DIFFERENTIABILITY OF LIPSCHITZ MAPS FROM METRIC MEASURE SPACES TO BANACH SPACES WITH THE RADON NIKODYM PROPERTY

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Abstract. In this paper we prove the differentiability of Lipschitz maps $X \to V$, where X is a complete metric measure space satisfying a doubling condition and a Poincaré inequality, and V denotes a Banach space with the Radon Nikodym Property (RNP). The proof depends on a new characterization of the differentiable structure on such metric measure spaces, in terms of directional derivatives in the direction of tangent vectors to suitable rectifiable curves.

Contents

1. Introduction

In this paper we will use the term *PI space* to refer to a λ-quasiconvex complete metric measure space (X, d^X, μ) satisfying a doubling

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condition

$$\mu(B_{2r}(x)) \le 2^\kappa \cdot \mu(B_r(x))\,, \tag{1.1}$$

and p-Poincaré inequality,

$$⨍_{B_r(x)} |f - f_{x,r}|\, d\mu \le \tau r \left(⨍_{B_{\lambda r}(x)} g^p \, d\mu \right)^{\frac{1}{p}}\,, \tag{1.2}$$

where $x \in X$, $r \in (0,\infty)$, f is a continuous function, g is an upper gradient for f,

$$⨍_A f\, d\mu := \frac{1}{\mu(A)} \int_A f\, d\mu\,,$$

$$f_{x,r} := ⨍_{B_r(x)} f\, d\mu\,;$$

see [HK96, Che99, Hei01]. We will also assume that the collection of measurable sets is the completion of the Borel σ-algebra with respect to μ (i.e. every subset of a set of measure zero is measurable) [Roy88, p.221]. Sometimes, by abuse of language, we just say that X is a PI space. The notation above, in particular the space X, the measure μ, as well as the constants κ and λ, will be maintained throughout the paper.

In [Che99], a differentiation theory for real valued Lipschitz functions on PI spaces was given. The notion of differentiation is expressed in terms of an atlas. An atlas consists of a countable collection $\{(U_\alpha, y^\alpha)\}_{\alpha\in\mathcal{A}}$ of *charts*, where the U_α's are measurable subsets, $\mu(X \setminus \bigcup_{\alpha\in\mathcal{A}} U_\alpha) = 0$, $y^\alpha : X \to \mathbf{R}^{k(\alpha)}$ is Lipschitz, and the charts satisfy certain additional conditions. We put $y^\alpha = (y_1^\alpha, \cdots, y_{k(\alpha)}^\alpha)$.

Let V denote a Banach space.

Definition 1.3. A Lipschitz map $f : X \to V$ is differentiable almost everywhere with respect to the atlas $\{(U_\alpha, y^\alpha)\}_{\alpha\in\mathcal{A}}$ if there is a collection

$$\left\{ \frac{\partial f}{\partial y_m^\alpha} : U_\alpha \longrightarrow V \right\}_{\alpha\in\mathcal{A},\ 1\le m\le k(\alpha)}$$

of Borel measurable functions uniquely determined μ-almost everywhere, such that for almost every $\underline{x} \in U_\alpha$,

$$f(x) = f(\underline{x}) + \sum_{m=1}^{k(\alpha)} \frac{\partial f}{\partial y_m^\alpha}(\underline{x})(y_m^\alpha(x) - y_m^\alpha(\underline{x})) + o(d^X(x,\underline{x}))\,. \tag{1.4}$$

We will say that f is differentiable at a specific point $\underline{x} \in X$ if (1.4) holds for that point.

The case $V = \mathbf{R}$ of Definition 1.3 was considered in [Che99]; one of the main results there [Che99, Theorem 4.38] was the existence of an atlas with respect to which every Lipschitz function $f : X \to \mathbf{R}$ is differentiable almost everywhere. We will fix such an atlas throughout the paper. It follows readily from the definitions that if (U_α, y^α) and $(\bar{U}_\alpha, \bar{y}^{\bar{\alpha}})$ are charts from two such atlases, then the matrix of partial derivatives $\frac{\partial y^\alpha_m}{\partial \bar{y}^{\bar{\alpha}}_{\bar{m}}}$ is defined and invertible almost everywhere in the overlap $U_\alpha \cap \bar{U}_{\bar{\alpha}}$. This also yields bi-Lipschitz invariant measurable tangent bundle TX.

The main result of this paper is:

Theorem 1.5. *Every Lipschitz map from X into a Banach space with the Radon-Nikodym Property is differentiable μ-almost everywhere.*

We recall that a Banach space V has the *Radon-Nikodym Property* (RNP) if every Lipschitz map $f : \mathbf{R} \to V$ is differentiable almost everywhere with respect to Lebesgue measure. Since $\mathbf{R}$ is an example of a PI space, Theorem 1.5 is optimal in the sense that the class of Banach space targets considered is maximal.

Just as in [Che99], [CK06], the differentiation theorem above imposes strong restrictions on PI spaces which bi-Lipschitz embed in RNP targets, and may therefore be used to deduce nonembedding theorems.

