TOPOLOGY OF SOBOLEV MAPPINGS II

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Abstract. We generalize and improve some recent results obtained by H. Brezis and Y. Y. Li [BL] concerning topologies of Sobolev mappings between Riemannian manifolds. We also settle two of their conjectures. In connections with the latter results, we find some global topological obstructions for smooth maps to be weakly sequentially dense or to be dense in the strong topology for Sobolev spaces of mappings. In fact, we obtain a necessary and sufficient condition for smooth maps to be dense in the strong topology, which corrects theorem 1 of [Be2].

Contents

1. Introduction

Throughout the paper, unless otherwise stated explicitly, we always assume M and N are compact smooth Riemannian manifolds without boundary and they are isometrically embedded into $\mathbb{R}^l$ and $\mathbb{R}^{\bar{l}}$ respectively. Denote $n = dimM$.

For any $1 \le p < \infty$, we consider the space of Sobolev mappings

$$W^{1,p}(M,N) = \left\{ u : u \in W^{1,p}\left(M, \mathbb{R}^{\bar{l}}\right), u(x) \in N \text{ for a.e. } x \in M \right\}, \tag{1.1}$$

with $d(u,v) = |u-v|_{W^{1,p}\left(M,\mathbb{R}^{\bar{l}}\right)}$ as the metric. In [BL], Brezis and Li initiated the study of path connectedness of the space $W^{1,p}(M,N)$. As in [BL], one defines $u \sim_p v$ for two maps $u, v \in W^{1,p}(M,N)$ if there exists a continuous path $w(\cdot) \in C\left([0,1], W^{1,p}(M,N)\right)$ such that $w(0) = u$ and $w(1) = v$. Then it was shown in [BL] that $W^{1,p}(M,N)$ is path connected when $1 \le p < 2$, $n \ge 2$ and N is connected. In fact, Brezis and Li showed that if $1 \le p < n$, and N is $([p]-1)$-connected, that is $\pi_i(N) = 0$ for $0 \le i \le [p]-1$, then $W^{1,p}(M,N)$ is path connected. On the other hand, they observed the following facts

(i) $W^{1,2}\left(S^1 \times \Lambda, S^1\right)$ is not path connected for any compact Riemannian manifold Λ with $\dim(\Lambda) \ge 1$. Similarly $W^{1,p}(S^n \times \Lambda, S^n)$ is not path connected for $p \ge n+1 \ge 2$.

(ii) $W^{1,p}(S^n, N)$ is path connected if $1 \le p < n$ and N is connected.

(iii) For any $m \ge 1$, $1 \le p < n+1$ and any connected N, $W^{1,p}(S^n \times B_1^m, N)$ is path connected.

One of the main results of the present work is the following (see Theorem 5.1)

Theorem 1.1. *Assume $1 \le p < n$, and $u, v \in W^{1,p}(M,N)$. Then $u \sim_p v$ if and only if u is $([p]-1)$-homotopic to v.*

For an accurate description of "$([p]-1)$-homotopy", one should refer to Definition 4.1. Roughly speaking, we say two maps $u, v \in W^{1,p}(M,N)$ are $([p]-1)$-homotopic, if for a generic $([p]-1)$-skeleton $M^{[p]-1}$ of M, $u|_{M^{[p]-1}}$ and $v|_{M^{[p]-1}}$ are homotopic. Note that on generic $([p]-1)$-skeletons, u and v are both in $W^{1,p}$ and hence they are essentially continuous. It, therefore, makes sense to say whether or not they are homotopic in the usual sense. It was proved by B. White in section 3 of [Wh2] that this definition does not depend on the specific choice of generic skeletons. With Theorem 1.1 we are able to reduce the question of path connectedness for $W^{1,p}(M,N)$ to a purely topological problem. For the latter the answers are standard in topology. Indeed we have (see Corollary 5.3)

Corollary 1.1. *Assume M and N are connected, and $1 \le p < n$. If there exists a $k \in \mathbb{Z}$ with $0 \le k \le [p]-1$, such that $\pi_i(M) = 0$ for $1 \le i \le k$, $\pi_i(N) = 0$ for $k+1 \le i \le [p]-1$, then $W^{1,p}(M,N)$ is path connected.*

Note that when $1 \le p < 2$, we may simply take $k = 0$. Hence $W^{1,p}(M,N)$ is always path connected as long as $n \ge 2$ and both M and N are connected. Corollary 1.1 generalizes theorem 0.2, theorem 0.3 and proposition 0.1 in [BL]. Recall that for any $1 \le q < p$, we have a map $i_{p,q} : W^{1,p}(M,N)/\sim_p \to W^{1,q}(M,N)/\sim_q$ defined in a natural way (see [BL]). Then another interesting implication of Theorem 1.1 is the following positive answer to the conjecture 2 (and its strengthened version conjecture 2′) of [BL] (see Corollary 5.1).

Corollary 1.2. *Assume $k \in \mathbb{N}$, $k \le q < p < k+1$. Then $i_{p,q}$ is a bijection.*

We now turn to the question whether a given map $u \in W^{1,p}(M,N)$ can be connected to a smooth map by a continuous path in $W^{1,p}(M,N)$. It was shown in theorem 0.4, theorem 0.5 in [BL] that either if $\dim M = 3$ and $\partial M \ne \emptyset$ (for any $1 \le p < \infty$ and any connected N) or if $N = S^1$ (any $1 \le p < \infty$ and any M), then any $u \in W^{1,p}(M,N)$ can be connected to a smooth map by a continuous path in $W^{1,p}(M,N)$. It was conjectured in [BL] that this is always the case for general smooth compact connected Riemannian manifolds. However, we find the issue is closely related to the question that whether such a map u can be weakly approximated by a sequence of smooth maps in $W^{1,p}(M,N)$. Recall two mapping spaces closely related to $W^{1,p}(M,N)$,

$$H_S^{1,p}(M,N) = \text{ the strong closure of } C^\infty(M,N) \text{ in } W^{1,p}(M,N);$$

$$H_W^{1,p}(M,N) = \{u : u \in W^{1,p}(M,N), \text{ there exists a sequence } u_i \in C^\infty(M,N)$$

$$\text{such that } u_i \rightharpoonup u \text{ in } W^{1,p}\left(M, \mathbb{R}^{\bar{l}}\right)\}.$$

