RECTIFIABILITY OF SETS OF FINITE PERIMETER IN CARNOT GROUPS: EXISTENCE OF A TANGENT HYPERPLANE

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Abstract. We consider sets of locally finite perimeter in Carnot groups. We show that if E is a set of locally finite perimeter in a Carnot group G then, for almost every $x \in G$ with respect to the perimeter measure of E, some tangent of E at x is a vertical halfspace. This is a partial extension of a theorem of Franchi-Serapioni-Serra Cassano in step 2 Carnot groups: they show in [18, 19] that, for almost every x, E has a unique tangent at x, and this tangent is a vertical halfspace.

1. Introduction

The differentiability properties of functions and the rectifiability properties of sets are classical themes of Real Analysis and Geometric Measure Theory, with many mutual connections. In the context of stratified Carnot groups, the first problem has been solved, within the category of Lipschitz maps, in a deep work of Pansu [37]; here we are interested in the second problem, in the class of sets E of locally finite perimeter: if we denote by $X_1, \ldots, X_m$ an orthonormal basis of the horizontal layer of the Lie algebra $\mathfrak{g}$ of left-invariant vector fields of the Carnot group $\mathbb{G}$, this class of sets is defined by the property that the distributional derivatives $X_1 \mathbb{1}_E, \ldots, X_m \mathbb{1}_E$ are representable by Radon measures in $\mathbb{G}$. This notion, which extends the classical one developed and deeply studied by De Giorgi in [12] and [13] (see also [3]), is compatible with the Carnot-Carathéodory (subriemannian) distance d induced by $X_1, \ldots, X_m$; in this context the total variation $|D\mathbb{1}_E|$ of the $\mathbb{R}^m$-valued measure $(X_1\mathbb{1}_E, \ldots, X_m\mathbb{1}_E)$ plays the role of surface measure associated to d. Our interest in this topic was also motivated by the recent papers [7], [8], where sets of finite perimeter in Carnot groups (and in particular in the Heisenberg groups) are used to study a new notion of differentiability for maps with values in L^1, with the aim of finding examples of spaces which cannot be bi-Lipschitz embedded into L^1.

The first basic properties of the class of sets of finite perimeter (and of BV functions as well), such as compactness, global and local isoperimetric inequalities, have been proved in [21]; then, in a series of papers [18, 19], Franchi, Serapioni and Serra Cassano made a more precise analysis of this class of sets, first in the Heisenberg groups $\mathbb{H}^n$ and then in all step 2 Carnot groups (using also some measure-theoretic properties proved, in a more general context, in [1], see also Theorem 4.16). As in the work of De Giorgi, the crucial problem is

Date: April 18, 2008.
1991 *Mathematics Subject Classification.* 28A75; 49Q15; 58C35.
Key words and phrases. Rectifiability, Carnot groups, Cacciopoli set, sets of finite perimeter.
The second author was partially supported by NSF grant DMS-0701515.

the analysis of tangent sets to E at a point $\bar{x}$, i.e. all limits

$$\lim_{i\to\infty} \delta_{1/r_i}(\bar{x}^{-1}E)$$

where $(r_i) \downarrow 0$ and convergence occurs locally in measure (here $\delta_r : \mathbb{G} \to \mathbb{G}$ denote the intrinsic dilations of the group). In [19] it is proved that for $|D\mathbb{1}_E|$-a.e. $\bar{x}$ there exists a unit vector $\nu_E(\bar{x}) \in \mathbf{S}^{m-1}$, that we shall call horizontal normal, such that

$$\sum_{i=1}^{m} \nu_{E,i}(\bar{x}) X_i \mathbb{1}_E \geq 0 \qquad \text{and} \qquad \sum_{i=1}^{m} \xi_i(\bar{x}) X_i \mathbb{1}_E = 0 \quad \forall \xi \perp \nu_E(\bar{x}). \tag{1.1}$$

We shall call these sets with constant horizontal normal (identified, in the coordinates relative to the basis $X_1, \ldots, X_m$, by the vector $\nu_E(\bar{x})$): the question is whether (1.1) implies additional information on the derivative of E along vector fields Y that do not belong to the horizontal layer: even though $m < n = \dim(\mathfrak{g})$, this can be expected, having in mind that the Lie algebra generated by $X_1, \ldots, X_m$ is the whole of $\mathfrak{g}$. The main result of [19] is the proof that, in all step 2 groups, (1.1) implies $[X_i, X_j]\mathbb{1}_E = 0$ for all i, $j = 1, \ldots, m$. As a consequence, up to a left translation E is really, when seen in exponential coordinates, an halfspace:

$$\left\{ x \in \mathbb{R}^n : \ \sum_{i=1}^{m} \nu_{E,i}(\bar{x}) x_i \geq 0 \right\}.$$

We shall call it vertical halfspace, keeping in mind that there is no dependence on the coordinates $x_{m+1}, \ldots, x_n$. This fact leads to a complete classification of the tangent sets and has relevant consequences, as in the classical theory, on the representation of $|D\mathbb{1}_E|$ in terms of the spherical Hausdorff measure and on the rectifiability, in a suitable intrinsic sense, of the measure-theoretic boundary of E, see [19] for more precise informations.

On the other hand, still in [19], it is proved that for general Carnot groups the conditions (1.1) do not characterize vertical halfspaces: an explicit example is provided in a step 3 group of Engel type (see also Section 7). Basically, because of this obstruction, the results of [19] are limited to step 2 groups.

The classification and even the regularity properties of sets E with a constant horizontal normal is a challenging and, so far, completely open question. However, recently we found a way to bypass this difficulty and, in this paper, we show the following result:

Theorem 1.2. *Suppose $E \subseteq \mathbb{G}$ has locally finite perimeter. Then, for $|D\mathbb{1}_E|$-a.e. $\bar{x} \in \mathbb{G}$ a vertical halfspace H belongs to the tangents to E at $\bar{x}$.*

Of course Theorem 1.2 does not provide yet a complete solution of the rectifiability problem: indeed, even though the direction $\nu_E(\bar{x})$ of the halfspace H depends on $\bar{x}$ only, we know that $\bar{x}^{-1}E$ is close on an infinitesimal sequence of scales to H, but we are not able to show that this happens on all sufficiently small scales. What is still missing is some monotonicity/stability argument that singles out halfspaces as the only possible tangents, wherever they are tangent (see also the discussion in Remark 5.5). In a similar context, namely the rectifiability of measures having a spherical density, this is precisely the phenomenon discovered by Preiss in [39]: we took some ideas from this paper, adapting them to the setting of Carnot groups, to obtain our result. For these reasons, the complete solution of the rectifiability problem seems to be related to the following question, or suitable stronger quantitative versions of it (we denote by $\mathrm{vol}_{\mathbb{G}}$ the Haar measure of the group and by e the identity of the group):

Question 1.3. *Let $E \subset \mathbb{G}$ be a set with a constant horizontal normal $\nu \in \mathbf{S}^{m-1}$ and let H be a vertical halfspace with the same horizontal normal. If*

$$\liminf_{R \to +\infty} \frac{\mathrm{vol}_{\mathbb{G}}\big((E \Delta H) \cap B_R(e)\big)}{\mathrm{vol}_{\mathbb{G}}\big(B_R(e)\big)} = 0,$$

then E is a vertical halfspace.

In order to illustrate the main ideas behind the proof of our result, let us call regular directions of E the vector fields Z in the Lie algebra $\mathfrak{g}$ such that $Z\mathbb{1}_E$ is representable by a Radon measure, and invariant directions those for which the measure is 0. Our strategy of proof rests mainly on the following observations: the first one (Proposition 4.7) is that the adjoint operator $\mathrm{Ad}_{\exp(Y)} : \mathfrak{g} \to \mathfrak{g}$ maps regular directions into regular directions whenever Y is an invariant direction. If

$$X := \sum_{i=1}^{m} \nu_{E,i}(\bar{x}) X_i \in \mathfrak{g},$$

we look at the vector space spanned by $\mathrm{Ad}_{\exp(Y)}(X)$, as Y varies among the invariant directions, and use this fact to show that any set with constant horizontal normal must have a regular direction Z not belonging to the vector space spanned by the invariant directions and X (which contains at least the horizontal layer). This is proved in Proposition 2.17 in purely geometric terms in general Lie gropus, and Proposition 2.18 provides a more explicit expression of the new regular directions generated, in Carnot groups, with this procedure.

Then, the second main observation is that if a regular direction Z for a set F has no component in the horizontal layer, then the tangents to F at $\bar{x}$ are invariant along a new direction depending on Z for most points $\bar{x}$; this follows (Lemma 5.8) by a simple scaling argument, taking into account that the Lie algebra dilations δ_r shrink more, as $r \downarrow 0$, in the non-horizontal directions. Therefore, at many points, a tangent to a set with constant horizontal normal has a new invariant direction. Having gained this new direction, this procedure can be restarted: the adjoint can be used to generate a new regular direction, then a tangent will have a new invariant direction, and so on.

In this way we show in Theorem 5.2 that, if we iterate the tangent operator sufficiently many times (the number depending on the Lie algebra stratification only) we do get a vertical halfspace. This means that we consider a tangent set E^1 to E at $\bar{x}$, then a tangent E^2 to E^1 at a suitable point $\bar{x}_1$ in the support of $|D\mathbb{1}_{E^1}|$, and so on. At this stage we borrow some ideas from [39] to conclude that, at $|D\mathbb{1}_E|$-a.e. point $\bar{x}$, iterated tangents are tangent to the initial set: this is accomplished in Section 6 and leads to the proof of Theorem 1.2.

Acknowledgements. We thank V. Magnani and A. Martini for some useful comments on a preliminary version of this paper.

2. Main notions

2.1. **Vector fields, divergence, X-derivative.** Throughout this section, we will denote by M a smooth differentiable manifold with topological dimension n, endowed with a n-differential volume form vol_M (eventually M will be a Lie group $\mathbb{G}$, and vol_M the right Haar measure).

For $x \in M$, the fiber T_xM of the tangent bundle TM is a derivation of germs of C^∞ functions at x (i.e., an $\mathbb{R}$-linear application from $C^\infty(x) \to \mathbb{R}$ that satisfies the Leibnitz rule). If $F : M \to N$ is smooth and $x \in M$, we shall denote by $dF_x : T_xM \to T_{F(x)}N$ its differential, defined as follows: the pull back operator $u \mapsto F_x^*(u) := u \circ F$ maps $C^\infty(F(x))$ into $C^\infty(x)$; thus, for $v \in T_xM$ we have that

$$dF_x(v)(u) := v(u \circ F)(x), \qquad u \in C^\infty(F(x))$$

defines an element of $T_{F(x)}N$.

We denote by $\Gamma(TM)$ the linear space of smooth vector fields, i.e. smooth sections of the tangent bundle TM; we will typically use the notation X, Y, Z to denote them. We use the notation $[X, Y]f := X(Yf) - Y(Xf)$ for the Lie bracket, that induces on $\Gamma(TM)$ an infinite-dimensional Lie algebra structure.

If $F : M \to N$ is smooth and invertible and $X \in \Gamma(TM)$, the push forward vector field $F_*X \in \Gamma(TN)$ is defined by the identity $(F_*X)_{F(x)} = dF_x(X_x)$. Equivalently,

$$(F_*X)u := [X(u \circ F)] \circ F^{-1} \qquad \forall u \in C^\infty(M). \tag{2.1}$$

The push-forward commutes with the Lie bracket, namely

$$[F_*X, F_*Y] = F_*[X, Y] \qquad \forall X,\, Y \in \Gamma(TM). \tag{2.2}$$

If $F : M \to N$ is smooth and σ is a smooth curve on M, then

$$dF_{\sigma(t)}(\sigma'(t)) = (F \circ \sigma)'(t), \tag{2.3}$$

where $\sigma'(t) \in T_{\sigma(t)}M$ and $(F \circ \sigma)'(t) \in T_{F(\sigma(t))}N$ are the tangent vector fields along the two curves, in M and N. If $u \in C^\infty(M)$, identifying $T_{u(p)}\mathbb{R}$ with $\mathbb{R}$ itself, given $X \in \Gamma(TM)$, we have

$$du_p(X) = X_p(u).$$

Now we use the volume form to define the divergence as follows:

$$\int_M Xu \, d\,\mathrm{vol}_M = -\int_M u \,\mathrm{div}\, X \, d\,\mathrm{vol}_M \qquad \forall u \in C_c^\infty(M). \tag{2.4}$$

When (M, g) is a Riemannian manifold and vol_M is the volume form induced by g, then an explicit expression of this differential operator can be obtained in terms of the components of X, and (2.4) corresponds to the divergence theorem on manifolds. We won't need either a Riemannian structure or an explicit expression of $\mathrm{div}\, X$ in the sequel, and for this reason we have chosen a definition based on (2.4): this emphasizes the dependence of $\mathrm{div}\, X$ on vol_M only. By applying this identity to a divergence-free vector field X, we obtain

$$\int_M uXv \, d\,\mathrm{vol}_M = -\int_M vXu \, d\,\mathrm{vol}_M \qquad \forall u,\, v \in C_c^\infty(M). \tag{2.5}$$

This motivates the following classical definition.

