SOME APPLICATIONS OF ℓ_p-COHOMOLOGY TO BOUNDARIES OF GROMOV HYPERBOLIC SPACES

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Abstract. We study quasi-isometry invariants of Gromov hyperbolic spaces, focussing on the ℓ_p-cohomology and closely related invariants such as the conformal dimension, combinatorial modulus, and the Combinatorial Loewner Property. We give new constructions of continuous ℓ_p-cohomology, thereby obtaining information about the ℓ_p-equivalence relation, as well as critical exponents associated with ℓ_p-cohomology. As an application, we provide a flexible construction of hyperbolic groups which do not have the Combinatorial Loewner Property, extending [Bou04] and complementing the examples from [BK]. Another consequence is the existence of hyperbolic groups with Sierpinski carpet boundary which have conformal dimension arbitrarily close to 1. In particular, we answer questions of Mario Bonk and John Mackay.

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1. Introduction

Background. In this paper we will be interested in quasi-isometry invariant structure in Gromov hyperbolic spaces, primarily structure which is reflected in the boundary. For some hyperbolic groups Γ, the topological structure of the boundary $\partial\Gamma$ alone contains substantial information: witness the JSJ decomposition encoded in the local cut point structure of the boundary [Bow98], and many situations where one can detect boundaries of certain subgroups $H \subset \Gamma$ by means of topological criteria. However, in many cases, for instance for generic hyperbolic groups, the topology reveals little of the structure of the group and is completely inadequate for addressing rigidity questions, since the homeomorphism group of the boundary is highly transitive. In these cases it is necessary to use the finer quasi-Mobius structure of the boundary and analytical invariants attached to it, such as modulus (Pansu, metric-measure, or combinatorial), ℓ_p-cohomology, and closely related quantities like the conformal dimension. The seminal work of Heinonen-Koskela [HK98], followed by [Che99, HKST01, Tys98, KZ08] indicates that when $\partial\Gamma$ is quasi-Mobius homeomorphic to a Loewner space (an Ahlfors Q-regular Q-Loewner space in the sense of [HK98]), there should be a rigidity theory resembling that of lattices in rank 1 Lie groups. This possibility is illustrated by [BP00, Xie06].

It remains unclear which hyperbolic groups Γ have Loewner boundary in the above sense. Conjecturally, $\partial\Gamma$ is Loewner if and only if it satisfies the Combinatorial Loewner Property [Kle06]. To provide some evidence of the abundance of such groups (modulo the conjecture), in [BK] we gave a variety of examples with the Combinatorial Loewner Property. On the other hand, it had already been shown in [BP03, Bou04] that there are groups Γ whose boundary is not Loewner, which can still be effectively studied using ℓ_p-cohomology and its cousins. Our main purpose in this paper is to advance the understanding of this complementary situation by providing new constructions of ℓ_p-cohomology classes, and giving a number of applications.

Setup. We will restrict our attention (at least in the introduction) to proper Gromov hyperbolic spaces which satisfy the following two additional conditions:

- (Bounded geometry) Every R-ball can be covered by at most $N = N(R, r)$ balls of radius $r \leq R$.
- (Nondegeneracy) There is a $C \in [0, \infty)$ such that every point x lies within distance at most C from all three sides of some ideal geodesic triangle Δ_x.

The visual boundary ∂X of such a space X is a compact, doubling, uniformly perfect metric space, which is determined up to quasi-Mobius homeomorphism by the quasi-isometry class of X. Conversely, every compact, doubling, uniformly perfect metric space is the visual boundary of a unique hyperbolic metric space as above, up to quasi-isometry (see Section 2).

To simplify the discussion of homological properties, we will impose (without loss of generality) the additional standing assumption that X is a simply connected metric simplicial complex with links of uniformly bounded complexity, and with all simplices isometric to regular Euclidean simplices with unit length edges.

Quasi-isometry invariant function spaces. Let X be a Gromov hyperbolic simplicial complex as above, with boundary ∂X.

We recall (see Section 3 and [Pan89, Gro93, Ele97, Bou04]) that for $p \in (1, \infty)$, the continuous (first) ℓ_p-cohomology $\ell_p H^1_{\text{cont}}(X)$ is canonically isomorphic to the space $A_p(\partial X)$ of continuous functions $u : \partial X \to \mathbb{R}$ which have a continuous extension $f : X^{(0)} \cup \partial X \to \mathbb{R}$ with p-summable coboundary:

$$\|df\|^p_{\ell_p} = \sum_{[vw] \in X^{(1)}} |f(v) - f(w)|^p < \infty \,.$$

Associated with the continuous ℓ_p-cohomology are several other quasi-isometry invariants:

(1) The ℓ_p-equivalence relation $\sim_p$ on ∂X, where $x \sim_p y$ iff $u(x) = u(y)$ for every $u \in A_p(\partial X)$.
(2) The infimal p such that $\ell_p H^1_{\text{cont}}(X) \simeq A_p(\partial X)$ is nontrivial. We will denote this by $p_{\neq 0}(X)$. Equivalently $p_{\neq 0}(X)$ is the infimal p such that $\sim_p$ has more than one coset.
(3) The infimum $p_{\text{sep}}(X)$ of the p such that $A_p(\partial X)$ separates points in ∂X, or equivalently, $p_{\text{sep}}(X)$ is the infimal p such that all cosets of $\sim_p$ are points.

These invariants were exploited in [BP03, Bou04, BK] due to their connection with conformal dimension and the Combinatorial Loewner Property. Specifically, when ∂X is approximately self-similar (e.g. if ∂X is the visual boundary of a hyperbolic group) then $p_{\text{sep}}(X)$ coincides with the Ahlfors regular conformal dimension of ∂X; and if ∂X has the Combinatorial Loewner Property then the two critical exponents $p_{\neq 0}(X)$ and $p_{\text{sep}}(X)$ coincide, i.e. for every $p \in (1, \infty)$, the function space $A_p(\partial X)$ separates points iff it is nontrivial (we refer the reader to Sections 2-3 for the precise statements and the relevant terminology).

Construction of nontrivial continuous ℓ_p-cohomology. The key results in this paper are constructions of nontrivial elements in the ℓ_p-cohomology. The general approach for the construction is inspired by [Bou04], and may be described as follows. Inside the Gromov hyperbolic complex X, we identify a subcomplex Y such that the relative cohomology of the pair $(X, X \setminus Y)$ reduces – essentially by excision – to the cohomology of Y relative to its frontier in X. Then we prove that the latter contains an abundance of nontrivial classes. This yields nontrivial classes in $\ell_p H^1_{\mathrm{cont}}(X)$ with additional control, allowing us to make deductions about the cosets of the ℓ_p-equivalence relation.

Let Y be a Gromov hyperbolic space satisfying our standing assumptions. We recall [Ele97] that if Y has Ahlfors Q-regular visual boundary ∂Y, then $A_p(\partial Y)$ contains the Lipschitz functions on ∂Y for any $p > Q$, and in particular it separates points. Our first result says that if $W \subset Y$ is a subcomplex with well-separated connected components, then for p slightly larger than Q, the relative cohomology $\ell_p H^1_{\mathrm{cont}}(Y, W)$ is highly nontrivial. In other words, at the price of increasing the exponent slightly, one can arrange for the representing functions $f : Y^{(0)} \to \mathbb{R}$ to be constant on the (0-skeleton of the) connected components of W, provided the components are far apart.

Theorem 1.1 (Corollary 4.6)**.** *Let Y be hyperbolic simply connected metric simplicial complex satisfying the assumptions above, and let* $\mathrm{Confdim}(\partial Y)$ *denotes the Ahlfors regular conformal dimension of ∂Y.*

For every $\alpha \in (0, 1)$, $C \geq 0$, there is a $D \geq 1$ with the following properties. Suppose $W \subset Y$ is a subcomplex of Y such that

1. *Every connected component of W is C-quasiconvex in Y.*
2. *For every $y \in W$ there is a complete geodesic $\gamma \subset W$ lying in the same connected component of W, such that* $\mathrm{dist}(y, \gamma) \leq C$.
3. *The distance between distinct components of W is at least D.*

Then for $p > \frac{1}{\alpha} \mathrm{Confdim}(\partial Y)$, there is an element of $\ell_p H^1_{\mathrm{cont}}(Y, W)$ represented by a function

$$f : Y^{(0)} \to \mathbb{R}$$

which takes distinct (constant) values in distinct components of W.

We prove Theorem 1.1 by translating it to an existence theorem for functions on ∂Y. It then reduces to the following theorem about Holder functions, which is of independent interest:

Theorem 1.2 (Theorem 4.1)**.** *For every $\alpha \in (0,1)$, there is a $D \in (4,\infty)$ with the following property. Suppose Z is a bounded metric space, and $\mathcal{C}$ is a countable collection of closed positive diameter subsets of Z such that the pairwise relative distance satisfies*

$$\Delta(C_1, C_2) = \frac{d(C_1, C_2)}{\min(\mathrm{diam}(C_1), \mathrm{diam}(C_2))} \geq D$$

for all $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$. Then there is a Holder function $u \in C^\alpha(Z)$ such that u is constant on every $C \in \mathcal{C}$, and takes distinct values on distinct elements of $\mathcal{C}$.

We remark that one can construct an example $Z, \mathcal{C}$ as in the theorem, where $Z = S^k$ and $\cup_{C\in\mathcal{C}} C$ has full Lebesgue measure. In this case, if $u : Z \to \mathbb{R}$ is a Lipschitz function which is constant on every $C \in \mathcal{C}$, then almost every point $z \in Z$ will be a point of differentiability of u and a point of density of some element $C \in \mathcal{C}$. Hence $Du = 0$ almost everywhere, and u is constant. This shows that it is necessary to take $\alpha < 1$.

Our second construction of ℓ_p-cohomology classes pertains to a special class of 2-complexes.

Definition 1.3. An *elementary polygonal complex* is a connected, simply connected, 2-dimensional cell complex Y whose edges are colored black or white, that enjoys the following properties:

- Y is a union of 2-cells which intersect pairwise in at most a single vertex or edge.
- Every 2-cell is combinatorially isomorphic to a polygon with even perimeter at least 6.
- The edges on the boundary of every 2-cell are alternately black and white.
- Every white edge has thickness one, and every black edge has thickness at least 2. Recall that the thickness of an edge is the number of 2-cells containing it.

The union of the white edges in Y is the *frontier* of Y, and is denoted $\mathcal{T}$; its connected components are trees.

A simple example of an elementary polygonal complex is the orbit $\Gamma(P)$ of a right-angled hexagon P in the hyperbolic plane, under the group Γ generated by reflections in 3 alternate sides $e_1, e_2, e_3 \subset \partial P$.

Elementary polygonal complexes are relevant for us because they turn up naturally as embedded subcomplexes $Y \subset X$, where X is a generic polygonal 2-complex, such that the pair $(Y, \mathcal{T})$ is the result of applying excision to the pair $(X, X \setminus Y)$.

For every elementary polygonal 2-complex Y, we define two invariants:

- $p_{\neq 0}(Y, \mathcal{T})$ is the infimum of the $p \in [1, \infty)$ such that $\ell_p H^1_{\mathrm{cont}}(Y, \mathcal{T})$ is nontrivial.
- $q_{\mathrm{sep}}(Y, \mathcal{T})$ is the infimum of the $p \in [1, \infty)$ such that $\ell_p H^1_{\mathrm{cont}}(Y, \mathcal{T})$ contains an element represented by a function $f : Y^{(1)} \to \mathbb{R}$ which takes distinct values on distinct components of $\mathcal{T}$.

Our main result about elementary polygonal complexes is the following pair of estimates on these invariants:

Theorem 1.4 (Corollary 6.6)**.** *Let Y be an elementary polygonal complex. Assume that the perimeter of every 2-cell of Y lies in $[2m_1, 2m_2]$ and that the thickness of every edge lies in $[k_1, k_2]$, with $m_1 \geq 3$ and $k_1 \geq 2$. Then:*

$$1 + \frac{\log(k_1 - 1)}{\log(m_2 - 1)} \leq p_{\neq 0}(Y, \mathcal{T}) \leq q_{sep}(Y, \mathcal{T}) \leq 1 + \frac{\log(k_2 - 1)}{\log(m_1 - 1)}.$$

Note that the estimate becomes sharp when the thickness and perimeter are constant.

Applications to spaces and groups. We now discuss some of applications given in the paper.

We first apply Theorem 1.1 to amalgams, generalizing the construction of [Bou04].