Theorem 1.6. *If X admits a bi-Lipschitz embedding in a Banach space with the RNP, then for μ-a.e. $x \in X$, every tangent cone at x is bi-Lipschitz homeomorphic to a Euclidean space.*

Discussion of the proof.

The proof of Theorem 1.5 exploits the framework introduced in [CK06], which involves inverse systems of finite dimensional Banach spaces and their inverse limits; see Section 2 for the relevant definitions. It was observed in [CK06] that any separable Banach space V can be realized as a subspace of an inverse limit space $V \subset \varprojlim W_i$, where $\varprojlim W_i$ is the inverse limit of an inverse system of finite dimensional Banach spaces. An advantage of this view point is that the differentiability theory for real valued Lipschitz functions leads immediately to a natural notion of a *weak derivative* of Lipschitz map $f : X \to V$, which is a map taking values in $\varprojlim W_i$.

It was pointed out in [CK08] that inverse limits $\varprojlim W_i$ of inverse systems of finite dimensional Banach spaces are precisely the duals of separable Banach spaces. Since V has the RNP, by a result of Ghoussoub-Maurey [GM84], one can choose an embedding $V \subset \varprojlim W_i$ as above, so that the pair $(\varprojlim W_i, V)$ has the *Asymptotic Norming Property* (ANP). The ANP was introduced by James-Ho [JH81], who showed that it implies the RNP, compare Section 3 and the appendix.

The first step in the proof of Theorem 1.5 is to show that if the weak derivative of f takes values in the subspace $V \subset \varprojlim\, W_i$, then f is differentiable μ-a.e. The argument for this is brief, and illustrates the smooth interaction between the inverse limit setup, the ANP, and basic theorems of measure theory (Egoroff's and Lusin's theorems). As another illustration of this smooth interaction, in the appendix we give a short proof of the James-Ho theorem [JH81].

The remainder of the proof, which appears in Section 4, is devoted to proving that the weak derivative of f takes values in V. A heuristic argument for this goes as follows. If $c : I \to X$ is a Lipschitz curve, then the composition $f \circ c$ is differentiable almost everywhere because V has the RNP. Hence the weak derivative of $f \circ c$ coincides with its usual derivative, and in particular, lies in V. When the curve c has a well-defined measurable velocity $c' : I \to TX$ and the chain rule is applicable, for almost every $t \in I$ the weak derivative evaluated on $c'(t)$ will be the same as $(f \circ c)'(t)$, which belongs to V. In this way one reduces the proof to showing that for a full measure set of points $x \in X$, the tangent space T_xX is spanned by the velocities of such curves c. This fact, which is of independent interest for the geometry of PI spaces, is established in Section 4.

Relation with previous work.

By using an embedding $V \subset \varprojlim\, W_i$ as described above, a version of Theorem 1.5 was proved in [CK06] for a class of separable targets with a property that was termed *Good Finite Dimensional Approximation* (GFDA). It was shown in [CK06] that separable dual spaces have the GFDA property. An essential ingredient in the proof of the differentiation theorem of [CK06] was to show that if V is a GFDA, then $V = \varprojlim\, W_i$. It follows trivially that for GFDA targets, the weak derivative lies in V; compare the discussion above. With the observation in [CK08] that inverse limits $\varprojlim W_i$ are just the duals of a separable

Banach spaces, it followed that the class of GFDA's is precisely the class of separable dual spaces, a strictly smaller class than that of separable spaces with the RNP (or equivalently, the ANP); see [MCO80], [Bou81], [Bou83]. Indeed, it was shown [CK08] that the GFDA condition is equivalent to the ANP supplemented by an additional condition.

2. Inverse limits and the ANP

In this section we briefly recall some facts from [CK06], [CK08].

Let V denote a separable subspace of the dual space Y^* of a separable Banach space Y. The pair (Y^*, V) is said to have the Asymptotic Norming Property (ANP) if for every sequence $\{v_k\} \subset V$ the conditions

$$v_k \overset{weak^*}{\longrightarrow} w \in Y^* ,$$

$$\|v_k\| \to \|w\| ,$$

imply *strong convergence*, i.e. $\lim_{k\to\infty} \|v_k - w\| = 0$. By [GM85], [JH81], a separable Banach space has the RNP if and only if it is isomorphic to some V which belongs to a pair (Y^*, V) with the ANP.

We observed in [CK08] that a Banach space is the dual of a separable Banach space if and only if it is isometric to the inverse limit $\varprojlim W_i$ of an inverse inverse system,

$$W_1 \overset{\theta_1}{\longleftarrow} W_2 \overset{\theta_2}{\longleftarrow} \ldots \overset{\theta_{i-1}}{\longleftarrow} W_i \overset{\theta_i}{\longleftarrow} \ldots , \tag{2.1}$$

where the W_i's are finite dimensional Banach spaces and the bonding maps θ_i are 1-Lipschitz. Such an inverse system will be called a *standard inverse system.*

For the remainder of the paper we consider only standard inverse systems.

We recall that by definition, $\varprojlim W_i$ consists of all sequences $(w_1, w_2, \ldots)$, where $w_i \in W_i$, the compatibility condition $\theta_i(w_{i+1}) = w_i$ holds for all I, and $\sup_i \|w_i\| < \infty$. The norm on $\varprojlim W_i$ is defined by $\|w\| = \lim_{i\to\infty} \|\pi_i(w)\|$.