Definition 2.6 (X-distributional derivative)**.** *Let $u \in L^1_{\mathrm{loc}}(M)$ and let $X \in \Gamma(TM)$ be divergence-free. We denote by Xu the distribution*

$$\langle Xu, v\rangle := -\int_M uXv\, d\,\mathrm{vol}_M, \qquad v \in C^\infty_c(M).$$

If $f \in L^1_{\mathrm{loc}}(M)$, we write $Xu = f$ if $\langle Xu, v\rangle = \int_M vf\, d\,\mathrm{vol}_M$ for all $v \in C^\infty_c(M)$. Analogously, if μ is a Radon measure in M, we write $Xu = \mu$ if $\langle Xu, v\rangle = \int_M v\, d\mu$ for all $v \in C^\infty_c(M)$.

According to (2.5) (still valid when $u \in C^1(M)$), the distributional definition of Xu is equivalent to the classical one whenever $u \in C^1(M)$.

In Euclidean spaces, the X-derivative of characteristic functions of nice domains can be easily computed (and of course the result could be extended to manifolds, but we won't need this extension).

2.2. ***X*-derivative of nice functions and domains.** If u is a C^1 function in $\mathbb{R}^n$, then Xu can be calculated as the scalar product between X and the gradient of u:

$$Xu = \langle X, \nabla u\rangle. \tag{2.7}$$

Assume that $E \subset \mathbb{R}^n$ is locally the sub-level set of the C^1 function f and that $X \in \Gamma(T\mathbb{R}^n)$ is divergence-free. Then, for any $v \in C^\infty_c(\mathbb{R}^n)$ we can apply the Gauss–Green formula to the vector field vX, whose divergence is Xv, to obtain

$$\int_E Xv\, dx = \int_{\partial E} \langle vX, \nu_E^{eu}\rangle\, d\mathscr{H}^{n-1},$$

where ν_E^{eu} is the unit (Euclidean) outer normal to E. This proves that

$$X\mathbb{1}_E = -\langle X, \nu_E^{eu}\rangle \mathscr{H}^{n-1} \llcorner \partial E.$$

However, we have an explicit formula for the unit (Euclidean) outer normal to E, it is $\nu_E^{eu}(x) = \nabla f(x)/|\nabla f(x)|$, so, by (2.7),

$$\begin{aligned}\langle X, \nu_E^{eu}\rangle &= \langle X, \frac{\nabla f}{|\nabla f|}\rangle \\ &= \frac{\langle X, \nabla f\rangle}{|\nabla f|} = \frac{Xf}{|\nabla f|}.\end{aligned}$$

Thus

$$X\mathbb{1}_E = -\frac{Xf}{|\nabla f|}\mathscr{H}^{n-1} \llcorner \partial E. \tag{2.8}$$

2.3. **Flow of a vector field.** Given $X \in \Gamma(TM)$ we can consider the associated flow, i.e., the solution $\Phi_X : M \times \mathbb{R} \to M$ of the following ODE

$$\begin{cases} \dfrac{d}{dt}\Phi_X(p,t) & = & X_{\Phi_X(p,t)} \\ \\ \Phi_X(p,0) & = & p. \end{cases} \tag{2.9}$$

Notice that the smoothness of X ensures uniqueness, and therefore the semigroup property

$$\Phi_X(x, t+s) = \Phi_X(\Phi_X(x,t), s) \qquad \forall t,\ s \in \mathbb{R},\ \forall x \in M \tag{2.10}$$

but not global existence; it will be guaranteed, however, in all cases considered in this paper. We obviously have

$$\frac{d}{dt}(u \circ \Phi_X)(p,t) = (Xu)(\Phi_X(p,t)) \qquad \forall u \in C^1(M). \tag{2.11}$$

An obvious consequence of this identity is that, for a C^1 function u, $Xu = 0$ implies that u is constant along the flow, i.e. $u \circ \Phi_X(\cdot, t) = u$ for all $t \in \mathbb{R}$. A similar statement holds even for distributional derivatives along vector fields: for simplicity let us state and prove this result for divergence-free vector fields only.

Theorem 2.12. *Let $u \in L^1_{\mathrm{loc}}(M)$ be satisfying $Xu = 0$ in the sense of distributions. Then, for all $t \in \mathbb{R}$, $u = u \circ \Phi_X(\cdot, t)$ vol_M-a.e. in M.*

Proof. Let $g \in C^1_c(M)$; we need to show that the map $t \mapsto \int_M gu \circ \Phi_X(\cdot, t)\, d\,\mathrm{vol}_M$ is independent of t. Indeed, the semigroup property (2.10), and the fact that X is divergence-free yield

$$\begin{aligned} & \int_M gu \circ \Phi_X(\cdot, t+s)\, d\,\mathrm{vol}_M - \int_M gu \circ \Phi_X(\cdot, t)\, d\,\mathrm{vol}_M \\ = & \int_M ug \circ \Phi_X(\cdot, -t-s)\, d\,\mathrm{vol}_M - \int_M ug \circ \Phi_X(\cdot, -t)\, d\,\mathrm{vol}_M \\ = & \int_M ug \circ \Phi_X(\Phi_X(\cdot, -s), -t)\, d\,\mathrm{vol}_M - \int_M ug \circ \Phi_X(\cdot, -t)\, d\,\mathrm{vol}_M \\ = & -s \int_M uX(g \circ \Phi_X(\cdot, -t))\, d\,\mathrm{vol}_M + o(s) = o(s). \end{aligned}$$

□

Remark 2.13. We notice also that the flow is vol_M-measure preserving (i.e. $\mathrm{vol}_M(\Phi_X(\cdot, t)^{-1}(A)) = \mathrm{vol}_M(A)$ for all Borel sets $A \subseteq M$ and $t \in \mathbb{R}$) if and only if $\mathrm{div}\, X$ is equal to 0. Indeed, if $f \in C^1_c(M)$, the measure preserving property gives that $\int_M f(\Phi_X(x,t))\, d\,\mathrm{vol}_M(x)$ is independent of t. A time differentiation and (2.11) then give

$$0 = \int_M \frac{d}{dt} f(\Phi_X(x,t))\, d\,\mathrm{vol}_M(x) = \int_M Xf(\Phi_X(x,t))\, d\,\mathrm{vol}_M(x) = \int_M Xf(y)\, d\,\mathrm{vol}_M(y).$$

Therefore $\int_M f\, \mathrm{div}\, X\, d\,\mathrm{vol}_M = 0$ for all $f \in C^1_c(M)$, and X is divergence-free. The proof of the converse implication is similar, and analogous to the one of Theorem 2.12.

2.4. **Lie groups.** Let $\mathbb{G}$ be a Lie group, i.e. a differentiable n-dimensional manifold with a smooth group operation. We shall denote by e the identity of the group, by $R_g(h) := hg$ the right translation, and by $L_g(h) := gh$ the left translation. We shall also denote by $\mathrm{vol}_{\mathbb{G}}$ the volume form and, at the same time, the right-invariant Haar measure.

Forced to make a choice, we follow the majority of the literature focusing on the *left* invariant vector fields. i.e. the vector fields $X \in \Gamma(T\mathbb{G})$ such that $(L_g)_*X = X$, so that $(dL_g)_x X = X_{L_g(x)}$ for all $x \in \mathbb{G}$. In differential terms, we have

$$X(f \circ L_g)(x) = Xf(L_g(x)) \qquad \forall x,\, g \in \mathbb{G}.$$

Thanks to (2.2) with $F = L_g$, the class of left invariant vector fields is easily seen to be closed under the Lie bracket, and we shall denote by $\mathfrak{g} \subseteq \Gamma(T\mathbb{G})$ the *Lie algebra* of left invariant vector fields. We will typically use the notations U, V, W to denote subspaces of $\mathfrak{g}$.

Note that, after fixing a vector $v \in T_e\mathbb{G}$, we can construct a left invariant vector field X defining $X_g := (L_g)_*v$ for any $g \in \mathbb{G}$. This construction is an isomorphism between the set $\mathfrak{g}$ of all left invariant vector fields and $T_e\mathbb{G}$, and proves that $\mathfrak{g}$ is a n-dimensional subspace of $\Gamma(T\mathbb{G})$.

Let $X \in \mathfrak{g}$ and let us denote, as usual in the theory, by $\exp(tX)$ the flow of X at time t starting from e (that is, $\exp(tX) := \Phi_X(e,t) = \Phi_{tX}(e,1)$); then, the curve $g\exp(tX)$ is the flow starting at g: indeed, since X is left invariant, setting for simplicity $\gamma(t) := \exp(tX)$ and $\gamma_g(t) := g\gamma(t)$, we have

$$\frac{d}{dt}\gamma_g(t) = \frac{d}{dt}(L_g(\gamma(t))) = (dL_g)_{\gamma(t)}\frac{d}{dt}\gamma(t) = (dL_g)_{\gamma(t)}X = X_{\gamma_g(t)}.$$

This implies that $\Phi_X(\cdot,t) = R_{\exp(tX)}$ and so the flow preserves the right Haar measure, and the left translation preserves the flow lines. By Remark 2.13 it follows that all $X \in \mathfrak{g}$ are divergence-free, and Theorem 2.12 gives

$$f \circ R_{\exp(tX)} = f \quad \forall t \in \mathbb{R} \qquad \Longleftrightarrow \qquad Xf = 0 \tag{2.14}$$

whenever $f \in L^1_{\mathrm{loc}}(\mathbb{G})$.

Before stating the next proposition, we recall the definition of the adjoint. For $k \in \mathbb{G}$, the conjugation map

$$\begin{aligned} C_k:\ \ \mathbb{G} &\to \ \ \mathbb{G} \\ g &\mapsto \ \ C_k(g) := kgk^{-1} \end{aligned} \tag{2.15}$$

is the composition of L_k with $R_{k^{-1}}$. The *adjoint* operator $k \mapsto \mathrm{Ad}_k$ maps $\mathbb{G}$ in $GL(\mathfrak{g})$ as follows:

$$\mathrm{Ad}_k(X) := (C_k)_*X, \text{ so that } \mathrm{Ad}_k(X)f(x) = X(f \circ C_k)(C_k^{-1}(x)). \tag{2.16}$$

The definition is well posed because $\mathrm{Ad}_k(X)$ is left invariant whenever X is left invariant: for all $g \in \mathbb{G}$ we have indeed

$$\mathrm{Ad}_k(X)(f \circ L_g)(x) = X(f \circ L_g \circ C_k)(k^{-1}xk) = X(f \circ R_{k^{-1}} \circ L_{gk})(k^{-1}xk) = X(f \circ R_{k^{-1}})(gxk).$$

On the other hand

$$\mathrm{Ad}_k(X)f(L_g(x)) = X(f \circ C_k)(k^{-1}gxk) = X(f \circ R_{k^{-1}})(gxk).$$

Proposition 2.17. *Assume that $\mathbb{G}$ is a connected, simply connected nilpotent Lie group. Let $\mathfrak{g}'$ be a Lie subalgebra of $\mathfrak{g}$ satisfying $\dim(\mathfrak{g}')+2 \leq \dim(\mathfrak{g})$, and assume that $W := \mathfrak{g}' \oplus \{\mathbb{R}X\}$ generates the whole Lie algebra $\mathfrak{g}$ for some $X \notin \mathfrak{g}'$. Then, there exists $k \in \exp(\mathfrak{g}')$ such that $\mathrm{Ad}_k(X) \notin W$.*

Proof. Note that $\mathfrak{g}'$ is a finite-dimensional sub-algebra and that exp is, under the simple connectedness assumption, a homeomorphism, hence $\mathbb{K} := \exp(\mathfrak{g}')$ is a closed (proper) Lie subgroup of $\mathbb{G}$. Therefore, we can consider the quotient manifold $\mathbb{G}/\mathbb{K}$, in fact the homogeneous space of right cosets: it consists of the equivalence classes of $\mathbb{G}$ induced by the relation

$$x \sim y \qquad \Longleftrightarrow \qquad y^{-1}x \in \mathbb{K}.$$

We shall denote by $\pi : \mathbb{G} \to \mathbb{G}/\mathbb{K}$ the canonical projection. The natural topology of $\mathbb{G}/\mathbb{K}$ is determined by the requirement that π should be continuous and open. Let $\mathfrak{m}$ denote some vector space of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{m}$. The sub-manifold $\exp(\mathfrak{m})$ is referred as a local cross section for $\mathbb{K}$ at the origin, and it can be used to give a differentiable structure to $\mathbb{G}/\mathbb{K}$. In fact, let $Z_1, \dots, Z_r$ be a basis of $\mathfrak{m}$, then the mapping

$$(x_1, \dots, x_r) \mapsto \pi(g \exp(x_1 Z_1 + \dots + x_r Z_r))$$

is a homeomorphism of an open set of $\mathbb{R}^r$ onto a neighborhood of $g\mathbb{K}$ in $\mathbb{G}/\mathbb{K}$. Then it is easy (see [24] for details) to see that with these charts, $\mathbb{G}/\mathbb{K}$ is an analytic manifold. In particular, π restrict to $\exp(\mathfrak{m})$ is a local diffeomorphism into $\mathbb{G}/\mathbb{K}$ and $d\pi(X) \neq 0$ since the projection of X on $\mathfrak{m}$ is non zero.

Notice that, by our assumption on the dimension of $\mathfrak{g}'$, the topological dimension of $\mathbb{G}/\mathbb{K}$ is at least 2. Now, if the statement were false, taking into account that $\mathrm{Ad}_k(\mathfrak{g}') \subseteq \mathfrak{g}'$, we would have $\mathrm{Ad}_k(W) \subseteq W$ for all $k \in \mathbb{K}$. By the definition of adjoint operator as composition of the differentials of right and left translations, the above would be equivalent to say that

$$(R_k)_*((L_{k^{-1}})_*(Y)) \in W \qquad \forall Y \in W,\ k \in \mathbb{K}.$$

Since the vector fields in W are left invariant (i.e. $(L_g)_*Y = Y$ for all $Y \in W$), this condition would say that W is $\mathbb{K}$-right invariant, and we can write this condition in the form $d(R_k)_x(W_x) \subset W_{xk}$ for all $x \in \mathbb{G}$ and $k \in \mathbb{K}$.