Theorem 1.5 (Corollary 5.3)**.** *Suppose A, B are hyperbolic groups, and we are given malnormal quasiconvex embeddings $C \hookrightarrow A$, $C \hookrightarrow B$. Suppose that there is a decreasing sequence $\{A_n\}_{n \in \mathbb{N}}$ of finite index subgroups of A such that $\cap_{n \in \mathbb{N}} A_n = C$, and set $\Gamma_n := A_n \star_C B$.*

1. *If $p_{sep}(A) < p_{sep}(B)$ then, for all $p \in (p_{sep}(A), p_{sep}(B)]$ and every n large enough, the ℓ_p-equivalence relation on $\partial\Gamma_n$ possesses a coset different from a point and the whole $\partial\Gamma$. In particular for n large enough, $\partial\Gamma_n$ does not admit the CLP.*
2. *If $p_{sep}(A) < p_{\neq 0}(B)$ then, for $p \in (p_{sep}(A), p_{\neq 0}(B))$ and every n large enough, the cosets of the ℓ_p-equivalence relation on $\partial\Gamma_n$ are single points and the boundaries of cosets gB, for $g \in \Gamma_n$.*

In particular, for large n, any quasi-isometry of Γ_n permutes the cosets gB, for $g \in \Gamma_n$.

We now sketch the proof of the theorem.

Let K_A, K_B, and K_C be finite 2-complexes with respective fundamental groups A, B, and C, such that there are simplicial embeddings $K_C \hookrightarrow K_A$, $K_C \hookrightarrow K_B$ inducing the given embeddings of fundamental groups. For every n, let $K_A^n \to K_A$ be the finite covering corresponding to the inclusion $A_n \subset A$, and fix a lift $K_C \hookrightarrow K_A^n$ of the embedding $K_C \hookrightarrow K_A$. We let K^n be the result of gluing K_A^n to K_B along the copies of K_C, so $\pi_1(K^n) \simeq A_n \star_C B = \Gamma_n$. The universal cover $\tilde{K}^n$ is a union of copies of the universal covers $\tilde{K}_A^n = \tilde{K}_A$ and $\tilde{K}_B$, whose incidence graph is the Bass-Serre tree of the decomposition $\Gamma_n = A_n \star_C B$. If we choose a copy $Y \subset \tilde{K}^n$ of $\tilde{K}_A$, then the frontier W_n of Y in $\tilde{K}^n$ breaks up into connected components which are stabilized by conjugates of C, where the minimal pairwise separation between distinct components of W_n tends to infinity as $n \to \infty$. Theorem 1.1 then applies, yielding nontrivial functions in $\ell_p H^1_{\text{cont}}(Y, W_n)$ for every $p > \text{Confdim}(A)$ and every n sufficiently large; by excision these give functions in $\ell_p H^1_{\text{cont}}(\tilde{K}^n) \simeq A_p(\partial\Gamma_n)$ which are constant on $\partial Z \subset \partial\Gamma_n$ for every copy $Z \subset \tilde{K}^n$ of $\tilde{K}_B$ in $\tilde{K}^n$. These functions provide enough information about the ℓ_p-equivalence relation $\sim_p$ to deduce (1) and (2).

As a corollary, we obtain examples of hyperbolic groups with Sierpinski carpet boundary which do not have the Combinatorial Loewner Property, and which are quasi-isometrically rigid. See Example 5.4. These examples answer a question of Mario Bonk.

Using our estimates for the critical exponents for elementary polygonal complexes, we obtain upper bounds for the Ahlfors regular conformal dimension of boundaries of a large class of 2-complexes.

Theorem 1.6 (Proposition 7.1)**.** *Let X be a simply connected hyperbolic 2-complex whose boundary is connected and approximately self-similar. Assume that X is a union of 2-cells, where 2-cells intersect pairwise in at most a vertex or edge.*

(1) *If the perimeter of every 2-cell is at least $n \geq 5$, the thickness of every edge lies in $[2, k]$, and the link of every vertex contains no circuit of length 3, then*

$$\text{Confdim}(\partial X) \leq 1 + \frac{\log(k-1)}{\log(n-3)}.$$

(2) *If the perimeter of every* 2*-cell is at least* $n \geq 7$ *and the thickness of every edge lies in* $[2, k]$*, then*

$$\mathrm{Confdim}(\partial X) \leq 1 + \frac{\log(k-1)}{\log(n-5)}.$$

To prove Theorem 1.6(1), by a straightforward consequence of [KK, Car], it suffices to show that the function space $A_p(\partial X)$ separates points in ∂X for $p > 1 + \frac{\log(k-1)}{\log(n-3)}$. To do this, we find that (a subdivision of) X contains lots of elementary polygonal complexes Y with thickness at most k and perimeter at least $2(n-2)$, such that excision applied to $(X, X \backslash Y)$ yields the pair $(Y, \mathcal{T})$. Theorem 1.4 then produces enough elements in $A_p(\partial X)$ to separate points.

As an application of Theorem 1.6 we obtain, for every $\epsilon > 0$, a hyperbolic group Γ with Sierpinski carpet boundary with the Combinatorial Loewner Property, such that $\mathrm{Confdim}(\partial\Gamma) < 1 + \epsilon$. See Example 7.2. This answers a question of John Mackay.

2. Preliminaries

This section is a brief presentation of some topics in geometric analysis that will be useful in the sequel. This includes the Ahlfors regular conformal dimension, the Combinatorial Loewner Property, approximately self-similar spaces, and few aspects of Gromov hyperbolic spaces.

Ahlfors regular conformal dimension. We refer to [MT10] for a detailed treatment of the conformal dimension and related subjects.

Recall that a metric space Z is called a *doubling metric space* if there is a constant $n \in \mathbb{N}$ such that every ball B can be covered by at most n balls of radius $\frac{r(B)}{2}$.

A space Z is *uniformly perfect* , if there exists a constant $0 < \lambda < 1$ such that for every ball $B(z, r)$ of Z with $0 < r \leq \operatorname{diam} Z$ one has $B(z, r) \setminus B(z, \lambda r) \neq \emptyset$.

A space Z is *Ahlfors Q-regular* (for some $Q \in (0, +\infty)$) if there is a measure μ on Z such that for every ball $B \subset (Z, d)$ of radius $0 < r \leq \mathrm{diam}(Z)$ one has $\mu(B) \asymp r^Q$.

Every compact, doubling, uniformly perfect metric space is quasi-Moebius homeomorphic to a Ahlfors regular meric space (see [Hei01]). This justifies the following definition.

Definition 2.1. Let Z be a compact, doubling, uniformly perfect metric space. The *Ahlfors regular conformal dimension* of Z is the infimum of the Hausdorff dimensions of the Ahlfors regular metric spaces which are quasi-Moebius homeomorphic to Z. We shall denote it by $\mathrm{Confdim}(Z)$.

The Combinatorial Loewner Property (CLP). The Combinatorial Loewner Property was introduced in [Kle06, BK]. We start with some basic related notions.

Let Z be a compact metric space, let $k \in \mathbb{N}$, and let $\kappa \geq 1$. A finite graph G_k is called a *κ-approximation of Z on scale k*, if it is the incidence graph of a covering of Z, such that for every $v \in G_k^0$ there exists $z_v \in Z$ with

$$B(z_v, \kappa^{-1}2^{-k}) \subset v \subset B(z_v, \kappa 2^{-k}),$$

and for $v, w \in G_k^0$ with $v \neq w$:

$$B(z_v, \kappa^{-1}2^{-k}) \cap B(z_w, \kappa^{-1}2^{-k}) = \emptyset.$$

Note that we identify every vertex v of G_k with the corresponding subset in Z. A collection of graphs $\{G_k\}_{k\in\mathbb{N}}$ is called a *κ-approximation* of Z, if for each $k \in \mathbb{N}$ the graph G_k is a κ-approximation of Z on scale k.

Let $\gamma \subset Z$ be a curve and let $\rho : G_k^0 \to \mathbb{R}_+$ be any function. The *ρ-length* of γ is

$$L_\rho(\gamma) = \sum_{v\cap\gamma\neq\emptyset} \rho(v).$$

For $p \geq 1$ the *p-mass* of ρ is

$$M_p(\rho) = \sum_{v\in G_k^0} \rho(v)^p.$$

Let $\mathcal{F}$ be a non-void family of curves in Z. We define the *G_k-combinatorial p-modulus* by

$$\mathrm{Mod}_p(\mathcal{F}, G_k) = \inf_\rho M_p(\rho),$$

where the infimum is over all *$\mathcal{F}$-admissible functions i.e.* functions $\rho : G_k^0 \to \mathbb{R}_+$ which satisfy $L_\rho(\gamma) \geq 1$ for every $\gamma \in \mathcal{F}$.

We denote by $\mathcal{F}(A, B)$ the family of curves joining two subsets A and B of Z and by $\mathrm{Mod}_p(A, B, G_k)$ its G_k-combinatorial p-modulus.

Suppose now that Z is a compact arcwise connected doubling metric space. Let $\{G_k\}_{k\in\mathbb{N}}$ be a κ-approximation of Z. In the following statement $\Delta(A,B)$ denotes the *relative distance* between two disjoint non degenerate continua $A, B \subset Z$ *i.e.*

$$\Delta(A,B) = \frac{\operatorname{dist}(A,B)}{\min\{\operatorname{diam} A, \operatorname{diam} B\}}. \tag{2.2}$$

Definition 2.3. Suppose $p > 1$. Then Z satisfies the *Combinatorial p-Loewner Property* if there exist two positive increasing functions ϕ, ψ on $(0, +\infty)$ with $\lim_{t\to 0} \psi(t) = 0$, such that for all disjoint non-degenerate continua $A, B \subset Z$ and for all k with $2^{-k} \leq \min\{\operatorname{diam} A, \operatorname{diam} B\}$ one has :

$$\phi(\Delta(A,B)^{-1}) \leq \operatorname{Mod}_p(A,B,G_k) \leq \psi(\Delta(A,B)^{-1}).$$

We say that Z satisfies the *Combinatorial Loewner Property* if it satisfies the Combinatorial p-Loewner Property for some $p > 1$.

The CLP is invariant under quasi-Moebius homeomorphisms, and a compact Ahlfors p-regular, p-Loewner metric space, satisfies the Combinatorial p-Loewner Property (see [BK] Th.2.6). It is conjectured in [Kle06] that if Z satisfies the CLP and is approximately self-similar (see below for the definition), then Z is quasi-Moebius homeomorphic to a regular Loewner space.

Approximately self-similar spaces. The following definition appears in [Kle06], [BK].

Definition 2.4. A compact metric space (Z,d) is called *approximately self-similar* if there is a constant $L_0 \geq 1$ such that if $B(z,r) \subset Z$ is a ball of radius $0 < r \leq \operatorname{diam}(Z)$, then there is an open subset $U \subset Z$ which is L_0-bi-Lipschitz homeomorphic to the rescaled ball $(B(z,r), \frac{1}{r}d)$.

Observe that approximately self-similar metric spaces are doubling and uniformly perfect. Examples include some classical fractal spaces like the square Sierpinski carpet and the cubical Menger sponge. Other examples are the visual boundaries of the hyperbolic spaces which admit an isometric properly discontinuous and cocompact group action [BK]. A further source of examples comes from expanding Thurston maps, [BM], [HP09].

The following result is due to S. Keith and the second (named) author [KK]. A proof is written in [Car].

Theorem 2.5. *Suppose Z is an arcwise connected, approximately self-similar metric space. Let $\{G_k\}_{k\in\mathbb{N}}$ be a κ-approximation of Z. Pick a positive constant d_0 that is small compared to the diameter of Z and to* *the constant L_0 of Definition 2.4. Denote by $\mathcal{F}_0$ the family of curves $\gamma \subset Z$ with* $\operatorname{diam}(\gamma) \geq d_0$. *Then*

$$\operatorname{Confdim}(Z) = \inf\{p \in [1, +\infty) \mid \lim_{k \to +\infty} \operatorname{Mod}_p(\mathcal{F}_0, G_k) = 0\}.$$

Hyperbolic spaces. Let X be a hyperbolic proper geodesic metric space and denote by ∂X its boundary at infinity. It carries a *visual metric*, *i.e.* a metric d for which there are constants $a > 1$, $C \geq 1$ such that for every $z, z' \in \partial X$, one has

$$C^{-1} a^{-L} \leq d(z, z') \leq C a^{-L}, \tag{2.6}$$

where L denotes the distance from x_0 (an origin in X) to a geodesic $(z, z') \subset X$. Moreover $X \cup \partial X$ is naturally a metric compactification of X. See *e.g.* [Gro87, BH99, KB02] for more details.

When X satifies the bounded geometry and nondegeneracy conditions of the introduction, then ∂X is a doubling uniformly perfect metric space.

We also notice that – thanks to Rip's construction (see [Gro87, BH99, KB02]) – every proper bounded geometry hyperbolic space is quasi-isometric to a contractible simplicial metric complex, with links of uniformly bounded complexity, and all simplices isometric to regular Euclidean simplices with unit length edges.

