B(\bar{p}, m^{-j})) \subset (\hat{\pi}^j)^{-1}(B(p, 3m^{-j}))\,.$$

Note that $B(p, 3m^{-j})$ is contained in a controlled number of 2-cells of X_j by Lemma 3.4(7), and the inverse image of each of these may be covered by at most 27 cells of Y_j by Lemma 3.4(8). Since a cell of Y_j has $\hat{d}^Y_\infty$-diameter $\leq m^{-j} < m \cdot r$, we are done.

(2). This follows immediately from (1) since regular maps preserve Ahlfors-regularity.

(3). By (1) the point inverses $(\hat{\pi}^\infty)^{-1}(\bar{p})$ have controlled cardinality; in particular, they are totally disconnected. Therefore $\hat{\pi}^\infty$ cannot decrease the topological dimension (see for example [Eng78, Thm. 1.24.4]); therefore, the topological dimension of X_∞ is at least that of $\mathbb{R}^2$.

One may give a more concrete proof that X_∞ has topological dimension ≥ 2 along the following lines. Let f be the restriction of $\hat{\pi}^\infty : \mathbb{R}^2 \to X_\infty$ to the boundary of $[0,1]^2$; note also that f is injective. To see this, suppose that $p, p' \in \partial[0,1]^2$ are distinct points with $f(p) = f(p')$. Then $x(p) = x_\infty(f(p)) = x_\infty(f(p')) = x(p')$, and by the definition of the equivalence relation we must have $x(p) = x(p') \in m^{-j}\mathbb{Z} \setminus m^{-(j-1)}\mathbb{Z}$ for some $j \in \mathbb{Z}$. In view of the definition of $\mathcal{R}_\infty$, by inspection we get a contradiction to the fact that $p, p' \in \partial[0,1]^2$. Letting $\gamma = f(\partial[0,1]^2) \subset X_\infty$ be the image, we apply the Tietze extension theorem to extend $g = f^{-1} : \gamma \to \partial[0,1]^2$ to a continuous map $\hat{g} : X_\infty \to \mathbb{R}^2$. By degree theory, the origin is a stable value of the composition

$$[0,1]^2 \xrightarrow{\hat{\pi}^\infty} X_\infty \xrightarrow{\hat{g}} \mathbb{R}^2 ,$$

so $\hat{g}$ cannot be approximated by a map to $\mathbb{R}^2 \setminus \{0\}$. Therefore X_∞ has topological dimension at least 2.

□

6. The upper bound on the Assouad-Nagata dimension

The proof of the upper bound on the topological dimension is more subtle than the proof of the lower bound. Note that the existence of a map $\mathbb{R}^2 \to X_\infty$ with finite point inverses is not by itself enough to imply that X_∞ has topological dimension 2: recall that the Peano curve $[0,1] \to [0,1]^2$ is a finite-to-one surjective map, showing that such maps can increase the topological dimension.

We now recall the definition of the Assouad-Nagata dimension. Let $\mathcal{C}$ be a cover of a metric space X. Then $\mathcal{C}$ is *r-bounded* if $\mathrm{diam}(C) \leq r$ for all $C \in \mathcal{C}$, and $\mathcal{C}$ has *r-multiplicity at most k* if every ball of radius r intersects at most k elements of $\mathcal{C}$.

Definition 6.1. [Ass82, LS05] The *Assouad-Nagata dimension* of a metric space X is the infimum of the integers $n \geq 0$ such that for some $c > 0$ and every $r > 0$, there is a cover $\mathcal{C}$ of X that is cr-bounded and has r-multiplicity at most $n+1$.

Note that the Assouad-Nagata dimension is bounded below by the topological dimension [LS05, Prop. 2.2].

Theorem 6.2. *The Assouad-Nagata dimension of X_∞ is at most 2.*

Proof. We exhibit "good coverings" in the sense of [LS05, Prop. 2.5](4): we will show that for every $k \in \mathbb{Z}$, there is a cover $\mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ of X_∞

such that $\mathcal{C}_i$ is m^{-k}-bounded and has $m^{-(k+1)}$-multiplicity 1 for all $i \in \{0,1,2\}$. This implies that the Assouad-Nagata dimension of X_∞ is at most 2.

Consider the family Ω_2 of 2-cells of X_k; from each $\sigma \in \Omega_2$ produce a subset c_σ by taking the closure of the subset of $\sigma^{(1)}$ obtained by removing those points $p \in \sigma^{(1)}$ which satisfy one of the following:

$$\begin{aligned} d_{\mathbb{R}}\left(x(p), \min x(\sigma^{(1)})\right) &\le m^{-(k+1)}/2, \\ d_{\mathbb{R}}\left(x(p), \max x(\sigma^{(1)})\right) &\le m^{-(k+1)}/2, \\ d_{S^1}\left(y_k(p), 0\right) &\le 3m_v^{-(k+1)}. \end{aligned} \tag{6.3}$$

Note that $\operatorname{diam} \hat{\pi}_k^\infty(c_\sigma) \le m^{-k}$. Consider now different cells $\sigma_0, \sigma_1 \in \Omega_2$; for $i = 0,1$ let $p_i \in \hat{\pi}_k^\infty(c_{\sigma_i})$; as c_{σ_i} does not meet the 1-skeleton of X_k, any cell τ of X_{k+1} intersecting $(\hat{\pi}_k^\infty)^{-1}(p_i)$ must intersect a given cell τ_i which belongs to the set of those subcells of $\hat{\pi}_k^{k+1}(\sigma_i)$ which meet $\hat{\pi}_k^{k+1}(c_{\sigma_i})$. Let τ_i' be a cell adjacent to τ_i; recall that X_{k+1} is obtained from X_k by quotienting by $(\hat{\pi}^k \circ \Phi^k)_* \mathcal{R}_0$; and thus (6.3) guarantees that either τ_0' and τ_1' are not adjacent or the sets $x(\tau_1')$ and $x(\tau_0')$ are at distance at least $m^{-(k+1)}$. Thus, applying Corollary 4.19 and the fact that x is 1-Lipschitz, we obtain

$$\hat{d}_\infty(\hat{\pi}_k^\infty(c_{\sigma_0}), \hat{\pi}_k^\infty(c_{\sigma_1})) \ge m^{-(k+1)}. \tag{6.4}$$

Let Ω_1 be the collection of 1-cells of X_k; from $e \in \Omega_1$ produce a subset c_e as follows: let $e^{(1)}$ be the subdivision of e in $X_k^{(1)}$ and let C_e be the collection of those points p which belong to a 2-cell of $X_k^{(1)}$ which intersects e; then, if e is vertical, c_e is obtained from C_e by taking the points $p \in C_e$ satisfying:

$$\begin{aligned} d_{\mathbb{R}}(x(p), x(e)) &\le m^{-(k+1)}/2 \\ d_{S^1}(y_k(\tau), 0) &\ge 5m_v^{-(k+1)}; \end{aligned} \tag{6.5}$$

if e is horizontal, c_e is obtained from C_e by taking the points $p \in C_e$ satisfying:

$$\begin{aligned} d_{\mathbb{R}}(x(p), \{\max x(e), \min x(e)\}) &\ge m^{-(k+1)} \\ d_{S^1}(y_k(p), 0) &\le 3m_v^{-(k+1)}. \end{aligned} \tag{6.6}$$

Let e_0, e_1 be distinct cells in Ω_1. Then either $x(e_0)$ and $x(e_1)$ are at distance $m^{-(k+1)}$ apart, or there are no adjacent cells τ_i of X_{k+1} such that $\tau_i \cap \hat{\pi}_k^{k+1}(c_{e_i}) \neq \emptyset$. Thus by Corollary 4.19 we conclude that:

$$\hat{d}_\infty\left(\hat{\pi}_k^{k+1}(e_0), \hat{\pi}_k^{k+1}(e_1)\right) \ge m^{-(k+1)}. \tag{6.7}$$

Let Ω_0 be the collection of vertices of X_k. For $v \in \Omega_0$ consider the set C_v consisting of those 2-cells of $X_k^{(1)}$ within combinatorial distance 5 from a cell containing v; then c_v consists of those points $p \in C_v$ satisfying:

$$\begin{aligned} d_{\mathbb{R}}(x(p), x(v)) &\leq m^{-(k+1)} \\ d_{S^1}(y_k(p), 0) &\leq 5m_v^{-(k+1)}. \end{aligned} \tag{6.8}$$

Then $\operatorname{diam} \hat{\pi}_k^\infty(c_v) \leq m^{-(k+1)}$. Let v_0 and v_1 distinct vertices. If $x(v_0) \neq x(v_1)$ then $\hat{d}_\infty(\hat{\pi}_k^\infty(c_{v_0}), \hat{\pi}_k^\infty(c_{v_1})) \geq m^{-(k+1)}$. Otherwise, there are no adjacent cells τ_i of X_{k+1} such that $\tau_i \cap \hat{\pi}_k^{k+1}(c_{v_i}) \neq \emptyset$. Thus Corollary 4.19 gives:

$$\hat{d}_\infty(\hat{\pi}_k^\infty(c_{v_0}), \hat{\pi}_k^\infty(c_{v_1})) \geq m^{-(k+1)}. \tag{6.9}$$

The families $\{\hat{\pi}_k^\infty(c_\sigma)\}_{\sigma\in\Omega_2}$, $\{\hat{\pi}_k^\infty(c_e)\}_{e\in\Omega_1}$ and $\{\hat{\pi}_k^\infty(c_v)\}_{e\in\Omega_0}$ provide a good covering of X_∞. □

7. The Poincaré inequality

In this section we prove that (X_∞, μ) satisfies a Poincaré inequality. Following Semmes [Sem], we do this by showing that any two points $p, q \in X_\infty$ can be joined by a "pencil" — a good measured family of curves. To obtain such a family, we first construct a string of horizontal galleries that connects p to q; we then convert this to a measured family of curves by replacing each 2-cell with the corresponding (appropriately normalized) measured family of horizontal geodesics, and then concatenating.

7.1. **Construction of galleries.**

Lemma 7.1. *For each $C > 0$ there is an $L = L(C)$ such that the following holds. For every $j \in \mathbb{Z}$, and every pair σ_0, σ_1 of 2-cells of X_j such that*

$$\hat{d}_\infty\big(\pi_j^\infty(\sigma_0), \pi_j^\infty(\sigma_1)\big) \leq Cm^{-j}; \tag{7.2}$$

there is a horizontal gallery T in X_j whose combinatorial length is at most L and which starts at σ_0 and ends at σ_1.

Proof. We first show that if $\sigma_0 \cap \sigma_1 \neq \emptyset$ there is a horizontal gallery S of length at most L_0 starting at σ_0 and ending at σ_1. If σ_0 and σ_1 share a vertical 1-cell we can just take $S = \{\sigma_0, \sigma_1\}$. If σ_0 and σ_1 share a horizontal 1-cell, we know by the definition of $\mathcal{R}_{j-1}$ that there is a horizontal gallery S of length at most $C_0 = C_0(m, m_v)$ which starts at σ_0 and ends at σ_1. If σ_0 and σ_1 share only a 0-cell, we know by Lemma 3.4 (5) that we can find a gallery S joining them and of length at most $C_1 = C_1(m, m_v)$. This can be turned into a horizontal gallery of length at most C_0C_1 by replacing consecutive cells which share a horizontal edge by a horizontal gallery of length at most C_0 connecting them. Thus one can take $L_0 = C_0C_1$.

We now turn to the general case. By Corollary 4.18 there is a chain T_0 in X_j consisting of at most $N(C)$ 2-cells, which starts at σ_0 and ends at σ_1. For any pair of consecutive 2-cells in T_0 we can use the discussion in the previous paragraph to construct a horizontal gallery connecting them. We thus obtain a horizontal gallery T of length at most $N(C)L_0$ starting at σ_0 and ending at σ_1. □

Next, using Lemma 7.1, we will show that one can connect any pair of points $p, q \in X_\infty$ by stringing together geometrically shrinking sequences of horizontal galleries. This is made precise in the following definition.