Now, let us consider the subspaces $d\pi_x(W_x)$ of $T_{\pi(x)}\mathbb{G}/\mathbb{K}$: they are all 1-dimensional, thanks to the fact that $\dim(W) = 1 + \dim(\mathfrak{g}')$, and they depend only on $\pi(x)$: indeed, $\mathbb{K}$-right invariance and the identity $\pi \circ R_k = \pi$ give

$$d\pi_x(Y_x) = d\pi_{xk}(d(R_k)_x(Y_x)) \in d\pi_{xk}(W_{xk})$$

for all $Y \in W$ and $k \in \mathbb{K}$. Therefore we can define a (smooth) 1-dimensional distribution $W/\mathbb{K}$ in $\mathbb{G}/\mathbb{K}$ by $(W/\mathbb{K})_y := d\pi_x(W_x)$, where x is any element of $\pi^{-1}(y)$. In particular $W/\mathbb{K}$ would be tangent to a 1-dimensional foliation $\mathcal{F}$ of $\mathbb{G}/\mathbb{K}$ that has at least codimension 1, since $\mathbb{G}/\mathbb{K}$ has at least dimension 2. Letting $\mathcal{F}'$ be the foliation of $\mathbb{G}$ whose leaves are the inverse images via π of leaves of $\mathcal{F}$, we find that still $\mathcal{F}'$ has codimension at least 1, and W is tangent to the leaves of $\mathcal{F}'$. But this contradicts the fact that W generates $\mathfrak{g}$: in fact, the only sub-manifold to which W could be tangent is all the manifold $\mathbb{G}$. □

In the following proposition we provide a characterization of the vector space spanned by $\mathrm{Ad}_{\exp(Y)}(X)$, where Y varies in a Lie subalgebra of $\mathfrak{g}$. This improved version of Proposition 2.18 was pointed out to us by V. Magnani.

Proposition 2.18. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, let $\mathfrak{g}' \subset \mathfrak{g}$ be a Lie algebra and let $X \in \mathfrak{g}$. Then*

$$\mathrm{span}\left(\{\mathrm{Ad}_{\exp(Y)}(X) : \ Y \in \mathfrak{g}'\}\right) = [\mathfrak{g}', X] + [\mathfrak{g}', [\mathfrak{g}', X]] + [\mathfrak{g}', [\mathfrak{g}', [\mathfrak{g}', X]]] + \cdots .$$

Proof. Let us denote by S the space $\mathrm{span}\left(\{\mathrm{Ad}_{\exp(Y)}(X) : \ Y \in \mathfrak{g}'\}\right)$. Obviously S contains X and all vector fields $\mathrm{Ad}_{\exp(rY)}(X)$ for $r \geq 0$ and $Y \in \mathfrak{g}'$. Now, denoting by $L(\mathfrak{g})$ the linear maps from $\mathfrak{g}$ to $\mathfrak{g}$, let us recall the formula (see [29], page 54) $\mathrm{Ad}_{\exp(Y)} = e^{\mathrm{ad}_Y}$, where $\mathrm{ad.} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is the operator $\mathrm{ad}_Y(X) = [Y, X]$ and the exponential e^A is defined for any $A \in L(\mathfrak{g})$, by $e^A := \sum_{i=0}^{\infty} A^i/i! \in L(\mathfrak{g})$. Therefore

$$\mathrm{Ad}_{\exp(Y)} X = X + [Y, X] + \frac{1}{2}[Y, [Y, X]] + \cdots \tag{2.19}$$

Let ν be the dimension of $\mathfrak{g}'$ and let $(Y_1, \dots, Y_\nu)$ be a basis of $\mathfrak{g}'$. Taking into account the identity (2.19), for all $Y = \sum_1^\nu r_j Y_j \in \mathfrak{g}'$, we define

$$\begin{aligned} \Phi(r_1, \dots, r_\nu) \;&:=\; \mathrm{Ad}_{\exp(\sum_1^\nu r_j Y_j)} X - X \\ &=\; \sum_{k=1}^{s-1} \frac{1}{k!} \Big(\sum_{j=1}^{\nu} r_j \ \mathrm{ad}\, Y_j\Big)^k X \\ &=\; \sum_{k=1}^{s-1} \frac{1}{k!} \sum_{j_1, \dots, j_k = 1}^{\nu} r_{j_1} \cdots r_{j_k} \big(\mathrm{ad}\, Y_{j_1} \cdots \mathrm{ad}\, Y_{j_k}\big) X \in S\,. \end{aligned}$$

Since this polynomial takes its values in S, it turns out that all its coefficients belong to S. In particular, we have

$$\mathrm{ad}\, Y_i(X) = \partial_{r_i} \Phi(0) \in S \quad \text{and} \quad \big(\mathrm{ad}\, Y_i\, \mathrm{ad}\, Y_j + \mathrm{ad}\, Y_j\, \mathrm{ad}\, Y_i\big) X = 2 \partial_{r_i} \partial_{r_j} \Phi(0) \in S\,.$$

The Jacobi identity can be read as $\mathrm{ad}_U\, \mathrm{ad}_W - \mathrm{ad}_W\, \mathrm{ad}_U = \mathrm{ad}_{[U,W]}$, so that

$$\big(\mathrm{ad}\, Y_i\, \mathrm{ad}\, Y_j + \mathrm{ad}\, Y_j\, \mathrm{ad}\, Y_i\big) X = 2\, \mathrm{ad}\, Y_i\, \mathrm{ad}\, Y_j X + \mathrm{ad}[Y_j, Y_i] X.$$

It follows that $\big(\mathrm{ad}\, Y_i\, \mathrm{ad}\, Y_j\big) X \in S$, and this proves that $[\mathfrak{g}', X] + [\mathfrak{g}', [\mathfrak{g}', X]] \subset S$. By induction, let us suppose that

$$\mathfrak{u}_{k-1} := [\mathfrak{g}', X] + [\mathfrak{g}', [\mathfrak{g}', X]] + \cdots + [\underbrace{\mathfrak{g}', [\mathfrak{g}', \cdots, [\mathfrak{g}'}_{(k-1) \text{ times}}, X] \cdots] \subset S$$

for some $k \geq 3$. In general we have

$$\partial_{r_{i_1}} \cdots \partial_{r_{i_k}} \Phi(0) = \frac{1}{k!} \sum_{\sigma} \big(\mathrm{ad}\, Y_{j_{\sigma(1)}} \cdots \mathrm{ad}\, Y_{j_{\sigma(k)}}\big) X \in S, \tag{2.20}$$

where the sum runs on all permutations σ of k elements. By the Jacobi identity

$$\big(\mathrm{ad}\, Y_{j_{\sigma(1)}} \cdots \mathrm{ad}\, Y_{j_{\sigma(k)}}\big) X - \big(\mathrm{ad}\, Y_{j_{\eta(1)}} \cdots \mathrm{ad}\, Y_{j_{\eta(k)}}\big) X \in \mathfrak{u}_{k-1}$$

if $\sigma\circ\eta^{-1}$ is a transposition. Then, by the inductive assumption, we can iterate transpositions in $\big(\operatorname{ad} Y_{j_{\sigma(1)}}\cdots\operatorname{ad} Y_{j_{\sigma(k)}}\big)X$ to write it as $\big(\operatorname{ad} Y_{j_1}\cdots\operatorname{ad} Y_{j_k}\big)X+W_\sigma$ with $W_\sigma\in S$. Then, from (2.20) we get $\big(\operatorname{ad} Y_{j_1}\cdots\operatorname{ad} Y_{j_k}\big)X\in S$. This shows that $\big(\operatorname{ad} Y_{j_1}\cdots\operatorname{ad} Y_{j_k}\big)X\in S$, so that $\mathfrak{u}_k\subset S$, and this proves the inclusion

$$[\mathfrak{g}',X]+[\mathfrak{g}',[\mathfrak{g}',X]]+[\mathfrak{g}',[\mathfrak{g}',[\mathfrak{g}',X]]]+\ldots\subset\operatorname{span}\{\operatorname{Ad}_{\exp Y}X\mid Y\in\mathfrak{g}'\}$$

Observing that the opposite inclusion trivially holds, we are led to our claim. □

2.5. **Carnot groups.** A Carnot group $\mathbb{G}$ of step $s\geq 1$ is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a step s stratification: this means that we can write

$$\mathfrak{g}=V_1\oplus\cdots\oplus V_s$$

with $[V_j,V_1]=V_{j+1}$, $i\leq j\leq s$, $V_s\neq\{0\}$ and $V_{s+1}=\{0\}$. We keep the notation $n=\sum_i\dim V_i$ for the topological dimension of $\mathbb{G}$, and denote by

$$Q:=\sum_{i=1}^{s}i\dim V_i$$

the so-called *homogeneous dimension* of $\mathbb{G}$. We denote by $\delta_\lambda:\mathfrak{g}\to\mathfrak{g}$ the family of inhomogeneous dilations defined by

$$\delta_\lambda(\sum_{i=1}^{s}v_i):=\sum_{i=1}^{s}\lambda^i v_i\qquad\lambda\geq 0$$

where $X=\sum_{i=1}^{s}v_i$ with $v_i\in V_i$, $1\leq i\leq s$. The dilations δ_λ belong to $GL(\mathfrak{g})$ and are uniquely determined by the homogeneity conditions

$$\delta_\lambda X=\lambda^k X\quad\forall X\in V_k,\qquad 1\leq k\leq s.$$

We denote by m the dimension of V_1 and we fix an inner product in V_1 and an orthonormal basis $X_1,\ldots,X_m$ of V_1. This basis of V_1 induces the so-called Carnot-Caratheodory left invariant distance d in $\mathbb{G}$, defined as follows:

$$d^2(x,y):=\inf\left\{\int_0^1\sum_{i=1}^{m}|a_i(t)|^2\,dt:\ \gamma(0)=x,\ \gamma(1)=y\right\},$$

where the infimum is made among all Lipschitz curves $\gamma:[0,1]\to\mathbb{G}$ such that $\gamma'(t)=\sum_1^m a_i(t)(X_i)_{\gamma(t)}$ for a.e. $t\in[0,1]$ (the so-called horizontal curves).

For Carnot groups, it is well known that the map $\exp:\mathfrak{g}\to\mathbb{G}$ is a diffeomorphism, so any element $g\in\mathbb{G}$ can represented as $\exp(X)$ for some unique $X\in\mathfrak{g}$, and therefore uniquely written in the form

$$\exp(\sum_{i=1}^{s}v_i),\qquad v_i\in V_i,\ 1\leq i\leq s. \tag{2.21}$$

This representation allows to define a family indexed by $\lambda \geq 0$ of intrinsic dilations $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, by

$$\delta_\lambda\big(\exp(\sum_{i=1}^s v_i)\big) := \exp\big(\sum_{i=1}^s \lambda^i v_i\big) \qquad \text{(i.e. } \exp\circ\delta_\lambda = \delta_\lambda \circ \exp.\text{)}$$

We have kept the same notation δ_λ for both dilations (in $\mathfrak{g}$ and in $\mathbb{G}$) because no ambiguity will arise. Obviously, $\delta_\lambda \circ \delta_\eta = \delta_{\lambda\eta}$, and the Baker-Campbell-Hausdorff formula gives

$$\delta_\lambda(xy) = \delta_\lambda(x)\delta_\lambda(y) \qquad \forall x,\, y \in \mathbb{G}.$$

Moreover, the Carnot-Caratheodory distance is well-behaved under these dilations, namely

$$d(\delta_\lambda x, \delta_\lambda y) = \lambda d(x, y) \qquad \forall x,\, y \in \mathbb{G}.$$

Besides $\delta_\lambda \circ \exp = \exp\circ\delta_\lambda$, another useful relation between dilations in $\mathbb{G}$ and dilations in $\mathfrak{g}$ is $\delta_\lambda X = (\delta_\lambda)_* X$, namely

$$X(u \circ \delta_\lambda)(g) = (\delta_\lambda X)u(\delta_\lambda g) \qquad \forall g \in \mathbb{G},\ \lambda \geq 0. \tag{2.22}$$

We have indeed

$$\begin{aligned} X(u\circ\delta_\lambda)(g) &= \frac{d}{dt} u\circ\delta_\lambda(g\exp(tX))\Big|_{t=0} = \frac{d}{dt}u(\delta_\lambda g \delta_\lambda \exp(tX))\Big|_{t=0} \\ &= \frac{d}{dt}u(\delta_\lambda g \exp(t\delta_\lambda X))\Big|_{t=0} = (\delta_\lambda X)u(\delta_\lambda g). \end{aligned}$$

3. Measure-theoretic tools

In this section we specify the notions of convergence used in this paper (at the level of sets and of measures), and point out some useful facts concerning Radon measures. The results quoted without an explicit reference are all quite standard, and can be found for instance in [3], and those concerning Hausdorff measures in metric spaces in [14] or [4].

Haar, Lebesgue and Hausdorff measures. Carnot groups are nilpotent and so unimodular, therefore the right and left Haar measures coincide, up to constant multiples. We fix one of them and denote it by $\mathrm{vol}_{\mathbb{G}}$.