3. ℓ_p-COHOMOLOGY

This section presents aspects of the ℓ_p-cohomology that will serve in the sequel. Only the first ℓ_p-cohomology will play a role. Therefore, instead of considering contractible simplicial complexes, we will content ourself with simply connected ones.

We consider in this section a simply connected metric simplicial complex X with links of uniformly bounded complexity, and all simplices isometric to regular Euclidean simplices with unit length edges. We suppose that it is a hyperbolic metric space, and we denote its visual boundary by ∂X.

First ℓ_p-Cohomology. For $k \in \mathbb{N}$ denote by $X^{(k)}$ the set of the k-simplices of X. For a countable set E and for $p \in [1, \infty)$, let $\ell_p(E)$ be the Banach space of p-summable real functions on E. The *k-th space of ℓ_p-cochains* is $C_p^{(k)}(X) := \ell_p(X^{(k)})$. The standard coboundary operator

$$d^{(k)} : C_p^{(k)}(X) \to C_p^{(k+1)}$$

is bounded because of the bounded geometry assumption on X. When $k = 0$, the operator $d^{(0)}$ is simply the restriction to $\ell_p(X^{(0)})$ of the differential operator d defined for every $f : X^{(0)} \to \mathbb{R}$ by

$$\forall a = (a_-, a_+) \in X^{(1)},\ df(a) = f(a_+) - f(a_-),$$

The *k-th ℓ_p-cohomology group* of X is

$$\ell_p H^k(X) = \ker d^{(k)} / \mathrm{Im}\ d^{(k-1)}.$$

Since X is simply connected, every 1-cocycle on X is the differential of a function $f : X^{(0)} \to \mathbb{R}$ which is unique up to an additive constant. Therefore in degree 1 we get a canonical isomorphism

$$\ell_p H^1(X) \simeq \{f : X^{(0)} \to \mathbb{R}\ ;\ df \in \ell_p(X^{(1)})\}/\ell_p(X^{(0)}) + \mathbb{R},$$

where $\mathbb{R}$ denotes the set of constant functions on $X^{(0)}$. In this paper we shall always represent $\ell_p H^1(X)$ via this isomorphism.

Equipped with the semi-norm induced by the ℓ_p-norm of df the topological vector space $\ell_p H^1(X)$ is a quasi-isometric invariant of X. Moreover if X satisfies a linear isoperimetric inequality, then $\ell_p H^1(X)$ is a Banach space, and $\ell_p H^1(X)$ injects in $\ell_q H^1(X)$ for $1 < p \leq q < +\infty$. See [Gro93], [Bou] for a proof of these results.

The *continuous first ℓ_p-cohomology group* of X is

$$\ell_p H^1_{\mathrm{cont}}(X) := \{[f] \in \ell_p H^1(X)\ ;\ f \text{ extends continuously to } X^{(0)} \cup \partial X\},$$

where $X^{(0)} \cup \partial X$ is the metric compactification of $X^{(0)}$.

Following P. Pansu we introduce the following quasi-isometric numerical invariant of X :

$$p_{\neq 0}(X) = \inf\{p \geq 1\ ;\ \ell_p H^1_{\mathrm{cont}}(X) \neq 0\}.$$

ℓ_p-Equivalence relations. For $[f] \in \ell_p H^1_{\mathrm{cont}}(X)$ denote by $f_\infty : \partial X \to \mathbb{R}$ its boundary extension. Following M. Gromov ([Gro93] p. 259, see also [Ele97], [Bou04]) we set

$$A_p(\partial X) := \{u : \partial X \to \mathbb{R}\ ;\ u = f_\infty \text{ with } [f] \in \ell_p H^1_{\mathrm{cont}}(X)\},$$

and we define the *ℓ_p-equivalence relation* on ∂X by :

$$x \sim_p y \Longleftrightarrow \forall u \in A_p(\partial X),\ u(x) = u(y).$$

This is a closed equivalence relation on ∂X which is invariant by the boundary extensions of the quasi-isometries of X. We set :

$$p_{\mathrm{sep}}(X) = \inf\{p \geq 1\ ;\ A_p(\partial X) \text{ separates points in } \partial X\}.$$

Equivalently $p_{\mathrm{sep}}(X)$ is the infimal p such that all cosets of $\sim_p$ are points.

Recall (from the introduction) that X is said *nondegenerate* if every $x \in X$ lies within uniformly bounded distance from all three sides of some ideal geodesic triangle. For nondegenerate spaces X every class $[f] \in \ell_p H^1_{\mathrm{cont}}(X)$ is fully determined by its boundary value f_∞ ([Str83], see also [Pan89], [BP03] Th.3.1). More precisely $[f] = 0$ if and only if f_∞ is constant. In particular we get :

Proposition 3.1. *Suppose X is nondegenerate. Then*

$$p_{\neq 0}(X) = \inf\{p \geq 1 \ ; \ (\partial X / \sim_p) \text{ is not a singleton}\}.$$

We also notice that under the assumption of the above proposition the ℓ_p-cosets are always connected ([BK] Prop. 10.1).

Relative ℓ_p-Cohomology. Let Y be a subcomplex of X. The *k-th space of relative ℓ_p-cochains of* (X, Y) is

$$C_p^{(k)}(X, Y) := \{\omega \in C_p^{(k)}(X) \ ; \ \omega_{|Y^{(k)}} = 0\}.$$

The *k-th relative ℓ_p-cohomology group* of (X, Y) is

$$\ell_p H^k(X, Y) = \frac{\ker(d^{(k)} : C_p^{(k)}(X, Y) \to C_p^{(k+1)}(X, Y))}{\mathrm{Im}(d^{(k-1)} : C_p^{(k-1)}(X, Y) \to C_p^{(k)}(X, Y))}.$$

An immediate property is the following excision principle :

Proposition 3.2. *Suppose $U \subset Y$ is a subset such that $Y \setminus U$ is a subcomplex of Y. Then for every $k \in \mathbb{N}$ the restriction map induces a canonical isomorphism*

$$\ell_p H^k(X, Y) \simeq \ell_p H^k(X \setminus U, Y \setminus U).$$

Since X is simply connected, by integrating every relative 1-cocycle, we obtain the following canonical isomorphism :

$$\begin{aligned} \ell_p H^1(X, Y) \ \simeq \ \{ & f : X^{(0)} \to \mathbb{R} \ ; \ df \in \ell_p(X^{(1)}) \text{ and } f_{|E^{(0)}} \text{ is constant} \\ & \text{on every connected component } E \text{ of } Y\}/\sim, \end{aligned}$$

where $f \sim g$ if and only if $f - g$ belongs to $\ell_p(X^{(0)}) + \mathbb{R}$. We will always represent $\ell_p H^1(X, Y)$ via this isomorphism.

We denote by $\ell_p H^1_{\mathrm{cont}}(X, Y)$ the subspace of $\ell_p H^1(X, Y)$ consisting of the classes $[f]$ such that f extends continuously to $X^{(0)} \cup \partial X$. We introduce two numerical invariants :

$$\begin{aligned} p_{\neq 0}(X, Y) &= \inf\{p \geq 1 \ ; \ \ell_p H^1_{\mathrm{cont}}(X, Y) \neq 0\} \\ q_{\mathrm{sep}}(X, Y) &= \inf\{p \geq 1 \ ; \ \ell_p H^1_{\mathrm{cont}}(X, Y) \text{ separates any pair} \\ & \qquad \text{of distinct connected components of } Y\}. \end{aligned}$$

In case one of the sets on the right hand side is empty, we just declare the corresponding invariant to be equal to $+\infty$. Obviously one has

$$p_{\neq 0}(X) \leq p_{\neq 0}(X,Y) \leq q_{\text{sep}}(X,Y).$$

A construction of cohomology classes. One of the goals of the paper is to construct ℓ_p-cohomology classes of X. We present here a construction that uses the relative cohomology of special subcomplexes of X.

Definition 3.3. A quasi-convex subcomplex $Y \subset X$ *decomposes* X, if it is simply connected and if the family $\{E_i\}_{i\in I}$ of the connected components of $\overline{X \setminus Y}$ is infinite and satisfies :

$$\operatorname{diam}(E_i \cup \partial E_i) \to 0 \text{ in } X \cup \partial X, \text{ when } i \to \infty. \tag{3.4}$$

A collection $\{Y_j\}_{j\in J}$ of quasi-convex subcomplexes of X *fully decomposes* X if every Y_j decomposes X, and if for every pair of distinct points $z_1, z_2 \in \partial X$, there is an $Y \in \{Y_j\}_{j\in J}$ whose associated connected component family contains distinct elements E_1, E_2, with $z_1 \in \partial E_1$ and $z_2 \in \partial E_2$.

The origin of Definition 3.3 lies in group amalgams, as illustrated by the following example.

Example 3.5. Let A, B, C three hyperbolic groups, suppose that A and B are non elementary and that C is a proper quasi-convex malnormal subgroup of A and B. Then the amalgamated product $\Gamma := A \star_C B$ is a hyperbolic group [Kap97]. Let K_A, K_B, and K_C be finite 2-complexes with respective fundamental groups A, B, and C, such that there are simplicial embeddings $K_C \hookrightarrow K_A$, $K_C \hookrightarrow K_B$ inducing the given embeddings of fundamental groups. We let K be the result of gluing K_A to K_B along the copies of K_C, so $\pi_1(K) \simeq A \star_C B = \Gamma$. The universal cover $\tilde{K}$ is a union of copies of the universal covers $\tilde{K}_A = \tilde{K}_A$ and $\tilde{K}_B$, whose incidence graph is the Bass-Serre tree of the decomposition $\Gamma = A \star_C B$. If we choose a copy $Y \subset \tilde{K}$ of $\tilde{K}_A$, then Y decomposes $\tilde{K}$. The frontier of Y in $\tilde{K}$ breaks up into connected components which are stabilized by conjugates of C.

The following result will serve repeatedly in the sequel.

Proposition 3.6. *Suppose that a quasi-convex subcomplex Y decomposes X and let $W := Y \setminus \operatorname{int}(Y)$ be its frontier in X. Then the restriction map*

$$\ell_p H^1_{cont}(X, \overline{X \setminus Y}) \to \ell_p H^1_{cont}(Y, W), \ [f] \mapsto [f_{|Y^{(0)}}],$$

is an isomorphism. In particular we have $p_{\neq 0}(X, \overline{X \setminus Y}) = p_{\neq 0}(Y, W)$ *and* $q_{sep}(X, \overline{X \setminus Y}) = q_{sep}(Y, W)$.

Proof. We apply the excision property (Proposition 3.2) to the complexes X, $\overline{X \setminus Y}$ and to the subset $U = X \setminus Y$. Since

$$(\overline{X \setminus Y}) \setminus U = (\overline{X \setminus Y}) \cap Y = Y \setminus \mathrm{int}(Y) = W,$$

we obtain that $\ell_p H^1(X, \overline{X \setminus Y})$ is isomorphic to $\ell_p H^1(Y, W)$. The complexes X and Y are simply connected, thus, via the canonical representation, the isomorphism simply writes $[f] \mapsto [f_{|Y^{(0)}}]$.

It remains to prove that if $f_{|Y^{(0)}}$ extends continuously to $Y^{(0)} \cup \partial Y$, then f extends continuously to $X^{(0)} \cup \partial X$. Let $\{E_i\}_{i \in I}$ be the connected component family of $\overline{X \setminus Y}$. Consider a sequence $\{x_n\}_{n \in \mathbb{N}} \subset X^{(0)}$ that converges in $X^{(0)} \cup \partial X$ to a point $x_\infty \in \partial X$. We wish to prove that $\{f(x_n)\}_{n \in \mathbb{N}}$ is a convergent sequence. We distinguish several cases.

If x_∞ belongs to $\partial X \setminus \partial Y$, then for n and m large enough the geodesic segments $[x_n, x_m]$ do not intersect Y. Therefore their vertices are all contained in the same E_i. Thus $f(x_n)$ is constant for n large enough.

Suppose now that $x_\infty \in \partial Y$. Denote by $(f_{|Y^{(0)}})_\infty$ the boundary extension of $f_{|Y^{(0)}}$ to ∂Y. We will prove that $f(x_n) \to (f_{|Y^{(0)}})_\infty(x_\infty)$. By taking a subsequence if necessary, it is enough to consider the following special cases :

(A) For every $n \in \mathbb{N}$, x_n belongs to $Y^{(0)}$,
(B) There is an $i \in I$ such that for every $n \in \mathbb{N}$, x_n belongs to E_i,
(C) There is a sequence $\{i_n\}_{n \in \mathbb{N}}$ of distinct elements of I such that $x_n \in E_{i_n}$.