Obviously we have

$$H_S^{1,p}(M,N) \subset H_W^{1,p}(M,N) \subset W^{1,p}(M,N)\,. \tag{1.2}$$

Whether the above inclusions in (1.2) are strict or not is a difficult question and has been studied by various authors. For the case $M = B^3$, $N = S^2$ and $p = 2$, it was shown in [BBC] that $H_W^{1,2}\left(B^3, S^2\right) = W^{1,2}\left(B^3, S^2\right)$. On the other hand, it is easy to check $H_S^{1,2}\left(B^3, S^2\right) \neq W^{1,2}\left(B^3, S^2\right)$. In fact, in [Be1], Bethuel gave a characterization of maps in $H_S^{1,2}\left(B^3, S^2\right)$. Recently, Hardt and Rivière [HR] proved a necessary and sufficient condition of maps in $H_S^{1,3}\left(B^4, S^2\right)$ in terms of certain quasi-mass of "minimal connections". For general manifolds M and N, some remarkable results were first established in [Be2] (see [Ha] for an alternative approach of the main result of [Be2] under some additional topological conditions). Recently some interesting progresses were made in [PR] for sequentially weak closure of smooth maps and geometric control on the so called "minimal connections". In general, it does not seem to be feasible to construct such "minimal connections" with geometric and analytic controls. Indeed, there is a global topological obstruction. More precisely we have (see Proposition 5.2 and Theorem 7.1)

Theorem 1.2. *Assume $1 \leq p < n$, $u \in W^{1,p}(M,N)$, and $h : K \to M$ is a Lipschitz rectilinear cell decomposition. Then u can be connected to a smooth map by a continuous path in $W^{1,p}(M,N)$ if and only if $u_{\#,p}(h)$ is extendible to M with respect to N. This topological condition on $u_{\#,p}(h)$ is also a necessary condition for u to be in $H_W^{1,p}(M,N)$.*

For the meaning of "$u_{\#,p}(h)$" and "extendible to M with respect to N" one should refer to Definition 2.2 and Remark 4.1. As a consequence of Theorem 1.2, we have (see Corollary 5.4 and the statement after Theorem 7.1)

Corollary 1.3. *Assume $1 \leq p < n$. Then every map in $W^{1,p}(M,N)$ can be connected by a continuous path in $W^{1,p}(M,N)$ to a smooth map if and only if M satisfies the $([p]-1)$-extension property with respect to N. The latter topological condition is also a necessary condition for $H_W^{1,p}(M,N)$ to be equal to $W^{1,p}(M,N)$.*

For the meaning of "$([p]-1)$-extension property with respect to N", one should refer to Definition 2.3. In particular, we have (see Remark 5.1)

Corollary 1.4. *Assume N is connected and $1 \leq p < n$. If either $[p] = 1$ or $[p] \geq 2$ and $\pi_i(N) = 0$ for $[p] \leq i \leq n-1$, then every map in $W^{1,p}(M,N)$ can be connected to a smooth map.*

We note that theorem 0.5 of [BL] follows from the Corollary 1.4. As for counterexamples to the conjecture 1 of [BL] and to the sequential weak density of $C^\infty(M,N)$ in $W^{1,p}(M,N)$ we have (see Corollary 5.5, Remark 5.2, and the discussions after Theorem 7.1)

Corollary 1.5. *Assume $m_1, m_2 \in \mathbb{N}$, $m_2 < m_1$.*

- *If $3 \leq p < 2m_2+2$, then there are maps in $W^{1,p}(\mathbb{CP}^{m_1}, \mathbb{CP}^{m_2})$ which cannot be connected to any smooth map by continuous paths in $W^{1,p}(\mathbb{CP}^{m_1}, \mathbb{CP}^{m_2})$. In addition $H_W^{1,p}(\mathbb{CP}^{m_1}, \mathbb{CP}^{m_2}) \neq W^{1,p}(\mathbb{CP}^{m_1}, \mathbb{CP}^{m_2})$;*

- *If $2 \le p < m_2+1$, then there are maps in $W^{1,p}(\mathbb{RP}^{m_1},\mathbb{RP}^{m_2})$ which cannot be connected to any smooth map by continuous paths in $W^{1,p}(\mathbb{RP}^{m_1},\mathbb{RP}^{m_2})$. In addition $H_W^{1,p}(\mathbb{RP}^{m_1},\mathbb{RP}^{m_2}) \neq W^{1,p}(\mathbb{RP}^{m_1},\mathbb{RP}^{m_2})$.*

In connection with Theorem 1.2 and Corollary 1.3, we have the following (see Conjecture 7.1)

Conjecture 1.1. *Assume $2 \le p < n$, $p \in \mathbb{N}$, and $h : K \to M$ is a Lipschitz rectilinear cell decomposition of M. If $u \in W^{1,p}(M,N)$ such that $u_{\#,p}(h)$ is extendible to M with respect to N, then $u \in H_W^{1,p}(M,N)$.*

One may also conjecture that if $2 \le p < n$, $p \in \mathbb{N}$, and M satisfies the $(p-1)$-extension property with respect to N, then $H_W^{1,p}(M,N) = W^{1,p}(M,N)$.

Finally we come to the question of strong density of smooth maps in $W^{1,p}(M,N)$. The following result was proved in [Be2].

Theorem ([Be2], pp153–154) *Let $1 \le p < n$. Smooth maps between M^n and N^k are dense in $W^{1,p}(M^n,N^k)$ if and only if $\pi_{[p]}(N^k) = 0$ ($[p]$ represents the largest integer less than or equal to p).*

Here we find this result has to be corrected. We have (see Theorem 6.3)

Theorem 1.3. *Let $1 \le p < n$. Smooth maps between M and N are dense in $W^{1,p}(M,N)$ if and only if $\pi_{[p]}(N) = 0$ and M satisfies the $([p]-1)$-extension property with respect to N.*

We note that without the $([p]-1)$-extension property of M with respect to N, the strong density of smooth maps in $W^{1,p}(M,N)$ is definitely false as seen from the cases $W^{1,3}(\mathbb{CP}^3,\mathbb{CP}^2)$ and $W^{1,2}(\mathbb{RP}^4,\mathbb{RP}^3)$ by Corollary 1.5 (see also [HnL1]). Theorem 1.3 has two interesting consequences (see Corollary 6.2 and Corollary 6.3)

Corollary 1.6. *Assume M and N are connected, $1 \le p < n$, k is an integer such that $0 \le k \le [p]-1$ and $\pi_i(M) = 0$ for $1 \le i \le k$, $\pi_i(N) = 0$ for $k+1 \le i \le [p]$. Then $H_S^{1,p}(M,N) = W^{1,p}(M,N)$.*

Corollary 1.7. *Assume N is connected, $1 \le p < n$, $\pi_i(N) = 0$ for $[p] \le i \le n-1$. Then $H_S^{1,p}(M,N) = W^{1,p}(M,N)$.*

Part (a) of theorem 1 in [Ha] is a special case of Corollary 1.6.

The present paper is written as follows. In Section 2, we introduce various basic concepts and notations for the topological aspects of our problem. One of the very crucial facts we used repeatedly in our proof is the homotopy extension theorem (property). We also discuss briefly k-homotopy of maps and a problem from obstruction theory. In the last part of Section 2 we discuss how a continuous homotopy can be replaced by a Lipschitz homotopy. Repeatedly applications of Fubini (and mean value) type theorems are used in the study of generic slices of Sobolev mappings in Section 3. Some quantitative controls of $W^{1,p}$ norm of maps when they are restricted to generic k-dimensional rectilinear cells are obtained. Some fine properties of Sobolev mappings such as approximately continuity and approximately differentiability (Federer-Ziemer, Calderon-Zygmund theorems) as well as area and co-area formulas are also briefly discussed. In Section 4, we discuss the k-homotopy property of $W^{1,p}(M,N)$ maps for $0 \le k \le [p]$. These issues were first studied carefully by B. White in [Wh1], [Wh2]. Here we use somewhat different arguments to obtain the main conclusions of [Wh2] as well as some generalizations.