We shall denote by $\mathscr{H}^k$ (resp. $\mathscr{S}^k$) the Hausdorff (resp. spherical Hausdorff) k-dimensional measure; these measures depend on the distance, and, unless otherwise stated, to build them we will use the Carnot-Caratheodory distance in $\mathbb{G}$ and the Euclidean distance in Euclidean spaces.

Using the left translation and scaling invariance of the Carnot-Caratheodory distance one can easily check that the Haar measures of $\mathbb{G}$ are a constant multiple of the spherical Hausdorff measure $\mathscr{S}^Q$ and of $\mathscr{H}^Q$. In exponential coordinates, all these measures are a constant multiple of the Lebesgue measure $\mathscr{L}^n$ in $\mathbb{R}^n$, namely

$$\mathrm{vol}_G\big(\{\exp(\sum_{i=1}^n x_i X_i) :\ (x_1, \dots, x_n) \in A\}\big) = c\mathscr{L}^n(A) \qquad \text{for all Borel sets } A \subseteq \mathbb{R}^n$$

for some constant c. Using this fact, one can easily prove that

$$\mathrm{vol}_{\mathbb{G}}(\delta_\lambda(A)) = \lambda^Q \,\mathrm{vol}_{\mathbb{G}}(A) \tag{3.1}$$

for all Borel sets $A \subseteq \mathbb{G}$.

The following implication will be useful: for μ nonnegative Radon measure, $t > 0$ and $B \subseteq \mathbb{G}$ Borel, we have

$$\limsup_{r\downarrow 0} \frac{\mu(B_r(x))}{\omega_k r^k} \geq t \ \ \forall x \in B \qquad \Longrightarrow \qquad \mu(B) \geq t\mathscr{S}^k(B),$$

where ω_k is the Lebesgue measure of the unit ball in $\mathbb{R}^k$ (it appears as a normalization constant in the definitions of $\mathscr{H}^k$ and $\mathscr{S}^k$, in order to ensure the identity $\mathscr{H}^k = \mathscr{S}^k = \mathscr{L}^k$ in $\mathbb{R}^k$). In particular we obtain that

$$\{x \in \mathbb{G} : \ \limsup_{r\downarrow 0} \frac{\mu(B_r(x))}{r^k} > 0\} \text{ is } \sigma\text{-finite with respect to } \mathscr{S}^k. \tag{3.2}$$

Characteristic functions, convergence in measure. For any set E we shall denote by $\mathbb{1}_E$ the characteristic function of E (1 on E, 0 on $\mathbb{G} \setminus E$); within the class of Borel sets of $\mathbb{G}$, the convergence we consider is the so-called *local convergence in measure* (equivalent to the $L^1_{\rm loc}$ convergence of the characteristic functions), namely:

$$E_h \to E \quad \Longleftrightarrow \quad \mathrm{vol}_{\mathbb{G}}\big(K \cap [(E_h \setminus E) \cup (E \setminus E_h)]\big) = 0 \quad \text{for all } K \subseteq \mathbb{G} \text{ compact.}$$

Radon measures and their convergence. The class $\mathcal{M}(\mathbb{G})$ of Radon measures in $\mathbb{G}$ coincides with the class of 0 order distributions in $\mathbb{G}$, namely those distributions T such that, for any bounded open set $\Omega \subseteq \mathbb{G}$ there exists $C(\Omega) \in [0, +\infty)$ satisfying

$$|\langle T, g\rangle| \leq C(\Omega) \sup |g| \qquad \forall g \in C^1_c(\Omega).$$

These distributions can be uniquely extended to $C_c(\mathbb{G})$, and their action can be represented, thanks to Riesz theorem, through an integral with respect to a σ-additive set function μ defined on bounded Borel sets. Thanks to this fact, the action of these distributions can be extended even up to bounded Borel functions with compact support. We will typically use both viewpoints in this paper (for instance the first one plays a role in the definition of distributional derivative, while the second one is essential to obtain differentiation results). If μ is a nonnegative Radon measure we shall denote

$$\mathrm{supp}\, \mu := \{x \in \mathbb{G} : \ \mu(B_r(x)) > 0 \ \ \forall r > 0\}.$$

The only convergence we use in $\mathcal{M}(\mathbb{G})$ is the weak* one induced by the duality with $C_c(\mathbb{G})$, namely $\mu_h \to \mu$ if

$$\lim_{h\to\infty} \int_{\mathbb{G}} u \, d\mu_h = \int_{\mathbb{G}} u \, d\mu \qquad \forall u \in C_c(\mathbb{G}).$$

Push-forward. If $f : \mathbb{G} \to \mathbb{G}$ is a proper Borel map, then $f^{-1}(B)$ is a bounded Borel set whenever B is a bounded Borel set. The push-forward measure $f_\sharp\mu$ is then defined by

$$f_\sharp\mu(B) := \mu(f^{-1}(B)).$$

In integral terms, this definition corresponds to

$$\int_{\mathbb{G}} u \, df_\sharp\mu := \int_{\mathbb{G}} u \circ f \, d\mu$$

whenever the integrals make sense (for instance u Borel, bounded and compactly supported).

Vector-valued Radon measures. We will also consider $\mathbb{R}^m$-valued Radon measures, representable as $(\mu_1, \ldots, \mu_m)$ with $\mu_i \in \mathcal{M}(\mathbb{G})$. The *total variation* of $|\mu|$ of an $\mathbb{R}^m$-valued measure μ is the smallest nonnegative measure ν defined on Borel sets of $\mathbb{G}$ such that $\nu(B) \geq |\mu(B)|$ for all bounded Borel set B; it can be explicitly defined by

$$|\mu|(B) := \sup\left\{\sum_{i=1}^{\infty} |\mu(B_i)| : \ (B_i) \text{ Borel partition of } B,\ B_i \text{ bounded}\right\}.$$

Push forward and convergence in $\mathcal{M}^m(\mathbb{G})$ can be defined componentwise. Useful relations between convergence and total variation are:

$$\liminf_{n\to\infty} |\mu_n|(A) \geq |\mu|(A) \qquad \text{for all } A \subseteq \mathbb{G} \text{ open,} \tag{3.3}$$

$$\sup_{n\to\infty} |\mu_n|(K) < +\infty \qquad \text{for all } K \subseteq \mathbb{G} \text{ compact,} \tag{3.4}$$

whenever $\mu_n \to \mu$ in $\mathcal{M}^m(\mathbb{G})$.

Asymptotically doubling measures. A nonnegative Radon measure μ in $\mathbb{G}$ is said to be *asymptotically* doubling if

$$\limsup_{r\downarrow 0} \frac{\mu(B_{2r}(x))}{\mu(B_r(x))} < +\infty \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{G}.$$

For asymptotically doubling measures all the standard results of Lebesgue differentiation theory hold: for instance, for any Borel set A, μ-a.e. point $x \in A$ is a *density point* of A, namely

$$\lim_{r\downarrow 0} \frac{\mu(A \cap B_r(x))}{\mu(B_r(x))} = 1.$$

The same result holds for *any* set A, provided we replace μ by the outer measure μ^*, defined for any $A \subseteq \mathbb{G}$ by

$$\mu^*(A) := \inf\{\mu(B) : B \text{ Borel, } B \supseteq A\}.$$

It follows directly from the definition that μ^* is subadditive. Moreover, let (B_n) be a minimizing sequence and let B the intersection of all sets B_n: then B is a Borel set, $B \supseteq A$ and $\mu^*(A) = \mu(B)$. Furthermore, for all Borel sets C we have $\mu^*(A \cap C) = \mu(B \cap C)$ (if not, adding the strict inequality $\mu^*(A \cap C) < \mu(B \cap C)$ to $\mu^*(A \setminus C) \leq \mu(B \setminus C)$ would give a contradiction). Choosing $C = B_r(x)$, with x density point of B, we obtain

$$\lim_{r\downarrow 0} \frac{\mu^*(A \cap B_r(x))}{\mu(B_r(x))} = \lim_{r\downarrow 0} \frac{\mu(B \cap B_r(x))}{\mu(B_r(x))} = 1.$$

This proves that the set of points of A that are not density points is contained in a μ–negligible Borel set. We will also be using in the proof of Theorem 6.4 the fact that μ^* is *countably* subadditive, namely $\mu^*(A) \leq \sum_i \mu^*(A_i)$ for all sequences (A_i) with $A \subseteq \cup_i A_i$.

We recall the following result, proved in Theorem 2.8.17 of [14]:

Theorem 3.5 (Differentiation)**.** *Assume that μ is asymptotically doubling and $\nu \in \mathcal{M}(\mathbb{G})$ is absolutely continuous with respect to μ. Then the limit*

$$f(x) := \lim_{r\downarrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))}$$

exists and is finite for μ-a.e. $x \in \operatorname{supp}\mu$.
In addition, $f \in L^1_{\mathrm{loc}}(\mu)$ and $\nu = f\mu$, i.e. $\nu(B) = \int_B f\,d\mu$ for all bounded Borel sets $B \subseteq \mathbb{G}$.

The proof given in [14] covers much more general situations; the reader already acquainted with the theory of differentiation with respect to doubling measures can easily realize that the results extend to asymptotically doubling ones by consider the *localized* (in $\mathbb{G}\times(0,+\infty)$) maximal operators:

$$M_{B,r}\nu(x) := \sup_{s\in(0,r)} \frac{\nu(B_s(x))}{\mu(B_s(x))}, \qquad x \in B,$$

where ν is any nonnegative Radon measure in $\mathbb{G}$. Thanks to the asymptotic doubling property, one can find a family of Borel sets $B_h \subseteq \operatorname{supp}\mu$ whose union covers $\mathbb{G}$, constants $C_h \geq 1$ and radii $r_h > 0$ such that $\mu(B_{3r}(x)) \leq C_h\mu(B_r(x))$ for $x \in B_h$ and $r \in (0, r_h)$. For the operators M_{B_h,r_h}, the uniform doubling property on B_h and a covering lemma yield the weak L^1 estimate $\mu(E \cap \{M_{B_h,r_h}\nu > t\}) \leq t^{-1}C_h\nu(E)$ (for $E \subseteq B_h$ Borel, $t > 0$). This leads to the differentiation result on all B_h, and then μ-a.e. on $\mathbb{G}$.

4. Sets of locally finite perimeter

In this section we recall a few useful facts about sets of finite perimeter, considering also sets whose derivative along non-horizontal directions is a measure.

Definition 4.1 (Regular and invariant directions)**.** *Let $f \in L^1_{\mathrm{loc}}(\mathbb{G})$.*
We shall denote by $\mathrm{Reg}(f)$ the vector subspace of $\mathfrak{g}$ made by vectors X such that Xf is representable by a Radon measure.
We shall denote by $\mathrm{Inv}(f)$ the subspace of $\mathrm{Reg}(f)$ corresponding to the vector fields X such that $Xf = 0$, and by $\mathrm{Inv}_0(f)$ the subset made by homogeneous directions, i.e.

$$\mathrm{Inv}_0(f) := \mathrm{Inv}(f) \cap \bigcup_{i=1}^{s} V_i.$$

Notice that, according to (2.14),

$$f \circ R_{\exp(tX)} = f \qquad \forall t \in \mathbb{R},\ X \in \mathrm{Inv}(f).$$

We will mostly consider regular and invariant directions of characteristic functions, therefore we set

$$\mathrm{Reg}(E) := \mathrm{Reg}(\mathbb{1}_E), \qquad \mathrm{Inv}(E) := \mathrm{Inv}(\mathbb{1}_E), \qquad \mathrm{Inv}_0(E) := \mathrm{Inv}_0(\mathbb{1}_E).$$

We can now naturally define halfspaces by requiring invariance along a codimension 1 space of directions, and monotonicity along the remaining direction; if this direction is horizontal, we call these sets *vertical halfspaces.*

Definition 4.2 (Vertical halfspaces)**.** *We say that a Borel set $H \subseteq \mathbb{G}$ is a* vertical halfspace *if $\mathrm{Inv}_0(H) \supseteq \cup_2^s V_i$, $V_1 \cap \mathrm{Inv}_0(H)$ is a codimension one subspace of V_1 and $X\mathbb{1}_H \geq 0$ for some $X \in V_1$, with $X\mathbb{1}_H \neq 0$.*

Since

$$\operatorname{span}\big(\operatorname{Inv}_0(H)\big) = \bigoplus_{i=1}^{s} V_i \cap \operatorname{Inv}_0(H), \tag{4.3}$$

we can equivalently say that H is an halfspace if $\operatorname{span}(\operatorname{Inv}_0(H))$ is a codimension 1 subspace of $\mathfrak{g}$, $V_1 \cap \operatorname{span}(\operatorname{Inv}_0(H))$ is a codimension 1 subspace of V_1 and $X\mathbb{1}_H \geq 0$ for some $X \in V_1$: indeed, (4.3) forces, whenever the codimension is 1, all subspaces $V_i \cap \operatorname{Inv}_0(H)$ to coincide with V_i, with just one exception.