Case (A) is obvious. In case (B), let $x'_n \in Y^{(0)}$ be a nearest point projection of x_n on $Y^{(0)}$. The geodesic segment $[x_n, x'_n]$ has no interior vertex in Y, and so it is contained in E_i. Thus $x'_n \in (E_i \cap Y^{(0)})$. Moreover $x'_n \to x_\infty$ when $n \to \infty$. Therefore we obtain :

$$f(x_n) = f(x'_n) = f_{|Y^{(0)}}(x'_n) \to (f_{|Y^{(0)}})_\infty(x_\infty).$$

In case (C), it follows from property (3.4) in Definition 3.3, that the subset E_{i_n} tends to the singleton $\{x_\infty\}$ in $X \cup \partial X$. For $n \in \mathbb{N}$ pick $y_n \in E_n \cap Y^{(0)}$. Then the sequence $\{y_n\}_{n \in \mathbb{N}}$ tends to x_∞. This leads to $f(x_n) = f_{|Y^{(0)}}(y_n) \to (f_{|Y^{(0)}})_\infty(x_\infty)$. $\square$

Corollary 3.7. (1) *Suppose that a subcomplex Y decomposes X, and let W be its frontier. Then one has :*

$$p_{\neq 0}(X) \leq p_{\neq 0}(Y, W).$$

(2) *Suppose that a subcomplex collection $\{Y_j\}_{j\in J}$ fully decomposes X, and let W_j be the frontier of Y_j. Then one has :*

$$p_{sep}(X) \leq \sup_{j\in J} q_{sep}(Y_j, W_j).$$

Proof of Corollary 3.7. (1). From a standard inequality and Proposition 3.6 one has

$$p_{\neq 0}(X) \leq p_{\neq 0}(X, \overline{X \setminus Y}) = p_{\neq 0}(Y, W).$$

(2). Assume the supremum is finite and let $p > \sup_{j\in J} q_{\text{sep}}(Y_j, W_j)$. Pick two distinct points $z_1, z_2 \in \partial X$. By definition there is an $Y \in \{Y_j\}_{j\in J}$ whose associated connected component family contains distinct elements E_1, E_2, with $z_1 \in \partial E_1$ and $z_2 \in \partial E_2$. Let W be the frontier of Y. From the choice of p there is a function $\varphi : Y^{(0)} \to \mathbb{R}$ with $[\varphi] \in \ell_p H^1_{\text{cont}}(Y, W)$, which takes different values on $E_1 \cap Y^{(0)}$ and $E_2 \cap Y^{(0)}$. By Proposition 3.6 there is an $[f] \in \ell_p H^1_{\text{cont}}(X, \overline{X \setminus Y})$ such that $[\varphi] = [f_{|Y^{(0)}}]$. Thus f_∞ belongs to $A_p(\partial X)$ and separates z_1 and z_2. □

Cohomology and the geometry of the boundary. The following result relates the ℓ_p-cohomology of X with the structure of the boundary ∂X, more precisely with the Ahlfors regular conformal dimension and the Combinatorial Loewner Property (see Section 2 for the definitions). It will serve as a main tool in the paper.

Theorem 3.8. *Assume that X is non-degenerate and that ∂X is connected and approximately self-similar, let $p \geq 1$. Then :*

(1) $p > \text{Confdim}(\partial X)$ *if and only if* $(\partial X/\sim_p) = \partial X$*; in particular*

$$p_{sep}(X) = \text{Confdim}(\partial X),$$

(2) *If ∂X satisfies the CLP, then for* $1 \leq p \leq \text{Confdim}(\partial X)$ *the quotient $\partial X/\sim_p$ is a singleton; in particular*

$$p_{\neq 0}(X) = p_{sep}(X) = \text{Confdim}(\partial X).$$

Proof. It follows immediately from Theorem 2.5 in combination with Cor.10.5 in [BK] and Proposition 3.1. □

4. A qualitative bound for $q_{\text{SEP}}(Y, W)$

In this section we make a qualitative connection between the relative invariant $q_{\text{sep}}(Y, W)$ defined in Section 3 and certain geometric properties of the pair (Y, W) (see Corollary 4.6). This relies on the following result which is also of independent interest. In the statement $\Delta(\cdot, \cdot)$ denotes the relative distance, see (2.2) for the definition.

Theorem 4.1. *For every $\alpha \in (0, 1)$ there is a $D \in (4, \infty)$ with the following property. Suppose Z is a bounded metric space, and $\mathcal{C}$ is a countable collection of closed positive diameter subsets of Z where $\Delta(C_1, C_2) \geq D$ for all $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$. Then there is a Holder function $u \in C^\alpha(Z)$ such that :*

(1) *$u\big|_C$ is constant for every $C \in \mathcal{C}$,*
(2) *If $C_1, C_2 \in \mathcal{C}$ and $u(C_1) = u(C_2)$, then $C_1 = C_2$.*

Remarks. 1) The countability of $\mathcal{C}$ is only used to obtain (2); one gets plenty of functions without countability.

2) The argument given in the proof of Prop.1.3 of [Bou04] shows that there is a function $\alpha : (0, \infty) \to (0, 1)$ with the following property. Let Z be bounded metric space, and let $\mathcal{C}$ be a countable collection of closed positive diameter subsets of Z with $\Delta(C_1, C_2) \geq D > 0$ for all $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$. Then for $\alpha = \alpha(D)$ there is a Holder function $u \in C^\alpha(Z)$ which satisfies items (1) and (2) above. Theorem 4.1 above asserts that we can choose the function α so that $\alpha(D) \to 1$ when $D \to \infty$.

To prove the theorem, we can assume that $\operatorname{diam}(Z) \leq 1$ by rescaling the metric of Z if necessary. Pick $\Lambda \in [4, D)$. Let $r_k = 2^{-k}$, and let $\mathcal{C}_k = \{C \in \mathcal{C} \ ; \ \operatorname{diam}(C) \in [r_{k+1}, r_k]\}$. Since $\operatorname{diam}(Z) \leq 1$ we have $\mathcal{C} = \cup_{k \geq 0} \mathcal{C}_k$. Given a Lipschitz function $v_{-1} : Z \to \mathbb{R}$, we will construct the Holder function u as a convergent series $\sum_{j=-1}^{\infty} v_j$ where for every $k \geq 0$:

- v_k is Lipschitz.
- v_k is supported in $\cup_{C \in \mathcal{C}_k} N_{\Lambda r_k}(C)$.
- For every $0 \leq j \leq k$ and $C \in \mathcal{C}_j$, the partial sum $u_k = \sum_{j=-1}^{k} v_j$ is constant on C.

Thus one may think of v_k as a "correction" which adjusts u_{k-1} so that it becomes constant on elements of $\mathcal{C}_k$.

Lemma 4.2. *Suppose $0 \leq i, j < k$, $i \neq j$, and there are $C_1 \in \mathcal{C}_i$, $C_2 \in \mathcal{C}_j$ such that*

$$\operatorname{dist}(N_{\Lambda r_i}(C_1), N_{\Lambda r_j}(C_2)) \leq r_k\,.$$

Then $|i - j| \geq \log_2(\frac{D}{6\Lambda})$.

Proof. We may assume $i < j$. We have

$$D \leq \Delta(C_1, C_2) \leq \frac{\operatorname{dist}(C_1, C_2)}{\left(\frac{r_j}{2}\right)} \implies \operatorname{dist}(C_1, C_2) \geq \frac{Dr_j}{2}$$

and

$$\operatorname{dist}(C_1, C_2) \leq \Lambda r_i + \Lambda r_j + r_k \leq 3\Lambda r_i$$

so

$$\frac{r_i}{r_j} \geq \frac{D}{6\Lambda} \implies j - i \geq \log_2\left(\frac{D}{6\Lambda}\right).$$

□

Let n be the integer part of $\log_2(\frac{D}{6\Lambda})$; we will assume that $n \geq 1$.

Let $\{L_j\}_{j\in\mathbb{Z}} \subset [0, \infty)$ be an increasing sequence such that $L_j = 0$ for all $j \leq -2$, and let $\hat{L}_k = \sum_{j=1}^{\infty} L_{k-jn}$ (this is a finite sum since $L_j = 0$ for $j \leq -2$).

Definition 4.3. The sequence $\{L_j\}$ is *feasible* if $L_k \geq \hat{L}_k$ for all $k \geq 0$.

As an example let $L_j = e^{\lambda j}$ for $j \geq -1$, and $L_j = 0$ for $j \leq -2$. Then $\{L_j\}$ is feasible if $e^{-\lambda n} < \frac{1}{2}$. In particular, we may take λ small when n is large.

Lemma 4.4. *Suppose $\{L_j\}$ is feasible, and $v_{-1} : Z \to \mathbb{R}$ is L_{-1}-Lipschitz. Then there is a sequence $\{v_k\}$, where for every $k \geq 0$:*

(1) *v_k is L_k-Lipschitz.*
(2) $\operatorname{Spt}(v_k) \subset N_{\Lambda r_k}(\cup_{C\in\mathcal{C}_k} C)$.
(3) $\|v_k\|_{C^0} \leq 2L_k r_k$.
(4) *For every $0 \leq j \leq k$, and every $C \in \mathcal{C}_j$, the partial sum $u_k = \sum_{j=-1}^{k} v_j$ is constant on C.*

Proof. Assume inductively that for some $k \geq 0$, there exist functions $v_{-1}, \ldots, v_{k-1}$ satisfying the conditions of the lemma. For every $C \in \mathcal{C}_k$, we would like to specify the constant value of the function u_k. To that end, choose a point $p_C \in C$ and some $u_C \in [u_{k-1}(p_C) - \hat{L}_k r_k, u_{k-1}(p_C) + \hat{L}_k r_k]$. Let

$$W_k = \left(Z \setminus (\cup_{C\in\mathcal{C}_k} N_{\Lambda r_k}(C))\right) \sqcup \left(\cup_{C\in\mathcal{C}_k} C\right),$$

and define $\bar{v}_k : W_k \to \mathbb{R}$ by

$$\bar{v}_k(x) = \begin{cases} 0 & x \in Z \setminus \cup_{C \in \mathcal{C}_k} N_{\Lambda r_k}(C) \\ u_C - u_{k-1}(x) & x \in C \in \mathcal{C}_k \ . \end{cases}$$

Sublemma 4.5. *One has :*

$$\mathrm{Lip}(\bar{v}_k) \leq \sum_{j=1}^{\infty} L_{k-jn} = \hat{L}_k \, .$$

Proof of the sublemma. Pick $x, y \in W_k$.

Case 1. Suppose $x, y \in C$ for some $C \in \mathcal{C}_k$. Then

$$\begin{aligned} \frac{|\bar{v}_k(x) - \bar{v}_k(y)|}{d(x,y)} &= \frac{|u_{k-1}(x) - u_{k-1}(y)|}{d(x,y)} \\ &\leq \sum \{\mathrm{Lip}(v_j) \mid j < k, \ \mathrm{Spt}(v_j) \cap C \neq \emptyset\} \\ &\leq \sum \{L_j \mid j < k, \ \mathrm{Spt}(v_j) \cap C \neq \emptyset\} \\ &\leq \sum \{L_j \mid j < k, \ \exists C_1 \in \mathcal{C}_j \text{ s.t. } N_{\Lambda r_j}(C_1) \cap C \neq \emptyset\} \\ &\leq \sum_{j \geq 1} L_{k-jn} = \hat{L}_k \, , \end{aligned}$$

since the sequence L_j is increasing, and consecutive elements of the set $\{j \ ; \ j < k, \ \exists C_1 \in \mathcal{C}_j \text{ s.t. } N_{\Lambda r_j}(C_1) \cap C \neq \emptyset\}$ differ by at least n, by Lemma 4.2.