Let $\pi_j : \varprojlim W_i \to W_j$ denote the natural projection. We may view the inverse limit $\varprojlim W_i$ as the dual space of the direct limit $\varinjlim W_i^*$ of the dual direct system

$$W_1^* \overset{\theta_1^*}{\to} W_2^* \overset{\theta_2^*}{\to} \ldots$$

Then a norm bounded sequence $\{v_k\} \subset \varprojlim W_i$ weak* converges to $v_\infty \in \varprojlim_i W_i$ if and only if the projected sequence $\{\pi_j(v_k)\} \subset W_j$ converges to $\pi_j(v_\infty)$, for all j.

3. Weak derivatives

Let T^*X denote the measurable cotangent bundle of the PI space X, and let $f : X \to V$ denote a Lipschitz map which is differentiable almost everywhere in the sense of Definition 1.3. The differential df is the bounded measurable section of $T^*X \otimes V$ whose expression in the canonical trivialization of $T^*X \otimes V$ over U_α is

$$df = \left(\frac{\partial f}{\partial y_1^\alpha}, \dots, \frac{\partial f}{\partial y_{k(\alpha)}^\alpha} \right) ; \tag{3.1}$$

compare (1.4).

Recall that by definition, the tangent bundle TX is the dual bundle of the cotangent bundle T^*X. It will be convenient to work with the derivative $D_x f : TX \to V$, which coincides with the differential df under the identification $\mathrm{Hom}(TX, V) \simeq T^*X \otimes V$. Thus,

$$D_x f(z) = \sum_{m=1}^{k(\alpha)} \frac{\partial f}{\partial y_m^\alpha}(x) z_m \, , \tag{3.2}$$

where $x \in U_\alpha$ and $z = \sum_m z_m \frac{\partial}{\partial y_m^\alpha} \in TX_x$.

Let $V \subset \varprojlim W_i$ denote any closed linear subspace and assume $f : X \to V$ is Lipschitz. Put $f_i = \pi_i \circ f$. For μ-a.e. $x \in X$, if $v \in T_x X$, the collection of directional derivatives $\{D_x f_i(v)\}$ determines a norm bounded compatible sequence in the inverse system $\{W_i\}$, and we thereby obtain a *weak derivative* $\{D_x f_i\} : TX \to \varprojlim W_i$. The weak derivative is *weakly measurable*, in the sense that its composition with $\pi_j : \varprojlim W_i \to W_j$ is measurable, for all j.

Remark 3.3. When a Lipschitz map $f : X \to V \subset \varprojlim W_i$ is differentiable almost everywhere, the weak derivative is the true derivative, i.e. for μ-a.e. $x \in X$, we have $D_x f = \{D_x f_i\}$. This follows readily from the definitions. In particular, it follows that in this case the weak derivative is a Borel measurable mapping.

Thus far in this section we have not explicitly invoked the Poincaré inequality. The next proposition, which is the converse of the measurability statement in the preceding remark, will make use of it.

Proposition 3.4. *Let $f : X \to V$ be a Lipschitz map, where V is an arbitrary Banach space, and suppose the weak derivative $\{Df_i\} : TX \to \varprojlim W_i$ is a Borel measurable mapping, with respect to the measurable vector bundle structure on TX. Then f is differentiable almost everywhere.*

Proof. Since $\{Df_i\} : TX \to \varprojlim W_i$ is measurable, by Lusin's theorem, for almost every $\underline{x} \in X$ there is an $\alpha \in \mathcal{A}$ and a measurable subset $A \subset U_\alpha$, such that $\underline{x} \in A$ is a density point of A, and $\{\frac{\partial f_i}{\partial y^\alpha_m}\} : U_\alpha \to \varprojlim W_i$ is continuous on A for all $m \in \{1, \ldots, k(\alpha)\}$.

Let ℓ denote the function on the right-hand side of (1.4), where the partial derivative $\frac{\partial f}{\partial y^\alpha_m}$ is replaced by

$$\{D_{\underline{x}} f_i\} \left(\frac{\partial}{\partial y^\alpha_m} \right) = \left\{ \frac{\partial f_i}{\partial y^\alpha_m}(\underline{x}) \right\} .$$

Put $\ell_i = \pi_i \circ \ell$. Then $D_x \ell$ is constant in the canonical local trivialization of TX on U_α, and by the assumed continuity of $\{\frac{\partial f_i}{\partial y^\alpha_m}\}$ on A, for all $x \in A$ and all i, we have

$$\lim_{x \to \underline{x}} \| D_x f_i - D_x \ell_i \| = 0 ,$$

where the convergence is uniform in i. Hence, by the Poincaré inequality applied to the function $f_i - \ell_i$, and the fact that f is Lipschitz, the quantity

$$\sup_{x \in B_r(\underline{x})} \frac{\| f_i(x) - \ell_i(x) \|}{r} \tag{3.5}$$

tends to zero as $r \to 0$, uniformly in i. Thus at $\underline{x}$, the weak derivative is a true derivative.