We have included this part of proof here not only to make the discussion clear and complete but also to facilitate our arguments in later sections. In Section 5, we first establish the equivalence between $u \sim_p v$ and that u is $([p]-1)$-homotopic to v (cf. Theorem 5.1). This leads to the proof of conjecture 2 and 2′ of [BL] as well as results which generalize those in [BL]. We also derive a necessary and sufficient condition for a map $u \in W^{1,p}(M,N)$ to be connected to a smooth map by a continuous path in $W^{1,p}(M,N)$. Thus we see the connection between the classical topological obstruction theory and the problem of connecting a Sobolev map to a smooth map in the Sobolev spaces $W^{1,p}(M,N)$. Section 6 is devoted to prove a corrected version of strong density theorem. To do so, we have to give another proof of the fact ([Be2], p154, theorem 2) that maps with canonical singularities ($R^{p,\infty}(M,N)$) are always strongly dense in $W^{1,p}(M,N)$ (see Theorem 6.1). Our proof is somewhat different from the one in [Be2]. This modification becomes necessary because we have troubles with the original proof, given in [Be2], with regarding to matching the boundary values when patching cubes for the case $n-p>1$. Moreover in studying the problem whether a specific map can be approximated in the strong topology by a sequence of smooth maps, we need the explicit construction in our proof of Theorem 6.1. As a consequence we know that for $1 \le p < n$, if $p \notin \mathbb{Z}$ or $p=1$ or $2 \le p < n$ but $p \in \mathbb{Z}$ and $\pi_p(N)=0$, then $H_S^{1,p}(M,N) = H_W^{1,p}(M,N)$ (see [Be2], Theorem 7.2 and [Hn]). The case $2 \le p < n$, $p \in \mathbb{Z}$ and $\pi_p(N) \neq 0$ is much more subtle. On the other hand, we have (see Theorem 7.2), for $1 \le p < n$, $H_S^{1,p}(M,N) = W^{1,p}(M,N)$ if and only if $\pi_{[p]}(N)=0$ and $H_W^{1,p}(M,N) = W^{1,p}(M,N)$. Our proof of Theorem 6.1 also relies on various analytical estimates, some of which were obtained in the earlier work of Bethuel [Be2]. The proof of the main theorem in Section 6 (Theorem 6.3) uses in a crucial way certain new deformations from the so-called dual skeletons, which is obviously motivated by the well known work of Federer and Fleming on normal and integral currents(see [Fe], in particular chapter 4). The construction of such deformations with the right analytical estimates is the key point of the whole proof. We note that the previously constructed deformations due to B. White [Wh1] (or that in [Ha]) do not seem to work for our purpose. Finally in Section 7, we discuss weak sequential density of smooth maps in Sobolev spaces. Several technical estimates concerning generic slices of Sobolev maps as well as estimates relative to the deformations constructed in Section 6 are included in the appendixes.

The present paper treats only compact manifolds without boundary. Essentially all the results discussed here can be generalized to the case that M has a smooth nonempty boundary ∂M. We shall return to these in a future article.

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2. Some preparations

For concepts of rectilinear cell complex and simplicial complex, we use those from [Whn] (see appendix II of [Whn], the notion of rectilinear cell complex used in this paper means the complex defined on p357 of [Whn]). [Mu] is also an excellent reference for basics in differential topology, but one needs to be careful with some small differences in definitions (the name rectilinear cell complex comes from [Mu], but the notion of rectilinear cell complex defined on p70 of [Mu] is different from the definition of complex on p357 of [Whn], the notion in [Mu] does not allow any subdivision of the proper face of any cell, but the notion in [Whn] does allow it, even though this kind of complex is not used in [Whn], see p357 of [Whn]). If after a rotation and a translation, a rectilinear cell is of the form $\prod_{i=1}^{d}\left[0,a^i\right]$, $a^i\geq 0$, then we say it is a cube. We have cubic complexes similar to simplicial complexes. By mimicking the notion of smooth triangulation of a manifold, we have the concepts of smooth cubeulation and smooth rectilinear cell decomposition of a manifold. In addition, if M is a smooth compact manifold, possibly with boundary, K is a finite simplicial complex, $h:|K|\to M$ is a bi-Lipschitz map, then we say $h:K\to M$ is a Lipschitz triangulation of M. Here $|K|$ is the polytope of K, that is, the union of all simplices in K. Similarly we have Lipschitz cubeulation and Lipschitz rectilinear cell decomposition of a smooth compact manifold.

2.1. **Homotopy extension property.** Homotopy extension theorem will play a crucial role in several of our proofs. We start with the following

Definition 2.1. *Let (X,A) be a topological pair and Y be a topological space. If every continuous map*

$$H_0:(X\times\{0\})\cup(A\times[0,1])\to Y$$

has a continuous extension to $H:X\times[0,1]\to Y$, then we say (X,A) satisfies the homotopy extension property with respect to Y (HEP w.r.t. Y).

If a topological pair (X,A) satisfies the homotopy extension property with respect to any topological space Y, then we say (X,A) satisfies the homotopy extension property (HEP).

For a general discussion of HEP (cofibration), one may refer to chapter I of [Hu] and chapter 6 of [Ma]. For basics in CW complex theory, one may refer [LW] and [Whd]. The following fact is well known and its proof may be found on p68 of [LW].

Proposition 2.1. *Let X be a CW complex and A be a subcomplex. Then (X,A) satisfies the homotopy extension property.*

Another version, which is more analytical, is also important to us (cf. p14 of [Hu]).

Proposition 2.2. *Let $Y\subset\mathbb{R}^n$ be a retraction of an open subset $V\subset\mathbb{R}^n$. Suppose X is a topological space such that $X\times[0,1]$ is normal, and $A\subset X$ is a closed subset, then (X,A) satisfies the homotopy extension property with respect to Y.*

Since we will need to use the construction in the proof of this latter proposition, we present the arguments here.