Case 2. There is a $C \in \mathcal{C}_k$ such that $x \in C$, and $y \notin C$. Then $d(x,y) \geq \Lambda r_k$. Reasoning as in Case 1, we have $|u_{k-1}(p_C) - u_{k-1}(x)| \leq \hat{L}_k r_k$. Hence

$$|\bar{v}_k(x)| = |u_C - u_{k-1}(x)| \leq |u_{k-1}(p_C) - u_{k-1}(x)| + \hat{L}_k r_k \leq 2\hat{L}_k r_k \, .$$

Therefore

$$|\bar{v}_k(x) - \bar{v}_k(y)| \leq |\bar{v}_k(x)| + |\bar{v}_k(y)| \leq 4\hat{L}_k r_k,$$

so, since $\Lambda \geq 4$,

$$\frac{|\bar{v}_k(x) - \bar{v}_k(y)|}{d(x,y)} \leq \frac{4}{\Lambda}\hat{L}_k \leq \hat{L}_k \, .$$

Thus the sublemma holds. $\square$

By McShane's extension lemma (see [Hei01]), there is an $\hat{L}_k$-Lipschitz extension $v_k : Z \to \mathbb{R}$ of $\bar{v}_k$, where $\|v_k\|_{C^0} \leq \|\bar{v}_k\|_{C^0} \leq 2\hat{L}_k r_k$. Since $\hat{L}_k \leq L_k$ by the feasibility assumption, v_k is L_k-Lipschitz and $\|v_k\|_{C^0} \leq 2L_k r_k$. The lemma holds by induction. $\square$

Proof of Theorem 4.1. To get the Holder bound, we use the feasible sequence $L_j = e^{\lambda j}$, and keep in mind that we may take λ close to 0 provided D is large. Suppose $x, y \in Z$ and $d(x,y) \in [r_{k+1}, r_k]$. Then

$$\begin{aligned}|u_{k-1}(x) - u_{k-1}(y)| &\leq r_k \sum \{L_j \mid j < k, \mathrm{Spt}(v_j) \cap \{x,y\} \neq \emptyset\} \\ &\leq r_k \hat{L}_k \leq L_k r_k\,.\end{aligned}$$

Also,

$$\begin{aligned}|\sum_{j\geq k} v_j(x) - \sum_{j\geq k} v_j(y)| &\leq 2\sum_{j\geq k} \|v_j\|_{C^0} \leq 4\sum_{j\geq k} L_j r_j \\ &= 4\sum_{j\geq k} e^{\lambda j} 2^{-j} \leq 8 L_k r_k\end{aligned}$$

when λ is small. So

$$\frac{|u(x)-u(y)|}{d(x,y)^\alpha} \leq \frac{9L_k r_k}{\left(\frac{r_k}{2}\right)^\alpha} = 9 \cdot 2^\alpha \cdot \left(e^\lambda \cdot 2^{\alpha-1}\right)^k$$

which is bounded independent of k when λ is small.

It is clear from the construction that if $\mathcal{C}$ is countable, then we may arrange that u takes different values on different C's. □

We now give an application to hyperbolic spaces. Again we consider Y a hyperbolic non-degenerate simply connected metric simplicial complex with links of uniformly bounded complexity, and all simplices isometric to regular Euclidean simplices with unit length edges.

Corollary 4.6. *Let Y be as above. For every $\alpha \in (0,1)$, $C \geq 0$, there is a $D \geq 1$ with the following properties. Suppose $W \subset Y$ is a subcomplex of Y such that*

(1) *Every connected component of W is C-quasiconvex in Y.*
(2) *For every $y \in W$ there is a complete geodesic $\gamma \subset W$ lying in the same connected component of W, such that $\mathrm{dist}(y,\gamma) \leq C$.*
(3) *The distance between distinct components of W is at least D.*

Then $q_{sep}(Y,W) \leq \frac{1}{\alpha}\,\mathrm{Confdim}(\partial Y)$.

The proof relies on Theorem 4.1 and on the following elementary lemma.

Lemma 4.7. *Let Y be a hyperbolic space and let d be a visual metric on ∂Y. For every $C \geq 0$ there are constants $A \geq 1$, $B \geq 0$ such that for every C-quasiconvex subsets $H_1, H_2 \subset Y$ with non empty limit sets, one has*

$$\Delta(\partial H_1, \partial H_2) \geq A \cdot a^{\frac{1}{2}\,\mathrm{dist}(H_1,H_2)} - B,$$

where a is the exponential parameter of d. Moreover A, B depend only on C, the hyperbolicity constant of Y, and the constants of the visual metric d.

Proof of the lemma. We denote the distance in Y by $|y_1 - y_2|$. Pick $h_1 \in H_1, h_2 \in H_2$ with $|h_1 - h_2| = \operatorname{dist}(H_1, H_2)$. If $\operatorname{dist}(H_1, H_2)$ is large enough – compared with C and the hyperbolicity constant of Y – then for every $y_1 \in H_1$, $y_2 \in H_2$ the subset $[y_1, h_1] \cup [h_1, h_2] \cup [h_2, y_2]$ is a quasi-geodesic segment with controlled constants. In particular the segment $[h_1, h_2]$ lies in a neighborhood of $[y_1, y_2]$ of controlled radius. Therefore $H_1 \cup [h_1, h_2] \cup H_2$ is a C'-quasiconvex subset, where C' depends only on C and the hyperbolicity constant of Y.

Pick an origin $y_0 \in Y$ and let p be a nearest point projection of y_0 on $H_1 \cup [h_1, h_2] \cup H_2$. We distinguish two cases.

Case 1 : $p \in [h_1, h_2]$. Set $L = \max_{i=1,2} \operatorname{dist}(p, H_i)$; one has with the relations (2.6) :

$$\Delta(\partial H_1, \partial H_2) \gtrsim \frac{a^{-|y_0 - p|}}{a^{-|y_0 - p| - L}} = a^L \geq a^{\frac{1}{2} \operatorname{dist}(H_1, H_2)}.$$

Case 2 : $p \in H_1 \cup H_2$. Suppose for example that p belongs to H_1. Since every geodesic joining H_1 to H_2 passes close by h_1 one has

$$\operatorname{dist}(\partial H_1, \partial H_2) \gtrsim a^{-|y_0 - h_1|}.$$

In addition every geodesic joining y_0 to H_2 passes close by p and thus close by h_1. Therefore :

$$\operatorname{diam}(\partial H_2) \lesssim a^{-|y_0 - h_1| - \operatorname{dist}(H_1, H_2)},$$

and so $\Delta(\partial H_1, \partial H_2) \gtrsim a^{\operatorname{dist}(H_1, H_2)}$. $\square$

Proof of Corollary 4.6. Let $\alpha \in (0, 1)$ and let d be a visual metric on ∂Y. ¿From the definition of the Ahlfors regular conformal dimension, and because $\frac{\alpha+1}{2\alpha} > 1$, there is an Ahlfors Q-regular metric δ on ∂Y, with $Q \leq \frac{\alpha+1}{2\alpha} \operatorname{Confdim}(\partial Y)$, and such that the identity map $(\partial Y, d) \to (\partial Y, \delta)$ is a quasi-Moebius homeomorphism. This last property implies that the relative distances associated to d and δ are quantitatively related (see [BK02] Lemma 3.2). Thus, thanks to the previous theorem and lemma, there is a constant $D \geq 1$ such that if $W \subset Y$ satisfies the conditions (1), (2), (3) of the statement, and if $\mathcal{H}$ denotes the family of its connected components, then, there is a function $u \in C^{\frac{\alpha+1}{2}}(\partial Y, \delta)$ with the following properties

- $u_{|\partial H}$ is constant for every $H \in \mathcal{H}$,

- If $H_1, H_2 \in \mathcal{H}$ and $u(\partial H_1) = u(\partial H_2)$, then $H_1 = H_2$.

We wish to extend u to a continuous function $f : Y^{(0)} \cup \partial Y \to \mathbb{R}$ that is constant on every $H \in \mathcal{H}$. To do so pick an origin $y_0 \in Y$ and observe that the nondegeneracy property of Y yields the existence of a constant $R \geq 0$ such that for every $y \in Y$ there is a $z \in \partial Y$ with $\mathrm{dist}(y, [y_0, z)) \leq R$.

Let $y \in Y^{(0)}$. If y belongs to a $H \in \mathcal{H}$, set $f(y) = u(\partial H)$. If not, pick a $z \in \partial Y$ such that $\mathrm{dist}(y, [y_0, z)) \leq R$ and define $f(y) = u(z)$.

We claim that f is a continuous function of $Y^{(0)} \cup \partial Y$. Let $a > 1$ be the exponential parameter of the visual metric d. Our hypothesis (2) implies that there is a constant $R' \geq 0$ so that for every $y \in Y^{(0)}$ there is a $z \in \partial Y$ with $\mathrm{dist}(y, [y_0, z)) \leq R'$ and $f(y) = u(z)$. Let $y_1, y_2 \in Y^{(0)}$; their mutual distance in $Y^{(0)} \cup \partial Y$ is comparable to a^{-L}, where $L = \mathrm{dist}(y_0, [y_1, y_2])$. The metrics d and δ being quasi-Moebius equivalent they are Holder equivalent (see [Hei01]). Thus $u \in C^{\beta}(\partial Y, d)$ for some $\beta > 0$; and we obtain :

$$|f(y_1) - f(y_2)| = |u(z_1) - u(z_2)| \lesssim a^{-\beta\, \mathrm{dist}(y_0, (z_1, z_2))} \lesssim a^{-\beta L}. \tag{4.8}$$

To establish the last inequality one notices that $\mathrm{dist}(y_0, (z_1, z_2)) - L$ is bounded by below just in terms of R' and the hyperbolicity constant of Y. The claim follows.

At present we claim that $df \in \ell_p(Y^{(1)})$ for $p > \frac{2Q}{\alpha+1}$. To see it, observe that u is a Lipschitz function of $(\partial Y, \delta^{\frac{\alpha+1}{2}})$ and $\delta^{\frac{\alpha+1}{2}}$ is an Ahlfors $\frac{2Q}{\alpha+1}$-regular metric on ∂Y which is quasi-Moebius equivalent to a visual metric. Therefore the claim follows from Elek's extension process [Ele97] (see [Bou] Prop.3.2 for more details).

Finally the function f defines an element of $\ell_p H^1_{\mathrm{cont}}(Y, W)$ for every $p > \frac{2Q}{\alpha+1}$, which separates every pair of distinct connected components of W. Thus

$$q_{\mathrm{sep}}(Y, W) \leq \frac{2Q}{\alpha + 1} \leq \frac{1}{\alpha} \, \mathrm{Confdim}(\partial Y).$$

$\square$

5. Applications to amalgamated products

This section uses Section 3 and Corollary 4.6 to construct examples of group amalgams whose ℓ_p-cohomology has specified behavior (see Corollary 5.3 especially).

Let A be a hyperbolic group and let $C \leqq A$ be a finitely presentable subgroup. Let K_A and K_C be finite 2-complexes with respective fundamental groups A and C, such that there is a simplicial embedding $K_A \hookrightarrow K_C$ inducing the given embedding of fundamental groups. Denote by $\tilde{K}_A$ the universal cover of K_A and pick a copy $\tilde{K}_C \subset \tilde{K}_A$ of the universal cover of K_C. The spaces and invariants $\ell_p H^1_{\mathrm{cont}}(Y, W)$, $p_{\neq 0}(Y)$, $p_{\mathrm{sep}}(Y)$, $p_{\neq 0}(Y, W)$, $q_{\mathrm{sep}}(Y, W)$, associated to the pair

$$(Y, W) = (\tilde{K}_A, \cup_{a \in A} a\tilde{K}_C),$$

will be denoted simply by $\ell_p H^1_{\mathrm{cont}}(A, C)$, $p_{\neq 0}(A)$, $p_{\mathrm{sep}}(A)$, $p_{\neq 0}(A, C)$ and $q_{\mathrm{sep}}(A, C)$.

We notice that $q_{\mathrm{sep}}(A, C)$ is finite when C is a quasi-convex malnormal subgroup (see Remark 2 after Theorem 4.1 and [Bou04] for more details).

Proposition 5.1. *Let A, B, C three hyperbolic groups, suppose that A and B are non elementary and that C is a proper quasi-convex malnormal subgroup of A and B. Let Γ be the amalgamated product $A *_C B$ (from* [Kap97] *it is hyperbolic). We have :*

1. *If $p_{\neq 0}(A, C) < p_{sep}(B)$ then for all $p \in (p_{\neq 0}(A, C), p_{sep}(B)]$ the ℓ_p-equivalence relation on $\partial\Gamma$ possesses a coset different from a point and the whole $\partial\Gamma$,*
2. *If $q_{sep}(A, C) < p_{\neq 0}(B)$ then for all $p \in (q_{sep}(A, C), p_{\neq 0}(B))$ the ℓ_p-equivalence relation on $\partial\Gamma$ is of the following form :*

$$x \sim_p y \iff x = y \text{ or } \exists g \in \Gamma \text{ such that } x, y \in g(\partial B).$$

Proof of Proposition 5.1. Item (2) is established in [Bou04] Th. 0.1. We provide here a more enlightening proof.

(1). Let $p \in (p_{\neq 0}(A, C), p_{\mathrm{sep}}(B)]$ as in the statement. From Example 3.5 and Corollary 3.7(1) we have $p > p_{\neq 0}(\Gamma)$. Hence the ℓ_p-cosets are different from $\partial\Gamma$. On the other hand we have the obvious inequality $p \leq p_{\mathrm{sep}}(B) \leq p_{\mathrm{sep}}(\Gamma)$. Thus Theorem 3.8(1) shows that the ℓ_p-equivalence relation on $\partial\Gamma$ admits a coset which is different from a singleton.