□

The proof the main theorem, modulo showing that the weak derivative lies in V.

For the remainder of the paper, we let $f : X \to V$ be as in the statement of Theorem 1.5, and $V \subset \varprojlim W_i$ be an embedding such that the pair $(\varprojlim W_i, V)$ has the ANP, as in Section 2.

Lemma 3.6. *Suppose the weak derivative $\{D_x f_i\}$ takes values in $V \subset \varprojlim W_i$ for μ-a.e. $x \in X$. Then f is differentiable almost everywhere.*

Proof. By Proposition 3.4, it suffices to show that the weak derivative is measurable. To verify this, it suffices to check that for every α and every finite measure subset $A_0 \subset U_\alpha$, there is a nearly full measure subset $A \subset A_0$ where $\{\frac{\partial f_i}{\partial y_m^\alpha}\} : A \to \varprojlim W_i$ is continuous.

Since the maps, $x \to \frac{\partial f_i}{\partial y_m^\alpha}$, $m = 1, \ldots, k(\alpha)$, are measurable, by Lusin's theorem, given any subset of finite measure, these functions are uniformly continuous, for all i, off a subset of arbitrarily small measure. By Egoroff's theorem and Lusin's theorem, the same holds for $\|\{\frac{\partial f_i}{\partial y_m^\alpha}\}\|$.

Since $\{D_x f_i\}$ takes values in V for μ-a.e. x, for any finite measure subset of X, we may invoke the Asymptotic Norming Property of $(\varprojlim W_i, V)$ to conclude that $\{D_x f_i\}$ is strongly continuous off a set of arbitrarily small measure.

□

4. Velocities of curves

To show that the weak derivative $\{Df_i\} : TX \to \varprojlim W_i$ takes values in V, and thereby complete the proof of Theorem 1.5, we will use directional derivatives along rectifiable curves, as indicated in the introduction. To formalize this, we need to make precise the notion of the velocity of a curve.

Velocities and the chain rule.

Let $X_0 \subset X$ be a full μ-measure subset such that for every $x \in X_0$, if $x \in U_\alpha \cap U_\beta$, then x is a point of differentiability of y_m^α with respect to the chart y^β, for all $m \in \{1, \ldots, k(\alpha)\}$.

Definition 4.1. If $c : I \to X$ is a Lipschitz curve, $t \in I$ is a point of differentiability of $y^\alpha \circ c$ for all $\alpha \in \mathcal{A}$, and $c(t) \in X_0$, then the *velocity of c at t* is defined to be the tangent vector

$$c'(t) = \sum_{m=1}^{k(\alpha)} (y_m^\alpha \circ c)'(t) \frac{\partial}{\partial y_m^\alpha} \in T_{c(t)}X\,.$$

Note that this definition makes sense because of the choice of the set X_0.

With this definition, the chain rule becomes:

Lemma 4.2. *Suppose $c : I \to X$ is a Lipschitz curve, and the velocity vector $c'(t) \in TX$ is defined.*

1. For any Lipschitz function $u : X \to \mathbf{R}$ which is differentiable with respect to the atlas $\{(U_\alpha, y^\alpha)\}$ at x, the derivative $(u \circ c)'(t)$ is defined, and

$$(u \circ c)'(t) = (D_{c(t)}u)(c'(t)) .$$

2. If $c(t) \in X$ is a point of weak differentiability of $f : X \to V$ (i.e. x is a point of differentiability of f_i for all i), and t is a point of differentiability of $f \circ c$, then we have the following chain rule relating the derivative of $f \circ c$ and the weak derivative of f:

$$(f \circ c)'(t) = \{(f_i \circ c)'(t)\} = \{(D_{c(t)} f_i)(c'(t))\} .$$

In particular, the weak derivative of f in the direction of the velocity vector $c'(t)$ lies in $V \subset \varprojlim W_i$.

The proof is straightforward.

Velocities span the tangent space.

Next we prove the following:

Theorem 4.3. *Fix a countable collection $\Phi = \{\phi_i : X \to V_i\}$ of Lipschitz maps into RNP Banach spaces. Let $\mathcal{V}_0$ be the collection of tangent vectors $v \in TX$ such that there is a 1-Lipschitz curve $c : I \to X$ and $t \in I$, where $v = c'(t)$, and t is a point of differentiability of $\phi_i \circ c$ for all i. Then there is a set $Z \subset X$ with $\mu(X \setminus Z) = 0$, such that $\mathcal{V}_0 \cap T_zX$ spans the fiber T_zX at every point $z \in Z$.*

Remark 4.4. Elsewhere we will show that for a full measure set of $x \in X$, there is a dense set of directions in $\mathcal{V}_0 \cap T_xX$, i.e. the set of rays in T_xX which intersect $\mathcal{V}_0$ is dense in T_xX. However, we will not need this finer result here.

Proof of Theorem 1.5 using Lemma 3.6 and Theorem 4.3. Applying Theorem 4.3 with $\Phi = \{f : X \to V\}$, we obtain a full measure subset $Z \subset X$ as in the Theorem. Let $W \subset Z$ be a full measure subset where f is weakly differentiable. Then for every $x \in W$, and every $v \in \mathcal{V}_0 \cap T_xX$, we have $\{(D_xf_i)(v)\} \in V$, by part 2 of Lemma 4.2.