Proof of Proposition 2.2. Given a continuous map

$$H_0:(X\times\{0\})\cup(A\times[0,1])\to Y,$$

by Tietze extension theorem we may find a continuous map $G:X\times[0,1]\to\mathbb{R}^n$ such that $G(x,0)=H_0(x,0)$ for $x\in X$, $G(a,t)=H_0(a,t)$ for $a\in A$ and $0\leq t\leq 1$. Now $U=G^{-1}(V)$ is open and $A\times[0,1]\subset U$, hence there exists an open set $W\supset A$

such that $W \times [0,1] \subset U$. Choose $\eta \in C(X,[0,1])$ such that $\eta|_A = 1$, $\eta|_{X \backslash W} = 0$. Let $r : V \to Y$ be the retraction map. Define $H(x,t) = r(G(x, t\eta(x)))$ for $x \in X$, $0 \le t \le 1$. Then H is the needed extension. □

Later on we also need the following

Definition 2.2. *Let A, X and Y be topological spaces, $i : A \to X$ be an embedding. Assume $(X, i(A))$ satisfies the HEP with respect to Y. Given α, a homotopy class of maps from A to Y. If for any representative f of α, $f \circ i^{-1}$ has a continuous extension to X, then we say α is extendible to X with respect to Y.*

2.2. **k-Homotopic maps and problems from obstruction theory.** We review now several basic definitions and facts concerning k-homotopy theory which has a lot to do with our main results.

Let X and Y be two topological spaces, $f, g \in C(X,Y)$. If f is homotopic to g as maps from X to Y, then we write $f \sim g$ *as maps from X to Y*. When it is clear what X and Y are, we simply write $f \sim g$.

Lemma 2.1. *Assume X and Y are topological spaces, X_1 and X_2 are CW complexes, $f, g \in C(X,Y)$, $\phi_i : X_i \to X$ is a homotopy equivalence for $i = 1,2$, $k \in \mathbb{Z}$, $k \ge 0$. If $f \circ \phi_1|_{X_1^k} \sim g \circ \phi_1|_{X_1^k}$, then $f \circ \phi_2|_{X_2^k} \sim g \circ \phi_2|_{X_2^k}$. Here X_i^k means the k-skeleton of X_i.*

Proof. Assume $\psi_i : X \to X_i$ is a homotopy inverse of ϕ_i. By the cellular approximation theorem (see p77 of [Whd]), we may find a cellular map $\varphi \in C(X_2, X_1)$ such that $\varphi \sim \psi_1 \circ \phi_2$. Then we have

$$f \circ \phi_2|_{X_2^k} \sim f \circ \phi_1 \circ \psi_1 \circ \phi_2|_{X_2^k} \sim f \circ \phi_1 \circ \varphi|_{X_2^k} \sim g \circ \phi_1 \circ \varphi|_{X_2^k}$$

$$\sim g \circ \phi_1 \circ \psi_1 \circ \phi_2|_{X_2^k} \sim g \circ \phi_2|_{X_2^k}.$$

□

Suppose X is homotopy equivalent to some CW complex X_0, and let $\phi : X_0 \to X$ be a homotopy equivalence. Given $f, g \in C(X,Y)$. We say f and g are k-homotopic as maps from X to Y, if $(f \circ \phi)|_{X_0^k} \sim (g \circ \phi)|_{X_0^k}$. Lemma 2.1 says the choice of X_0 and ϕ plays no role. Usually we write $f \sim_k g$ as maps from X to Y or simply $f \sim_k g$ when it is clear what X and Y are. It is easy to see that k-homotopicity between maps is an equivalence relation.

Similar to homotopy equivalence, we have k-homotopy equivalence between special topological spaces. Indeed, let X and Y be two topological spaces. Assume both X and Y are homotopy equivalent to some CW complexes, and $k \in \mathbb{Z}$ is given with $k \ge 0$. If we can find $\phi \in C(X,Y)$, $\psi \in C(Y,X)$ such that $\psi\phi \sim_k id_X$ and $\phi\psi \sim_k id_Y$, then we say X and Y are k-homotopy equivalent.

The classical obstruction theory deals with the extension problem for maps. The following problem is closely related to our discussion.

> *Let X be a CW complex, Y be a topological space, $k \in \mathbb{Z}$, $k \ge 0$. Given a $f \in C(X^{k+1}, Y)$. We want to know whether there exists a $g \in C(X,Y)$ such that $g|_{X^k} = f|_{X^k}$, that is whether $f|_{X^k}$ has a continuous extension to the whole X.*

We have the following

Lemma 2.2. *Let X, Y and Z be topological spaces, X and Y be endowed with CW complex structures $\left(X^j\right)_{j\in\mathbb{Z}}$ and $\left(Y^j\right)_{j\in\mathbb{Z}}$ respectively, $k\in\mathbb{Z}$, $k\geq 0$. If X is $(k+1)$-homotopy equivalent to Y and for every $f_0\in C\left(X^{k+1},Z\right)$, $f_0|_{X^k}$ has a continuous extension to the whole X, then for any $f\in C\left(Y^{k+1},Z\right)$, $f|_{Y^k}$ has a continuous extension to Y.*

Proof. We may find $\phi\in C(X,Y)$ and $\psi\in C(Y,X)$ such that $\psi\phi\sim_{k+1} id_X$, $\phi\psi\sim_{k+1} id_Y$. By the cellular approximation theorem, we may assume ϕ and ψ are both cellular. Let i be the map from Y^k to Y^{k+1} such that $i(y)=y$ for every $y\in Y^k$.

We claim that $\phi\psi\sim i$ as maps from Y^k to Y^{k+1}. In fact since $\phi\psi\sim_{k+1} id_Y$, we may find a continuous map H_0 from $Y^{k+1}\times[0,1]$ to Y, such that $H_0(y,0)=\phi(\psi(y))$, $H_0(y,1)=y$ for any $y\in Y^{k+1}$. By the cellular approximation theorem we may find a cellular map H from $Y^{k+1}\times[0,1]$ to Y such that $H(y,0)=\phi(\psi(y))$, $H(y,1)=y$ for any $y\in Y^{k+1}$. Since $H\left(Y^k\times[0,1]\right)\subset Y^{k+1}$, the claim follows. Next, for any given $f\in C\left(Y^{k+1},Z\right)$, we define $f_0(x)=f(\phi(x))$ for $x\in X^{k+1}$. Then we may find $g_0\in C(X,Z)$ such that $g_0|_{X^k}=f_0|_{X^k}$. Set $g=g_0\circ\psi$, by the above claim we see $g|_{Y^k}\sim f|_{Y^k}$. It follows from Proposition 2.1 that $f|_{Y^k}$ has a continuous extension to Y. □

Now let us introduce the following

Definition 2.3. *Let X and Y be topological spaces where X possesses some CW complex structure, and $k\in\mathbb{Z}$, $k\geq 0$. If for some CW complex structure $\left(X^j\right)_{j\in\mathbb{Z}}$ of X, every $f\in C\left(X^{k+1},Y\right)$, $f|_{X^k}$ has a continuous extension to X, then we say X satisfies the k-extension property with respect to Y.*

By Lemma 2.2, we see the k-extension property does not depend on the particular choice of CW complex structure on X. This fact will be useful to us later in constructions of various examples. In other words, it suffices to check this property for a particular CW complex structure of X.