(2). Let $p \in (q_{\mathrm{sep}}(A, C), p_{\neq 0}(B))$ as in the statement. Let $u \in A_p(\partial\Gamma)$. Its restriction to every $g(\partial B)$ $(g \in \Gamma)$ is constant since $p < p_{\neq 0}(B)$. Thus every $g(\partial B)$ is contained in a $\sim_p$-coset.

Conversely to establish that

$$x \sim_p y \Longrightarrow x = y \text{ or } \exists g \in \Gamma \text{ such that } x, y \in g(\partial B),$$

we will exhibit some elements of $A_p(\partial\Gamma)$. We consider finite simplicial complexes K_A, K_B, K_C, K, with respective fundamental group A, B, C, Γ, as described in Example 3.5. Let $\tilde{K}$ be the universal cover of K and choose copies $\tilde{K}_A, \tilde{K}_B \subset \tilde{K}$ of the universal covers of K_A, K_B. We know from Example 3.5 that $\tilde{K}_A$ decomposes $\tilde{K}$.

Pick two distinct subcomplexes $g\tilde{K}_B, g'\tilde{K}_B \subset \tilde{K}$ and a geodesic $\gamma \subset \tilde{K}$ joining them. It passes through a subcomplex $g''\tilde{K}_A$. Applying $(g'')^{-1}$ if necessary, we may assume that $g'' = 1$. Therefore $g\tilde{K}_B$ and $g'\tilde{K}_B$ lie in different components of $\overline{\tilde{K} \setminus \tilde{K}_A}$. Since $p > q_{\mathrm{sep}}(A, C)$, there is an element $[\varphi] \in \ell_p H^1_{\mathrm{cont}}(A, C)$ represented by a function $\varphi : \tilde{K}_A^{(0)} \to \mathbb{R}$ which takes distinct (constant) values in distinct component of the frontier of $\tilde{K}_A$. Its extension $f : \tilde{K}^{(0)} \to \mathbb{R}$ defined in Proposition 3.6, is such that f_∞ belongs to $A_p(\partial\Gamma)$ and separates $g(\partial B)$ from $g'(\partial B)$. $\square$

From Theorem 3.8(2), and since $\sim_p$ is invariant under the boundary extensions of the quasi-isometries of Γ, we get :

Corollary 5.2. *Let A, B, C, Γ be as in the previous proposition.*

1. *If $p_{\neq 0}(A, C) < p_{sep}(B)$ then $\partial\Gamma$ does not admit the CLP.*
2. *If $q_{sep}(A, C) < p_{\neq 0}(B)$ then any quasi-isometry of Γ permutes the cosets gB, for $g \in \Gamma$. More precisely, the image of a coset gB by a quasi-isometry of Γ lies within bounded distance (quantitatively) from a unique coset $g'B$.*

In combination with Corollary 4.6 this leads to:

Corollary 5.3. *Let A, B, C be as in the previous proposition and suppose that there is a decreasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite index subgroups of A such that $\cap_{n\in\mathbb{N}} A_n = C$. Set $\Gamma_n := A_n *_C B$.*

1. *If $p_{sep}(A) < p_{sep}(B)$ then, for all $p \in (p_{sep}(A), p_{sep}(B)]$ and every n large enough, the ℓ_p-equivalence relation on $\partial\Gamma_n$ possesses a coset different from a point and the whole $\partial\Gamma$. In particular for n large enough, $\partial\Gamma_n$ does not admit the CLP.*
2. *If $p_{sep}(A) < p_{\neq 0}(B)$ then, for $p \in (p_{sep}(A), p_{\neq 0}(B))$ and every n large enough, the cosets of the ℓ_p-equivalence relation on $\partial\Gamma_n$ are single points and the boundaries of cosets gB, for $g \in \Gamma_n$. In particular, for large n, any quasi-isometry of Γ_n permutes the cosets gB, for $g \in \Gamma_n$.*

Proof. According to Proposition 5.1 and Corollary 5.2 it is enough to prove that

$$\mathrm{limsup}_n\ q_{\mathrm{sep}}(A_n, C) \leq p_{\mathrm{sep}}(A).$$

We consider the simplicial complexes $\tilde{K}_A, \tilde{K}_C$ introduced in the beginning of the section, and we set for $n \in \mathbb{N}$:

$$(Y, W_n) := (\tilde{K}_A, \cup_{a \in A_n} a\tilde{K}_C).$$

Then just by definition we have: $q_{\mathrm{sep}}(A_n, C) = q_{\mathrm{sep}}(Y, W_n)$. On the other hand the hypothesis $\cap_{n\in\mathbb{N}} A_n = C$ implies that the minimal pairwise separation between distinct components of W_n tends infinity as $n \to \infty$. It follows from Corollary 4.6 that

$$\mathrm{limsup}_n \; q_{\mathrm{sep}}(Y, W_n) \leq \mathrm{Confdim}(\partial A).$$

With Theorem 3.8(1) we get the expected limit. □

Example 5.4. It is now possible to answer M. Bonk's question about examples of approximately self-similar Sierpinski carpets without the CLP. Pick two hyperbolic groups A and B whose boundaries are homeomorphic to the Sierpinski carpet, and which admit an isomorphic peripheral subgroup C. Assume that $\mathrm{Confdim}(\partial A) < \mathrm{Confdim}(\partial B)$ and that there is a sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite index subgroups of A such that $\cap_{n\in\mathbb{N}} A_n = C$. Such examples can be found among hyperbolic Coxeter groups, because quasi-convex subgroups of Coxeter groups are separable. See Example 8.3 and [HW10]. Then for n large enough the boundary of $A_n *_C B$ is homeomorphic to the Sierpinski carpet. Moreover, according to Corollary 5.3, it doesn't admit the CLP for n large enough.

Example 5.5. Let M, M' and N be closed hyperbolic (*i.e.* constant curvature -1) manifolds with $1 \leq \dim(N) < \dim(M) < \dim(M')$. Suppose that M and M' contain as a submanifold an isometric totally geodesic copy of N. The group $\pi_1(N)$ is separable in $\pi_1(M)$ (see [Ber00]). For the standard hyperbolic space $\mathbb{H}^k$ of dimension $k \geq 2$, one has by [Pan89] : $\mathrm{Confdim}(\partial\mathbb{H}^k) = p_{\neq 0}(\mathbb{H}^k) = k - 1$. Therefore the assumptions in items (1) and (2) of Corollary 5.3 are satisfied with $A = \pi_1(M)$, $B = \pi_1(M')$ and $C = \pi_1(N)$. It follows that the manifold M admits a finite cover M_n containing an isometric totally geodesic copy of N, such that the space $K := M_n \sqcup_N M'$ possesses the following properties :

- For $p \in (\dim M - 1, \dim M' - 1)$, by letting $\tilde{K}$ be the universal cover of K, the cosets of the ℓ_p-equivalence relation on $\partial\tilde{K}$ are points and the boundaries of lifts of M'. In particular $\partial\tilde{K}$ doesn't satisfy the CLP,
- Every quasi-isometry of $\tilde{K}$ permutes the lifts of $M' \subset K$.

The second property may also be proven using the topology of the boundary, or using coarse topology. By combining topological arguments with the zooming method of R. Schwartz [Sch95], one can deduce from the second property that every quasi-isometry of $\tilde{K}$ lies within bounded distance from an isometry.

6. Elementary polygonal complexes

In this section we compute the invariants $p_{\neq 0}(X, \mathcal{H})$ and $q_{\mathrm{sep}}(X, \mathcal{H})$ in some very special cases. They will serve in the next sections to obtain upper bounds for the conformal dimension and for the invariant $p_{\neq 0}(X)$.

Definition 6.1. A *polygonal complex* is a connected simply connected 2-cell complex X of the following form :

- Every 2-cell is isomorphic to a polygon with at least 3 sides.
- Every pair of 2-cells shares at most a vertex or an edge.

The number of sides of a 2-cell is called its *perimeter*, the number of 2-cells containing an edge is called its *thickness*.

Definition 6.2. An *elementary polygonal complex* is a polygonal complex Y whose edges are colored black or white, that enjoys the following properties :

- Every 2-cell has even perimeter at least 6.
- The edges on the boundary of every 2-cell are alternately black and white.
- Every white edge has thickness 1, and every black edge has thickness at least 2.

We will equip every elementary polygonal complex Y with a length metric of negative curvature, by identifying every 2-cell with a constant negative curvature right angled regular polygon of unit length edges; in particular Y is a Gromov hyperbolic metric space (quasi-isometric to a tree). The union of the white edges of Y is called its *frontier*. The frontier is a locally convex subcomplex of Y, and hence every connected component is a $CAT(0)$ space, i.e. subtree of Y. We call such components *frontier trees* of Y.

The (unique) elementary polygonal complex, whose 2-cells have constant perimeter $2m$ and whose black edges have constant thickness k ($m \geq 3, k \geq 2$), will be denoted by $Y_{m,k}$.

We denote by $\mathcal{T}$ the frontier of Y, and we consider the associated invariants : $p_{\neq 0}(Y,\mathcal{T}), q_{\mathrm{sep}}(Y,\mathcal{T})$. When $Y = Y_{m,k}$ we write $\mathcal{T}_{m,k}$, $p_{m,k}$, $q_{m,k}$ for brevity.

Theorem 6.3. *For $m \geq 3$ and $k \geq 2$ one has :*

$$p_{m,k} = q_{m,k} = 1 + \frac{\log(k-1)}{\log(m-1)}.$$

Proof. We first establish the upper bound

$$q_{m,k} \leq 1 + \frac{\log(k-1)}{\log(m-1)}. \tag{6.4}$$

For this purpose we will construct some elements in $\ell_p H^1_{\mathrm{cont}}(Y_{m,k},\mathcal{T}_{m,k})$. Set $Y_m := Y_{m,2}$ for simplicity. Observe that Y_m is a planar polygonal complex.

For any choice of 2-cells $c \subset Y_{m,k}$, $d \subset Y_m$ there is an obvious continuous polygonal map $r_{c,d} : Y_{m,k} \to Y_m$ sending c to d and such that for every $y_0 \in c$ and every $y \in Y_{m,k}$ one has $|r_{c,d}(y_0) - r_{c,d}(y)| = |y_0 - y|$.

Observe that the frontier of Y_m is an union of disjoint geodesics. We will define a continuous map $\varphi : Y_m \cup \partial Y_m \to \mathbb{H}^2 \cup \mathbb{S}^1$ which maps every frontier geodesic of Y_m to an ideal point in $\mathbb{S}^1$. To do so, fix a 2-cell $d \subset Y_m$. At first we define $\varphi\big|_d$ so that its image is a regular ideal m-gon. In other words φ collapses every white edge of d to an ideal point in $\mathbb{S}^1$, and these ideal points are m regularly distributed points in $\mathbb{S}^1$. For $n \in \mathbb{N}$, let P_n be the union of the 2-cells of Y_m whose combinatorial distance to d is less than or equal to n. By induction we define φ on the subcomplex P_n, so that

(i) The images of the 2-cells of Y_m form a tesselation of $\mathbb{H}^2$ by ideal m-gons,
(ii) The images of the frontier geodesics of Y_m, passing through the subcomplex P_n, are $m(m-1)^{n-1}$ ideal points regularly distributed in $\mathbb{S}^1$.

Let $u : \mathbb{S}^1 \to \mathbb{R}$ be a Lipschitz function. Since $\varphi(Y_m^{(0)}) \subset \mathbb{S}^1$, the composition $u \circ \varphi \circ r_{c,d}$ is well defined on $Y_{m,k}^{(0)}$. Let $f : Y_{m,k}^{(0)} \to \mathbb{R}$ be this function. By construction its restriction to every frontier tree $T \subset \mathcal{T}_{m,k}$ is constant. Moreover f extends continuously to $Y_{m,k}^{(0)} \cup \partial Y_{m,k}$ since $r_{c,d}$ does. It remains to estimate the p-norm of df. Let C be the Lipschitz constant of u. There are $m(m-1)^n(k-1)^n$ black edges at the frontier of the subcomplex $r_{c,d}^{-1}(P_n)$. If a is such an edge, one has from property (ii) above :

$$|df(a)| \leq \frac{2\pi C}{m(m-1)^{n-1}}.$$

Therefore :

$$\|df\|_p^p \lesssim \sum_{n\in\mathbb{N}} \frac{(m-1)^n(k-1)^n}{(m-1)^{pn}} = \sum_{n\in\mathbb{N}} \big((m-1)^{1-p}(k-1)\big)^n.$$

Thus $[f]$ belongs to $\ell_p H^1_{\mathrm{cont}}(Y_{m,k}, \mathcal{T}_{m,k})$ for $(m-1)^{1-p}(k-1) < 1$ *i.e.* for

$$p > 1 + \frac{\log(k-1)}{\log(m-1)}.$$

In addition, by varying the 2-cell c and the fonction u, one obtains functions f that separate any given pair of distinct frontier trees $T, T' \subset \mathcal{T}_{m,k}$. Inequality (6.4) now follows.