Since $\mathcal{V}_0 \cap T_xX$ spans T_xX, it follows that $\{(D_x f_i)(T_xX)\} \subset V$. Hence f is differentiable almost everywhere by Lemma 3.6.

We now turn to the proof of Theorem 4.3.

Let $\mathcal{V}$ be the (fiberwise) span of $\mathcal{V}_0$, i.e. $\mathcal{V} \cap T_xX = \operatorname{span}(\mathcal{V}_0 \cap T_xX)$.

We begin with a preview of the argument. We will first show that $\mathcal{V}$ defines a measurable sub-bundle of TX. If there is a positive measure set of points $x \in X$ where $\dim(\mathcal{V} \cap T_xX) < \dim T_xX$, then by Lusin's theorem, we may pass to a positive measure subset A of some U_α with the same property, where in addition $\mathcal{V}$ lies in a continuous codimension 1 sub-bundle E of the $k(\alpha)$-dimensional bundle $TX\big|_A$, where the continuity is defined with respect to the bundle chart induced by y^α. If p is a density point of A, and u is a linear combination of coordinates $y_1^\alpha, \ldots, y_{k(\alpha)}^\alpha$ whose derivative at p has kernel $E \cap T_pX$, then one finds that there is an upper gradient ρ for u such that

$$\lim_{\substack{x \to p, \\ x \in A}} \rho(x) = 0.$$

This implies that the average of ρ over $B_r(p)$ tends to zero as $r \to 0$, which contradicts the nondegeneracy of u.

We now give the details. Our first step is:

Lemma 4.5. *The sub-bundle $\mathcal{V} \subset TX$ is measurable.*

Prior to proving the lemma, we recall some facts about Suslin sets [Fed69]. A subset of a metric space is *Suslin* if it is the continuous image of a Borel subset of a complete separable metric space. Suslin sets in a complete, σ-finite Borel regular measure space such as X, are μ-measurable; this is why we assumed that the measure μ is complete. Note that if Z is a complete separable metric space, then the image of a Suslin set $S \subset Z$ under a Suslin measurable mapping $\tau : Z \to X$ is also Suslin (because the graph of τ is a Suslin subset of $Z \times X$).

Proof. In brief, the proof is a straightforward application of the facts about Suslin sets recalled above.

It suffices to show that for each α, the restriction of $\mathcal{V}$ to U_α is measurable. We may assume that U_α is Borel measurable, and that it is contained in the set X_0 defined before Definition 4.1.

Let Γ denote the space of 1-Lipschitz maps $c : [0,1] \to X$, equipped with the compact-open topology. Then Γ is a complete, separable metric space, since X is complete and doubling. Consider the collection $S_0 \subset \Gamma \times [0,1]$ of pairs (c,t) such that $c(t) \in U_\alpha$, and the composition $\phi_i \circ c : [0,1] \to V_i$ is differentiable at t for all i. This is easily seen to be a Borel set. Also, the map $\sigma : S_0 \to \mathbf{R}^{k(\alpha)}$ which sends $(c,t) \in S_0$ to

$$(D_{c(t)}y^\alpha)(c'(t)) = (y^\alpha \circ c)'(t) \in \mathbf{R}^{k(\alpha)}$$

is Borel measurable.

For $j \in \{1,\ldots,k(\alpha)\}$, let T_j be the set of points $x \in U_\alpha$ where the fiber $\mathcal{V} \cap T_xX$ has dimension j. We claim that T_j is a Suslin subset of X. To see this, let S_1 be the set of j-tuples $((c_1,t_1),\ldots,(c_j,t_j)) \in S_0^j$ such that $c_1(t_1) = \ldots = c_j(t_j)$; this is a closed subset of S_0^j. Then let $S_2 \subset U_\alpha \times \wedge^j \mathbf{R}^{k(\alpha)}$ be the image of S_1 under the Borel map $S_1 \to U_\alpha \times \wedge^j \mathbf{R}^{k(\alpha)}$ which sends $((c_1,t_1),\ldots,(c_j,t_j))$ to

$$(c_1(t_1), \sigma((c_1,t_1)) \wedge \ldots \wedge \sigma((c_j,t_j)) \in U_\alpha \times \wedge^j \mathbf{R}^{k(\alpha)}\,.$$

Then $\cup_{k \geq j} T_k$ – the set of points where the fiber of $\mathcal{V}$ has dimension at least j – is the projection of $S_2 \cap (U_\alpha \times (\wedge^j \mathbf{R}^{k(\alpha)} \setminus \{0\}))$ to U_α, and is therefore a Suslin set and μ-measurable. It follows that T_j is μ-measurable for all $j \in \{1,\ldots,k(\alpha)\}$.

Let $\bar{T}_j \subset T_j$ be a full measure Borel subset of T_j. Then $\cup_{j=1}^{k(\alpha)} \bar{T}_j$ has full measure in U_α.