2.3. **From continuous maps to Lipschitz maps.** Let X be a compact metric space with metric denoted as d. For any function $f:X\to\mathbb{R}$, we set

$$|f|_{\infty,X}=\sup_{x\in X}|f(x)|,\quad [f]_{Lip(X)}=\sup_{x_1,x_2\in X,\ x_1\neq x_2}\frac{|f(x_1)-f(x_2)|}{d(x_1,x_2)}.$$

We simply write $|f|_\infty$ and $[f]_{Lip}$ when it is clear what X is. Define

$$Lip(X,\mathbb{R})=\left\{f:X\to\mathbb{R}:[f]_{Lip(X)}<\infty\right\},$$

it is a Banach space under the norm

$$|f|_{Lip(X)}=|f|_{\infty,X}+[f]_{Lip(X)}.$$

It is always convenient to replace usual continuous homotopies by Lipschitz homotopies when the image spaces are compact smooth manifolds as in present article. We describe a few elementary results below which will be sufficient for our purposes.

Lemma 2.3. *Let X be a compact metric space. Then $Lip(X,\mathbb{R})$ is dense in $C(X,\mathbb{R})$ under the uniform convergence topology.*

Proof. Indeed this follows easily from Stone-Weierstrass theorem. But we may also give a direct proof. Given a $f \in C(X, \mathbb{R})$. For any $a \in \mathbb{R}$, $a > 0$, define

$$f_a(x) = \min_{y \in X} (f(y) + a \cdot d(x, y)) \qquad \text{for any } x \in X.$$

We easily check $[f_a]_{Lip} \le a$ and $|f_a - f|_\infty \to 0$ as $a \to \infty$. □

Proposition 2.3. *Let X be a compact metric space. Then we have*

(1) *$Lip(X, N)$ is dense in $C(X, N)$ under the uniform convergence topology.*
(2) *For any $f \in C(X, N)$, there exists a $g \in Lip(X, N)$ such that $f \sim g$.*
(3) *For any $f, g \in Lip(X, N)$, if $f \sim g$, then there exists a continuous path in $Lip(X, N)$, namely $H \in C([0,1], Lip(X, N))$, such that $H(0) = f$, $H(1) = g$. Usually we write the latter statement as $f \sim_{Lip} g$.*

Proof. Choose $\varepsilon > 0$ small enough such that

$$V_{2\varepsilon} = \left\{ y : y \in \mathbb{R}^{\bar{l}},\ \operatorname{dist}(y, N) < 2\varepsilon \right\}$$

is a tubular neighborhood of N. Let $\pi : V_{2\varepsilon} \to N$ be the nearest point projection map, which is smooth because of the smallness of ε.

Given any $f \in C(X, N)$. By Lemma 2.3 we may find $f_j \in Lip\left(X, \mathbb{R}^{\bar{l}}\right)$ such that f_j converges to f uniformly. For j large enough, we have $f_j(X) \subset V_\varepsilon$. Let $g_j = \pi \circ f_j$. Then $g_j \in Lip(X, N)$ and g_j converges uniformly to f. This proves (1).

Given any $f \in C(X, N)$, choose a $g \in Lip(X, N)$ such that $|f - g|_\infty \le \varepsilon$. Let

$$H(x, t) = \pi((1-t) f(x) + t g(x)) \qquad \text{for } x \in X,\ 0 \le t \le 1.$$

Then H is a homotopy from f to g. This proves (2).

Given $f, g \in Lip(X, N)$ such that $f \sim g$, let $G : X \times [0,1] \to N$ be a continuous map such that $G(x, 0) = f(x)$, $G(x, 1) = g(x)$ for $x \in X$. Choose $\delta > 0$ small enough such that for $x_1, x_2 \in X$, $t_1, t_2 \in [0,1]$, we have $|G(x_1, t_1) - G(x_2, t_2)| \le \frac{\varepsilon}{8}$ when $d(x_1, x_2) + |t_1 - t_2| \le \delta$. Let $G_t : X \to N$ be defined by $G_t(x) = G(x, t)$ for $x \in X$. Choose $m \in \mathbb{N}$ such that $1/m < \delta$. For $1 \le k \le m - 1$, choose $L_{k/m} \in Lip(X, N)$ such that $|L_{k/m}(x) - G_{k/m}(x)| \le \frac{\varepsilon}{8}$ for any $x \in X$. Set $L_0 = f$, $L_1 = g$. For any $0 \le k \le m-1$, $t \in \left[\frac{k}{m}, \frac{k+1}{m}\right]$, $x \in X$, set

$$L(t)(x) = (k + 1 - mt) L_{k/m}(x) + (mt - k) L_{(k+1)/m}(x),$$

Clearly $L \in C\left([0,1], Lip\left(X, \mathbb{R}^{\bar{l}}\right)\right)$. Let $H(t)(x) = \pi(L(t)(x))$ for $x \in X$, $0 \le t \le 1$. Then clearly

$$|H(t_2) - H(t_1)|_\infty \le c(N) |L(t_2) - L(t_1)|_\infty. \tag{2.1}$$

On the other hand, $\pi|_{V_\varepsilon}$ clearly has a smooth extension $\overline{\pi} : \mathbb{R}^{\bar{l}} \to \mathbb{R}^{\bar{l}}$, which satisfies $\overline{\pi}(y) = 0$ for all y outside a big ball. For $0 \le t_1, t_2 \le 1$, $x_1, x_2 \in X$, we have

$$\begin{aligned}
&|(H(t_2)(x_2) - H(t_1)(x_2)) - (H(t_2)(x_1) - H(t_1)(x_1))| \qquad (2.2)\\
&= |\overline{\pi}(L(t_2)(x_2)) - \overline{\pi}(L(t_2)(x_1)) - \overline{\pi}(L(t_1)(x_2)) + \overline{\pi}(L(t_1)(x_1))| \\
&= \Big| \int_0^1 \overline{\pi}'((1-s) L(t_2)(x_1) + s L(t_2)(x_2)) (L(t_2)(x_2) - L(t_2)(x_1)) \, ds \\
&\quad - \int_0^1 \overline{\pi}'((1-s) L(t_1)(x_1) + s L(t_1)(x_2)) (L(t_1)(x_2) - L(t_1)(x_1)) \, ds \Big|
\end{aligned}$$

$\leq c(N)\left[L(t_2)-L(t_1)\right]_{Lip} d(x_1,x_2)+c(N)\left[L(t_2)\right]_{Lip}\left|L(t_2)-L(t_1)\right|_\infty d(x_1,x_2)$.
Inequalities (2.1) and (2.2) together implies $H\in C([0,1],Lip(X,N))$ and hence we get (3). □

3. Generic slices of Sobolev functions

One of the technical steps in our proofs involves restrictions of given Sobolev maps to various lower dimensional skeletons in general positions. Thus we have to obtain analytic controls on generic slices of Sobolev functions.