To establish the theorem it remains to prove :

$$p_{m,k} \geq 1 + \frac{\log(k-1)}{\log(m-1)}. \tag{6.5}$$

To do so we consider, for $m, k, \ell \geq 3$, the polygonal complex $\Delta_{m,k,\ell}$ defined by the following properties :

- Every 2-cell has perimeter $2m$.
- Every edge is colored black or white, and the edge colors on the boundary of each 2-cell are alternating.
- The thickness of every black edge is k, while the thickness of white edges is ℓ.
- The link of every vertex is the full bipartite graph with $k+\ell$ vertices.

As with elementary polygonal complexes, we metrize $\Delta_{m,k,\ell}$ so that each cell is a regular right-angled hyperbolic polygon, and hence $\Delta_{m,k,\ell}$ is a right-angled Fuchsian building. The union of its white edges is a disjoint union of totally geodesic trees in $\Delta_{m,k,\ell}$. By cutting $\Delta_{m,k,\ell}$ along these trees, one divides $\Delta_{m,k,\ell}$ into subcomplexes, each isometric to $Y_{m,k}$. Let $Y \subset \Delta_{m,k,\ell}$ be one such subcomplex. It decomposes $\Delta_{m,k,\ell}$. Denote by $\mathcal{T}$ its frontier. By Corollary 3.7 one gets that $p_{\neq 0}(\Delta_{m,k,\ell}) \leq p_{\neq 0}(Y, \mathcal{T}) = p_{m,k}$. On the other hand $\partial\Delta_{m,k,\ell}$ is known to admit the CLP (see [BP00, BK]). Hence, with Theorem 3.8(2) we obtain :

$$p_{m,k} \geq p_{\neq 0}(\Delta_{m,k,\ell}) = \mathrm{Confdim}(\partial\Delta_{m,k,\ell}).$$

From [Bou00] formula (0.2), one has $\mathrm{Confdim}(\partial\Delta_{m,k,\ell}) = 1 + \frac{1}{x}$ where x is the unique positive number which satisfies :

$$\frac{(k-1)^x + (\ell-1)^x}{\big(1+(k-1)^x\big)\big(1+(\ell-1)^x\big)} = \frac{1}{m}.$$

By an easy computation we get that

$$\lim_{\ell\to+\infty} \mathrm{Confdim}(\partial\Delta_{m,k,\ell}) = 1 + \frac{\log(k-1)}{\log(m-1)}.$$

Inequality 6.5 follows. □

Corollary 6.6. *Let Y be an elementary polygonal complex. Assume that the perimeter of every 2-cell of Y lies in $[2m_1, 2m_2]$ and that the thickness of every black edge lies in $[k_1, k_2]$, with $m_1 \geq 3$ and $k_1 \geq 2$. Then :*

$$1 + \frac{\log(k_1-1)}{\log(m_2-1)} \leq p_{\neq 0}(Y,\mathcal{T}) \leq q_{sep}(Y,\mathcal{T}) \leq 1 + \frac{\log(k_2-1)}{\log(m_1-1)}.$$

Proof. We first establish the last inequality. By adding 2-cells to Y if necessary, we obtain an elementary polygonal complex Y', whose black edges are of constant thickness k_2, and whose 2-cell perimeters are larger than or equal to $2m_1$. Let $\mathcal{T}'$ be its frontier. The cellular embedding $Y \to Y'$ induces a restriction map $\ell_pH^1_{\mathrm{cont}}(Y',\mathcal{T}') \to \ell_pH^1_{\mathrm{cont}}(Y,\mathcal{T})$, which leads to $q_{\mathrm{sep}}(Y,\mathcal{T}) \leq q_{\mathrm{sep}}(Y',\mathcal{T}')$.

We now compare $q_{\mathrm{sep}}(Y',\mathcal{T}')$ with q_{m_1,k_2}. Observe that any polygon P of perimeter larger than $2m_1$ contains a $2m_1$-gon whose black edges are contained in black edges of P, and whose white edges are contained in the interior of P. With this observation one can construct embeddings $\varphi : Y_{m_1,k_2} \to Y'$ in such a way that $\varphi(Y_{m_1,k_2})$ decomposes Y' and Proposition 3.6 yields a monomorphism

$$\ell_pH^1_{\mathrm{cont}}(Y_{m_1,k_2},\mathcal{T}_{m_1,k_2}) \hookrightarrow \ell_pH^1_{\mathrm{cont}}(Y',\mathcal{T}').$$

For $p > q_{m_1,k_2}$, by varying the embedding φ, one obtains elements of $\ell_pH^1_{\mathrm{cont}}(Y',\mathcal{T}')$ that separate any given pair of distinct components of $\mathcal{T}'$. Therefore $q_{\mathrm{sep}}(Y',\mathcal{T}') \leq q_{m_1,k_2}$, and the last inequality of Corollary 6.6 follows from Theorem 6.3.

The first inequality can be proved in a similar way. □

7. Applications to polygonal complexes

Building on earlier Sections 3 and 6, this section derives some results on the conformal dimension of polygonal complex boundaries (Proposition 7.1).

Let X be a polygonal complex. Notice that if the perimeter of every 2-cell is larger than or equal to 7, then X admits a length metric of negative curvature by identifying every 2-cell with a constant negative curvature regular polygon of unit length edges and of angles $\frac{2\pi}{3}$.

If every 2-cell has perimeter at least 5, and the link of of every vertex contains no circuit of length 3, then X admits a length metric of negative curvature by identifying every 2-cell with a constant negative curvature regular polygon of unit length edges and of angles $\frac{\pi}{2}$.

The following result gives a upper bound for the conformal dimension of such polygonal complexes. In some cases a lower bound can be obtained by M. Gromov's method of "round trees", see [Gro93] p.207, [Bou95], [Mac].

Proposition 7.1. *Let X be a polygonal complex of negative curvature as above; suppose that ∂X is connected and approximately self-similar.*

(1) *If the perimeter of every 2-cell is larger than or equal to $n \geq 5$, the thickness of every edge lies in $[2, k]$, and the link of every vertex contains no circuit of length 3, then*
$$\mathrm{Confdim}(\partial X) \leq 1 + \frac{\log(k-1)}{\log(n-3)}.$$
(2) *If the perimeter of every 2-cell is larger than or equal to $n \geq 7$ and the thickness of every edge lies in $[2, k]$, then*
$$\mathrm{Confdim}(\partial X) \leq 1 + \frac{\log(k-1)}{\log(n-5)}.$$

Proof. We shall construct a family of embedded elementary polygonal complexes that fully decomposes X. The statement will then follow from Corollaries 3.7(2), 6.6 and Theorem 3.8(1).

(1). At first we associate to every edge $e \subset X$ a *rooted directed tree* defined by the following process :

(A) Join by a directed segment the middle of e to the center O of every 2-cell c containing e,
(B) Connect by a directed segment the center O to the middle of every edge e' of c that is not adjacent nor equal to e,

(C) Restart the process with e' and every 2-cell distinct from c that contains e'.

Let T_e be the resulting graph in X. We claim that T_e is a bi-Lipschitz embedded tree in X. To do so, we first notice that from its definition every edge of T_e admits a direction. A path in T_e will be called a *directed path* if its edges are all directed in the same way. Pick a directed path $\gamma \subset T_e$. Since X is right angled, item (B) in combination with angle considerations implies that the union of the 2-cells met by γ is a convex subset of X. Therefore two different directed paths in T_e issuing from the same point have distinct endpoints. Thus T_e is an embedded tree in X. Moreover every backtrack free path in T_e is the union of at most two directed paths making an angle bounded away from 0. Since X is non positively curved the claim follows now easily.

Pick an $0 < \epsilon \leq \frac{1}{4}$ and consider the ϵ-neighborhood Y_e of T_e in X. This is a bi-Lipschitz embedded elementary polygonal complex. The perimeter of every of its 2-cell is larger than or equal to $2(n-2)$, and the thickness of its edges is smaller than or equal to k. Obviously the family $\{Y_e\}_{e \in X^{(1)}}$ fully decomposes X. Therefore according to Theorem 3.8(1), Corollaries 3.7(2) and 6.6 we get

$$\operatorname{Confdim}(\partial X) = p_{\text{sep}}(X) \leq \sup_{e \in X^{(1)}} q_{\text{sep}}(Y_e, \mathcal{T}_e) \leq 1 + \frac{\log(k-1)}{\log(n-3)},$$

where $\mathcal{T}_e$ denotes the frontier of Y_e.

(2). The method is the same apart from a slight modification in the construction process of the rooted directed trees. Item (B) becomes :

(B') Connect O to the middle of every edge e' of c whose combinatorial distance to e is at least 2.

We claim that T_e is again a bi-Lipschitz embedded tree in X. To see this, consider again a directed path $\gamma \subset T_e$ and the 2-cells $c_1, ..., c_n \subset X$ successively met by γ. Their union is not convex in X but a slight modification is. Indeed pick $i \in \{1, ..., n-1\}$, let x be one of the vertices of the segment $c_i \cap c_{i+1}$, and let $y \in c_i \setminus c_{i+1}$, $z \in c_{i+1} \setminus c_i$ be the vertices adjacent to x. Denote by σ_x the convex hull of the subset $\{x, y, z\} \subset X$. By considering the link at x, one sees that it either degenerates to the union $[yx] \cup [xz]$, or it is a simplex contained in the unique 2-cell which contains x, y, z. Since X is a non positively curved polygonal complex with unit length edges and $\frac{2\pi}{3}$ angles, the angles of σ_x at y and z are smaller than $\frac{\pi}{6}$. Moreover item (B') implies that

$y \notin c_{i-1}$ and $z \notin c_{i+2}$. It follows from angle considerations that the union

$$(\cup_i c_i) \cup (\cup_x \sigma_x)$$

defines a locally convex isometric immersion of an abstract CAT(0) space into X, and it is therefore a global embedding with convex image.

The rest of the proof is similar to case (1). Here the 2-cells of the associated elementary polygonal complex Y_e have perimeter larger than or equal to $2(n-4)$ because of (B'). $\square$

Example 7.2. Assume that the link of every vertex is the 1-skeleton of the k-dimensional cube and that the perimeter of every 2-cell is equal to $n \geq 5$. Then Proposition 7.1(1) in combination with the lower bound established in [Bou95] p.140 gives

$$1 + \frac{\log(k-1)}{\log(n-3) + \log 15} \leq \mathrm{Confdim}(\partial X) \leq 1 + \frac{\log(k-1)}{\log(n-3)}.$$

When the link is the 1-skeleton of the 3-dimensional cube ∂X is homeomorphic to the Sierpinski carpet. For even n, examples of such complexes are provided by Davis complexes of Coxeter groups; their boundaries admit the CLP (see [BK] 9.4). Therefore one obtains examples of Sierpinski carpet boundaries satisfying the CLP, whose conformal dimension is arbitrarily close to 1. This answers a question of J. Mackay.

Example 7.3. Assume now that the link of every vertex is the full bipartite graph with $k + k$ vertices and that the perimeter of every 2-cell is equal to $n \geq 5$. From [Bou00] one has

$$\mathrm{Confdim}(\partial X) = 1 + \frac{\log(k-1)}{\log\left(\frac{n}{2} - 1 + \sqrt{(\frac{n}{2}-1)^2 - 1}\right)},$$

which is pretty close to the upper bound obtained in Proposition 7.1. We emphasize that the Hausdorff dimensions of the visual metrics on ∂X do not give in general such precise upper bounds. For example it follows from [LL10] that if every 2-cell of X is isometric to the same right angled polygon $P \subset \mathbb{H}^2$, then the Hausdorff of the associated visual metric is larger than

$$1 + \frac{\mathrm{length}(\partial P)}{\mathrm{area}(P)} \log(k-1).$$

8. Applications to Coxeter groups

This applies earlier results for the invariants $\mathrm{Confdim}(\partial\Gamma)$ and $p_{\neq 0}(\Gamma)$ to Coxeter groups Γ (Corollary 8.1 and Proposition 8.2)

Recall that a group Γ is a *Coxeter group* if it admits a presentation of the form

$$\Gamma = \langle s,\ s \in S \mid s^2 = 1,\ (st)^{m_{st}} = 1 \text{ for } s \neq t \rangle,$$

with $|S| < +\infty$, and $m_{st} \in \{2, 3, ..., +\infty\}$. To such a presentation one associates a finite simplicial graph L whose vertices are the elements $s \in S$ and whose edges join the pairs (s, t) such that $s \neq t$ and $m_{st} \neq +\infty$. We label every edge of L with the corresponding integer m_{st}. The labelled graph L is called the *defining graph* of Γ. The valence of a vertex $s \in L^{(0)}$ will be denoted by $\mathrm{val}(s)$.