Fix $j \in \{1,\ldots,k(\alpha)\}$. Let $G(j,k(\alpha))$ denote the Grassman manifold of j-planes in $\mathbf{R}^{k(\alpha)}$. Then there is a well-defined map $\gamma_j : \bar{T}_j \to G(j,k(\alpha))$ such that for every $x \in \bar{T}_j$, the fiber $\mathcal{V} \cap T_xX$ maps under $D_xy^\alpha : T_xX \to \mathbf{R}^{k(\alpha)}$ to the subspace $\gamma_j(x)$ of $\mathbf{R}^{k(\alpha)}$. To see that γ_j is measurable, pick an open subset of $G(j,k(\alpha))$, and observe that its inverse image in $\bar{T}_j$ is a Suslin set, using a construction similar to the one above.

$\square$

We now return to the proof of Theorem 4.3.

Suppose $\dim(\mathcal{V} \cap T_xX)$ is strictly smaller than $\dim T_xX$ for a positive measure set of points $x \in X$. Then for some index $\alpha \in \mathcal{A}$, there is a measurable subset $A \subset U_\alpha$ with $\mu(A) \in (0,\infty)$, where the strict inequality $\dim(\mathcal{V} \cap T_xX) < k(\alpha)$ holds. By Lusin's theorem, we may assume without loss of generality that $\mathcal{V}|A$ is contained in a codimension 1 sub-bundle E of $TX|A$, where E is a continuous sub-bundle relative to the bundle charts given by y^α, i.e. the fiber of E at $x \in A$ is the kernel of $\psi(x) \circ D_x y^\alpha : T_x X \to \mathbf{R}$, for some continuous map $\psi : A \to (\mathbf{R}^{k(\alpha)})^* \setminus \{0\}$. It follows from the defining property of our atlas $\{(U_\alpha, y^\alpha)\}$ – specifically the almost everywhere uniqueness of coefficients appearing in (1.4) – that we may also assume that for every $p \in A$, every nontrivial linear combination of the coordinate functions y^α_m has nonzero pointwise upper Lipschitz constant at p.

Choose $p \in A$, and put $\bar{\psi} = \psi(p)$.

Lemma 4.6. *There is a continuous function $\zeta : A \to [0, \infty)$ such that $\zeta(x) \to 0$ as $x \to p$, and*

$$|D_x(\bar{\psi} \circ y^\alpha)(v)| \leq \zeta(x),$$

for every $v \in \mathcal{V}_0 \cap T_x X$.

Proof. Let c and t be as in the definition of $\mathcal{V}_0$, so that $c'(t) = v$. Then $D_x(\bar{\psi} \circ y^\alpha)(v) = \bar{\psi}((D_x y^\alpha)(v))$. Also, the vector $(D_x y^\alpha)(v)$ has uniformly bounded norm since y^α is Lipschitz, and it lies in the hyperplane $\ker \psi(x) \subset \mathbf{R}^{k(\alpha)}$, which approaches $\ker \bar{\psi}$ as $x \to p$. The lemma follows. $\square$

Define a function $\rho : X \to [0, \infty)$ by $\rho(x) = \zeta(x)$ if $x \in A$, and $\rho(x) = L$ otherwise, where L is the Lipschitz constant of $\bar{\psi} \circ y^\alpha$. We claim that ρ is an upper gradient for $\bar{\psi} \circ y^\alpha$. To see this, we need only show that if $c : I \to X$ is a 1-Lipschitz curve, then for almost every $t \in I$, we have

$$|\bar{\psi} \circ y^\alpha \circ c)'(t)| \leq \rho \circ c(t)\,.$$

If $t \in I$ is such that $c(t) \notin A$, then this obviously holds, since $\bar{\psi} \circ y^\alpha \circ c$ is L-Lipschitz. If $t \in I$ is a point such that $c(t) \in A$ and the derivatives $(y^\alpha \circ c)'(t)$ and $(\phi_i \circ c)'(t)$ are defined for all i, then $c'(t)$ is defined and lies in $\mathcal{V}_0$. Therefore the chain rule applies, and

$$|(\bar{\psi} \circ y^\alpha \circ c)'(t)| = |\bar{\psi}((D_{c(t)} y^\alpha)(c'(t))| \leq \zeta(c(t)) = \rho \circ c(t)$$

by Lemma 4.6. The remaining points $t \in I$ have measure zero.

Now let $p \in A$ be a density point of A. Applying the Poincaré inequality to $\psi \circ y^\alpha$ on balls $B(p, r)$, using the fact that p is a density point of Z and $\psi \circ y^\alpha$ is Lipschitz, we conclude the pointwise upper

Lipschitz constant of $\psi \circ y^\alpha$ at p is zero:

$$\limsup_{r\to 0} \frac{\sup\ \{\psi \circ y^\alpha(x) - \psi \circ y^\alpha(p) \mid x \in B(p,r)\}}{r} = 0.$$

This is a contradiction to the choice of A, which completes the proof of the theorem.

□

5. A new characterization of the minimal upper gradient

In this section will give a new characterization of the minimal upper gradient. We then apply this to give a different proof of Theorem 4.3. It will also play a role in the proof of the stronger version of Theorem 4.3 alluded to in Remark 4.4.