Let K be a finite rectilinear cell complex, $1\leq p<\infty$. Then we define

$$\mathcal{W}^{1,p}(K,\mathbb{R})=\Big\{f: f:|K|\to\mathbb{R} \text{ is a Borel function such that } f|_\Delta\in W^{1,p}(\Delta,\mathbb{R}) \text{ and the trace } T(f|_\Delta)=f|_{Bd(\Delta)}, \text{ for any } \Delta\in K\Big\}.$$

Here $Bd(\Delta)$ denotes the boundary of Δ. We also write

$$|f|_{\mathcal{W}^{1,p}(K)}=\sum_{\Delta\in K}|f|_\Delta|_{W^{1,p}(\Delta)}.$$

If $f\in\mathcal{W}^{1,p}(K,\mathbb{R})$, $k\in\mathbb{Z}$, $0\leq k<p$, then there exists a unique $g\in C\left(|K^k|,\mathbb{R}\right)$ such that for any $\Delta\in K^k$, we have $f|_\Delta=g|_\Delta$ $\mathcal{H}^d$ a.e. on Δ, with $d=\dim(\Delta)$. Here K^k is the complex of all cells in K with dimension less than or equal to k. We also remark that, whenever necessary, we use the following equivalence relation for Borel functions $f,g:|K|\to\mathbb{R}$, that is, f and g are equivalent if and only if for any $\Delta\in K$, $f|_\Delta=g|_\Delta$ $\mathcal{H}^d$ a.e. on Δ, here $d=\dim(\Delta)$.

In the future, we also need a similar function space as follows. Let K be a finite rectilinear cell complex, $m=\dim K$, $1\leq p<\infty$. Assume K satisfies

$$|K|=\bigcup_{\Delta\in K,dim(\Delta)=m}\Delta.$$

If $f:|K|\to\mathbb{R}$ is a Borel function such that

- $f|_\Delta\in W^{1,p}(\Delta,\mathbb{R})$ for any $\Delta\in K$ with $\dim(\Delta)=m$;
- For any $\Sigma\in K$ with $\dim(\Sigma)=m-1$, $\Sigma\subset Bd(\Delta_i)$, $\dim(\Delta_i)=m$ for $i=1,2$, we have $T(f|_{\Delta_1})|_\Sigma=T(f|_{\Delta_2})|_\Sigma$,

then we say f lies in $\widetilde{W}^{1,p}(K,\mathbb{R})$, and we write

$$|f|_{\widetilde{W}^{1,p}(K)}=\sum_{\Delta\in K,dim(\Delta)=m}|f|_\Delta|_{W^{1,p}(\Delta)}.$$

For convenience, we also make a convention that, whenever necessary, we always fix a suitable representative of an equivalence class of measurable functions.

Lemma 3.1. *Assume $1\leq p<\infty$, and $u\in W^{1,p}(B_1^m,\mathbb{R})$ with the trace $T(u)=f\in Lip(\partial B_1,\mathbb{R})$. Then there exists a sequence $u_i\in Lip\left(\overline{B_1},\mathbb{R}\right)$ such that $u_i|_{\partial B_1}=f$ and $u_i\to u$ in $W^{1,p}(B_1,\mathbb{R})$.*

Proof. This is a well known fact, but because the way it is proved is going to be used many times in the future, we present it here. For any $0<\delta<1$, we define

$$u_\delta(x)=\begin{cases} u(x/(1-\delta)), & \text{for } |x|\leq 1-\delta; \\ f(x/|x|), & \text{for } 1-\delta\leq|x|\leq 1. \end{cases}$$

Then $u_\delta\in W^{1,p}(B_1)$ and $u_\delta\to u$ in $W^{1,p}(B_1)$ as $\delta\to 0^+$. Hence we may assume for some $\delta\in(0,1)$, $u(x)=f(x/|x|)$ for $1-\delta\leq|x|\leq 1$. Choose $\eta\in C_c^\infty(B_1,\mathbb{R})$

such that $\eta|_{B_{1-\delta/2}} = 1$, $\eta|_{B_1 \backslash B_{1-\delta/3}} = 0$ and $0 \leq \eta \leq 1$. Choose a mollifier $\rho \in C_c^\infty(\mathbb{R}^m, \mathbb{R})$ such that $\rho \geq 0$, $\rho|_{\mathbb{R}^m \backslash B_1} = 0$ and $\int_{\mathbb{R}^m} \rho(x)\, dx = 1$. Let $\rho_\varepsilon(x) = \frac{1}{\varepsilon^m} \rho\left(\frac{x}{\varepsilon}\right)$. For $\varepsilon > 0$ small enough, let v_ε be defined on $B_{1-\delta/4}$ by $v_\varepsilon(x) = \int_{B_1} \rho_\varepsilon(x - y) u(y) dy$. Now set $w_\varepsilon(x) = \eta(x) v_\varepsilon(x) + (1 - \eta(x)) u(x)$. Then clearly we have $w_\varepsilon \in Lip\left(\overline{B_1}, \mathbb{R}\right)$ and $w_\varepsilon \to u$ in $W^{1,p}(B_1, \mathbb{R})$ as $\varepsilon \to 0^+$. □

Let Δ be a rectilinear cell, $y \in \mathrm{Int}(\Delta)$. Then for any $x \in \Delta$, we set

$$|x|_{y,\Delta} = \inf\{t : t > 0, x \in y + t(\Delta - y)\}. \tag{3.1}$$

This is the usual Minkowski functional of Δ with respect to y. When it is clear what y is, we simply write $|x|_\Delta$ instead of $|x|_{y,\Delta}$.

Lemma 3.2. *Assume K is a finite rectilinear cell complex, $1 \leq p < \infty$. Then,*

- *$Lip(|K|, \mathbb{R})$ is dense in $\mathcal{W}^{1,p}(K, \mathbb{R})$.*
- *Define a space $\mathcal{E} = \mathcal{W}^{1,p}(K, \mathbb{R}) \cap C(|K|, \mathbb{R})$ with norm*

$$|f|_{\mathcal{E}} = |f|_{\mathcal{W}^{1,p}(K)} + |f|_{\infty,|K|}.$$

Then $Lip(|K|, \mathbb{R})$ is dense in $\mathcal{E}$.