Definition 7.3. Let $p, q \in X_\infty$. A **string of galleries** connecting p to q consists of four sequences $\{\sigma_j\}_{j\geq j_0}$, $\{\tau_j\}_{j\geq j_0}$, $\{\mathcal{G}_j\}_{j>j_0}$, $\{\mathcal{T}_j\}_{j>j_0}$ which satisfy the following additional conditions for some constant C_0:

- **(G1):** $\hat{d}_\infty(p, q) \approx_{C_0} m^{-j_0}$;
- **(G2):** σ_j and τ_j are cells of X_j and $\sigma_{j_0} = \tau_{j_0}$;
- **(G3):** We have the following control on the distances from p and q:

$$\begin{aligned} \hat{d}_\infty\big(p, \pi_j^\infty(\sigma_j)\big) &\leq C_0 m^{-j}; \\ \hat{d}_\infty\big(q, \pi_j^\infty(\tau_j)\big) &\leq C_0 m^{-j}; \end{aligned} \tag{7.4}$$

- **(G4):** For each $j \geq j_0$ the vertical faces of σ_j and τ_j are ordered: we denote those of σ_j by $e_j^{(0)}$, $e_j^{(1)}$, and those of τ_j by $f_j^{(0)}$ and $f_j^{(1)}$. We also require $e_{j_0}^{(1)} = f_{j_0}^{(0)}$;
- **(G5):** $\mathcal{G}_j$ (resp. $\mathcal{T}_j$) is a collection of m horizontal galleries $\{\mathcal{G}_j^{(1)}, \cdots, \mathcal{G}_j^{(m)}\}$ (resp. $\{\mathcal{T}_j^{(1)}, \cdots, \mathcal{T}_j^{(m)}\}$); let $\{e_{j-1,1}^{(0)}, \cdots, e_{j-1,m}^{(0)}\}$ (resp. $\{f_{j-1,1}^{(1)}, \cdots, f_{j-1,m}^{(1)}\}$) denote the 1-cells in the first subdivision of $e_{j-1}^{(0)}$ (resp. $f_{j-1}^{(1)}$); then $\mathcal{G}_j^{(i)}$ (resp. $\mathcal{T}_j^{(i)}$) is a horizontal gallery of at most C_0 cells which connects $e_j^{(1)}$ (resp. $\pi_{j-1}(f_{j-1,i}^{(1)})$) to $\pi_{j-1}(e_{j-1,1}^{(0)})$ (resp. $f_j^{(1)}$).

Lemma 7.5. *Suppose $p, q \in X_\infty$ and $\hat{d}_\infty(p, q) \in [m^{-j_0-1}, m^{-j_0})$. Then there is a string of galleries connecting p to q where the constant C_0 does not depend on the pair p, q.*

Proof. As $\hat{d}_\infty(p,q) < m^{-j_0}$ by Proposition 4.12 we conclude that:

$$d_{j_0}\left((\pi^\infty_{j_0})^{-1}(p), (\pi^\infty_{j_0})^{-1}(q)\right) < 3m^{-j_0}, \tag{7.6}$$

and thus $(\pi^\infty_{j_0-1})^{-1}(p)$ and $(\pi^\infty_{j_0-1})^{-1}(q)$ intersect adjacent cells of X_{j_0-1}; we can therefore find a C_0 independent of p, q and a single cell $\sigma_{j_0} = \tau_{j_0}$ of X_{j_0} such that (7.4) holds with $j = j_0$. For each j choose σ_j and τ_j such that $(\pi^\infty_j)^{-1}(p) \cap \sigma_j \neq \emptyset$ and $(\pi^\infty_j)^{-1}(q) \cap \tau_j \neq \emptyset$. We now choose a first and a last vertical edge for each σ_j and τ_j as in **(G4)**. We now construct the collection of galleries $\{\mathcal{G}_j\}_{j>j_0}$ and $\{\mathcal{T}_j\}_{j>j_0}$. By symmetry we just focus on the construction of $\mathcal{G}_j$. By the choice of the cells $\{\sigma_j\}_{j\ge j_0}$ we have that:

$$d_j\left(\sigma_j, \pi_{j-1}(\sigma_{j-1})\right) \le C_0 m^{-j}; \tag{7.7}$$

therefore, by possibly enlarging C_0, for each $i \in \{1, \cdots, m\}$ we can choose by Lemma 7.1 a horizontal gallery consisting of at most C_0 cells

$$\mathcal{G}^{(i)}_j = \{\theta_1, \cdots, \theta_L\} \subset X_j \tag{7.8}$$

which joins $e^{(1)}_j$ to $\pi_{j-1}(e^{(0)}_{j-1,i})$. □

Corollary 7.9. *Let $p, q \in X_\infty$ be such that $\hat{d}_\infty(p,q) \in [m^{-j_0-1}, m^{-j_0})$; then there is a universal constant C_1 such that, for each $M > j_0$, there is a horizontal gallery $\mathcal{G}_M = \{\sigma_0, \cdots, \sigma_L\}$ consisting of 2-cells of X_M and whose length is at most $C_0 m^{-j_0}$, where one has:*

$$\begin{aligned} &\hat{d}_\infty(\pi^\infty_M(\sigma_0), p) \le C_0 m^{-M};\\ &\hat{d}_\infty(\pi^\infty_M(\sigma_L), q) \le C_0 m^{-M}. \end{aligned} \tag{7.10}$$

Proof. We use Lemma 7.5 to take a string of galleries connecting p to q of length at most $C(C_0)m^{-j_0}$, where C_0 is the constant in Lemma 7.5. We now truncate this string to obtain a horizontal gallery in X_M:

$$\begin{aligned} T = \Big\{ &\sigma_M, \mathcal{G}^{(1)}_M, \pi_{M-1}(\sigma_{M-1}), \pi_{M-1}(\mathcal{G}^{(1)}_{M-1}), \cdots,\\ &\pi^M_{j_0}(\sigma_{j_0}) = \pi^M_{j_0}(\tau_{j_0}), \pi^M_{j_0+1}(\mathcal{T}^{(1)}_{j_0+1}), \cdots, \tau_M \Big\}; \end{aligned} \tag{7.11}$$

we now inductively modify T to increase the minimal generation of cells in T to end up with a gallery consisting of only 2-cells of X_M. At the first step, we take the end cell σ of $\mathcal{G}^{(1)}_{j_0+1}$ in X_{j_0+1}, and the first cell τ of $\mathcal{T}^{(1)}_{j_0+1}$ in X_{j_0+1}; as both cells intersect some vertical faces of a pair of 2-cells in $\pi^{M-1}_{j_0}(\sigma_{j_0})$, we can find a horizontal gallery S in X_{j_0+1} connecting σ to τ whose length is at most $C(H)m^{-j_0}$. We then replace

$\pi^M_{j_0}(\sigma_{j_0})$ by $\pi^M_{j_0+1}(S)$. A similar process is applied now to replace the cells of T of generation j_0+1 by cells of generation j_0+2. The increase in length is of a factor $C(H)m^{-j_0-1}$ for each cell of generation j_0+1. Continuing inductively, one obtains a gallery T consisting of cells of generation M and whose length is $\le C(C_0, H, m)m^{-j_0}$. □

7.2. **Constructing Semmes pencils.**

Definition 7.12. A measured family of curves $(\Gamma, I_\Gamma, \Omega_\Gamma, \nu_\Gamma)$ is a measurable function $\Gamma : I_\Gamma \times \Omega_\Gamma \to X$, where I_Γ is an interval of $\mathbb{R}$, such that ν_Γ is a probability measure on Ω_Γ, and there is an L such that for each $\omega \in \Omega_\Gamma$ the map $t \mapsto \Gamma(t, \omega)$ is a Lipschitz curve in X of Lipschitz constant at most L.

We say that a measured family of curves $(\Gamma, I_\Gamma, \Omega_\Gamma, \nu_\Gamma)$ joins a set S_0 to a set S_1 if for ν_Γ-a.e. $\omega \in \Omega_\Gamma$ one has $\Gamma(\min I_\Gamma, \omega) \in S_0$ and $\Gamma(\max I_\Gamma, \omega) \in S_1$. Given a subinterval $I \subset I_\Gamma$ we denote by Γ_I the restriction $\Gamma : I \times \Omega_\Gamma \to X$.

To a measured family of curves $(\Gamma, I_\Gamma, \Omega_\Gamma, \nu_\Gamma)$ we associate a measure μ_Γ by:

$$\mu_\Gamma = \Gamma|_\#(\mathcal{L}^1 \times \nu_\Gamma). \tag{7.13}$$

In the following we will often simply write Γ instead of $(\Gamma, I_\Gamma, \Omega_\Gamma, \nu_\Gamma)$. We finally define the support of Γ by:

$$\operatorname{spt}\Gamma = \{p \in X_\infty : \forall r > 0\, \nu_\Gamma(\{\omega : \Gamma(I_\Gamma, \omega) \cap B(p, r) \neq \emptyset\}) > 0\}. \tag{7.14}$$

Definition 7.15. For a metric measure space (X, μ) recall the definition of the Riesz potential μ_p centered at p:

$$\mu_p = \frac{d(p, \cdot)}{\mu(B(p, d(p, \cdot)))}\, \mu. \tag{7.16}$$

We will denote the Riesz potential of (X, μ_∞) centered on $p \in X_\infty$ by $\mu_{\infty,p}$. To prove the Poincaré inequality and to establish a bound on the analytic dimension we rely on the following theorem.

Theorem 7.17. *There is a universal constant C depending only on (m, m_v) such that the following holds.*

- **(C1):** *For $p, q \in X_\infty$, there is a measured family of curves Γ joining p to q;*
- **(C2):** *The Lipschitz constant of Γ is at most C and $\mathcal{L}^1(I_\Gamma) \le C\hat{d}_\infty(p, q)$;*

(C3): *The measure μ_Γ is controlled by the Riesz potentials centered at p and q:*

$$\mu_\Gamma \leq C(\mu_{\infty,p} \llcorner B(p, Cd(p,q)) + \mu_{\infty,q} \llcorner B(q, Cd(p,q))). \tag{7.18}$$

Proof. Using Lemma 7.5, we take a string of galleries $S = \{ \{\sigma_j\}_{j\geq j_0}, \{\tau_j\}_{j\geq j_0}, \{\mathcal{G}_j\}_{j>j_0}, \{\mathcal{T}_j\}_{j>j_0} \}$ connecting p to q. The idea of the proof is based on the following observations: first, one can use S to string together horizontal segments to obtain Lipschitz paths joining p to q. Secondly, the Fubini representations of the measures on the cells of S give rise to a "natural" transverse measure on the family of quasi-geodesics connecting p to q. To ease the exposition, we have divided the proof in two steps. In the first we discuss how to build a measured family of curves joining $\pi^\infty_{j_0}(\sigma_{j_0})$ to p. In the second we join together two measured families of curves connecting the points p and q to $\pi^\infty_{j_0}(\sigma_{j_0}) = \pi^\infty_{j_0}(\tau_{j_0})$.

Step 1: Construction of a measured family of curves joining $\pi^\infty_{j_0}(\sigma_{j_0})$ to p.

We now construct a measured family of curves $(\Gamma_p, I_p, \Omega_p, \nu_p)$ joining $\pi^\infty_{j_0}(\sigma_{j_0})$ to p. We let:

$$I_p = \left[0, \sum_{j_0\leq s} m^{-s}\right], \tag{7.19}$$

and introduce the intervals:

$$\begin{aligned} A_i &= \left[\sum_{j_0\leq s<i} m^{-s}, \sum_{j_0\leq s<i} m^{-s} + \frac{m^{-i}}{2}\right] \quad (i \geq j_0) \\ B_i &= \left[\sum_{j_0\leq s<i-1} m^{-s} + \frac{m^{-i+1}}{2}, \sum_{j_0\leq s<i} m^{-s}\right] \quad (i > j_0). \end{aligned} \tag{7.20}$$

Note that by convention a sum over an empty set of indices is taken to be 0; for example, $A_{j_0} = [0, m^{-j_0}/2]$ and $B_{j_0+1} = [m^{-j_0}/2, m^{-j_0}]$. For later convenience let $A_i = [a_i^{(0)}, a_i^{(1)}]$ and $B_i = [b_i^{(0)}, b_i^{(1)}]$. We choose a cell $\hat{\sigma}_{j_0} \subset Y_{j_0}$ such that $\hat{\pi}^{j_0}(\hat{\sigma}_{j_0}) = \sigma_{j_0}$ and let $\hat{e}^{(0)}_{j_0}$ and $\hat{e}^{(1)}_{j_0}$ denote its vertical faces. Let Ω_p denote the subset of $\hat{e}^{(1)}_{j_0}$ obtained by removing points p' such that $y(p') = km_v^{-j}$ for $(k,j) \in \mathbb{Z}^2$. Note that Ω_p has full $\mathcal{H}^1$-measure in $\hat{e}^{(1)}_{j_0}$. Let $\nu_p = \mathcal{H}^1 \llcorner \Omega_p$.

We define a map $\hat{H}_{j_0} : A_{j_0} \times \Omega_p \to \hat{\sigma}_{j_0}$ by letting $\hat{H}_{j_0}(\cdot, p')$ denote the unique constant velocity horizontal curve in $\hat{\sigma}_{j_0}$ starting at p' and ending in $\hat{e}^{(0)}_{j_0}$. We let $\Gamma_p|A_{j_0}\times\Omega_p=\hat{\pi}^{\infty}_{j_0}\circ\hat{H}_{j_0}$. Then Fubini's Theorem implies that:

$$(\hat{H}_{j_0})_{\#}(\mathcal{L}^1\times\nu_p \llcorner A_{j_0}\times\Omega_p)=m_v^{j_0}\mathcal{L}^2 \llcorner \hat{\sigma}_{j_0}. \tag{7.21}$$

Note that the Lipschitz constant of $\hat{H}_{j_0}$ is at most 2.