Generalized upper gradients and minimal upper gradients.

Recall that if $u : X \to \mathbf{R}$ is a Lipschitz function, then a Borel measurable function $g : X \to [0,\infty]$ is a *generalized upper gradient* if there is a sequence of Lipschitz functions $u_k : X \to \mathbf{R}$, and a sequence $g_k \in L^p_{\mathrm{loc}}(X)$ such that g_k is an upper gradient for u_k for all k, and $u_k \stackrel{L^p_{\mathrm{loc}}}{\to} u$, $g_k \stackrel{L^p_{\mathrm{loc}}}{\to} g$ [Che99, Section 2]. This is equivalent to being a *p-weak upper gradient*, i.e. satisfing the usual upper gradient condition for all but a set of curves of zero p-modulus [Sha00, Che99]. It was shown in [Che99] that every Lipschitz function $u : X \to \mathbf{R}$ has a *minimal upper gradient*, which is a generalized upper gradient $g_f : X \to \mathbf{R}$ with the property that every generalized upper gradient g satisfies $g \geq g_f$ almost everywhere.

Negligible sets.

Let $\mathcal{C}$ denote the space of 1-Lipchitz curves $c : [0,1] \to X$. With the metric $d(c_1, c_2) = \max_{t\in[0,1]}\ d^X(c_1(t), c_2(t))$, the space $\mathcal{C}$ is complete and separable, as is $\mathcal{C} \times [0,1]$ equipped with the product metric.

Definition 5.1. A subset $N \subset \mathcal{C} \times [0,1]$ will be called *negligible* if it is Borel, and for all $c \in \mathcal{C}$,

$$\mathcal{L}(N \cap (\{c\} \times [0,1])) = 0\,,$$

where $\mathcal{L}$ denotes Lebesgue measure on $[0,1]$. Clearly, a countable union of negligible sets is negligible.

Theorem 5.2. *Let $f : X \to \mathbf{R}$ be a Lipschitz function with minimal upper gradient g_f, and let $N \subset \mathcal{C} \times [0,1]$ be a negligible set. Define a function $\hat{g}_f : X \to [0,\infty)$ by letting $\hat{g}_f(x)$ be the supremum of the set*

$$\{\, |(f \circ c)'(t)| \;\mid\; (c,t) \in \mathcal{C} \times [0,1] \setminus N,\;\; c(t) = x,\;\; \text{and } (f \circ c)'(t) \text{ exists}\}$$

if it is nonempty, and 0 otherwise. Then $\hat{g}_f$ coincides with g_f almost everywhere.

Proof. We begin by showing that the function $\hat{g}_f$ defined above is μ-measurable. For this it suffices to show that $\hat{g}_f^{-1}((a,\infty))$ is a Suslin set for all $a \in [0,\infty)$.

Let $\pi : \mathcal{C} \times [0,1] \to X$ be the map $\pi((c,t)) = c(t)$. Then π is continuous, and $\hat{g}_f^{-1}((a,\infty))$ is the image under π of the Borel set

$$\{(c,t) \in \mathcal{C} \times [0,1] \setminus N \;\mid\; (f \circ c)'(t) \text{ exists},\; |(f \circ c)'(t)| \in (a,\infty)\}\;;$$

hence $\hat{g}_f^{-1}((a,\infty))$ is Suslin, as claimed.

Let $g : X \to \mathbf{R}$ be a Borel measurable function such that $g \geq \hat{g}_f$, and $g = \hat{g}_f$ almost everywhere. Notice that g is an upper gradient for f, because if $c \in \mathcal{C}$, then for a.e. $t \in [0,1]$, the derivative $(f \circ c)'(t)$ exists, $(c,t) \notin N$, and

$$|(f \circ c)'(t)| \leq \hat{g}_f \circ c(t) \leq g \circ c(t)\,.$$

Therefore $\hat{g}_f \geq g_f$ almost everywhere, since g_f is a minimal upper gradient.

Observe that $\hat{g}_f \leq \operatorname{Lip} f$ everywhere; this follows from the fact that if $c \in \mathcal{C}$ and $(f \circ c)'(t)$ exists, then $\operatorname{Lip}_{c(t)}(f) \geq |(f \circ c)'(t)|$ because c is 1-Lipschitz. By [Che99, Thm 6.1], we have $g_f = \operatorname{Lip} f$ almost everywhere, and hence $\hat{g}_f \leq g_f$ almost everywhere. Thus $\hat{g}_f = g_f$ almost everywhere. □

Remark 5.3. The full strength of Theorem 5.2 is not used in the application given below. It would be sufficient to know that $g \geq C\, g_f$ μ-a.e., for some $C \in (0,\infty)$ which does not depend on f, and this is considerably easier to prove, see [Che99, Prop. 4.26].

An alternate proof of Theorem 4.3.

Returning to the setting of Theorem 4.3, let $N \subset \mathcal{C} \times [0,1]$ be the negligible set of pairs (c,t) such that one of the compositions $\{y^\alpha \circ c\}_{\alpha \in \mathcal{A}}$ or $\{\phi_i \circ c\}_{\phi_i \in \Phi}$ is not differentiable at t.