Proof. We use induction to prove the first assertion. In fact, it is clearly true when $\dim K = 0$. Assume it has been proved for $\dim K = m - 1$ for some $m \geq 1$. Now assume $\dim K = m$. Given any $u \in \mathcal{W}^{1,p}(K, \mathbb{R})$, we may find a sequence of maps $f_i \in Lip\left(|K^{m-1}|, \mathbb{R}\right)$ such that $f_i \to u|_{|K^{m-1}|}$ in $\mathcal{W}^{1,p}\left(K^{m-1}, \mathbb{R}\right)$. For any $\Delta \in K \backslash K^{m-1}$, we pick up a point $y_\Delta \in \mathrm{Int}(\Delta)$. Since Δ is bi-Lipschitz to $\overline{B_1^m}$ by the obvious map, from the proof of Lemma 3.1 we may assume for some $\delta \in (0, 1)$, for each $\Delta \in K \backslash K^{m-1}$, one has

$$u(x) = u\left(y_\Delta + \frac{x - y_\Delta}{|x|_\Delta}\right) \quad \text{for } x \in \Delta \text{ with } 1 - \delta \leq |x|_\Delta \leq 1.$$

Choose a $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta|_{(-\infty, 1-\delta/2]} = 1$, $\eta|_{[1-\delta/3, \infty)} = 0$. Let u_i be defined as

$$u_i(x) = \begin{cases} f_i(x), & x \in |K^{m-1}|; \\ \eta(|x|_\Delta) u(x) + (1 - \eta(|x|_\Delta)) f_i\left(y_\Delta + \frac{x - y_\Delta}{|x|_\Delta}\right), & x \in \Delta, \Delta \in K \backslash K^{m-1}. \end{cases}$$

Then clearly $u_i \in \mathcal{W}^{1,p}(K, \mathbb{R})$ and $u_i \to u$ in $\mathcal{W}^{1,p}(K)$. By using Lemma 3.1 on each $\Delta \in K \backslash K^{m-1}$ we get u_i can be approximated in $\mathcal{W}^{1,p}(K)$ by functions in $Lip(|K|, \mathbb{R})$, hence so is u. The proof of the second assertion is exactly the same as the first one. □

Henceforth till the end of this section we shall assume M is a n-dimensional Riemannian manifold without boundary, $\Omega \subset M$ is a domain with compact closure and Lipschitz boundary. Assume the parameter space P is a m-dimensional Riemannian manifold, Q is a d-dimensional Riemannian manifold without boundary and $D \subset Q$ is a domain with compact closure and Lipschitz boundary, and the dimensions satisfy $d + m \geq n$.

Given a map $H : \overline{D} \times P \to M$, we assume H satisfies

- (H_1) $H \in Lip(\overline{D} \times P)$ and $[H(\cdot, \xi)]_{Lip(\overline{D})} \leq c_0$ for any $\xi \in P$.
- (H_2) There exists a positive number c_1 such that the n dimensional Jacobian $J_H(x, \xi) \geq c_1$, $\mathcal{H}^{d+m}$ a.e. $(x, \xi) \in \overline{D} \times P$.

(H_3) There exists a positive number c_2 such that $\mathcal{H}^{d+m-n}\left(H^{-1}(y)\right) \leq c_2$ for $\mathcal{H}^n$ a.e. $y \in M$.

For convenience we use H^x and H_ξ to denote maps defined by $H^x(\xi) = H_\xi(x) = H(x,\xi)$.

Lemma 3.3. *Given a map $H : \overline{D} \times P \to M$ satisfying (H_1), (H_2) and (H_3). Then for any Borel function $\chi : M \to \widetilde{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ with $\chi \geq 0$, we have*

$$\int_P d\mathcal{H}^m(\xi) \int_D \chi(H_\xi(x))\, d\mathcal{H}^d(x) \leq c_1^{-1} c_2 \int_M \chi(y)\, d\mathcal{H}^n(y).$$

Especially for any Borel subset $E \subset M$, we have

$$\int_P \mathcal{H}^d\left(H_\xi^{-1}(E)\right) d\mathcal{H}^m(\xi) \leq c_1^{-1} c_2 \mathcal{H}^n(E).$$

If in addition $\mathcal{H}^n(E) = 0$, then $\mathcal{H}^d\left(H_\xi^{-1}(E)\right) = 0$ for $\mathcal{H}^m$ a.e. $\xi \in P$.

Proof. By the coarea formula (see p258 [Fe] or section 10 and 12 of [Si]) we have

$$\int_P d\mathcal{H}^m(\xi) \int_D \chi(H_\xi(x))\, d\mathcal{H}^d(x) \leq c_1^{-1} \int_{D\times P} \chi(H(x,\xi))\, J_H(x,\xi)\, d\mathcal{H}^{d+m}(x,\xi)$$

$$= c_1^{-1} \int_M \chi(y)\, \mathcal{H}^{d+m-n}\left(H^{-1}(y)\right) d\mathcal{H}^n(y) \leq c_1^{-1} c_2 \int_M \chi(y)\, d\mathcal{H}^n(y).$$

Note here we need condition (H_1) to insure the validity of coarea formula quoted above. Though the coarea formula is true for a larger class of Sobolev maps (see [MSZ]), the present form is sufficient for our purposes. □

Lemma 3.4. *Assume $1 \leq p < \infty$, $f \in W^{1,p}(\Omega, \mathbb{R})$, and $H : \overline{D} \times P \to \overline{\Omega} \subset M$ is a map satisfying $(H_1), (H_2)$ and (H_3). Then*

(1) *There exists a Borel set $E \subset P$ such that $\mathcal{H}^m(E) = 0$ and for any $\xi \in P \backslash E$,*
   (i) *$f \circ H_\xi \in W^{1,p}(D)$;*
   (ii) *f is approximately differentiable at $H_\xi(x)$ for $\mathcal{H}^d$ a.e. $x \in D$, in addition,*

$$d^{ap}(f \circ H_\xi)_x = d^{ap} f_{H_\xi(x)} \circ (H_\xi)_{*,x} \quad \text{for } \mathcal{H}^d \text{ a.e. } x \in D,$$

   here $(H_\xi)_{*,x}$ denotes the tangent map of H_ξ at x.

(2) *If $f_i \in Lip\left(\overline{\Omega}, \mathbb{R}\right)$ satisfies $f_i \to f$ in $W^{1,p}(\Omega)$, then there exists a subsequence $f_{i'}$ and a Borel set $E \subset P$ such that $\mathcal{H}^m(E) = 0$ and for any $\xi \in P \backslash E$, $f_{i'} \circ H_\xi \to f \circ H_\xi$ in $W^{1,p}(D)$.*

(3) *If we define $\tilde{f}$ by $\tilde{f}(\xi) = f \circ H_\xi$ for any $\xi \in P$, then $\tilde{f} \in L^p\left(P, W^{1,p}(D)\right)$, in addition*

$$|\tilde{f}|_{L^p(P,W^{1,p}(D))} \leq c|f|_{W^{1,p}(\Omega)},$$

   here c depends only on p, c_0, c_1 and c_2.