As there is a uniform bound on the number of cells of Y_{j_0} that project to σ_{j_0}, we conclude from (7.21) that:

$$(\Gamma_p)|_{\#}(\mathcal{L}^1\times\nu_p \llcorner A_{j_0}\times\Omega_p)\approx m_v^{j_0}\mu \llcorner \operatorname{spt}(\Gamma_p|A_{j_0}\times\Omega_p). \tag{7.22}$$

Let $\mathcal{G}_{j_0+1}=\{\mathcal{G}^{(i)}_{j_0+1}\}_{1\le i\le m}$ and note that for $p'\in\Omega_p$ there is a unique $i=i(p')$ such that the point $\hat{\pi}^{j_0+1}_{j_0}\circ\hat{H}_{j_0}(a^{(1)}_{j_0},p')$ belongs to the last cell of $\mathcal{G}^{(i)}_{j_0+1}$. We now write $\mathcal{G}^{(i)}_{j_0+1}=\{\theta_1,\cdots,\theta_L\}$ and for each $1\le k\le L$ choose a cell $\hat{\theta}_k\subset Y_{j_0+1}$ such that $\hat{\pi}^{j_0+1}(\hat{\theta}_k)=\theta_k$. Intuitively, the collection of cells $C_{j_0+1}=\{\theta_k\}_{1\le k\le L}$ induces a *broken horizontal path* $\hat{J}(\cdot,p'):B_{j_0+1}\to Y_{j_0+1}$. More precisely, observe that $\hat{\theta}_L\cap(\hat{\pi}^{\infty}_{j_0+1})^{-1}(\hat{\pi}^{\infty}_{j_0}(\hat{H}_{j_0}(a^{(1)}_{j_0},p'))$ consists of a single point p'_{θ_L}. Then there is a unique horizontal constant velocity path $\xi_{\theta_L}:[b^{(0)}_{j_0+1},b^{(0)}_{j_0+1}+L^{-1}m^{-j_0}/2]\to\hat{\theta}_L$ which starts at p'_{θ_L} and ends on the vertical 1-cell of $\hat{\theta}_L$ other than the one containing p'_{θ_L}. Let $q'_{\theta_L}=\xi_{\theta_L}(b^{(0)}_{j_0+1}+L^{-1}m^{-j_0}/2)$. Then again $\hat{\theta}_{L-1}\cap(\hat{\pi}^{\infty}_{j_0+1})^{-1}(\hat{\pi}^{\infty}_{j_0+1}(q'_{\theta_L}))$ consists of a single point $p'_{\theta_{L-1}}$. This happens because $p'\in\Omega_p$ forces the points p'_{θ_L} and q'_{θ_L} to lie outside the 0-skeleton of the cells containing them. We then observe that there is a unique horizontal constant velocity path $\xi_{\theta_{L-1}}:[b^{(0)}_{j_0+1}+L^{-1}m^{-j_0}/2,b^{(0)}_{j_0+1}+2L^{-1}m^{-j_0}/2]\to\hat{\theta}_{L-1}$ which starts at $p'_{\theta_{L-1}}$ and ends on the 1-cell opposite to the one containing $p'_{\theta_{L-1}}$. The construction of the paths $\xi_{\theta_{L-2}},\cdots,\xi_{\theta_1}$ continues by backward induction, constructing ξ_{θ_k} knowing $q'_{\theta_{k+1}}$ as we did for θ_{L-1}, and we omit the details. We then let for $1\le k\le L$:

$$\hat{J}(\cdot,p')|(b^{(0)}_{j_0+1}+kL^{-1}m^{-j_0}/2,b^{(0)}_{j_0+1}+(k+1)L^{-1}m^{-j_0}/2)=\xi_{k-L}, \tag{7.23}$$

and extend $\hat{J}(\cdot,p')$ to B_{j_0+1} so that it is continuous from the left and $\hat{J}(b^{(0)}_{j_0+1},p')=p'_{\theta_L}$. We then let

$$\Gamma_p|B_{j_0+1}\times\Omega_p=\hat{\pi}^{\infty}_{j_0+1}\circ\hat{J}. \tag{7.24}$$

Note that $\Gamma_p(\cdot,p')|B_{j_0+1}$ is a Lipschitz curve joining $\hat{\pi}^{\infty}_{j_0}(\hat{H}_{j_0}(a^{(1)}_{j_0},p'))$ to a point on the last face $e^{(1)}_{j_0+1}$ of σ_{j_0+1}, and whose Lipschitz constant is at most $2L \leq 2C$. Taking into account that for $i_1 \neq i_2$ the cardinality of the set:

$$\left\{(\theta,\theta') \in \mathcal{G}^{(i_1)}_{j_0+1} \times \mathcal{G}^{(i_2)}_{j_0+1} : \mu\left(\hat{\pi}^{\infty}_{j_0+1}(\theta) \cap \hat{\pi}^{\infty}_{j_0+1}(\theta')\right) > 0\right\} \tag{7.25}$$

is uniformly bounded, an argument similar to the one which yielded (7.22) gives:

$$(\Gamma_p)|_\#(\mathcal{L}^1 \times \nu_p \llcorner B_{j_0+1} \times \Omega_p) \approx m_v^{j_0+1}\mu \llcorner \operatorname{spt}(\Gamma_p|B_{j_0+1} \times \Omega_p). \tag{7.26}$$

The construction then continues by induction on the intervals A_i, B_i. Note that the choice of Ω_p guarantees that, knowing $\Gamma_p(\cdot,p')|\bigcup_{s\leq i} A_s \cap \bigcup_{s=j_0+1}^{i} B_s$, there is a unique extension $\Gamma_p|B_{i+1}\times\Omega_p$ which is obtained using steps similar to those we used to construct $\Gamma_p|B_{j_0+1}\times\Omega_p$. The point is again that $(\hat{\pi}^{\infty}_{i+1})^{-1}(\Gamma(a_i^{(1)},p'))$ does not intersect the 0-skeleton of Y_{i+1} and so the extension to B_{i+1} is uniquely determined. In particular, one has also the following analogue of (7.26):

$$(\Gamma_p)|_\#(\mathcal{L}^1 \times \nu_p \llcorner B_{i+1} \times \Omega_p) \approx m_v^{i}\mu \llcorner \operatorname{spt}(\Gamma_p|B_{i+1} \times \Omega_p). \tag{7.27}$$

Similarly, knowing $\Gamma_p(\cdot,p')|\bigcup_{s\leq i} A_i \cap \bigcup_{s\leq i+1} B_i$, there is a unique extension $\Gamma_p|A_{i+1}\times\Omega_p$ which is constructed similarly to how we constructed $\Gamma_p|A_{j_0}\times\Omega_p$. Moreover, one has the following analogue of (7.22):

$$(\Gamma_p)|_\#(\mathcal{L}^1 \times \nu_p \llcorner A_{i+1} \times \Omega_p) \approx m_v^{i+1}\mu \llcorner \operatorname{spt}(\Gamma_p|A_{i+1} \times \Omega_p). \tag{7.28}$$

We finally let $\Gamma_p(\sum_{s=j_0}^{\infty} m^{-s},\cdot) = p$ because $\Gamma_p(a_i^{(1)},p') \to p$ and $\Gamma_p(b_i^{(1)},p') \to p$ uniformly in p' as $i \nearrow \infty$. Thus $\Gamma_p(\cdot,p')$ is a Lipschitz curve of Lipschitz constant at most $2C$ which joins $\hat{\pi}^{\infty}_{j_0}(p')$ to p. Note that because of the choice of the string of galleries S (compare (7.4)), there is a universal constant C_0 such that:

$$\operatorname{spt}\Gamma_p|(A_i \cup B_i)\times\Omega_p \subset B(p, C_0 m^{-i}). \tag{7.29}$$

Let μ_{Γ_p} denote the measure $(\Gamma_p)|_\#(\mathcal{L}^1 \times \nu_p \llcorner [0,\sum_{s=j_0}^{\infty} m^{-s}] \times \Omega_p)$. Thus (7.28) and (7.27) imply that:

$$\frac{d\mu_{\Gamma_p}}{d\mu} \mid (X_\infty \setminus B(p, C_0 m^{-j})) \lesssim \sum_{i=-\infty}^{j} m_v^i \lesssim m_v^j, \tag{7.30}$$

which gives

$$\mu_{\Gamma_p} \lesssim \mu_{\infty,p} \llcorner B(p, C m^{-j_0}), \tag{7.31}$$

where $\mu_{\infty,p}$ denotes the Riesz potential centered on p.

Step 2: Joining two measured families of curves. We use *Step 1* to build measured families of curves Γ_p and Γ_q joining $\pi^{\infty}_{j_0}(\sigma_{j_0}) = \pi^{\infty}_{j_0}(\tau_{j_0})$

to p and q respectively. Note that by condition **(G4)** in the properties of the string S, both Γ_p and Γ_q start at $\pi^\infty_{j_0}(e^{(1)}_{j_0}) = \pi^\infty_{j_0}(f^{(0)}_{j_0})$, where $e^{(1)}_{j_0}$ denotes the last face of σ_{j_0} and $f^{(0)}_{j_0}$ the first face of τ_{j_0}.

We now want to concatenate the reverse of Γ_p and Γ_q. In fact, Γ_p is defined on $I_p \times \Omega_p$ and Γ_q on $I_q \times \Omega_p$ where Ω_p is a full-measure subset of $e^{(1)}_{j_0}$ and for $p' \in \Omega_p$ we have $\Gamma_p(\min I_p, p') = \Gamma_q(\min I_q, p') = p'$. Note also that the transverse measures ν_p and ν_q agree; we can thus concatenate the reverse of Γ_p and Γ_q obtaining a measured family of curves Γ_{pq}. More precisely let:

$$I_{\Gamma_{pq}} = I_p \cup (I_q + \max I_p - \min I_q), \tag{7.32}$$

and define:

$$\Gamma_{pq}(t,\omega) = \begin{cases} \Gamma_p(\max I_{\Gamma_p} - t) & \text{if } t \in [0, \mathcal{L}^1(I_{\Gamma_p})] \\ \Gamma_q(t - \min I_{\Gamma_q} + \max I_{\Gamma_p}) & \text{if } t \in [\mathcal{L}^1(I_{\Gamma_p}), \mathcal{L}^1(I_{\Gamma_p}) + \mathcal{L}^1(I_{\Gamma_q})]. \end{cases} \tag{7.33}$$

For the measure ν_{pq} we take ν_p. Now **(C1)** follows from the choice of the string S, and **(C2)** follows since Γ_p and Γ_q are $2C$-Lipschitz and because $I_{\Gamma_{pq}}$ is given by (7.32). Note that:

$$\mu_\Gamma = (\Gamma_{pq})|_\#(\mathcal{L}^1 \times \nu_{pq} \llcorner I_{\Gamma_{pq}} \times \Omega_p) = \mu_{\Gamma_p} + \mu_{\Gamma_q}, \tag{7.34}$$

and so we get (7.18) in **(C3)** by using (7.30). □

Theorem 7.35. *The metric measure space (X_∞, μ) admits a $(1,1)$-Poincaré inequality.*

Proof. The $(1,1)$-Poincaré inequality can be proven by appealing to a result [Sem, Thm. 1.22]. As (X_∞, μ) is Ahlfors-regular[2], one has to show the existence of a universal constant C such that the following holds: for any pair of points $p, q \in X_\infty$ and any pair (u, g), u being a real-valued Borel function on X_∞ and g an upper gradient of u, one has the estimate:

$$|u(p) - u(q)| \le C \left(\int_{B(p,Cd(p,q))} g\, d\mu_{\infty,p} + \int_{B(q,Cd(p,q))} g\, d\mu_{\infty,q} \right). \tag{7.36}$$

Let Γ be a measured family of curves joining p to q as in Theorem 7.17. Then for each $\omega \in \Omega_\Gamma$ one has:

$$|u(p) - u(q)| \le \int_{I_\Gamma} g(\Gamma(t,\omega))\, dt; \tag{7.37}$$

[2]actually by the discussion in [Hei01, Chap. 4] it is enough to assume μ doubling

integrating (7.37) in ν_Γ one obtains:

$$|u(p) - u(q)| \leq \int g \, d\mu_\Gamma, \tag{7.38}$$

and (7.36) follows from (7.18). □

8. The analytic dimension

In this section we prove parts (4) and (5) of Theorem 1.1, see Theorem 8.5.