Let us recall that m denotes the dimension of V_1, and that $X_1, \ldots, X_m$ is a given orthonormal basis of V_1. With this notation, vertical halfspaces can be characterized as follows:

Proposition 4.4 (Characterization of vertical halfspaces)**.** *$H \subseteq \mathbb{G}$ is a vertical halfspace if and only if there exist $c \in \mathbb{R}$ and a unit vector $\nu \in \mathbf{S}^{m-1}$ such that $H = H_{c,\nu}$, where*

$$H_{c,\nu} := \exp\big(\{\sum_{i=1}^{m} a_i X_i + \sum_{i=2}^{s} v_i : \ v_i \in V_i, \ a \in \mathbb{R}^m, \ \sum_{i=1}^{m} a_i \nu_i \leq c\}\big). \tag{4.5}$$

Proof. Let us denote by $\nu \in \mathbf{S}^{m-1}$ the unique vector such that the vector $Y = \sum_i \nu_i X_i$ is orthogonal to all invariant directions in V_1. Let us work in exponential coordinates, with the function

$$(x_1, \ldots, x_n) \mapsto \exp(\sum_{i=1}^{n} x_i v_i),$$

and let $\tilde{H} \subset \mathbb{R}^n$ be the set H in these coordinates. Here $(v_1, \ldots, v_n)$ is a basis of $\mathfrak{g}$ compatible with the stratification: this means that, if m_i are the dimensions of V_i, with $1 \leq i \leq s$, $l_0 = 0$ and $l_i = \sum_1^i m_j$, then $v_{l_{i-1}+1}, \ldots, v_{l_i}$ is a basis of V_i. By the Baker-Campbell-Hausdorff formula, in these coordinates the vector fields v_i correspond to ∂_{x_i} for $l_{s-1}+1 \leq i \leq l_s = n$, and Theorem 2.12 gives that $\mathbb{1}_{\tilde{H}}$ does not depend on $x_{l_{s-1}+1}, \ldots, x_n$. For $l_{s-2}+1 \leq i \leq l_{s-1}$ the vector fields $v_i - \partial_{x_i}$, still in these coordinates, are given by the sum of polynomials multiplied by ∂_{x_j}, with $l_{s-1}+1 \leq j \leq l_s$. As a consequence $\partial_{x_i}\mathbb{1}_{\tilde{H}} = 0$ and we can apply Theorem 2.12 again to obtain that $\mathbb{1}_{\tilde{H}}$ does not depend on $x_{l_{s-2}+1}, \ldots, x_{l_{s-1}}$ either. Continuing in this way we obtain that $\mathbb{1}_{\tilde{H}}$ depends on $(x_1, \ldots, x_{m_1})$ only. Furthermore, $\sum_i \xi_i \partial_{x_i}\mathbb{1}_{\tilde{H}}$ is equal to 0 if $\xi \perp \nu$, and it is nonnegative if $\xi = \nu$. Then, a classical Euclidean argument (it appears in De Giorgi's rectifiability proof [13], see also the proof of this result in Theorem 3.59 of [3]) shows that $\mathbb{1}_{\tilde{H}}$ depends on $\sum_1^m \nu_i x_i$ only, and it is a monotone function of this quantity. This immediately gives (4.5). □

Remark 4.6. An analogous computation in exponential coordinates shows that $\operatorname{Inv}(f) = \mathfrak{g}$ if and only if f is equivalent to a constant.

In the next proposition we point out useful stability properties of $\operatorname{Reg}(f)$ and $\operatorname{Inv}(f)$.

Proposition 4.7. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{G})$. Then* $\operatorname{Reg}(f)$, $\operatorname{Inv}(f)$, $\operatorname{Inv}_0(f)$ *are invariant under left translations, and* $\operatorname{Inv}_0(f)$ *is invariant under intrinsic dilations. Moreover:*

(i) $\operatorname{Inv}(f)$ *is a Lie subalgebra of* $\mathfrak{g}$ *and* $[\operatorname{Inv}_0(f), \operatorname{Inv}_0(f)] \subset \operatorname{Inv}_0(f)$*;*

(ii) *If* $X \in \mathrm{Inv}(f)$ *and* $k = \exp(X)$, *then* Ad_k *maps* $\mathrm{Reg}(f)$ *into* $\mathrm{Reg}(f)$ *and* $\mathrm{Inv}(f)$ *into* $\mathrm{Inv}(f)$. *More precisely*

$$\mathrm{Ad}_k(Y)f = (R_{k^{-1}})_\sharp Y f \qquad \forall Y \in \mathrm{Reg}(f). \tag{4.8}$$

Proof. The proof of the invariance is simple, so we omit it.

(i) We simply notice that for all X, $Y \in \mathrm{Inv}(f)$ we have

$$\int_{\mathbb{G}} f[X,Y]g \, d\,\mathrm{vol}_{\mathbb{G}} = -\langle Xf, Yg\rangle + \langle Yf, Xg\rangle = 0 \qquad \forall g \in C_c^\infty(\mathbb{G}).$$

The second stated property follows by the fact that $[V_i, V_j] \subset V_{i+j}$.

(ii) Let $Y \in \mathrm{Reg}(f)$ and $Z = \mathrm{Ad}_k(Y)$. For $g \in C_c^\infty(\mathbb{G})$ and $k \in \mathbb{G}$ we have (taking into account the left invariance of Y)

$$Zg(x) = Y(g \circ C_k)(C_k^{-1}(x)) = Y(g \circ R_{k^{-1}})(L_k \circ C_{k^{-1}}(x)) = Y(g \circ R_{k^{-1}})(R_k(x)).$$

Therefore $(Zg) \circ R_{k^{-1}} = Y(g \circ R_{k^{-1}})$ and a change of variables gives

$$\int_{\mathbb{G}} f Zg \, d\,\mathrm{vol}_{\mathbb{G}} = \int_{\mathbb{G}} f \circ R_{k^{-1}} Y(g \circ R_{k^{-1}}) \, d\,\mathrm{vol}_{\mathbb{G}}.$$

Now, if $k = \exp(X)$ with $X \in \mathrm{Inv}(f)$, we have $f \circ R_{k^{-1}} = f$, and this gives (4.8). □

Remark 4.9. Let $X \in \mathrm{Reg}(f)$ and assume that $Xf \geq 0$; then, combining (2.19) with (4.8), we obtain

$$Xf + \sum_{i=1}^{s-1} \frac{t^i}{i!} \mathrm{ad}_Y^i(X) f \geq 0 \qquad \forall t \in \mathbb{R},\ \forall Y \in \mathrm{Inv}(f).$$

Since t can be chosen arbitrarily large, this implies that

$$\mathrm{ad}_Y^{s-1}(X) f \geq 0 \qquad \forall Y \in \mathrm{Inv}(f).$$

In particular, if s is even, by applying the same inequality with $-Y$ in place of Y we get

$$\mathrm{ad}_Y^{s-1}(X) \in \mathrm{Inv}(f). \tag{4.10}$$

Definition 4.11 (Sets of locally finite perimeter)**.** *The main object of investigation of this paper is the class of sets of* locally finite perimeter, *i.e. those Borel sets* E *such that* $X\mathbb{1}_E$ *is a Radon measure for any* $X \in V_1$.

Still using the orthonormal basis of V_1, for $f \in L^1_{\mathrm{loc}}(\mathbb{G})$ with $X_i f \in \mathcal{M}(\mathbb{G})$ we can define the $\mathbb{R}^m$-valued Radon measure

$$Df := (X_1 f, \ldots, X_m f). \tag{4.12}$$

Two very basic properties that will play a role in the sequel are:

$$Df = 0 \qquad \Longrightarrow \qquad f \text{ is (equivalent to) a constant} \tag{4.13}$$

(4.14)

$$\sup_n \int_\Omega |f_n| \, d\,\mathrm{vol}_{\mathbb{G}} + |Df_n|(\Omega) < +\infty \ \ \forall \Omega \Subset \mathbb{G} \quad \Longrightarrow \quad (f_n) \text{ relatively compact in } L^1_{\mathrm{loc}}.$$

The proof of the first one can be obtained combining Proposition 4.7 (that gives that $\mathrm{Inv}(f) = \mathfrak{g}$ with Remark 4.6). The second one has been proved in [21].

Definition 4.15 (De Giorgi's reduced boundary). *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. We denote by $\mathscr{F}E$ the set of points $x \in \operatorname{supp}|D\mathbb{1}_E|$ where:*

(i) *the limit* $\nu_E(x) = (\nu_{E,1}(x), \ldots, \nu_{E,m}(x)) := \lim_{r\downarrow 0} \dfrac{D\mathbb{1}_E(B_r(x))}{|D\mathbb{1}_E|(B_r(x))}$ *exists;*

(ii) $|\nu_E(x)| = 1$.

The following result has been obtained in [1].

Theorem 4.16. *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. Then $|D\mathbb{1}_E|$ is asymptotically doubling, and more precisely the following property holds: for $|D\mathbb{1}_E|$-a.e. $x \in \mathbb{G}$ there exists $\bar{r}(x) > 0$ satisfying*

$$l_{\mathbb{G}} r^{Q-1} \leq |D\mathbb{1}_E|(B_r(x)) \leq L_{\mathbb{G}} r^{Q-1} \qquad \forall r \in (0, \bar{r}(x)), \tag{4.17}$$

with $l_{\mathbb{G}}$ and $L_{\mathbb{G}}$ depending on $\mathbb{G}$ only. As a consequence $|D\mathbb{1}_E|$ is concentrated on $\mathscr{F}E$, i.e., $|D\mathbb{1}_E|(\mathbb{G} \setminus \mathscr{F}E) = 0$.

Actually the result in [1] is valid in all Ahlfors Q-regular metric spaces for which a Poincaré inequality holds (in this context, obviously including all Lie groups, still the measure $|D\mathbb{1}_E|$ makes sense, see [32]); (4.17) also implies that the measure $|D\mathbb{1}_E|$ can also be bounded from above and below by the spherical Hausdorff measure $\mathscr{S}^{Q-1}$, namely

$$\frac{l_{\mathbb{G}}}{\omega_{Q-1}} \mathscr{S}^{Q-1}(A \cap \mathscr{F}E) \leq |D\mathbb{1}_E|(A) \leq \frac{L_{\mathbb{G}}}{\omega_{Q-1}} \mathscr{S}^{Q-1}(A \cap \mathscr{F}E) \tag{4.18}$$

for all Borel sets $A \subseteq \mathbb{G}$ (since $\mathscr{H}^k \leq \mathscr{S}^k \leq 2^k \mathscr{H}^k$, similar inequalities hold with $\mathscr{H}^{Q-1}$). In general doubling metric spaces, where no natural dimension Q exists, the asymptotic doubling property of $|D\mathbb{1}_E|$ and a suitable representation of it in terms of Hausdorff measures have been obtained in [2].

5. Iterated tangents are halfspaces

In this section we show that if we iterate sufficiently many times the tangent operator we do get a vertical halfspace. Let us begin with a precise definition of tangent set.

Definition 5.1 (Tangent set). *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter and $x \in \mathscr{F}E$. We denote by $\operatorname{Tan}(E, x)$ all limit points, in the topology of local convergence in measure, of the translated and rescaled family of sets $\{\delta_{1/r}(x^{-1}E)\}_{r>0}$ as $r \downarrow 0$.*
If $F \in \operatorname{Tan}(E, x)$ we say that F is tangent to E at x. We also set

$$\operatorname{Tan}(E) := \bigcup_{x \in \mathscr{F}E} \operatorname{Tan}(E, x)$$

It is also useful to consider *iterated* tangents; to this aim, still for $x \in \mathscr{F}E$, we define $\operatorname{Tan}^1(E, x) := \operatorname{Tan}(E, x)$ and

$$\operatorname{Tan}^{k+1}(E, x) := \bigcup \left\{ \operatorname{Tan}(F) : \ F \in \operatorname{Tan}^k(E, x) \right\}.$$

The result we shall prove in this section is an intermediate step towards Theorem 1.2:

Theorem 5.2. *Let $E \subseteq \mathbb{G}$ be a set with locally finite perimeter. Then, for $|D\mathbb{1}_E|$-a.e. $x \in \mathbb{G}$ we have (with the notation (4.5))*

$$H_{0,\nu_E(x)} \in \mathrm{Tan}^k(E,x) \qquad \text{with} \qquad k := 1 + 2(n-m).$$

Notice that, by Theorem 4.16, we need only to consider points $x \in \mathscr{F}E$. Our starting point is the following proposition, obtained in [19], showing that the tangent set at points in the reduced boundary is always invariant along codimension 1 subspace of V_1, and monotone along the remaining horizontal direction.

Proposition 5.3. *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. Then, for all $\bar{x} \in \mathscr{F}E$ the following properties hold:*

(i) $0 < \liminf_{r\downarrow 0} |D\mathbb{1}_E|(B_r(\bar{x}))/r^{Q-1} \leq \limsup_{r\downarrow 0} |D\mathbb{1}_E|(B_r(\bar{x}))/r^{Q-1} < +\infty$;

(ii) $\mathrm{Tan}(E,\bar{x}) \neq \emptyset$ *and, for all* $F \in \mathrm{Tan}(E,\bar{x})$, *we have that* $e \in \mathrm{supp}\,|D\mathbb{1}_F|$ *and*

$$\nu_F = \nu_E(\bar{x}) \qquad |D\mathbb{1}_F|\text{-a.e. in } \mathbb{G}.$$

In particular $V_1 \cap \mathrm{Inv}_0(F)$ coincides with the codimension 1 subspace of V_1

$$\left\{ \sum_{i=1}^m a_i X_i : \sum_{i=1}^m a_i \nu_{E,i}(\bar{x}) = 0 \right\}$$

and, setting, $X_x := \sum_{i=1}^m \nu_{E,i}(\bar{x})(X_i)_x \in \mathfrak{g}$, $X\mathbb{1}_F$ is a nonnegative Radon measure.