Using Proposition 7.1 we will deduce :

Corollary 8.1. *Suppose Γ is a Coxeter group with defining graph L.*

(1) *If for every $s \in L^{(0)}$, $(s, t) \in L^{(1)}$ one has $\mathrm{val}(s) \in [2, k]$, $m_{st} \geq m \geq 3$ and L contains no circuit of length 3, then Γ is hyperbolic and*
$$\mathrm{Confdim}(\partial\Gamma) \leq 1 + \frac{\log(k-1)}{\log(2m-3)}.$$
(2) *If for every $s \in L^{(0)}$, $(s, t) \in L^{(1)}$ one has $\mathrm{val}(s) \in [2, k]$, $m_{st} \geq m \geq 4$, then Γ is hyperbolic and*
$$\mathrm{Confdim}(\partial\Gamma) \leq 1 + \frac{\log(k-1)}{\log(2m-5)}.$$

The above corollary shows that global bounds for the valence and the integers m_{st} yield upper bounds for the conformal dimension. In contrast, the following result asserts that local bounds are enough to obtain upper bounds for $p_{\neq 0}(\Gamma)$.

Proposition 8.2. *Suppose Γ is a hyperbolic Coxeter group with defining graph L. For $s \in L^{(0)}$ set $m_s := \inf_{(s,t)\in L^{(1)}} m_{st}$.*

(1) *Suppose that there an $s \in L^{(0)}$ with $\mathrm{val}(s) \geq 2$, $m_s \geq 3$ which does not belong to any length 3 circuit of L. Then*
$$p_{\neq 0}(\Gamma) \leq 1 + \frac{\log(\mathrm{val}(s) - 1)}{\log(m_s - 1)}.$$

(2) *Suppose that there is an* $s \in L^{(0)}$ *with* $\mathrm{val}(s) \geq 2$, $m_s \geq 5$. *Then*

$$p_{\neq 0}(\Gamma) \leq 1 + \frac{\log(\mathrm{val}(s) - 1)}{\log(m_s - 3)}.$$

Remarks. The definition of the invariant $p_{\neq 0}(\Gamma)$ given in Section 3 requires Γ to be hyperbolic. However the above proposition extends to non hyperbolic Coxeter groups as well. In this case the conclusion is simply that $\ell_p H^1(\Gamma) \neq 0$ for p larger than the right hand side.

The proofs of Corollary 8.1 and Proposition 8.2 rely on the fact that every Coxeter group Γ has a properly discontinuous, cocompact, isometric action on a $CAT(0)$ cellular complex X called the Davis complex of Γ. We list below some of its properties (see [Dav08] Chapters 7 and 12 for more details).

For $I \subset S$, denote by Γ_I the subgroup of Γ generated by I. It is again a Coxeter group; its defining graph is the maximal subgraph of L whose vertex set is I. By attaching a simplex to every subset $I \subset S$ such that Γ_I is finite, one obtains a simplicial complex ΣL whose 1-skeleton is the graph L. The Davis complex X enjoys the following properties :

- The 1-skeleton of X identifies naturally with the Cayley graph of (Γ, S); in particular every edge of X is labelled by a generator $s \in S$,
- Every k-cell is isometric to a k-dimensional Euclidean polytope,
- The link of every vertex is isomorphic to the simplicial complex ΣL,
- For $(s,t) \in L^{(1)}$ the corresponding 2-cells of X are regular Euclidean polygons of perimeter $2m_{st}$ and unit length edges alternately labelled s and t. More generally if $I \subset S$ spans a k-simplex in ΣL, then the corresponding k-cells of X are isometric to the Euclidean polytope which is the Davis complex of the finite Coxeter group Γ_I,
- For every $I \subset S$, the Davis complex of Γ_I has a canonical isometric embedding in X.

Proof of Corollary 8.1. By decomposing Γ as a free product where each factor is a Coxeter group associated to a connected component of L, we can and will restrict ourself to the case L is connected.

For $I \subset S$, the hypotheses in (1) or (2) imply that the subgroup Γ_I is finite if and only if $|I| = 1$ or 2. Therefore one has $\Sigma L = L$ and the Davis' complex X is a polygonal complex. It clearly satisfies the hypothesis (1) or (2) in Proposition 7.1 with $n = 2m$. The corollary follows. □

Proof of Proposition 8.2. (1). Pick $s \in L^{(0)}$ as in the statement, set $J := \{t \in S \ ; \ (s,t) \in L^{(1)}\}$ and consider the Coxeter subgroup $A \leqq \Gamma$ generated by $\{s\} \cup J$. Its Davis complex, denoted by X_A, isometrically embeds in X.

Since there is no length 3 circuit containing s, the defining graph of A consists only of segments joining s to its neighbour vertices $t \in J$. Therefore X_A is an elementary polygonal complex; its black edges are those labelled by s and the white ones are labelled by an element of J. The perimeter of its 2-cells is at least $2m_s \geq 6$ and the thickness of its black edges is equal to $\mathrm{val}(s)$. Denote by $\mathcal{T}_A$ its frontier.

The absence of a length 3 circuit containing s implies that for every $t \in J$ and $K \subset S$ with $\{s,t\} \subsetneqq K$, the subgroup Γ_K is infinite. From the previous description of the Davis complex X we obtain that $X_A \backslash \mathcal{T}_A$ is an open subset of X. Therefore X_A decomposes X, since X is simply connected. With Corollaries 3.7(1) and 6.6 we get

$$p_{\neq 0}(X) \leq p_{\neq 0}(X_A, \mathcal{T}_A) \leq 1 + \frac{\log(\mathrm{val}(s) - 1)}{\log(m_s - 1)}.$$

(2). Let s and J be as in case (1). Consider the union of the 2-cells of X that correspond to the family of edges $(s,t) \in L^{(1)}$ with $t \in J$. Let Ξ be one of its connected components. It is a 2-cell complex whose universal cover is an elementary polygonal complex. We color its edges black and white : the black edges are those labelled by s and the white ones those labelled by an element of J. The perimeter of its 2-cells is larger than or equal to $2m_s \geq 10$ and the thickness of its black edges is equal to $\mathrm{val}(s)$.

Since $m_s \geq 5$ we see that for every $t \in J$ and $K \subset S$ with $\{s,t\} \subsetneqq K$, the subgroup Γ_K is infinite. Thus Ξ minus the union of its white edges is an open subset of X. We will construct an elementary polygonal complex $Y \subset \Xi$ that decomposes X. The statement will then follow from Corollaries 3.7(1) and 6.6 again.

The construction of Y uses a variant of the method presented in the proof of Proposition 7.1. We associate to every black edge $e \subset \Xi$ a *rooted directed tree* defined by the following process :

(A) Join by a directed segment the middle of e to the center O of every 2-cell c containing e,

(B) Connect by a directed segment the center O to the middle of every black edge e' of c that is distinct from e and from the two nearest ones,

(C) Restart the process with e' and every 2-cell distinct from c that contains e'.

Let T_e be the resulting graph in Ξ. We claim that T_e is a bi-Lipschitz embedded tree in X. To see it we first modify the constant curvature of every 2-cell of Ξ so that their angles become equal to $\frac{3\pi}{4}$. Since the original angles were at least equal to $\pi - \frac{\pi}{m_s} \geq \frac{3\pi}{4}$ and since $\frac{3\pi}{4} + \frac{3\pi}{4} + \frac{\pi}{2} = 2\pi$, the complex X remains nonpositively curved.

Consider a directed path $\gamma \subset T_e$ as defined in the proof of Proposition 7.1, and let $c_1, ..., c_n \subset \Xi$ be the 2-cells successively met by γ. Their union is not convex in X, but a slight modification is. Indeed, pick $i \in \{1, ..., n-1\}$, let x be one of the vertices of the segment $c_i \cap c_{i+1}$, and let $y \in c_i \setminus c_{i+1}$, $z \in c_{i+1} \setminus c_i$ be the vertices adjacent to x. Consider the geodesic simplex $\sigma_x \subset X$ whose vertices are x, y, z. Since X is non positively curved with unit length edges and $\frac{3\pi}{4}$ angles, the angles of σ_x at y and z are smaller than $\frac{\pi}{4}$. Moreover item (B) implies that for two consecutive segments $c_{i-1} \cap c_i$ and $c_i \cap c_{i+1}$ the associated simplices are disjoint (they are separated by a black edge of c_i at least). It follows from angle considerations that

$$(\cup_i c_i) \cup (\cup_x \sigma_x)$$

is a convex subset of X. The rest of the argument and of the construction is similar to the proof of Proposition 7.1. Here the 2-cells of the associated elementary polygonal complex Y have perimeter larger than or equal to $2(m_s - 2)$ because of the second item of the construction process. $\square$

Example 8.3. When a hyperbolic group boundary $\partial\Gamma$ satisfies the CLP one knows from Theorem 3.8(2) that $p_{\neq 0}(\Gamma) = \mathrm{Confdim}(\partial\Gamma)$. So Proposition 8.2 yields an upper bound for the conformal dimension of CLP Coxeter boundaries. For example consider the following hyperbolic Coxeter group

$$\Gamma = \langle s_1, \ldots, s_4 \mid s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1 \text{ for } i \neq j \rangle,$$

where the order m_{ij} is finite for all $i \neq j$ and $\sum_{i \neq j} \frac{1}{m_{ij}} < 1$ for all $j \in \{1, ..., 4\}$. The associated graph is the 1-skeleton of the tetrahedron. The visual boundary is homeomorphic to the Sierpinski carpet, so its conformal dimension is larger than 1 [Mac10]. Moreover it admits the CLP [BK]. Define

$$m = \max_{1 \le i \le 4} (\min_{j \neq i} m_{ij}).$$

If $m \geq 5$ then from Proposition 8.2(2) we get that

$$\mathrm{Confdim}(\partial \Gamma) \leq 1 + \frac{\log 2}{\log(m-3)}.$$

In particular if we choose an $i \in \{1, ..., 4\}$, fix the orders $\{m_{jk}\}_{j \neq i, k \neq i}$, and let the $\{m_{ij}\}_{j \neq i}$ go to $+\infty$, the conformal dimension tends to 1. We obtain in such a way a family of Coxeter groups with Sierpinski carpet boundaries of different conformal dimensions, which all contain an isomorphic peripheral subgroup, namely the subgroup generated by $\{s_j\}_{j \neq i}$. Existence of groups with these properties was evoked in Example 5.4.

Example 8.4. The previous examples are quite special because they satisfy the CLP independently of the choice of the coefficients m_{st}. In general the CLP for Coxeter group boundaries is very sensitive to the coefficients m_{st}. The reason is the following: suppose that the graph of a hyperbolic Coxeter group Γ decomposes as $L = L_1 \cup L_2$ where L_i $(i = 1, 2)$ is a *flag* subgraph of L (*i.e.* every edge of L with both endpoints in L_i belongs to L_i). Denote by Γ_i the Coxeter group with defining graph L_i, and assume that :

- There is a vertex $s \in L_1$ such that no edges issuing from s belong to L_2,
- The Coxeter group Γ_2 satisfies $\mathrm{Confdim}(\partial\Gamma_2) > 1$.

Then if the coefficients of the edges issuing from s are large enough compared to $\mathrm{Confdim}(\partial\Gamma_2)$, we get from Proposition 8.2(2) and Theorem 3.8(1) :

$$p_{\neq 0}(\Gamma) < \mathrm{Confdim}(\partial\Gamma_2) \leq \mathrm{Confdim}(\partial\Gamma),$$

and so, according to Theorem 3.8(2), the CLP fails for $\partial\Gamma$.

Concrete examples are provided, for instance, by Coxeter groups whose defining graphs are complete bipartite. Let $L(k, \ell)$ be the full bipartite graph with k black vertices and ℓ white vertices ($k \geq 3$, $\ell \geq 3$). When all the coefficients m_{st} are equal to the same integer $m \geq 3$ then it is known that the corresponding Coxeter group boundary admits the CLP [BP00, BK]. Now suppose that the number k of black vertices is at least equal to 4. Pick one black vertex and decompose $L(k,\ell)$ accordingly :

$$L(k,\ell) = L(1,\ell) \cup L(k-1,\ell).$$

Next, choose the coefficients $m_{st} \geq 3$ arbitrarily on the edges of the second factor graph and let the coefficients of the first one go to $+\infty$. Since we have $k-1 \geq 3$, the boundary of the second factor group is homeomorphic to the Menger sponge; thus its conformal dimension is larger than 1 [Mac10]. Therefore the above discussion applies and the CLP fails.

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