We let $\mathrm{Lip_b}(X)$ denote the set of bounded real-valued Lipschitz functions defined on X, which is a Banach algebra with norm given by:

$$\max\left(\|f\|_\infty, \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)}\right). \tag{8.1}$$

Definition 8.2. Consider a bounded linear operator $D : \mathrm{Lip_b}(X) \to L^\infty(\mu)$ where μ is a Radon measure on the metric space X. Suppose that $F : X \to Y$ is Lipschitz; then one obtains the **pushforward** $F_\# D : \mathrm{Lip_b}(Y) \to L^\infty(F_\#\mu)$ of D as follows; given $u \in \mathrm{Lip_b}(Y)$, $F_\# Du$ is determined by the requirement:

$$\int_Y g \, F_\# Du \, dF_\# u = \int_X g \circ F \, D(u \circ F) \, d\mu \quad (\forall g \in L^1(F_\#\mu)). \tag{8.3}$$

The collection of metric measure spaces $\{(X_j, \mu_j)\}_{j \in \mathbb{Z}}$ and maps $\{\pi_j\}_{j \in \mathbb{Z} \cup \{\infty\}}$ gives rise to a direct system of operators $\{D_j\}_{j \in \mathbb{Z}}$ as follows. Let $\hat{D}$ be the horizontal derivative operator:

$$\hat{D} : \mathrm{Lip_b}(\mathbb{R}^2) \to L^\infty(\mathcal{L}^2); \tag{8.4}$$

we then let $D_j = \hat{\pi}^j_\# \hat{D}$ for $j \in \mathbb{Z} \cup \{\infty\}$. Note that if $k \geq j$, $\hat{\pi}^k = \pi^k_j \circ \hat{\pi}^j$ and so $\pi^k_{j,\#} D_j = D_k$. In fact, the operators D_j are derivations in the sense of Weaver [Wea00], i.e. satisfy a product rule and a weak* continuity axiom. These properties are easy to verify, but will not be used in the following, except in the alternative argument given in Remark 8.24.

Theorem 8.5. *(X_∞, μ) has analytic dimension 1 and (X_∞, x_∞) is a differentiability chart. Moreover, (5) in Theorem 1.1 holds.*

Proof. Let f be a Lipschitz function; as the analytic dimension is a local property, we will assume f to be bounded. Consider an approximate continuity point p of $D_\infty f$ and fix $M \in \mathbb{N}$; using that p is an approximate continuity point of $D_\infty f$, for each $\varepsilon > 0$ one can choose

$r_0(\varepsilon)$ such that, for any $r \le r_0$, any $i \le M + \lfloor \log_m(1/r) \rfloor$ and any cell $\hat{\sigma}_i$ of Y_i satisfying

$$\hat{\pi}_i^\infty(\hat{\sigma}_i) \subset B(p, 2Cr), \tag{8.6}$$

one has:

$$\fint_{\hat{\pi}_i^\infty(\hat{\sigma}_i)} |D_\infty f(p) - D_\infty f| \, d\mu \le \varepsilon. \tag{8.7}$$

Let $q \in X_\infty$ satisfy $d(p, q) \le r$. We let:

$$u = f - f(p) - D_\infty f(p)(x_\infty - x_\infty(p)); \tag{8.8}$$

we will show that:

$$|u(p) - u(q)| \lesssim (m^{-M} + \varepsilon) r, \tag{8.9}$$

from which the Theorem follows as M and ε are arbitrary.

Let $M_r = \lfloor \log_m(1/r) \rfloor + M$; we use Corollary 7.9 to obtain a horizontal gallery $\mathcal{G}_{M_r} = \{\sigma_1, \cdots, \sigma_{L_r}\}$ of 2-cells of X_{M_r} such that $\pi_{M_r}^\infty(\mathcal{G}_{M_r})$ "almost connects" p to q. More precisely, one has:

$$\begin{aligned} d(\pi_{M_r}^\infty(\sigma_1), p) &\le Crm^{-M} \\ d(\pi_{M_r}^\infty(\sigma_{L_r}), q) &\le Crm^{-M}; \end{aligned} \tag{8.10}$$

and

$$L_r m^{-M} r \le Cr, \tag{8.11}$$

where C is a universal constant.

For $j \in \mathbb{Z}$ let S_j denote the 1-skeleton of Y_j; then S_j is $\hat{\pi}^j$-saturated, i.e. $(\hat{\pi}^j)^{-1}(\hat{\pi}^j(S_j)) = S_j$; as S_j is $\mathcal{L}^2$-null, we conclude that $\hat{\pi}^j(S_j)$ is μ_j-null. Note that $\hat{\pi}^j$ restricts to a homeomorphism mapping $Y_j \setminus S_j$ onto $X_j \setminus \hat{\pi}^j(S_j)$; in particular, $\hat{\pi}^j$ maps open cells of Y_j homeomorphically onto open cells of X_j. Let $v \in \mathrm{Lip_b}(X_j)$ and σ be an open cell of X_j; using the definition of pushforward, we conclude that $D_j v$ coincides μ_j-a.e. with the horizontal derivative $\hat{D}(v \circ \hat{\pi}^j)$, i.e.:

$$D_j v = \hat{D}(v \circ \hat{\pi}^j) \circ (\hat{\pi}^j)^{-1} \quad (\text{on } \sigma). \tag{8.12}$$

Consider two adjacent cells σ_i, σ_{i+1} of $\mathcal{G}_{M_r}$; as σ_i and σ_{i+1} share a vertical face, the Fubini representation of $\mu_{M_r} \llcorner (\sigma_i \cup \sigma_{i+1})$ and the fundamental Theorem of Calculus imply

$$\left| \fint_{\sigma_i} (u \circ \pi^\infty_{M_r})\, d\mu_{M_r} - \fint_{\sigma_{i+1}} (u \circ \pi^\infty_{M_r})\, d\mu_{M_r} \right| \lesssim m^{-M} r \fint_{\sigma_i \cup \sigma_{i+1}} |D_{M_r}(u \circ \pi^\infty_{M_r})|\, d\mu_{M_r}. \tag{8.13}$$

The goal is now to replace the derivative $D_{M_r}(u \circ \pi^\infty_{M_r})$ with $D_\infty u$ in the right hand side of (8.13). Fix a 2-cell σ of X_{M_r} and let $\hat\sigma$ be the 2-cell of Y_{M_r} such that $\hat\pi^{M_r}(\hat\sigma) = \sigma$. The map $\hat\pi^{M_r}$ restricts to a homeomorphism between the interiors of $\hat\sigma$ and σ; we now define $g \in L^1(\mu_{M_r})$ by letting:

$$g = \begin{cases} \frac{\chi_\sigma}{\mu_{M_r}(\sigma)} \left(\operatorname{sgn}(D_{M_r}(u \circ \pi^\infty_{M_r}))\right) & \text{on the interior of } \sigma \\ 0 & \text{elsewhere;} \end{cases} \tag{8.14}$$

then we have:

$$g \circ \hat\pi^{M_r} = \frac{\chi_{\hat\sigma}}{\mathcal{L}^2(\hat\sigma)} \operatorname{sgn}(D_{M_r}(u \circ \pi^\infty_{M_r})) \circ \hat\pi^{M_r}, \tag{8.15}$$

and using the definition of pushforward (8.3) we get:

$$\fint_\sigma |D_{M_r}(u \circ \pi^\infty_{M_r})|\, d\mu_{M_r} \le \fint_{\hat\sigma} \left| \hat D(u \circ \hat\pi^\infty) \right| d\mathcal{L}^2. \tag{8.16}$$

Let $\hat S$ be the union of the 1-skeleta of Y_j for $j \in \mathbb{Z}$; then $\hat S$ is $\mathcal{L}^2$-null and $\hat\pi^\infty$-saturated so that $\mu_\infty(\hat\pi^\infty(\hat S)) = 0$. Now $\hat\pi^\infty$ restricts to a homeomorphism between $\mathbb{R}^2 \setminus \hat S$ and $X_\infty \setminus \hat\pi^\infty(\hat S)$; in particular, we can define $g \in L^1(\mu_\infty)$ as:

$$g = \begin{cases} \frac{\chi_{\hat\pi^\infty(\hat\sigma)}}{\mu_\infty(\hat\pi^\infty(\hat\sigma))} \left(\operatorname{sgn} \hat D(u \circ \hat\pi^\infty)\right) \circ (\hat\pi^\infty)^{-1} & \text{on } X_\infty \setminus \hat\pi^\infty(\hat\sigma) \\ 0 & \text{elsewhere;} \end{cases} \tag{8.17}$$

then:

$$g \circ \hat\pi^\infty = \frac{\chi_{\hat\sigma}}{\mathcal{L}^2(\hat\sigma)} \operatorname{sgn} \hat D(u \circ \hat\pi^\infty) \tag{8.18}$$

as an element of $L^1(\mathcal{L}^2)$; therefore, using the definition of pushforward (8.3), we get:

$$\fint_{\hat\sigma} \left| \hat D(u \circ \hat\pi^\infty) \right| d\mathcal{L}^2 \le \fint_{\hat\pi^\infty(\hat\sigma)} |D_\infty u|\, d\mu_\infty. \tag{8.19}$$

Combining (8.19), (8.16), (8.13) we get:
(8.20)

$$\left|\fint_{\sigma_i}(u\circ\pi^\infty_{M_r})\,d\mu_{M_r}-\fint_{\sigma_{i+1}}(u\circ\pi^\infty_{M_r})\,d\mu_{M_r}\right|\lesssim m^{-M}r\fint_{\hat\pi^\infty(\sigma_i\cup\sigma_{i+1})}|D_\infty u|\,d\mu_\infty,$$

and using (8.7) we conclude that:

$$\left|\fint_{\sigma_i}(u\circ\pi^\infty_{M_r})\,d\mu_{M_r}-\fint_{\sigma_{i+1}}(u\circ\pi^\infty_{M_r})\,d\mu_{M_r}\right|\lesssim m^{-M}r\varepsilon. \tag{8.21}$$

Summing over i and using (8.11) we obtain:
(8.22)

$$\left|\fint_{\pi^\infty_{M_r}(\sigma_1)}u\,d\mu_\infty-\fint_{\pi^\infty_{M_r}(\sigma_{Lr})}u\,d\mu_\infty\right|=\left|\fint_{\sigma_1}u\circ\pi^\infty_{M_r}\,d\mu_{M_r}-\fint_{\sigma_{Lr}}u\circ\pi^\infty_{M_r}\,d\mu_{M_r}\right|\le C\varepsilon r;$$

using (8.10) and that u is Lipschitz, we obtain:

$$|u(p)-u(q)|\le 2C\,\mathrm{LIP}(u)rm^{-M}+\varepsilon Cr, \tag{8.23}$$

which gives (8.9).

Finally, the horizontal derivative operator D_∞ is associated with the pushforward under $\hat\pi^\infty$ of the family $\hat\Gamma$ of horizontal lines of $\mathbb{R}^n$ equipped with the obvious measure; now (8.16) can be interpreted as saying that this family of lines gives a universal Alberti representation in the sense of Bate. Note also that the horizontal geodesics are really *gradient curves* for f, in the sense that f decreases/increases along them with *optimal speed* in the sense of [Sch13], see also Remark 8.28. □

Remark 8.24. There is an alternative approach to the proof that the analytic dimension is 1, which uses directly the measured families of curves constructed in Theorem 7.17. This approach is based on Weaver derivations. By [Sch14, Thm. 5.9] it suffices to show that for each Lipschitz function f the inequality:

$$\operatorname{Lip} f\lesssim|D_\infty f| \tag{8.25}$$

holds μ_∞-a.e. Let p be a Lebesgue point for $|D_\infty f|$. Given a point q, we consider the measured family of curves Γ joining p to q which was constructed in Theorem 7.17; by inspecting the construction of Γ and using estimates like (8.16), (8.19) one concludes that there is a universal constant C such that:

$$|f(p)-f(q)|\le C\int_{B(p,Cd(p,q))\cup B(q,Cd(p,q))}|D_\infty f|\,(d\mu_{\infty,p}+d\mu_{\infty,q}); \tag{8.26}$$

recall that $\mu_{\infty,p}(B(p,Cd(p,q)))\approx\mu_{\infty,q}(B(p,Cd(p,q)))\approx d(p,q)$. As p is a Lebesgue point for $|D_\infty f|$, the fact that μ_∞ is doubling implies that:

$$\begin{aligned}\lim_{q\to p} \fint_{B(p,Cd(p,q))} |D_\infty f|\, d\mu_{\infty,p} &= |D_\infty f|(p)\\ \lim_{q\to p} \fint_{B(q,Cd(p,q))} |D_\infty f|\, d\mu_{\infty,q} &= |D_\infty f|(q).\end{aligned} \tag{8.27}$$

Thus, dividing (8.26) by $d(p,q)$ and taking the $\limsup$ as $q \to p$, (8.25) follows.