In groups of step 2, in [19] it is proved that constancy of ν_E characterizes vertical subspaces. We provide here a different proof of this fact, based on the properties of the adjoint operator, and in particular on Remark 4.9.

Proposition 5.4. *Let $E \subset \mathbb{G}$ be a set with locally finite perimeter, and assume that ν_E is (equivalent to) a constant. Then, if $\mathbb{G}$ is a step 2 group, E is a vertical halfspace.*

Proof. Let us denote by ξ the constant value of ν_E, and set $X := \sum_i \xi_i X_i$. Then $X\mathbb{1}_E \geq 0$ and $\mathrm{Inv}(E)$ contains all vectors $Y = \sum_i \eta_i X_i$ with $\eta \in \mathbb{R}^m$ perpendicular to ξ. From (4.10) we get $[Y,X]\mathbb{1}_E = 0$ for any $Y \in \mathrm{Inv}(E) \cap V_1$, and since these commutators, together with the commutators $\{[Y_1,Y_2] : Y_i \in \mathrm{Inv}(E) \cap V_1\}$, span the whole of V_2, the proof is achieved. □

Remark 5.5. The following simple example, that we learned from F. Serra Cassano, shows that the sign condition is essential for the validity of the classification result, even in the first Heisenberg group $\mathbb{H}^1$. Choosing exponential coordinates (x,y,t), and the vector fields $X_1 := \partial_x + 2y\partial_t$ and $X_2 := \partial_y - 2x\partial t$, the function

$$f(x,y,t) := g(t + 2xy)$$

(with g smooth) satisfies $X_1 f = 4yg'(t+2xy)$ and $X_2 f = 0$. Therefore the sets $E_t := \{f < t\}$ are X_2-invariant and are not halfspaces. The same example can be used to show that there is no local version of Proposition 5.4, because the sets E_t locally may satisfy $X_1\mathbb{1}_{E_t} \geq 0$ or $X_1\mathbb{1}_{E_t} \leq 0$ (depending on the sign of g' and y), but are not locally halfspaces.

The non-locality appears also in our argument: indeed, the proof of (4.10) depends on the sign condition of $\mathrm{Ad}_{\exp(tX_2)}(X_1)\mathbb{1}_E$ with t arbitrarily large, and this is the right translate, by $\exp(tX_2)$, of $X_1\mathbb{1}_E$. The proof given in [19] depends, instead, on the possibility of joining two different points in $\mathbb{H}^1$ by following integral lines of X_2 in both directions, and integral lines of X_1 in just one direction: an inspection of the proof reveals that these paths can not be confined in a bounded region, even if the initial and final point are confined within a small region. In this sense, Proposition 5.4 could be considered as a kind of Liouville theorem.

Let $f \in L^1_{\rm loc}(\mathbb{G})$ and $X \in \mathfrak{g}$; then, for all $r > 0$ we have the identity

$$\delta_{1/r}X(f\circ\delta_r) = r^{-Q}(\delta_{1/r})_\sharp(Xf) \tag{5.6}$$

in the sense of distributions. Indeed, writing in brief $X_r := \delta_{1/r}X$, if $g \in C^\infty_c(\mathbb{G})$, from (2.22) we get $X_r(g\circ\delta_r) = (Xg)\circ\delta_r$; as a consequence (3.1) gives

$$\begin{aligned}\langle X_r(f\circ\delta_r), g\rangle &= -\int_{\mathbb{G}}(f\circ\delta_r)X_r g\,d\,\mathrm{vol}_{\mathbb{G}} = -r^{-Q}\int_{\mathbb{G}} f(X_r g)\circ\delta_{1/r}\,d\,\mathrm{vol}_{\mathbb{G}} \\ &= -r^{-Q}\int_{\mathbb{G}} fX(g\circ\delta_{1/r})\,d\,\mathrm{vol}_Q = \langle r^{-Q}(\delta_{1/r})_\sharp(Xf), g\rangle.\end{aligned} \tag{5.7}$$

The first crucial lemma shows that if $X \in \mathrm{Reg}(E)$ belongs to $\oplus_2^s V_i$, then the tangents to E at $|D\mathbb{1}_E|$-a.e. x are invariant under Y, where Y is the "higher degree part" of X induced by the stratification of $\mathfrak{g}$. The underlying reason for this fact is that the intrinsic dilations behave quite differently in the X direction and in the horizontal direction.

Lemma 5.8. *Let F be a set with locally finite perimeter, $X \in \mathrm{Reg}(F)$, $\mu = X\mathbb{1}_F$ and assume that $X = \sum_{i=2}^l v_i$ with $v_i \in V_i$ and $l \le s$. Then, for $|D\mathbb{1}_F|$-a.e. x, $v_l \in \mathrm{Inv}_0(L)$ for all $L \in \mathrm{Tan}(F,x)$.*

Proof. ¿From (3.2) we know that the set N of points x such that $\limsup_{r\downarrow 0} r^{2-Q}|\mu|(B_r(x))$ is positive is σ-finite with respect to $\mathscr{S}^{Q-2}$, and therefore $\mathscr{S}^{Q-1}$-negligible and $|D\mathbb{1}_F|$-negligible (recall (4.18)). We will prove that the statement holds at any $x \in (\mathscr{F}F)\setminus N$ and we shall assume, up to a left translation, that $x = e$. Given any $g \in C^1_c(\mathbb{G})$, let R be such that $\mathrm{supp}(g) \subsetneq B_R(e)$; (5.6) with $f = \mathbb{1}_F$ gives

$$\int_{\mathbb{G}} \mathbb{1}_{\delta_{1/r}F}X_r g\,d\,\mathrm{vol}_{\mathbb{G}} = r^{l-Q}\int_{\mathbb{G}} g\circ\delta_{1/r}\,d\mu$$

with $X_r := r^l\delta_{1/r}X$, so that $X_r \to v_l$ as $r \downarrow 0$. Now, notice that $l \ge 2$, and that the right hand side can be bounded with

$$\sup|g| r^{l-Q}|\mu|(B_{Rr}(e)) = O(r^{l-Q})o(r^{Q-2}) = o(1).$$

So, passing to the limit as $r \downarrow 0$ along a suitable sequence, we obtain that $v_l\mathbb{1}_L = 0$ for all $L \in \mathrm{Tan}(F,e)$. □

The invariance of Inv_0 under left translations and scaling shows that $\mathrm{Inv}_0(F)$ contains $\mathrm{Inv}_0(E)$ for all $F \in \mathrm{Tan}(E)$. Let us define codimension of $\mathrm{Inv}_0(E)$ in $\mathfrak{g}$ as the codimension of its linear span; we know that this codimension is at least 1 (because the codimension within V_1 is 1) for *all* tangent sets, and it is equal to 1 precisely for vertical halfspaces, thanks to Proposition 4.4.

The second crucial lemma shows that, when the codimension of $\mathrm{Inv}_0(E)$ in $\mathfrak{g}$ is at least 2, a double tangent *strictly* increases, at $|D\mathbb{1}_E|$-a.e. point, the set $\mathrm{Inv}_0(E)$. The strategy is to find first a tangent set F with $\mathrm{Reg}(F) \supsetneq \mathrm{span}(\mathrm{Inv}_0(E))$ (this is based on the geometric Proposition 2.17 and Proposition 4.7) and then on the application of the previous lemma, which turns a regular direction of F into an invariant homogeneous direction of a tangent to F.

Lemma 5.9 (Improvement of $\mathrm{Inv}_0(E)$)**.** *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter and assume that*

$$\dim\big(\mathrm{span}(\mathrm{Inv}_0(E))\big) \leq n-2.$$

Then, for all $\bar{x} \in \mathscr{F}E$, $\mathrm{Inv}_0(L) \supsetneq \mathrm{Inv}_0(E)$ for some $L \in \mathrm{Tan}^2(E, \bar{x})$.

Proof. **(Step 1)** We show first the existence of $Z \in \mathfrak{g} \setminus [\mathrm{span}(\mathrm{Inv}_0(E)) + V_1]$ such that $Z \in \mathrm{Reg}(F)$ for all $F \in \mathrm{Tan}(E, \bar{x})$. To this aim, we apply Proposition 2.17 with $\mathfrak{g}' := \mathrm{span}(\mathrm{Inv}_0(E))$ (recall that, by Proposition 4.7(i), $\mathfrak{g}'$ is a Lie algebra) and $X := \sum_1^m \nu_{E,i}(\bar{x}) X_i$ to obtain $Y \in \mathfrak{g}'$ such that

$$Z := \mathrm{Ad}_{\exp(Y)}(X) \notin \mathrm{span}(\mathrm{Inv}_0(E)) \oplus \{\mathbb{R}X\} = \mathrm{span}(\mathrm{Inv}_0(E)) + V_1.$$

Then, since $\mathrm{Inv}_0(F)$ contains $\mathrm{Inv}_0(E)$ for all $F \in \mathrm{Tan}(F, \bar{x})$, we have that $Y \in \mathrm{Inv}(F)$, therefore Proposition 4.7(ii) shows that $Z \in \mathrm{Reg}(F)$ for all $F \in \mathrm{Tan}(E, \bar{x})$.

(Step 2) Now, let $F \in \mathrm{Tan}(E, \bar{x})$, $Z \notin \mathrm{span}(\mathrm{Inv}_0(E)) + V_1$ given by the previous step, and set $\mu = Z\mathbb{1}_F$. Possibly removing from Z its horizontal component we can write $Z = v_{i_1} + \cdots + v_{i_l}$ with $i_j \geq 2$ and $v_{i_j} \in V_{i_j}$. Then, $v_{i_k} \notin \mathrm{Inv}_0(E)$ for at least one $k \in \{1, \ldots, l\}$, and let us choose the largest one with this property. Then, setting $Z' = v_{i_1} + \cdots + v_{i_k}$, since $v_{i_j} \in \mathrm{Inv}_0(E) \subseteq \mathrm{Inv}_0(F)$ for all $k < j \leq l$, we still have $Z'\mathbb{1}_F = \mu$. By Lemma 5.8 we can find $L \in \mathrm{Tan}(F)$ with $v_{i_k}\mathbb{1}_L = 0$, i.e. $v_{i_k} \in \mathrm{Inv}_0(L)$. Since $v_{i_k} \notin \mathrm{Inv}_0(E)$, we have proved that $\mathrm{Inv}_0(L)$ strictly contains $\mathrm{Inv}_0(E)$. $\square$

Proof of Theorem 5.2. Recall that $m = \dim(V_1)$. Sets in $\mathrm{Tan}(E, \bar{x})$ are invariant, thanks to Proposition 5.3, in at least $m-1$ directions. Let us define

$$i_k := \max\left\{\dim\big(\mathrm{span}(\mathrm{Inv}_0(F))\big) : \ F \in \mathrm{Tan}^k(E, \bar{x})\right\}.$$

Then $i_1 \geq m-1$ and we proved in Lemma 5.9 that $i_{k+2} > i_k$ as long as there exists $F \in \mathrm{Tan}^k(E, \bar{x})$ with $\dim\big(\mathrm{span}(\mathrm{Inv}_0(F))\big) \leq n-2$. By iterating k times, with $k \leq 2(n-m)$, the tangent operator we find $F \in \mathrm{Tan}^k(E, \bar{x})$ with $\dim\big(\mathrm{span}(\mathrm{Inv}_0(F))\big) \geq n-1$.

We know from Proposition 5.3 that $e \in \mathrm{supp}\,|D\mathbb{1}_F|$, that the codimension of $\mathrm{Inv}_0(F)$ is exactly 1, and precisely that

$$V_1 \cap \mathrm{Inv}_0(F) = \left\{ \sum_{i=1}^m a_i X_i : \ \sum_{i=1}^m a_i \nu_{E,i}(\bar{x}) = 0 \right\}$$

and that $\sum_i \nu_{E,i}(\bar{x}) X_i \mathbb{1}_F \geq 0$. Therefore Proposition 4.4 gives $F = H_{0,\nu_E(\bar{x})}$.

6. Iterated tangents are tangent

In this section we complete the proof of Theorem 1.2. Taking into account the statement of Theorem 5.2, we need only to prove the following result.

Theorem 6.1. *Let $E \subseteq \mathbb{G}$ be a set with locally finite perimeter. Then, for $|D\mathbb{1}_E|$-a.e. $x \in \mathbb{G}$ we have*
$$\bigcup_{k=2}^{\infty} \mathrm{Tan}^k(E,x) \subseteq \mathrm{Tan}(E,x).$$

In turn, this result follows by an analogous one involving tangents to measures, proved in [39] in the Euclidean case; we just adapt the argument to Carnot groups and to vector-valued measures. In the sequel we shall denote by $I_{x,r}(y) := \delta_{1/r}(x^{-1}y)$ the composition $\delta_{1/r} \circ L_{x^{-1}}$.

We say that a measure $\mu \in \mathcal{M}^m(\mathbb{G})$ is *asymptotically q-regular* if

$$0 < \liminf_{r \downarrow 0} \frac{|\mu|(B_r(x))}{r^q} \leq \limsup_{r \downarrow 0} \frac{|\mu|(B_r(x))}{r^q} < +\infty \qquad \text{for } |\mu|\text{-a.e. } x \in \mathbb{G}. \tag{6.2}$$

Notice that asymptotically q-regular measures are asymptotically doubling, and that the perimeter measure $|D\mathbb{1}_E|$ is asymptotically $(Q-1)$-regular, thanks to Theorem 4.16.