For each $\alpha \in \mathcal{A}$, and each rational $k(\alpha)$-tuple $(a_1, \ldots, a_{k(\alpha)}) \in \mathbb{Q}^{k(\alpha)}$, let $h_{(a_1,\ldots,a_{k(\alpha)})} : X \to \mathbf{R}$ be the function $\hat{g}_f$ defined as in Theorem 5.2, with $f = \sum_m \, a_m \, y^\alpha_m$.

Now let $Z \subset X$ be a full measure Borel set such that:

(1) $Z \subset X_0$, where $X_0 \subset X$ is the set defined before Definition 4.1.
(2) For all $\alpha \in \mathcal{A}$, $(a_1, \ldots, a_{k(\alpha)}) \in \mathbb{Q}^{k(\alpha)}$, and $x \in Z$, the function $h_{(a_1,\ldots,a_{k(\alpha)})}(x)$ coincides with the pointwise upper Lipschitz constant of the function $f = \sum_m \, a_m \, y^\alpha_m$ at every point in Z.
(3) The function $h_{(a_1,\ldots,a_{k(\alpha)})}$ is approximately continuous at every point in Z.

Suppose that the $\mathcal{V}_0 \cap T_x X$ does not span the fiber $T_x X$ at some point $x \in U_\alpha \cap Z$. Then there is a nonzero $k(\alpha)$-tuple $(b_1, \ldots, b_{k(\alpha)}) \in \mathbf{R}^{k(\alpha)}$ such that $\sum_m b_m \, (D_x y^\alpha_m)(v) = 0$ for all $v \in \mathcal{V}_0 \cap T_x X$. The chain rule (Lemma 4.2) implies that $\hat{g}_u(x) = 0$, where $u := \sum_m b_m \, y^\alpha_m$. If $\{a^j\} \subset \mathbb{Q}^{k(\alpha)}$ is a sequence converging to $b = (b_1, \ldots, b_{k(\alpha)})$, then $h_{a^j}(x) \to 0$. Therefore the upper pointwise Lipschitz constant of $\sum_m \, a^j_m \, y^\alpha_m$ at x tends to zero as well. Hence the nontrivial linear combination $\sum_m \, b_m \, y^\alpha_m$ has zero pointwise upper Lipschitz constant at x, which contradicts the fact that the differentials of the y^α_m's are independent on $U_\alpha \cap Z$.

□

Appendix. The ANP implies the RNP (An alternate proof of the theorem of James-Ho [JH81])

Using the formalism of inverse limit spaces, in this appendix we will give a short direct proof that the ANP implies that a Lipschitz function $f : I \to V \subset \varprojlim W_i$, is differentiable a.e., provided the separable space V has the ANP; compare [JH81].

Let $I \subset \mathbf{R}$ denote a finite interval. Given $\eta > 0$, by Lusin's theorem (respectively, Egoroff's and Lusin's theorems) there exists $A \subset I$ such that $\mathcal{L}(I \setminus A) < \eta$ and in addition, $D_x f_i$ (for all i) and $\|\{D_x f_i\}\|$ are uniformly continuous on A. It suffices to show that f is differentiable, with derivative $D_x f = \{D_x f_i\}$, for every density point $\underline{x}$ of A.

Put

$$\Delta_\epsilon f(\underline{x}) = \frac{f(\underline{x} + \epsilon) - f(\underline{x})}{\epsilon} \, .$$

Then for all i,

$$\pi_i(\Delta_\epsilon f(\underline{x})) = \frac{f_i(\underline{x}+\epsilon) - f_i(\underline{x})}{\epsilon}\,,$$

$$\pi_i(\Delta_\epsilon f(\underline{x})) = \frac{1}{\epsilon} \cdot \int_{\underline{x}}^{\underline{x}+\epsilon} D_x f_i\, dx\,. \tag{5.4}$$

Taking the norm of both sides of (5.4) and using $\|D_x f_i\| \leq \|\{D_x f_i\}\|$, gives

$$\|\pi_i(\Delta_\epsilon f(\underline{x}))\| \leq \frac{1}{\epsilon} \cdot \int_{\underline{x}}^{\underline{x}+\epsilon} \|\{Df_i\}\|\, dx\,. \tag{5.5}$$

Letting $\epsilon \to 0$, then $i \to \infty$ in (5.4) and using that $D_x f_i$ is continuous on A and that $\underline{x}$ is a density point of A, shows that $\Delta_\epsilon f$ converges weak* to $\{D_{\underline{x}} f_i\}$ as $\epsilon \to 0$. Letting $i \to \infty$, then $\epsilon \to 0$ in (5.5) and using that $\|\{D_x f_i\}\|$ is continuous on A and $\underline{x}$ is a density point of A, shows that $\limsup_{i\to\infty} \|\Delta_\epsilon f\| \leq \|\{D_{\underline{x}} f_i\}\|$. Since $\Delta_\epsilon f(\underline{x}) \in V$, the ANP implies that $\Delta_\epsilon f(\underline{x})$ converges strongly to $\{D_{\underline{x}} f_i\}$.

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