Remark 8.28. Let f be Lipschitz and g_f denote its generalized minimal upper gradient. By [Che99, Secs. 5,6] (see [CKS] for a simplified proof) we know that $g_f = \operatorname{Lip} f$ μ_∞-a.e. However, in these examples one can obtain a direct argument under the following lines. As we are dealing with a PI space we already know that $\operatorname{Lip} f \le Cg_f$; moreover, as the Sobolev space $H^{1,p}(\mu_\infty)$ is reflexive for $p > 1$, one can always assume that g_f is the L^p-limit of a sequence g_k where g_k is an upper gradient of a Lipschitz function f_k, and where $f_k \to f$ in $H^{1,p}(\mu_\infty)$ and $\operatorname{Lip} f_k \to \operatorname{Lip} f$ μ-a.e. It thus suffices to show that whenever g is an upper gradient for f, one has $g \ge \operatorname{Lip} f$ μ-a.e. Without loss of generality we can assume $g \in L^1_{\mathrm{loc}}(\mu_\infty)$. We secondly observe that, as x_∞ is a chart function, we have $\operatorname{Lip} f = |D_\infty f|$ μ-a.e. Now the Fubini representation of the measure $\mathcal{L}^2$ on $\mathbb{R}^2$ descends to a similar representation of the measure μ_∞ in terms of the horizontal geodesics of X_∞. If p is a Lebesgue point for g and $|D_\infty f|$, for any $\varepsilon > 0$ we can find a horizontal line γ with $\gamma(0)$ a Lebesgue point for $g \circ \gamma$ and $D_\infty f \circ \gamma$, $d(\gamma(0), p) \le \varepsilon$, and $g(\gamma(0)) \in [g(p) - \varepsilon, g(p) + \varepsilon]$, and $|D_\infty f|(\gamma(0)) \in [|D_\infty f|(\gamma(0)) - \varepsilon, |D_\infty f|(\gamma(0)) + \varepsilon]$. Applying Lebesgue differentation to $g \circ \gamma$ and $D_\infty f \circ \gamma$ at 0 one gets $g(\gamma(0)) \ge |D_\infty f|(\gamma(0))$.

9. The $n = 2$ case of Theorem 1.1 concluded

We now verify (6) of Theorem 1.1 in the $n = 2$ case.

Let $\{(\lambda_k X_\infty, p_k)\}$ and (Z, z) be as in the statement of Theorem 1.1. For every k, the map $\hat{\pi}^\infty : (\mathbb{R}^n, \lambda_k d_\alpha) \to X_\infty$ is a rescaling of a David-Semmes regular map, so it is a David-Semmes regular map with uniform constants. Therefore there is an N such that for every k, there exists points $\hat{p}_{k,1}, \ldots, \hat{p}_{k,\ell}$ ($\ell \le N$), such that $(\hat{\pi}^\infty)^{-1}(B(p_k, r)) \subset \cup_i B(p_{k,i}, Cr)$ for all $r \in (0, \infty)$. After passing to a subsequence we may assume that ℓ is constant, and that for each $1 \le i \le \ell$, the pointed maps $\hat{\pi}^\infty : (\mathbb{R}^n, \lambda_k \hat{d}^Y_\infty, \hat{p}_{k,i}) \to (\lambda_k X_\infty, p_k)$ Gromov-Hausdorff converge as $k \to \infty$ to a Lipschitz map $\phi_i : (W_i, w_i) \to (Z, z)$. Then the ϕ_i's are David-Semmes regular, and $\cup_i \operatorname{Im}(\phi_i) = Z$.

As ϕ_i is a light map, the topological dimension of (Z, z) is at least n. Note that the properties of satisfying a $(1,1)$-Poincaré inequality, being Ahlfors Q-regular, and having Assouad-Nagata dimension $\leq n$, with uniform constants, all pass to Gromov-Hausdorff limits. Therefore (2) and (3) hold.

Part (4) can be verified in several different ways.

One approach is to implement either of the arguments of Section 8 by passing the ingredients — the horizontal galleries of Corollary 7.9, the measured families of curves Γ of Theorem 7.17 and the derivation D_∞ — to the limit space Z. More specifically, in adapting Remark 8.24, one can associate to Γ and D_∞ normal 1-currents in the sense of Lang and use compactness of normal currents [Lan11, Thm. 5.4].

A second approach is to exploit the symmetry of X_∞. Using the self-similarity of X_∞ induced by the affine transformation $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ we may assume, without loss of generality, that the scale factor λ_k is 1 for all k. Also, note that the action $\mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ preserves the equivalence relation $\mathcal{R}_\infty$ on the open set $\{p \in \mathbb{R}^2 \mid x(p) \notin \mathbb{Z}\}$, and induces an action on $\{p \in X_\infty \mid x_\infty(p) \notin \mathbb{Z}\}$ which is a local isometry. In particular, for every point $p \in X_\infty$ with $x_\infty(p) \notin \mathbb{Z}$, if $r = \operatorname{dist}(x_\infty(p), \mathbb{Z})$, then the ball $B(p, r) \subset X_\infty$ is measure-preserving isometric to the ball $B(\hat{\pi}^\infty(\hat{q}), r) \subset X_\infty$ for some $\hat{q} \in (0,1)^2$. This property passes to the Gromov-Hausdorff limit, allowing one to see that $\mathcal{H}^Q$-a.e. point of Z lies in a ball that is isometric to a ball in X_∞ itself, and in particular the analytic dimension is 1.

A third approach involves rescaling the direct system of cell complexes $\{X_j\}$ and passing to a pointed limit, which is another pointed direct system with similar properties. See Section 11 — especially Subsection 11.4 — for more details.

10. The $n > 2$ case of Theorem 1.1

In this section we will discuss the higher-dimensional version of the examples treated in Sections 3-5.

Pick $n > 2$. We will imitate the $n = 2$ construction, but where the last $(n-1)$ coordinates of $\mathbb{R}^n$ will play the rôle of the y-coordinate in $\mathbb{R}^2$. Thus the notation (x, y) will henceforth mean that $y \in \mathbb{R}^{n-1}$. The words "horizontal" and "vertical" will have obvious interpretations in this new setup.

Let $m = 1 + 3(n-1)$, and pick $m_v \gg m$. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation $\Phi(x, y) = (m^{-1}x, m_v^{-1}y)$. Let Y_0 be the cell structure on $\mathbb{R}^n$ coming from the tiling by unit cubes, and for $j \in \mathbb{Z}$ let Y_j be the image of Y_0 under Φ^j.

Following the $n = 2$ construction, we define an equivalence relation $\mathcal{R}$ on $\mathbb{R}^n$ that is generated by the identifications of certain pairs of $(n-1)$-cells of Y_1 by vertical translation. For each $k \in \mathbb{Z}$, $d \in \{2, \ldots, n\}$, we perform gluings within the three vertical hyperplanes

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = k + m^{-1}(1 + 3(d-2) + (i-1))\}_{i\in\{1,2,3\}} ,$$

so as to enable horizontal galleries to "jump" in the x_d-coordinate direction. To define these gluings, for each $k, \ell \in \mathbb{Z}$, $2 \leq d \leq n$, $1 \leq i \leq 3$, we let

(10.1)
$$a_{k,\ell,d,i} = \left\{ (x_1, \ldots, x_n) \;\middle|\; \begin{array}{c} x_1 = m^{-1}(1 + 3(d-2) + (i-1)) \,, \\ x_d \in [(3\ell + i - 1)m_v^{-1}, (3\ell + i)m_v^{-1}] \end{array} \right\} ,$$

(10.2)
$$a'_{k,\ell,d,i} = \left\{ (x_1, \ldots, x_n) \;\middle|\; \begin{array}{c} x_1 = m^{-1}(1 + 3(d-2) + (i-1)) \,, \\ x_d \in [(3\ell + i)m_v^{-1}, (3\ell + i + 1)m_v^{-1}] \end{array} \right\} ,$$

and we identify $a_{k,\ell,d,i}$ with $a'_{k,\ell,d,i}$ by the vertical translation $p \mapsto p + m_v^{-1}e_d$.

The definition of $\mathcal{R}_\infty$ and the pseododistance $\hat{d}_\infty$ on the quotient X_∞ remain the same as before. Then $(X_\infty, \hat{d}_\infty)$ is the metric space of Theorem 1.1 for general n.

The verification of the assertions in Theorem 1.1 for $(X_\infty, \hat{d}_\infty)$ proceeds along the same steps, with appropriate modifications, a few of which we indicate here:

- A gallery is a chain of cells where two consecutive cells share an $(n-1)$-face.
- (cf. Lemma 3.3(1)) If $p_0, p_1 \in X_j$ are distinct points with $\pi_j(p_0) = \pi_j(p_1)$, then $p_i = \hat{\pi}^j(\hat{p}_i)$ for a unique point $\hat{p}_i \in \mathbb{R}^n$, and the pair $\hat{p}_0, \hat{p}_1$ is contained in $\ell \cap (\hat{\sigma}_0 \cup \hat{\sigma}_1)$, where $\ell \subset \mathbb{R}^n$ is vertical line $\ell \subset \mathbb{R}^n$ parallel to one the d^{th}-coordinate axis for some $2 \leq d \leq n$, and $\hat{\sigma}_0, \hat{\sigma}_1$ is a vertical gallery the $(n-1)$-skeleton of Y_{j+1} (i.e. a pair of vertical $(n-1)$-cells that share an $(n-2)$-cell).

- If σ is an n-cell of X_j (or Y_j), then it has two vertical $(n-1)$-faces τ_0, τ_1, where $x_j(\tau_1) = x_j(\tau_0) + m^{-j}$; these replace the vertical 1-cells in the $n = 2$ case.
- The cosets of $\mathcal{R}_j$ lie in orbits of $m^{-j}\mathbb{Z} \times m_v^{-j}\mathbb{Z}^{n-1}$, so the maps $\hat{y}_j$ and y_j take values in $\mathbb{R}^{n-1}/m_v^{-j}\mathbb{Z}^{n-1}$ instead of $\mathbb{R}/m_v^{-j}\mathbb{Z}$.
- For the proof that $(X_\infty, \hat{d}_\infty)$ has Assouad-Nagata dimension $\leq n$, one builds $(n+1)$ "good families" of subsets by working with cells of dimension 0 through n.

In addition to the minor points above, there are changes to the proof of Proposition 4.12 that require more care. In Steps 1 and 2 of the proof, rather than using the distance to $0 \in S^1(m_v^{-j})$, one uses the distance to the $(n-2)$-skeleton in the torus $T^{n-1}(m_v^{-j}) = \mathbb{R}^{n-1}/m_v^{-j}\mathbb{Z}^{n-1}$. However, the conclusion of Step 2 only says that $\pi_i^{-1}(\bar{\sigma}_{i_3})$ lies at controlled distance from the $(n-2)$-skeleton of $T^{n-1}(m_v^{-j})$. To proceed, one has to inductively exclude the possibility that $\pi_i^{-1}(\bar{\sigma}_{i_3})$ lies far from the $(n-3)$-skeleton of $T^{n-1}(m_v^{-j})$, etc. This is done using a variation on Steps 1 and 2, where one uses the distance from the $(n-k)$-skeleton of $T^{n-1}(m_v^{-j})$, for $k \geq 3$, and a variation of Lemma 4.9.

11. Generalizations

In this section we consider generalizations of the examples given in previous sections. It is not our intention to be exhaustive — it is clear that one can further generalize the contruction in different ways. Our main purpose is to illustrate that the same overall scheme of proof applies to a much broader class of examples, and to clarify the logical structure of the proofs by identifying the essential properties needed.

11.1. **Admissible systems.** Before giving the precise definition, we begin with some observations about the examples discussed in earlier sections. These are direct limits of direct systems

$$\ldots \xrightarrow{\pi_{-1}} X_0 \xrightarrow{\pi_0} X_1 \xrightarrow{\pi_1} \ldots \xrightarrow{\pi_{j-1}} X_j \xrightarrow{\pi_j} \ldots$$

where the X_j's are cell complexes. Crucial to the analysis is the distinction between horizontal and vertical directions, and the fact that the first coordinate function on $\mathbb{R}^n$ descends to a compatible family of functions. We will axiomatize this by requiring cells to have distinguished characteristic maps that induce the horizontal/vertical structure.

Fix integers $n \geq 2$ and $2 \leq m \leq m_v$. Let $\{Y_j\}_{j\in\mathbb{Z}}$ be a family of tilings of $\mathbb{R}^n$, where Y_j is a tiling by translates of the parallelopiped $[0, m^{-j}] \times [0, m_v^{-j}]^{n-1}$, and for all $j \in \mathbb{Z}$, the tiling Y_{j+1} is subdivision of Y_j. Let G be the group of isometries $g : \mathbb{R}^n \to \mathbb{R}^n$ of the form $g = \mathrm{id}_{\mathbb{R}} \times h$ for some isometry $h : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$. Let $G_j \subset G$ be the subgroup that preserves the cell structure of Y_j; hence for all $g \in G_j$, the linear part $L(g)$ preserves the set of coordinate vectors $\{\pm e_i\}_{2 \leq i \leq n}$.

Definition 11.1. An **admissible direct system** is a tuple consisting of:

- A direct system of cell complexes indexed by the integers
$$\ldots \xrightarrow{\pi_{-1}} X_0 \xrightarrow{\pi_0} \ldots\ldots \xrightarrow{\pi_{j-1}} X_j \xrightarrow{\pi_j} \ldots$$
- A collection $\{x_j : X_j \to \mathbb{R}\}$ of continuous maps.
- For each cell σ of X_j, a collection Φ_σ of distinguished characteristic maps $\phi : \hat{\sigma} \to \sigma$, where $\hat{\sigma}$ is a cell of Y_j.
- A Radon measure μ_j on X_j for all $j \in \mathbb{Z}$.