Definition 6.3 (Tangents to a measure)**.** *Let $\mu \in \mathcal{M}^m(\mathbb{G})$ be asymptotically q-regular. We shall denote by* $\mathrm{Tan}(\mu,x)$ *the family of all measures $\nu \in \mathcal{M}^m(\mathbb{G})$ that are weak* limit points as $r \downarrow 0$ of the family of measures $r^{-q}(I_{x,r})_\sharp \mu$.*

Theorem 6.4. *Let $\mu \in \mathcal{M}^m(\mathbb{G})$ be asymptotically q-regular. Then, for $|\mu|$-a.e. x, the following property holds:*
$$\mathrm{Tan}(\nu,y) \subseteq \mathrm{Tan}(\mu,x) \qquad \forall \nu \in \mathrm{Tan}(\mu,x),\ y \in \operatorname{supp}|\nu|.$$

The connection between Theorem 6.1 and Theorem 6.4 rests on the following observation:
$$L \in \mathrm{Tan}(F,x) \qquad \Longleftrightarrow \qquad D\mathbb{1}_L \in \mathrm{Tan}(D\mathbb{1}_F,x) \setminus \{0\} \tag{6.5}$$
for all $x \in \mathscr{F}F$. The implication $\Rightarrow$ in (6.5) is easy, because a simple scaling argument gives
$$L = \lim_{i \to \infty} \delta_{1/r_i}(x^{-1}F) \qquad \Longrightarrow \qquad D\mathbb{1}_L = \lim_{i\to\infty} r_i^{1-Q}(I_{x,r_i})_\sharp D\mathbb{1}_F. \tag{6.6}$$
Therefore $L \in \mathrm{Tan}(F,x)$ implies $D\mathbb{1}_L \in \mathrm{Tan}(D\mathbb{1}_F,x)$; clearly $D\mathbb{1}_L \neq 0$ because $x \in \mathscr{F}F$.

Now we prove the harder implication $\Leftarrow$ in (6.5): assume, up to a left translation, that $x = e$, and that $D\mathbb{1}_L \neq 0$ is the weak* limit of $r_i^{1-Q}(I_{e,r_i})_\sharp D\mathbb{1}_F$, with $r_i \downarrow 0$; now, set $F_i := \delta_{1/r_i}F$, so that $D\mathbb{1}_{F_i} = r_i^{1-Q}(I_{e,r_i})_\sharp D\mathbb{1}_F$, and by the compactness properties of sets of finite perimeter (see (4.14)) assume with no loss of generality that $F_i \to L'$ locally in measure, so that $L' \in \mathrm{Tan}(F,e)$. Then $r_i^{1-Q}(I_{e,r_i})_\sharp D\mathbb{1}_F = D\mathbb{1}_{F_i}$ weakly* converge to $D\mathbb{1}_{L'}$: indeed, the convergence in the sense of distributions is obvious, and since the total variations are locally uniformly bounded, we have weak* convergence as well. It follows that $D\mathbb{1}_L = D\mathbb{1}_{L'}$. Since $\mathbb{1}_L - \mathbb{1}_{L'}$ has zero horizontal distributional derivative, by (4.13) it must be (equivalent to) a constant; this can happen only when either $L = L'$ or $L = \mathbb{G} \setminus L'$; but the second possibility is ruled out because it would imply that $D\mathbb{1}_L = -D\mathbb{1}_{L'}$ and that $D\mathbb{1}_L = 0$. This proves that $L = L' \in \mathrm{Tan}(F, e)$.

Proof. **(of Theorem 6.1)** At any point $x \in \mathscr{F}E$ where the property stated in Theorem 6.4 holds with $\mu = D\mathbb{1}_E$ we may consider any $F \in \mathrm{Tan}(E, x)$ and $L \in \mathrm{Tan}(F, y)$ for some $y \in \mathscr{F}F$; then, by (6.5) we know that $D\mathbb{1}_F \in \mathrm{Tan}(D\mathbb{1}_E, x)$ and $D\mathbb{1}_L \in \mathrm{Tan}(D\mathbb{1}_F, y) \setminus \{0\}$; as a consequence, Theorem 6.4 gives $D\mathbb{1}_L \in \mathrm{Tan}(D\mathbb{1}_E, x) \setminus \{0\}$, hence (6.5) again gives that $L \in \mathrm{Tan}(E, x)$. This proves that $\mathrm{Tan}^2(E, x) \subseteq \mathrm{Tan}(E, x)$, and therefore $\mathrm{Tan}^3(E, x) \subseteq \mathrm{Tan}^2(E, x)$, and so on. $\square$

The rest of this section is devoted to the proof of Theorem 6.4. We will follow with minor variants (because we are dealing with vector-valued measures) the proof given in Mattila's book [31]. Before proceeding to the proof of Theorem 6.4 we state a simple lemma.

Lemma 6.7. *Assume that $A \subset \mathbb{G}$ and $a \in A$ is a density point for A relative to $|\mu|^*$, i.e.*

$$\lim_{r \downarrow 0} \frac{|\mu|^*(B_r(a) \cap A)}{|\mu|(B_r(a))} = 1. \tag{6.8}$$

If, for some $r_i \downarrow 0$ and $\lambda_i \geq 0$ the measures $\lambda_i (I_{a,r_i})_\sharp \mu$ weakly converge to ν, then*

$$\lim_{i \to \infty} \frac{d(a\delta_{r_i} y, A)}{r_i} = 0 \qquad \forall y \in \operatorname{supp} |\nu|.$$

Proof. Let $\tau := d(y, e)$ and let us argue by contradiction. If the statement were false, τ would be positive and there would exist $\epsilon \in (0, \tau)$ such that $d(a\delta_{r_i} y, A) > \epsilon r_i$ for infinitely many values of i. Possibly extracting a subsequence, let us assume that this happens for all i: we know that

$$B_{\epsilon r_i}(a\delta_{r_i} y) \subseteq \mathbb{G} \setminus A \tag{6.9}$$

and since $\epsilon < \tau$ we have

$$B_{\epsilon r_i}(a\delta_{r_i} y) \subseteq B_{\tau r_i}(a\delta_{r_i} y) \subseteq B_{2\tau r_i}(a). \tag{6.10}$$

Now use, in this order, the definition of density point, (6.9), (6.10) and (3.3) to get

$$\begin{aligned}
1 &= \lim_{i \to \infty} \frac{|\mu|^*(B_{2\tau r_i}(a)) \cap A)}{|\mu|(B_{2\tau r_i}(a))} \leq \limsup_{i \to \infty} \frac{|\mu|(B_{2\tau r_i}(a) \setminus B_{\epsilon r_i}(a\delta_{r_i} y))}{|\mu|(B_{2\tau r_i}(a))} \\
&= \limsup_{i \to \infty} \frac{|\mu|(B_{2\tau r_i}(a))) - |\mu|(B_{\epsilon r_i}(a\delta_{r_i} y))}{|\mu|(B_{2\tau r_i}(a))} = 1 - \liminf_{i \to \infty} \frac{|\mu|(B_{\epsilon r_i}(a\delta_{r_i} y))}{|\mu|(B_{2\tau r_i}(a))} \\
&= 1 - \liminf_{i \to \infty} \frac{(I_{a,r_i})_\sharp |\mu|(B_\epsilon(y))}{(I_{a,r_i})_\sharp |\mu|(B_{2\tau}(e))} \leq 1 - \frac{\liminf_{i \to \infty} |\lambda_i (I_{a,r_i})_\sharp \mu|(B_\epsilon(y))}{\limsup_{i \to \infty} |\lambda_i (I_{a,r_i})_\sharp \mu|(B_{2\tau}(e))} \\
&\leq 1 - \frac{|\nu|(B_\epsilon(y))}{\limsup_{i \to \infty} |\lambda_i (I_{a,r_i})_\sharp \mu|(B_{2\tau}(e))}.
\end{aligned}$$

But, $|\nu|(B_\epsilon(y)) > 0$ because $y \in \operatorname{supp} |\nu|$, and the $\limsup$ is finite by (3.4). This contradiction concludes the proof of the lemma. $\square$

Proof of Theorem 6.4. For ν, $\nu' \in \mathcal{M}^m(\mathbb{G})$, define

$$d_R(\nu,\nu') := \sup\left\{\int_{\mathbb{G}} \phi\, d\nu - \int_{\mathbb{G}} \phi\, d\nu' :\ \phi \in \mathcal{D}_R\right\},$$

where

$$\mathcal{D}_R := \{\phi \in C_c(B_R(e)) :\ \sup|\phi| \le 1 \quad \text{and} \quad |\phi(x)-\phi(y)| \le d(x,y)\ \forall x,\, y \in \mathbb{G}\}.$$

It is well known, and easy to check, that d_R induces the weak* convergence in all bounded sets of $\mathcal{M}^m(B_R(e))$. We define a distance $\bar{d}$ in $\mathcal{M}^m(\mathbb{G})$ by

$$\bar{d}(\mu,\nu) := \sum_{R=1}^{\infty} 2^{-R} \min\bigl\{1, d_R(\mu,\nu)\bigr\}.$$

Let x be a point where the limsup in (6.2) is finite; now we check that, for all infinitesimal sequences $(r_i) \subset (0,+\infty)$, we have

$$\nu = \text{weak}^* - \lim_{i\to\infty} r_i^{-q}(I_{x,r_i})_\sharp\mu \qquad \Longleftrightarrow \qquad \lim_{i\to\infty} \bar{d}(\nu, r_i^{-q}(I_{x,r_i})_\sharp\mu) = 0. \tag{6.11}$$

The implication $\Rightarrow$ is obvious, because $\bar{d}$-convergence is equivalent to d_R-convergence for all R, and all weakly*-convergent sequences are locally uniformly bounded (see (3.4)). The implication $\Leftarrow$ is analogous, but it depends on our choice of x, which ensures the property

$$\sup_{i\in\mathbb{N}} r_i^{-q}|(I_{x,r_i})_\sharp\mu|(B_R(e)) = \sup_{i\in\mathbb{N}} r_i^{-q}|\mu|(B_{Rr_i}(x)) \le R^q \limsup_{r\downarrow 0} \frac{|\mu|(B_r(x))}{r^q} < +\infty.$$

This property ensures that $r_i^{-q}(I_{x,r_i})_\sharp\mu$ is bounded in all $\mathcal{M}^m(B_R(e))$ for all $R > 0$, and enables to pass from d_R-convergence to weak* convergence in all balls $B_R(e)$.

Thanks to the equivalence stated in (6.11), by a diagonal argument it suffices to prove that, for $|\mu|$-a.e. x, the following property holds: for all $\nu \in \mathrm{Tan}(\mu,x)$, $y \in \operatorname{supp}|\nu|$ and $r > 0$ we have $r^{-q}(I_{y,r})_\sharp\nu \in \mathrm{Tan}(\mu,x)$. But since the operation $\sigma \mapsto r^{-q}(I_{e,r})_\sharp\sigma$ is easily seen to map $\mathrm{Tan}(\mu,x)$ into $\mathrm{Tan}(\mu,x)$, and $I_{y,r} = I_{e,r}\circ I_{y,1}$, we need just to show that:

(*) for $|\mu|$-a.e. x the following property holds: for all $\nu \in \mathrm{Tan}(\mu,x)$ and all $y \in \operatorname{supp}|\nu|$, we have $(I_{y,1})_\sharp\nu \in \mathrm{Tan}(\mu,x)$.

Heuristically, this property holds at "Lebesgue" points of the multivalued map $x \mapsto \mathrm{Tan}(\mu,x)$, thanks to the identity

$$I_{\delta_{1/r}(x^{-1}y),1} \circ I_{x,r} = I_{y,r}. \tag{6.12}$$

Indeed, this identity implies that tangents to μ at x on the scale r are close to tangents to μ at y on the scale r when $d(x,y) \ll r$.

Let us consider the set R of points where the property (*) fails: for all $x \in R$ there exist a measure $\nu \in \mathrm{Tan}(\mu,x)$ and a point $y \in \operatorname{supp}|\nu|$ such that $(I_{y,1})_\sharp\nu \notin \mathrm{Tan}(\mu,x)$. This implies, thanks to the implication $\Leftarrow$ in (6.11), the existence of integers z, $k \ge 1$ such that the measure $(I_{y,1})_\sharp\nu$ is $1/k$ far (relative to $\bar{d}$) from the set $r^{-q}(I_{x,r})_\sharp\mu :\ r \in (0,1/z)\}$. Set

$$A_{z,k} := \{x \in \mathbb{G} :\ \exists \nu \in \mathrm{Tan}(\mu,x),\ \exists y \in \operatorname{supp}|\nu| \text{ such that}$$
$$\overline{d}((I_{y,1})_\sharp\nu, r^{-q}(I_{x,r})_\sharp\mu) > 1/k,\ \ \forall r \in (0,1/z)\}.$$

Since R is contained in the union of these sets, to conclude the proof it suffices to show that $|\mu|^*(A_{z,k}) = 0$ for any z, $k \geq 1$.

Suppose by contradiction $|\mu|^*(A_{z,k}) \neq 0$ for some z, $k \geq 1$ and let us fix these two parameters; it is not difficult to check that we can cover the space $\mathcal{M}^m(\mathbb{G})$ with a family $\{B_l\}$ of sets satisfying

$$\bar{d}(\nu, \nu') < \frac{1}{2k} \qquad \forall \nu,\, \nu' \in B_l. \tag{6.13}$$

Let us now consider the sets

$$\begin{aligned} A_{z,k,l} := \{x \in \mathbb{G} :\ \exists \nu \in \operatorname{Tan}(\mu, x),\ \exists y \in \operatorname{supp}|\nu| \text{ such that } (I_{y,1})_\sharp \nu \in B_l, \\ \overline{d}((I_{y,1})_\sharp \nu, r^{-q}(I_{x,r})_\sharp \mu) > 1/k,\ \ \forall r \in (0, 1/z)\big\}\,. \end{aligned}$$

Since $\cup_l A_{z,k,l}$ contains $A_{z,k}$ and $|\mu|^*$ is countably subadditive, at least one of these sets satisfies $|\mu|^*(A_{z,k,l}) > 0$. Let us fix l with this property, and let us denote $A_{z,k,l}$ by A.