The tuple is required to satisfy the following conditions for some constants Δ, H:

(Ax1): X_j is connected, is a union of its closed n-cells, and all links have cardinality at most Δ.

(Ax2): Compatibility of distinguished characteristic maps:

- **(Ax2a):** (Compatibility with G) For any cell σ of X_j, any two elements $\phi_0 : \hat{\sigma}_0 \to \sigma$, $\phi_1 : \hat{\sigma}_1 \to \sigma$ of Φ_σ agree up to precomposition with some $g \in G_j$ such that $g(\hat{\sigma}_0) = \hat{\sigma}_1$. In particular, the notions of vertical and horizontal cells in Y_j descend to well-defined notions for cells in X_j.
- **(Ax2b):** (Compatibility with face restrictions) The restriction of any distinguished characteristic map $\phi : \hat{\sigma} \to \sigma$ to a face of $\hat{\sigma}$ is a distinguished characteristic map.
- **(Ax2c):** (Compatibility with subdivision) If σ is a cell of X_j and $\phi : \hat{\sigma} \to \sigma$ belongs to Φ_σ, then for any cell $\hat{\sigma}'$ of Y_{j+1} contained in $\hat{\sigma}$, the composition $\pi_j \circ \phi\big|_{\hat{\sigma}'} : \hat{\sigma}' \to \sigma' \subset X_{j+1}$ is a distinguished characteristic map of some cell σ' of X_{j+1}. Moreover all distinguished characteristic maps of X_{j+1} may be obtained in this way.
- **(Ax2d):** (Compatibility with x_j) For every cell σ of X_j and every $\phi : \hat{\sigma} \to \sigma$ in Φ_σ, $x_j \circ \phi = x$, where $x : \mathbb{R}^n \to \mathbb{R}$ is the first coordinate function.

(Ax3): $\pi_j : X_j^{(1)} \to X_{j+1}$ is a surjective cellular map, where $X_j^{(1)}$ denotes the subdivision of X_j defined by restricting distinguished characteristic maps $\phi : \hat{\sigma} \to \sigma$ to cells $\hat{\sigma}' \in Y_{j+1}$.

(Ax4): (Fibers have controlled vertical diameter) For every j, and every 0-cell $\bar{v}$ of X_{j+1}, any two elements v_0, v_1 of the inverse image $\pi_j^{-1}(\bar{v})$ are contained in a vertical edge path of $X_j^{(1)}$ of combinatorial length at most H.

(Ax5): (Gallery accessibility) For every j, and any two adjacent n-cells σ_0, σ_1 of X_j, there is a horizontal gallery from σ_0 to σ_1 of combinatorial length at most Δ.

(Ax6): Compatibility of measures:

(Ax6a): $(\pi_j)_\# \mu_j = \mu_{j+1}$ for all $j \in \mathbb{Z}$.

(Ax6b): (Lebesgue measure on cells) For every j, every n-cell σ of X_j, and every $\phi_\sigma \in \Phi_\sigma$, the restriction $\mu_j \llcorner \sigma$ agrees with the pushforward of the Lebesgue measure, $(\phi_\sigma)_\# \mathcal{L}^n$, up to a constant w_σ.

(Ax6c): (Doubling) For any two adjacent n-cells σ, σ of X_j, the weights w_σ, $w_{\sigma'}$ agree to within a factor of at most Δ.

Remark 11.2. As with the examples from Sections 2 and 10, one way of constructing an admissible direct system is by defining an increasing sequence $\ldots \subset \mathcal{R}_{j-1} \subset \mathcal{R}_j \subset \mathcal{R}_{j+1} \subset \ldots$ of equivalence relations on $\mathbb{R}^n$, such that $\mathcal{R}_j$ respects the cell structure on Y_j and the quotients $X_j = \mathbb{R}^n/\mathcal{R}_j$ satisfy Definition 11.1.

For any admissible system we define a pseododistance $\hat{d}_\infty$ on the direct limit X_∞ as follows. We let $\hat{d}_\infty$ be the largest pseudodistance on X_∞ such that, for every $j \in \mathbb{Z}$ and every cell $\hat{\sigma}$ of X_j, the projection to X_∞ has $\hat{d}_\infty$-diameter at most m^{-j}. In general this need not define a metric (see the next example), so we form a metric space $(\bar{X}_\infty, \bar{d}_\infty)$, the **limit space** of the admissible system, by collapsing subsets of X_∞ of zero $\hat{d}_\infty$-diameter to points. We denote by $\bar{\pi}_j^\infty$ the composition $X_j \overset{\pi_j^\infty}{\to} X_\infty \to \bar{X}_\infty$. The pushforward of μ_j under $\bar{\pi}_j^\infty$ is independent of j by Axiom **(Ax6)**, and it thus defines the natural measure μ_∞ on $\bar{X}_\infty$.

Example 11.3. To illustrate how the pseudodistance $\hat{d}_\infty$ can fail to be a distance, we construct a direct system $\{X_j\}$ that satisfies all the conditions of Definition 11.1 except **(Ax5)**, such that the psuedodistance $\hat{d}_\infty$ is not a distance. It is not hard to modify this example to obtain an admissible system with the same property.

Let $n = 2$, $m = 3$, $m_v = 6$. For every $k, \ell \in \mathbb{Z}^2$, we define a pair of 2-cells

$$a_{k,\ell} = \left[\frac{1}{3} + k, \frac{2}{3} + k\right] \times \left[\frac{1}{6} + \ell, \frac{2}{6} + \ell\right] ,$$

$$a'_{k,\ell} = \left[\frac{1}{3}+k, \frac{2}{3}+k\right] \times \left[\frac{4}{6}+\ell, \frac{5}{6}+\ell\right] .$$

Now let $\mathcal{R}$ be the equivalence relation on $\mathbb{R}^2$ obtained by identifying $a_{k,\ell}$ with $a'_{k,\ell}$ by the vertical translation $p \mapsto p+\frac{1}{2}e_2$, for every k, ℓ. We now define $\mathcal{R}_j$, X_j, $\hat{\pi}^j$, etc. as in Section 3. Then $\{X_j = \mathbb{R}^2/\mathcal{R}_j\}$ is the desired direct system. Note that there is a Cantor set in the vertical line $\{x = \frac{1}{2}\}$ that maps to a subset of X_∞ with zero $\hat{d}_\infty$-diameter. On the other hand if $\hat{p} \in \mathbb{R}^2$, $p = \hat{\pi}^\infty(p)$, then the set $(\hat{\pi}^\infty)^{-1}(p)$ is countable, being a countable union of the sets $(\hat{\pi}^j)^{-1}(\hat{\pi}^j(\hat{p}))$, each of which is finite.

Most of Theorem 1.1 generalizes verbatim to admissible systems:

Theorem 11.4. *For every n, m, H, there is an $\underline{m_v} = \underline{m_v}(n,m,H)$ such that if $m_v \geq \underline{m_v}$, then:*

1. *$(\bar{X}_\infty, \bar{d}_\infty, \mu_\infty)$ is a complete doubling metric measure space satisfying a $(1,1)$-Poincaré inequality.*
2. *$\bar{X}_\infty$ has topological and Assouad-Nagata dimension n.*
3. *$(\bar{X}_\infty, \bar{d}_\infty, \mu_\infty)$ has analytic dimension 1.*
4. *(1)-(3) also hold for any pointed measured Gromov-Hausdorff limit of any sequence of rescalings of $(\bar{X}_\infty, \bar{d}_\infty, \mu_\infty)$.*

Part (1) of Theorem 1.1 does not generalize directly to admissible systems, because, in particular, $(\bar{X}_\infty, \bar{d}_\infty)$ is not Ahlfors-regular in general. To formulate a modified statement, we introduce another metric on the X_j's, which plays the role of the metric d_α on $\mathbb{R}^n$.

Definition 11.5. For all $j \in \mathbb{Z}$, let d_j be the largest pseudodistance on X_j such that $d_j \leq \hat{d}_j$, and every cell of $X_j^{(k)}$ has d_j-diameter at most $m^{-(j+k)}$.

Theorem 11.6. *For every $j \in \mathbb{Z}$:*

1. *(X_j, d_j, μ_j) is a doubling metric measure space.*
2. *$\bar{\pi}_j^\infty : (X_j, d_j) \to (\bar{X}_\infty, \bar{d}_\infty)$ is a Lipschitz light map (see Definition 11.7); in particular, the point inverses of $\bar{\pi}_j^\infty$ are uniformly totally disconnected (i.e. have Assouad-Nagata dimension 0).*
3. *Let $Q = 1 + \frac{(n-1)\log m_v}{\log m}$. If for every j, the inverse image under $\pi_j : X_j \to X_{j+1}$ of every open n-cell of X_{j+1} is a single open n-cell of $X_j^{(1)}$, then (X_j, d_j) is Ahlfors Q-regular, and $\bar{\pi}_j^\infty : (X_j, d_j) \to (\bar{X}_\infty, \bar{d}_\infty)$ is a David-Semmes regular map.*

We recall here the notion of Lipschitz light map [CK13, Def. 1.14].

Definition 11.7. A Lipschitz map $f : X \to Y$ is **Lipschitz light** if there is some $C > 0$ such that for every bounded subset $W \subset Y$ the $\mathrm{diam}(W)$-components of $f^{-1}(W)$ have diameter at most $C\,\mathrm{diam}(W)$. Note that a Lipschitz light map is also a continuous light map.

11.2. **The proof of Theorem 11.4.** Overall, the proof of parts (1)-(3) of Theorem 11.4 follows closely that of Theorem 1.1. However, some modifications are needed and these are explained in this Subsection. In Subsection 11.4 we give the proof of item (4), i.e. the stability under taking weak tangents. Note that even for the specific examples considered in the previous sections a cleaner treatment is obtained in the more general context of admissible systems.

The proof of Proposition 4.12 remains valid using a map y_j which takes values in the quotient $\tilde{T}_j$ of $\mathbb{R}^{n-1} \simeq \{0\} \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$ by the action of G_j, which is a quotient of the torus $T^{n-1} = \mathbb{R}^{n-1}/m_v^{-j}\mathbb{Z}^{n-1}$ by the finite group of orthogonal transformations of $\mathbb{R}^{n-1}$ that preserves the subset $\{\pm e_d\}_{2\le d\le n}$. Note that in the case $n \ge 3$ one must also modify *Steps 1 and 2* of that argument by using the distances from the $\{0, 1, \cdots, n-2\}$-skeleta of $\tilde{T}_j$ (compare Section 10).

The proof that the Assouad-Nagata dimension of $\bar{X}_\infty$ is at most n can be carried out as in Theorem 6.2 by working with the cells of X_j of dimensions $1, \cdots, n$. The fact that the topological dimension (and hence the Assouad-Nagata dimension) is at least n can be deduced by adapting either of the arguments in Lemma 5.2.

To prove the Poincaré inequality, one can essentially follow the argument of Theorem 7.35.

We now turn to the proof of assertion (3) in Theorem 11.4, i.e. that $(\bar{X}_\infty, \bar{d}_\infty, \mu_\infty)$ has analytic dimension 1. We first define a direct system of derivations $\{D_j : \mathrm{Lip}_b(X_j, d_j) \to L^\infty(X_j, \mu_j)\}$ as follows. For every bounded d_j-Lipschitz function $u_j : X_j \to \mathbb{R}$ we let $D_j u_j$ be the function in $L^\infty(\mu_j)$ such that, for every n-cell σ of X_j, we have

$$D_j(u_j \circ \phi_\sigma) = \frac{\partial(u_j \circ \phi_\sigma)}{\partial x}$$

for every distinguished characteristic map $\phi_\sigma \in \Phi_\sigma$; this is well-defined because of **(Ax2a)**. Note that the family $\{D_j\}$ is compatible with the projections $\{\pi_j^k\}$, and by pushforward we get a well-defined derivation $D_\infty : \mathrm{Lip}_b(\bar{X}_\infty) \to L^\infty(\bar{X}_\infty, \mu_\infty)$. With this setup, the first part of the proof of Theorem 8.5 (or using Weaver derivations as sketched in Remark 8.24) carries over. The justification after (8.13) requires modifications for general admissible systems, however, as it is based on the fact that for μ_∞-a.e. point, the point inverse $(\pi_j^\infty)^{-1}(p)$ contains only one point. Instead, one may use the fact that for a Lipschitz function $f : \bar{X}_\infty \to \mathbb{R}$, the function $D_j(f \circ \bar{\pi}_j^\infty)$ is a.e. constant on the fibers of $\bar{\pi}_j^\infty$, which is established in Lemmas 11.8, 11.13, and 11.19.