Since $|\mu|^*(A) > 0$ and $|\mu|$ is asymptotically doubling, we can find $a \in A$ which is a density point of A relative to $|\mu|^*$. ¿From now on also the point a will be fixed, and so an associated measure $\nu_a \in \operatorname{Tan}(\mu, a)$, a point $y_a \in \operatorname{supp}|\nu_a|$ satisfying $(I_{y_a,1})_\sharp \nu_a \in B_l$ and

$$\overline{d}((I_{y_a,1})_\sharp \nu_a, r^{-q}(I_{a,r})_\sharp \mu) > \frac{1}{k},\ \ \forall r \in (0, 1/m). \tag{6.14}$$

We can also write $\nu_a = \lim_{i\to\infty} r_i^{-q}(I_{a,r_i})_\sharp \mu$, for suitable $r_i \downarrow 0$, and clearly (6.14) implies that $y_a \neq e$.

Let us consider the points $a \cdot \delta_{r_i} y_a$ and their distance from A and take $a_i \in A$ such that $\operatorname{dist}(a\delta_{r_i} y_a, a_i) \leq \operatorname{dist}(a\delta_{r_i} y_a, A) + r_i/i$. Lemma 6.7 yields that $\operatorname{dist}(a\delta_{r_i} y_a, a_i) = o(r_i)$ as $i \to \infty$, and so $\delta_{1/r_i}(a^{-1}a_i) \to y_a$. Now, (6.12) shows that $I_{\delta_{1/r_i}(a^{-1}a_i),1} \circ I_{a,r_i} = I_{a_i,r_i}$, so that

$$\begin{aligned} \lim_{i\to\infty} r_i^{-q}(I_{a_i,r_i})_\sharp \mu &= \lim_{i\to\infty} r_i^{-q}(I_{\delta_{1/r_i}(a^{-1}a_i),1})_\sharp (I_{a,r_i})_\sharp \mu \\ &= \lim_{i\to\infty} (I_{\delta_{1/r_i}(a^{-1}a_i),1})_\sharp \big(r_i^{-q}(I_{a,r_i})_\sharp \mu\big) = (I_{y_a,1})_\sharp \nu_a. \end{aligned}$$

So, we can fix i sufficiently large such that $r_i < 1/z$ and

$$\overline{d}(r_i^{-q}(I_{a_i,r_i})_\sharp \mu, (I_{y_a,1})_\sharp \nu_a) < \frac{1}{2k}. \tag{6.15}$$

Since $a_i \in A = A_{z,k,l}$, we can find a measure $\nu' \in \operatorname{Tan}(\mu, a_i)$ and a point $y' \in \operatorname{supp}|\nu'|$ with $(I_{y',1})_\sharp \nu' \in B_l$ such that

$$\frac{1}{k} < \overline{d}(r_i^{-q}(I_{a_i,r_i})_\sharp \mu, (I_{y',1})_\sharp \nu').$$

By applying the triangle inequality we obtain

$$\frac{1}{k} < \overline{d}(r_i^{-q}(I_{a_i,r_i})_\sharp \mu, (I_{y_a,1})_\sharp \nu_a) + \overline{d}((I_{y_a,1})_\sharp \nu_a, (I_{y',1})_\sharp \nu') < \frac{1}{2k} + \frac{1}{2k},$$

where we used (6.15) and our choice (6.13) of B_l. The contradiction ends the proof of the theorem.

7. The Engel cone example

In this section we revisit the example in [19] of a set with a constant normal which is not a vertical halfspace, and we show why the improvement procedure does not work, at least at some points, in this case.

7.1. The Engel group. Let us recall the definition of Engel Lie algebra and group.

Let $\mathbb{E}$ be the Carnot group whose Lie algebra is $\mathfrak{g} = V_1 \oplus V_2 \oplus V_3$ with $V_1 = \operatorname{span}\{X_1, X_2\}$, $V_2 = \{\mathbb{R}X_3\}$ and $V_3 = \{\mathbb{R}X_4\}$, the only non zero commutation relations being

$$[X_1, X_2] = -X_3, \qquad [X_1, X_3] = -X_4. \tag{7.1}$$

In exponential coordinates we can identify $\mathbb{E}$ with $\mathbb{R}^4$, and the group law takes the form

$$x \cdot y = H(\sum_{i=1}^{4} x_i X_i, \sum_{i=1}^{4} y_i X_i),$$

where H is given by the Campbell–Hausdorff formula

$$H(X, Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] - [Y, [X, Y]]).$$

Still in exponential coordinates, an explicit representation of the vector fields is:

$$\begin{aligned}
X_1 &= \partial_1, \\
X_2 &= \partial_2 - x_1 \partial_3 + \frac{x_1^2}{2} \partial_4, \\
X_3 &= \partial_3 - x_1 \partial_4, \\
X_4 &= \partial_4.
\end{aligned}$$

Clearly $\mathbb{E}$ is a Carnot group with step $s = 3$, topological dimension $n = 4$, homogeneous dimension $Q = 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 7$, and dimension of the horizontal layer $m = 2$. From now on, we shall use exponential coordinates to denote the elements of the group.

7.2. A cone in the Engel group. For any $\alpha > 0$, let $P = P_\alpha : \mathbb{R}^4 \to \mathbb{R}$ be the polynomial

$$P(x) = \alpha x_2^3 + 2x_4,$$

whose gradient is

$$\nabla P(x) = \left(0, 3\alpha x_2^2, 0, 2\right).$$

In particular all level sets $\{P = c\}$ of P are obviously graphs of smooth functions depending on (x_1, x_2, x_3). The derivative of P is particularly simple along the vector fields of the horizontal layer: indeed, we have

$$X_1 P(x) \;=\; \partial_1(\alpha x_2^3 + 2x_4) = 0$$

and

$$\begin{aligned}
X_2 P(x) &= [\partial_2 - x_1 \partial_3 + \frac{x_1^2}{2} \partial_4](\alpha x_2^3 + 2x_4) \\
&= 3\alpha x_2^2 + x_1^2 \geq 0.
\end{aligned}$$

Hence

$$X_1P(x) = 0, \quad X_2P(x) = x_1^2 + 3\alpha x_2^2 \qquad \forall x \in \mathbb{R}^4. \tag{7.2}$$

We define

$$C := \{x \in \mathbb{R}^4 : \ P(x) \leq 0\},$$

whose boundary ∂C is the set $\{P = 0\}$. Notice that, due to the (intrinsic) homogeneity of degree 3 of the polynomial, the set C is a cone, i.e. $\delta_r C = C$ for all $r > 0$.

We shall denote by $\nu_C^{eu}(x) = \nabla P(x)/|\nabla P(x)|$ the unit (Euclidean) outer normal to C. We also have the expansion

$$\begin{aligned} |\nabla P|(x) &= \sqrt{4 + 9\alpha^2 x_2^4} \\ &= 2 + \frac{9}{2}\alpha^2 x_2^4 + O(d^4(x,0)). \end{aligned}$$

Thanks to Subsection 2.2 the set C has locally finite perimeter, and more precisely we have the formula (2.8) (throughout this section $\mathscr{H}^k$ is the Hausdorff measure induced by the Euclidean distance)

$$Z\mathbb{1}_C = -\frac{ZP}{|\nabla P|}\mathscr{H}^3 \llcorner \partial C \qquad \forall Z \in \mathfrak{g}. \tag{7.3}$$

In particular (7.2) and (7.3) give

$$D\mathbb{1}_C = (X_1\mathbb{1}_C, X_2\mathbb{1}_C) = (0,1)X_2\mathbb{1}_C = -\frac{x_1^2 + 3\alpha x_2^2}{|\nabla P(x)|}(0,1)\mathscr{H}^3 \llcorner \partial C.$$

It follows that

$$|D\mathbb{1}_C| = \frac{x_1^2 + 3\alpha x_2^2}{|\nabla P(x)|}\mathscr{H}^3 \llcorner \partial C \tag{7.4}$$

and that the horizontal normal, that is the vector field $\nu_C = (0,1)$, is constant, so that *all* points of $\operatorname{supp}|D\mathbb{1}_C|$ belong to $\mathcal{F}C$.

Since we proved in Lemma 5.8 that non-horizontal regular directions Z for E give rise, after blow-up, to invariant directions, at least at points $\bar{x}$ where $|Z\mathbb{1}_E|(B_r(\bar{x}))/r^{Q-2}$ is infinitesimal as $r \downarrow 0$, and since the cone is self-similar under blow-up at $\bar{x} = 0$, it must happen that $|Z\mathbb{1}_C|(B_r(0))/r^{Q-2}$ is not infinitesimal as $r \downarrow 0$ for any non-horizontal regular directions Z (actually, for the cone C, *all* directions are regular). Let us show explicitly this fact for $Z := \mathrm{Ad}_{\exp(X_1)}(X_2)$: taking into account the commutator relations (7.1) and

$$\begin{aligned} Z := \mathrm{Ad}_{\exp(X_1)} X_2 &= X_2 + [X_1, X_2] + \frac{1}{2}[X_1,[X_1,X_2]] \\ &= X_2 - X_3 + \frac{1}{2}X_4 \\ &= \partial_2 - x_1\partial_3 + \frac{x_1^2}{2}\partial_4 - \partial_3 + x_1\partial_4 + \partial_4 \\ &= \partial_2 - (1+x_1)\partial_3 + \frac{(1+x_1)^2}{2}\partial_4. \end{aligned}$$

We can now compute the derivative along the vector field Z:

$$\begin{aligned} ZP(x) &= \left[\partial_2 - (1+x_1)\partial_3 + \frac{(1+x_1)^2}{2}\partial_4\right](\alpha x_2^3 + 2x_4) \\ &= 3\alpha x_2^2 + (1+x_1)^2 \\ &= 1 + O(d(x,0)). \end{aligned}$$

Intuitively, the quotient $|Z\mathbb{1}_C|(B_r(0))/|D\mathbb{1}_C|(B_r(0))$ tends to $+\infty$ as $r \downarrow 0$ because of the relations (7.3) and (7.4), and the fact that $ZP(0) \neq 0$ (notice that the factor $|\nabla P|$ is close to 2 near to the origin). Let us make a more precise analysis: according to the ball-box theorem, balls $B_r(0)$ are comparable to the boxes

$$Q_r := [-r,r]^2 \times [-r^2,r^2] \times [-r^3,r^3],$$

so we will compute the density on these boxes, rather than on balls. We shall assume, for the sake of simplicity, that $\alpha \in (0,2]$. The homogeneity of C and the fact that $0 \in \mathscr{F}C$ give $|D\mathbb{1}_C|(Q_r) = cr^6$ for some positive constant c. The function

$$x_4 = -\alpha x_2^3 := g(x_1,x_2,x_3),$$

whose graph is ∂C, has absolute value strictly less than r^3, thus $Q_r \cap \partial C$ is the graph of g on the "basis" $[-r,r]^2 \times [-r^2,r^2]$ of the box Q_r. Moreover, since g has zero gradient at the origin,

$$\begin{aligned} \mathscr{H}^3(Q_r \cap \partial C) &= \int_{[-r,r]^2\times[-r^2,r^2]} \sqrt{1+|\nabla g|^2}\, d\mathscr{L}^3 \\ &\sim \int_{[-r,r]^2\times[-r^2,r^2]} 1\, d\mathscr{L}^3 \\ &= \mathscr{L}^3([-r,r]^2 \times [-r^2,r^2]) = 8r^4. \end{aligned}$$

¿From (7.3) we obtain $|Z\mathbb{1}_C|(Q_r) = 4r^4 + o(r^4)$ and we conclude that $|Z\mathbb{1}_C|(B_r(0))/|D\mathbb{1}_C|(B_r(0)) \sim r^{-2}$.

7.3. **Other constant normal sets in the Engel group.** We present here another family of sets that have constant horizontal normal. This time we have a dependence on two parameters a, $b \in \mathbb{R}$. Let $P_{a,b} : \mathbb{R}^4 \to \mathbb{R}$ be the polynomial

$$P_{a,b}(x) = 2ax_4 - bx_3 + x_2.$$

Since $\partial_2 P_{a,b} \neq 0$, all level sets $\{P_{a,b} = c\}$ of $P_{a,b}$ are obviously smooth manifolds.

Note that when both a and b are zero, the sub-level sets are vertical halfspaces. In general, the derivatives along the vector fields of the horizontal layer are

$$X_1 P(x) = 0, \quad X_2 P(x) = ax_1^2 + bx_1 + 1 \qquad \forall x \in \mathbb{R}^4. \tag{7.5}$$

So, if (a,b) is close to $(1,0)$ then $ax_1^2 + bx_1 + 1$ is a perturbation of $x_1^2 + 1$ that is strictly greater than 0. Thus $X_2 P(x) > 0$ for any (a,b) in a neighborhood of $(1,0)$. In other words the sub-level sets have constant horizontal normal. However, these sets are not cones, except when they are vertical halfspaces.

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