Lemma 11.8. *Fix $p_\infty \in \bar{X}_\infty$ and $k \geq j$; let $S = \{\sigma_0, \cdots, \sigma_L\}$ be a chain of n-cells of $X_j^{(k-j)}$ such that each cell of S intersects $(\bar{\pi}_j^\infty)^{-1}(p_\infty)$. Assume that:*

$$p_j \in \sigma_0 \cap (\bar{\pi}_j^\infty)^{-1}(p_\infty); \tag{11.9}$$

if m_v is sufficiently large there is a universal constant C_0 such that:

$$d_j(p_j, \sigma_L) \leq 3C_0 m^{-k}. \tag{11.10}$$

Moreover, let $k \geq j$ and denote by σ_k the n-cell of $X_k^{(1)}$ containing $\pi_j^k(p_j)$. Assume that any vertical gallery in $X_k^{(1)}$ from σ_k to the horizontal $(n-1)$-skeleton of X_k has length at least $10H$. Then $(\bar{\pi}_k^\infty)^{-1}(p_\infty)$ is entirely contained in the open n-cell of X_k that contains $\pi_j^k(p_j)$.

Proof. Let $S = \{\sigma_0, \cdots, \sigma_L\}$ be a chain of n-cells of $X_j^{(k-j)}$ such that each cell of S intersects $(\bar{\pi}_j^\infty)^{-1}(p_\infty)$. Fix $p_j \in \sigma_0 \cap (\bar{\pi}_j^\infty)^{-1}(p_\infty)$. By Corollary 4.19, $\pi_j^k(S)$ lies in the star of a cell of X_k; by Axiom **(Ax4)** we conclude that $\pi_j^{k-1}(S)$ lies in a combinatorial ball of radius C_0 in $X_{k-1}^{(1)}$ which is centered on an n-cell of $X_{k-1}^{(1)}$. Consider now an n-cell σ of $X_j^{(k-1-j)}$; by Axiom **(Ax2c)** the map $\pi_j^{k-1} : X_j^{(k-j)} \to X_{k-1}^{(1)}$ is a combinatorial isomorphism when restricted to σ. Thus, for $0 \leq i \leq L$ we have:

$$\begin{aligned} &d_{\mathbb{R}}(x_j(\sigma_0), x_j(\sigma_i)) \leq 2m^{-k} \\ &d_{\tilde{T}_j}(y_j(\sigma_0), y_j(\sigma_i)) \leq 2C_0 m_v^{-k}; \end{aligned} \tag{11.11}$$

therefore, if m_v is sufficiently large compared to C_0 and m we must have:

$$d_j(\sigma_0, \sigma_i) \leq 3C_0 m^{-k}. \tag{11.12}$$

Let now σ_k denote the n-cell of $X_k^{(1)}$ containing $\pi_j^k(p_j)$ and assume that any vertical gallery in $X_k^{(1)}$ from σ_k to the horizontal $(n-1)$-skeleton of $X_k^{(1)}$ has length at least $10H$. Let $q \in (\bar{\pi}_k^\infty)^{-1}(p_\infty)$ and choose a cell σ_k' of $X_k^{(1)}$ which contains q and intersects $(\pi_k^\infty)^{-1}(p_\infty)$. As both cells $\pi_k(\sigma_k)$ and $\pi_k(\sigma_k')$ meet $(\bar{\pi}_{k+1}^\infty)^{-1}(p_\infty)$, by Corollary 4.19 they must be adjacent. However, as σ'_k intersects $(\pi_k^\infty)^{-1}(p_\infty)$, by Axiom **(Ax4)** there is a vertical gallery in $X_k^{(1)}$ of length at most H from σ_k to a cell τ adjacent to σ'_k. Thus q lies in the open cell of X_k containing $\pi_j^k(p_j)$. □

Lemma 11.13. *Let $\{g_j : X_j \to \mathbb{R}\}_{j\in\mathbb{Z}}$ be a family of bounded measurable functions, and suppose that $\{g_j\}$ is compatible with projection in the sense that, for all $j \le k$, one has $g_k \circ \pi_j^k = g_j$ μ_j-a.e. Moreover, let $g_\infty \in L^\infty(\bar{X}_\infty, \mu_\infty)$ be the function satisfying:*

$$\bar{\pi}^\infty_{j\#}(g_j\mu_j) = g_\infty \mu_\infty. \tag{11.14}$$

Then for every $j \in \mathbb{Z}$, and μ_∞-a.e. $p \in \bar{X}_\infty$, if $\tilde{\mu}_j(p)$ is the disintegration of μ_j with respect to $\bar{\pi}_j^\infty$, then g_j is $\tilde{\mu}_j(p)$-a.e. equal to $g_\infty(p)$ on $(\bar{\pi}_j^\infty)^{-1}(p)$.

Proof. For $\eta \in [1, \lfloor \frac{m_v}{3H} \rfloor]$ and $k \ge j$ let $S_{\eta,k}$ be the set of points $q \in X_j$ such that:

(a) q does not belong to the $(n-1)$-skeleton of $X_j^{(k-j)}$ for any $k \ge j$.
(b) If σ is the n-cell of $X_k^{(1)}$ containing $\pi_j^k(q)$, then any vertical gallery in $X_k^{(1)}$ from σ to the horizontal $(n-1)$-skeleton of X_k has length at least ηH.

If $q \in S_{\eta,k}$ and $\eta \ge 10$, by Lemma 11.8 the fibre $(\bar{\pi}_k^\infty)^{-1}(\bar{\pi}_j^\infty(q))$ is entirely contained in the open n-cell of X_k which contains $\pi_j^k(q)$.

We now look at the set of points S in X_j such that (b) occurs infinitely often:

$$S_\eta = \bigcap_{k\ge 1} \bigcup_{k' \ge k} S_{\eta,k'}; \tag{11.15}$$

then $\mu_j(S_\eta^c) = 0$. Consider now the subset $\Omega \subset \bar{X}_\infty$:

$$\Omega = \left\{ p \in \bar{X}_\infty : (\bar{\pi}_j^\infty)^{-1}(p) \cap S_{10}^c \neq \emptyset \right\}; \tag{11.16}$$

then by Lemma 11.8 $(\bar{\pi}_j^\infty)^{-1}(\Omega) \subset S_9^c$ and so $\mu_\infty(\Omega) = 0$. Thus for μ_∞-a.e. $p \in \bar{X}_\infty$ we can assume that $(\bar{\pi}_j^\infty)^{-1}(p) \subset S_{10}$.

As the Lebesgue Differentiation Theorem holds in X_j, for μ_∞-a.e. $p \in \bar{X}_\infty$ there is a subset $T_p \subset (\bar{\pi}_j^\infty)^{-1}(p)$ of full $\tilde{\mu}_j(p)$ measure such that $T_p \subset S_{10}$ and every $q \in T_p$ is an approximate continuity point of g_j. Let $q, q' \in T_p$, and $\varepsilon > 0$. For any $k_0 \ge j$ we can find $k \ge k_0$ such that $q \in S_{10,k}$; let $\hat{\sigma}$, $\hat{\sigma}'$ be the n-cells of $X_j^{(k-j)}$ which contain q, q'

respectively. Provided k_0 is sufficiently large, we will have

$$\max\left(\left|\fint_{\hat\sigma} g_j\,d\mu_j - g_j(q)\right|, \left|\fint_{\hat\sigma'} g_j\,d\mu_j - g_j(q)\right|\right) < \varepsilon\,. \tag{11.17}$$

As $q \in S_{10,k}$, we have $\pi^k_j(\hat\sigma) = \pi^k_j(\hat\sigma')$, and then

$$\fint_{\hat\sigma} g_j\,d\mu_j = \fint_{\pi^k_j(\hat\sigma)} g_k\,d\mu_k = \fint_{\pi^k_j(\hat\sigma')} g_k\,d\mu_k = \fint_{\hat\sigma'} g_j\,d\mu_j\,. \tag{11.18}$$

Therefore $|g_j(q) - g_j(q')| < 2\varepsilon$, and as ε was arbitrary, we have $g_j(q) = g_j(q')$. Now the lemma follows because (11.14) and the Disintegration Theorem imply that for μ_∞-a.e. $p \in \bar X_\infty$ $g_\infty(p)$ is the $\tilde\mu_j(p)$ average of g_j. □

Lemma 11.19. *For $j \in \mathbb{Z}$ and $p \in \bar X_\infty$, we let $\tilde\mu_j(p)$ denote the disintegration of μ_j with respect to $\bar\pi^\infty_j$. Let $u : \bar X_\infty \to \mathbb{R}$ be Lipschitz, and $u_j = u \circ \bar\pi^\infty_j$. Let $D_\infty u \in L^\infty(\bar X_\infty, \mu_\infty)$ be the function satisfying:*

$$\bar\pi^\infty_{j\#}(D_j u_j \cdot \mu_j) = D_\infty u \cdot \mu_\infty. \tag{11.20}$$

Then for μ_∞-a.e. $p \in X_\infty$, one has:

$$D_j u_j = D_\infty u(p) \quad (\tilde\mu_j(p)\text{-a.e.}) \tag{11.21}$$

Proof. We apply Lemma 11.13; thus it suffices to show that for $k \geq j$ one has:

$$D_k u_k \circ \pi^k_j = D_j u_j \quad (\mu_j\text{-a.e.}) \tag{11.22}$$

Let S be the set of points $q \in X_j$ which do not belong to the $(n-1)$-skeleton of $X_j^{(m-j)}$ for any $m \geq j$. Note that

$$\mu_j(S) = \mu_k(\pi^k_j(S)) = 0. \tag{11.23}$$

Now, for μ_j-a.e. q, $D_j u_j(q)$ equals $(u_j \circ \gamma_q)'(0)$ where γ_q is a unit-speed horizontal segment with $\gamma_q(0) = q$ and along which $x_j \circ \gamma_q$ is non-decreasing. As the fibres of π^k_j are finite, we also have that for μ_j-a.e. q, $D_k u_k(\pi^k_j(q))$ equals $(u_k \circ \gamma_{\pi^k_j(q)})'(0)$ where $\gamma_{\pi^k_j(q)}$ is a unit-speed horizontal segment with $\gamma_{\pi^k_j(q)}(0) = \pi^k_j(q)$, and along which $x_k \circ \gamma_{\pi^k_j(q)}$ is non-decreasing. If $q \in S$ for some $\delta > 0$ one has:

$$\pi^k_j(\gamma_q(t)) = \gamma_{\pi^k_j(q)}(t) \quad (|t| \leq \delta); \tag{11.24}$$

as $u_j = u_k \circ \pi^k_j$ we conclude that (11.22) holds. □

11.3. **Proof of Theorem 11.6.**

Proof of Theorem 11.6. We first prove (1). Note that Axioms **(Ax6)**, **(Ax1)** imply the existence of a $C = C(\Delta, n)$ (independent of j) such that:
(11.25)

$$\mu_j\left(B_{X_j}(p_j, r)\right) \geq C^{-1}\mu_j\left(B_{X_j}(p_j, 2r)\right) \quad (\forall r \leq 4m^{-j+1}, p_j \in X_j).$$

Assume that $r > m^{-j}$, and let k be such that $m^{-k} < r \leq m^{-k+1}$; and choose a point $p_{k-1} \in (\pi^j_{k-1})^{-1}(p_j)$. As π^j_{k-1} is 1-Lipschitz:

$$B_{X_{k-1}}(p_{k-1}, r) \subset (\pi^j_{k-1})^{-1}\left(B_{X_j}(p_j, r)\right); \tag{11.26}$$

on the other hand, Proposition 4.12 implies that:

$$\begin{aligned}(\pi^j_{k-1})^{-1}\left(B_{X_j}(p_j, 2r)\right) &\subset (\bar{\pi}^\infty_{k-1})^{-1}\left(B_{\bar{X}_\infty}(\bar{\pi}^\infty_j(p_j), 2r)\right) \\ &\subset B_{X_{k-1}}(p_{k-1}, 4m^{-k+1});\end{aligned} \tag{11.27}$$

thus, combining (11.26), (11.27) one gets:

$$\begin{aligned}\mu_j\left(B_j(p_j, r)\right) &\geq \mu_{k-1}\left(B_{X_{k-1}}(p_{k-1}, r)\right) \\ &\geq C^{-2-\log_2 m}\mu_{k-1}\left(B_{X_{k-1}}(p_{k-1}, 4m^{-k+1})\right) \\ &\geq C^{-2-\log_2 m}\mu_j\left(B_j(p_j, 2r)\right).\end{aligned} \tag{11.28}$$

We now prove (2), i.e. that $\bar{\pi}^\infty_j$ is a Lipschitz-light map. We will show that there is a universal constant C such that for each $(k, j, p_\infty) \in \mathbb{Z}^2 \times \bar{X}_\infty$, $(\bar{\pi}^\infty_j)^{-1}(p_\infty)$ can be covered by a family of sets $\{\Omega_\alpha\}_\alpha$ such that:

$$\operatorname{diam}\Omega_\alpha \leq Cm^{-k}, \tag{11.29}$$

and

$$d(\Omega_\alpha, \Omega_\beta) \geq m^{-k} \quad (\alpha \neq \beta). \tag{11.30}$$

Note that by Corollary 4.19 any two points of $(\bar{\pi}^\infty_j)^{-1}(p_\infty)$ must belong to adjacent cells of X_j, so the case of interest is $k > j$.

For each n-cell σ of $X^{(k-j)}_j$ intersecting $(\bar{\pi}^\infty_j)^{-1}(p_\infty)$ let Ω_σ denote the set of points of $(\bar{\pi}^\infty_j)^{-1}(p_\infty)$ that can be connected to σ using a chain $S = \{\sigma_0, \cdots, \sigma_L\}$ of cells of $X^{(k-j)}_j$, such that each σ_i intersects $(\bar{\pi}^\infty_j)^{-1}(p_\infty)$. From (11.10) in Lemma 11.8 we conclude that:

$$\operatorname{diam}\Omega_\sigma \leq 6C_0 m^{-k}. \tag{11.31}$$

On the other hand, if $\Omega_\sigma \neq \Omega_{\sigma'}$, then Ω_σ and $\Omega_{\sigma'}$ do not intersect adjacent cells of $X^{(k-j)}_j$ and hence:

$$d_j(\Omega_\sigma, \Omega_{\sigma'}) \geq m^{-k}. \tag{11.32}$$

We now turn to the proof of (3). We first observe that if σ is an n-cell of X_j, one has:

$$\mu_j(\sigma) = w_\sigma m^{-jQ}, \tag{11.33}$$

where w_σ is the weight in Axiom **(Ax6b)**. Let σ_0, σ_1 be n-cells of X_j. If

$$d_j(\sigma_0, \sigma_1) \le m^{-j}, \tag{11.34}$$

the cells σ_0, σ_1 are adjacent and hence

$$\frac{w_{\sigma_0}}{w_{\sigma_1}} \le \Delta \tag{11.35}$$

by Axiom **(Ax6c)**. Assume now that $d_j(\sigma_0, \sigma_1) > m^{-j}$. As X_j is connected (**(Ax1)**) there is an $N \ge 2$ such that:

$$(N-1)m^{-j} < d_j(\sigma_0, \sigma_1) \le N m^{-j}; \tag{11.36}$$

let $k < 0$ be such that $m^{-k-1} < N \le m^{-k}$. Using Proposition 4.12 we see that:

$$d_k\left((\pi^j_{j+k})^{-1}(\sigma_0), (\pi^j_{j+k})^{-1}(\sigma_1)\right) \le 3m^{-j-k}. \tag{11.37}$$

Now by Axiom **(Ax3)**

$$\pi^j_{j+k} : X^{(-k)}_{j+k} \to X_j \tag{11.38}$$

is a surjective cellular map; moreover, we are assuming that π^j_{j+k} is injective on the complement of the $(n-1)$-skeleton of $X^{(-k)}_{j+k}$. Thus, if $\mathring{\sigma}_i$ denotes the interior of σ_i, there is a unique n-cell $\tilde{\sigma}_i$ of X_{j+k} such that:

$$(\pi^j_{j+k})^{-1}(\mathring{\sigma}_i) \cap \tilde{\sigma}_i \ne \emptyset. \tag{11.39}$$

By (11.37) $\tilde{\sigma}_0$, $\tilde{\sigma}_1$ are at combinatorial distance at most 3 and hence:

$$\frac{w_{\tilde{\sigma}_0}}{w_{\tilde{\sigma}_1}} \le \Delta^3; \tag{11.40}$$

but $w_{\sigma_i} = w_{\tilde{\sigma}_i}$ and so the weights w_{σ_0} and w_{σ_1} are comparable up to a uniformly bounded multiplicative factor. We therefore find a universal constant C such that, for each $j \in \mathbb{Z}$, each $p_j \in X_j$ and each $r \le 3m^{-j+1}$ one has:

$$\mu_j\left(B_{X_j}(p_j, r)\right) \approx_C r^{-jQ}. \tag{11.41}$$

Suppose now that $m^{-k} < r \le m^{-k+1}$ and let $p_k \in (\pi^j_k)^{-1}(p_j)$; then, arguing as in the proof of (1), we obtain:

$$B_{X_{k-1}}(p_{k-1}, m^{-k}) \subset (\pi^j_{k-1})^{-1}\left(B_{X_j}(p_j, r)\right) \subset B_{X_{k-1}}(p_{k-1}, 3m^{-k+1}); \tag{11.42}$$

therefore, by enlarging C, we have that (11.41) holds also for $r > 3m^{-j+1}$. This proves that each (X_j, d_j, μ_j) is Ahlfors-regular, where the constant in the Ahlfors-regularity condition is independent of j.

We now show that $\bar{\pi}_j^\infty$ is David-Semmes regular (where the constants again do not depend on j). Let $k < 0$; as the map in (11.38) is injective on the complement of the $(n-1)$-skeleton of $X_{j+k}^{(-k)}$, and as by **(Ax1)** there is a uniform bound Δ on the cardinality of each link of X_j, we conclude that there is a universal constant $C = C(\Delta)$ such that for each $p_j \in X_j$ one has that $(\pi_{j+k}^j)^{-1}(p_j)$ has cardinality at most C. As k and j are arbitary, we conclude that for $p_\infty \in X_\infty$ also $(\pi_j^\infty)^{-1}(p_\infty)$ has cardinality at most C. Fix now $\bar{p}_\infty \in \bar{X}_\infty$ and let $\{\Omega_\alpha\}$ be a family of subsets of X_j which cover $(\bar{\pi}_j^\infty)^{-1}(\bar{p}_\infty)$, and which were obtained in the proof of (2). In constructing the $\{\Omega_\alpha\}_\alpha$ there was the freedom to choose a scale m^{-k}, which in this case we take to be m^{-j}, so that (11.29), (11.30) hold with $k = j$. For each Ω_α there is a $p_{\infty,\sigma} \in [\bar{p}_\infty]$ such that:

$$(\pi_j^\infty)^{-1}(p_{\infty,\sigma}) \cap \Omega_\alpha \neq \emptyset. \tag{11.43}$$

Fix now one $p_\infty \in [\bar{p}_\infty]$. Then by Corollary 4.19 the sets $(\pi_j^\infty)^{-1}(p_\infty)$ and $(\pi_j^\infty)^{-1}(p_{\infty,\sigma})$ must intersect adjacent cells of X_j and so $(\pi_j^\infty)^{-1}(p_\infty) \cap \Omega_\alpha \neq \emptyset$. As distinct Ω_α's are disjoint, we conclude that the cardinality of the set $\{\Omega_\alpha\}$ is at most C. Let now $\bar{q}_\infty \in B_{\bar{X}_\infty}(\bar{p}_\infty, m^{-j})$; then $(\bar{\pi}_j^\infty)^{-1}(\bar{q}_\infty)$ is contained in a $3m^{-j}$-neighbourhood of $\bigcup_\alpha \Omega_\alpha$; as each set Ω_α intersects $(\pi_j^\infty)^{-1}(p_\infty)$ we conclude that:

$$(\bar{\pi}_j^\infty)^{-1}(\bar{q}_\infty) \subset \bigcup_{p \in (\pi_j^\infty)^{-1}(p_\infty)} B(p, 6(C_0+1)m^{-j}); \tag{11.44}$$

thus the David-Semmes regularity condition holds with constants $3(C_0+1)$ and C. □

11.4. **Preservation of admissibility under limits.**

We now turn to the proof of (4) in Theorem 11.4, i.e. the stability of assertions (1)–(3) under the operation of taking weak tangents. This is an immediate consequence of the following lemma and (1)-(3) of Theorem 11.4.

Lemma 11.45. *Any weak tangent of an admissible system is, modulo rescaling, measure-preserving isometric to the limit space of some admissible system.*

Proof. Let (Y, ν, p) be a weak tangent of $(\bar{X}_\infty, \mu_\infty)$. Thus there exists a sequence of basepoints $\{p_\alpha\}$ in $\bar{X}_\infty$, as well as sequences $\{\lambda_\alpha\}$, $\{\lambda'_\alpha\}$ of scale factors, such that the sequence $\{(\lambda_\alpha \bar{X}_\infty, \lambda'_\alpha \mu_\infty, p_\alpha)\}$ converges in the pointed measured Gromov-Hausdorff topology to the pointed (doubling) metric measure space (Y, ν, p).

For each α, we have $\lambda_\alpha = a_\alpha m^{j_\alpha}$ for unique elements $j_\alpha \in \mathbb{Z}$ and $a_\alpha \in [1, m)$. After passing to a subsequence, we may assume that a_α converges to some $a_\infty \in [1, m]$. Without loss of generality, we may replace λ_α with m^{j_α}, since the resulting sequence will converge to the same limit, modulo rescaling by a_∞^{-1}.

Now observe that for every α, the rescaled metric measure space $(\lambda_\alpha \bar{X}_\infty, \lambda'_\alpha \mu_\infty)$ is measure-preserving isometric to the limit space of an admissible system obtained from $\{(X_j, \mu_j)\}$ by shifting the indices by j_α, and rescaling the measures; moreover this new admissible system satisfies Definition 11.1 where the constants m, m_v, H, Δ are independent of α. Thus Lemma 11.45 is reduced to the following lemma. $\square$

Lemma 11.46. *Suppose $\{(X_{j,\alpha}, \mu_{j,\alpha})\}_{\alpha \in \mathbb{N}}$ is a sequence of admissible systems with uniform constants, and $q_{j,\alpha} \in X_{j,\alpha}$ are projection compatible basepoints chosen such that*

$$0 < \liminf_{\alpha} \mu_{0,\alpha}(B(q_{0,\alpha}, 1)) \leq \limsup_{\alpha} \mu_{0,\alpha}(B(q_{0,\alpha}, 1)) < \infty \,. \tag{11.47}$$

Then after passing to a subsequence, the sequences of pointed admissible systems converge in a natural sense to a pointed admissible system $(X_{j,\infty}, \mu_{j,\infty}, q_{j,\infty})$ whose limit space $(\bar{X}_{\infty,\infty}, \mu_{\infty,\infty}, q_{\infty,\infty})$ is measure-preserving isometric to the pointed measured Gromov-Hausdorff limit of the sequence of pointed limit spaces $\{(\bar{X}_{\infty,\alpha}, \mu_{\infty,\alpha}, q_{\infty,\alpha})\}$.

Proof. This is a consequence of standard finiteness/compactness arguments applied to the controlled-geometry complexes of the admissible systems, so we will be brief.

Pick $j \in \mathbb{Z}$. Since the parameter Δ is independent of α, for any N, there are only finitely many possibilities for the combinatorial N-ball centered at $q_{j,\alpha}$, up to a homeomorphism preserving the collection of distinguished characteristic maps. Therefore, after passing to a subsequence, there is a pointed cell-complex $(X_{j,\infty}, q_{j,\infty})$ with a collection of distinguished characteristic maps, such that for all N and large α, there is a homeomorphism $\Psi_{j,\alpha,N}$ from the combinatorial N-ball in $X_{j,\alpha}$ centered at $q_{j,\alpha}$ to the combinatorial N-ball in $X_{j,\infty}$ centered at $q_{j,\infty}$, such that $\Psi_{j,\alpha,N}$ respects distinguished characteristic maps,

$\Psi_{j,\alpha,N}(q_{j,\alpha}) \to q_{j,\infty}$ as $\alpha \to \infty$, and for all $N' \geq N$ the maps $\Psi_{j,\alpha,N}$, $\Psi_{j,\alpha,N'}$ are compatible on the N-balls for large α.

Using (11.47), we get that the $\mu_{j,\alpha}$-measure of any n-cell of $X_{j,\alpha}$ containing $q_{j,\alpha}$ is controlled. Therefore, passing to a subsequence again, there is a family of measures $\{\mu_{j,\infty}\}$ and a compatible system of projection maps $\{\pi^k_{j,\infty} : X_{j,\infty} \to X_{k,\infty}\}$ which define an admissible system, such that the maps $\{\Psi_{j,\alpha,N}\}$ are asymptotically measure-preserving and compatible with projection.

It follows from Proposition 4.12 that for all R, the $\hat{d}_{j,\alpha}$-ball $B(q_{j,\alpha}, R) \subset X_{j,\alpha}$ is a Gromov-Hausdorff approximation to within error $\lesssim m^{-j}$ of the $\bar{d}_{\infty,\alpha}$-ball $B(q_{\infty,\alpha}, R) \subset \bar{X}_{\infty,\alpha}$, so the maps $\{\Psi_{j,\alpha,N}\}$ induce the pointed measured Gromov-Hausdorff convergence

$$(\bar{X}_{\infty,\alpha}, \bar{d}_{\infty,\alpha}, \mu_{\infty,\alpha}, q_{\infty,\alpha}) \longrightarrow (\bar{X}_{\infty,\infty}, \bar{d}_{\infty,\infty}, \mu_{\infty,\infty}, q_{\infty,\infty})\,.$$

□

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