Proof. From p233 of [EG] or p214 of [Fe] we know there exists a Borel set X_0 such that $\mathcal{H}^n(X_0) = 0$ and for any $x \in \Omega \backslash X_0$, f is approximately differentiable at x, f_i is differentiable at x. For $x \in \Omega \backslash X_0$, $d^{ap} f(x)$ and $df_i(x)$ has already been defined. For $x \in X_0$, we simply set $d^{ap} f(x) = 0$, $df_i(x) = 0$. From Lemma 3.3 we may find a Borel set $E_1 \subset P$ such that $\mathcal{H}^m(E_1) = 0$ and $\mathcal{H}^d\left(H_\xi^{-1}(X_0)\right) = 0$ for $\xi \in P \backslash E_1$. On the other hand, from Lemma 3.3 we know

(3.2)
$$\int_P d\mathcal{H}^m(\xi) \int_D \left(|f_i(H_\xi(x)) - f(H_\xi(x))|^p + |(df_i)_{H_\xi(x)} - d^{ap} f_{H_\xi(x)}|^p\right) d\mathcal{H}^d(x)$$

$$\leq c_1^{-1} c_2 \int_\Omega \left(|f_i(y) - f(y)|^p + |(df_i)_y - d^{ap} f_y|^p\right) d\mathcal{H}^n(y) \to 0 \quad \text{as } i \to \infty.$$

Hence we may find a subsequence $f_{i'}$ and a Borel set $E_2 \subset P$ such that $\mathcal{H}^m(E_2) = 0$ and for any $\xi \in P \backslash E_2$,

$$\int_D \left(|f_{i'}(H_\xi(x)) - f(H_\xi(x))|^p + |(df_{i'})_{H_\xi(x)} - d^{ap} f_{H_\xi(x)}|^p\right) d\mathcal{H}^d(x) \to 0. \tag{3.3}$$

Then for any $\xi \in P \backslash (E_1 \cup E_2)$, we have $f_{i'} \circ H_\xi \to f \circ H_\xi$ in $L^p(D)$, also for $\mathcal{H}^d$ a.e. $x \in D$, f is approximately differentiable at $H_\xi(x)$, $f_{i'}$ is differentiable at $H_\xi(x)$ and $df_{i'}|_{H_\xi(\cdot)} \to d^{ap} f|_{H_\xi(\cdot)}$ in $L^p(D)$, which clearly implies $(df_{i'})_{H_\xi(\cdot)} \circ (H_\xi)_{*,\cdot} \to d^{ap} f_{H_\xi(\cdot)} \circ (H_\xi)_{*,\cdot}$ in $L^p(D)$. Hence we have $f \circ H_\xi \in W^{1,p}(D)$ and $f_{i'} \circ H_\xi \to f \circ H_\xi$ in $W^{1,p}(D)$, $d^{ap}(f \circ H_\xi)_x = d^{ap} f_{H_\xi(x)} \circ (H_\xi)_{*,x}$ for $\mathcal{H}^d$ a.e. $x \in D$. This implies $\tilde{f}_{i'} \to \tilde{f}$ $\mathcal{H}^m$ a.e. on P, and hence $\tilde{f}$ is Lebesgue measurable. In addition, we have

$$\int_P |\tilde{f}(\xi)|^p_{W^{1,p}(D)} d\mathcal{H}^m(\xi) \tag{3.4}$$

$$= \int_P d\mathcal{H}^m(\xi) \int_D \left(|f(H_\xi(x))|^p + |d^{ap} f_{H_\xi(x)} \circ (H_\xi)_{*,x}|^p\right) d\mathcal{H}^d(x)$$

$$\leq c \int_\Omega \left(|f(y)|^p + |d^{ap} f_y|^p\right) d\mathcal{H}^n(y),$$

here c depends only on p, c_0, c_1 and c_2. This clearly implies Lemma 3.4. □

Corollary 3.1. *Let $1 \leq p < \infty$, $f \in W^{1,p}(\Omega, \mathbb{R})$, K be a finite rectilinear cell complex, $H : |K| \times P \to \overline{\Omega} \subset M$ be a map such that $H|_{\Delta \times P}$ satisfies (H_1), (H_2) and (H_3) for any $\Delta \in K$. Then there exists a Borel set $E \subset P$ such that $\mathcal{H}^m(E) = 0$ and for any $\xi \in P \backslash E$, we have $f \circ H_\xi \in \mathcal{W}^{1,p}(K)$, in addition, the map $\tilde{f} \in L^p\left(P, \mathcal{W}^{1,p}(K)\right)$, where $\tilde{f}(\xi) = f \circ H_\xi$ for $\xi \in P$.*

Proof. Choose a sequence $f_i \in Lip\left(\overline{\Omega}, \mathbb{R}\right)$ such that $f_i \to f$ in $W^{1,p}\left(\overline{\Omega}\right)$. Then we may find a Borel set $E \subset P$ and a subsequence $f_{i'}$ such that $\mathcal{H}^m(E) = 0$ and for any $\xi \in P \backslash E$, we have

- $f \circ H_\xi|_\Delta \in W^{1,p}(\Delta)$ for any $\Delta \in K$;
- $f_{i'} \circ H_\xi|_\Delta \to f \circ H_\xi|_\Delta$ in $W^{1,p}(\Delta)$, for any $\Delta \in K$.

Since $T(f_{i'} \circ H_\xi|_\Delta) = f_{i'} \circ H_\xi|_{Bd(\Delta)}$, by taking a limit we get $T(f \circ H_\xi|_\Delta) = f \circ H_\xi|_{Bd(\Delta)}$. □

We also have the following interpolation inequality for the curve-linear case, which is an easy consequence of the classical Gagliado-Nirenberg-Sobolev's inequality.

Lemma 3.5. *Assume* $H : \overline{D} \times P \to \overline{\Omega} \subset M$ *is a map satisfying* (H_1), (H_2) *and* (H_3), $d < q < p < \infty$, *and* $f \in W^{1,p}(\Omega, \mathbb{R})$. *Then*

$$\left(\int_P |f \circ H_\xi|^p_{L^\infty(D)} d\mathcal{H}^m(\xi)\right)^{\frac{1}{p}} \le c\left(|df|^{\frac{q}{p}}_{L^p(\Omega)} |f|^{1-\frac{q}{p}}_{L^p(\Omega)} + |f|_{L^p(\Omega)}\right).$$

Here c *is a positive constant depending only on* p, q, D, c_0, c_1 *and* c_2.

Proof. By the usual Sobolev inequality, for any $\phi \in Lip(\overline{D}, \mathbb{R})$, we have

$$|\phi|_{L^\infty(D)} \le c(q, D)\left(|d\phi|_{L^q(D)} + |\phi|_{L^q(D)}\right). \tag{3.5}$$

Since $p/q > 1$, for any $\phi \in Lip(\overline{D}, \mathbb{R})$, applying (3.5) to $|\phi|^{\frac{p}{q}}$, we get

$$|\phi|^{\frac{p}{q}}_{L^\infty(D)} \le c(p, q, D)\left(||\phi|^{\frac{p}{q}-1} d\phi|_{L^q(D)} + |\phi|^{\frac{p}{q}}_{L^p(D)}\right).$$

Taking qth power on both sides and applying Holder's inequality to the right hand side, we get

$$|\phi|^p_{L^\infty(D)} \le c(p, q, D)\left(|d\phi|^q_{L^p(D)} |\phi|^{p-q}_{L^p(D)} + |\phi|^p_{L^p(D)}\right). \tag{3.6}$$

A simple approximation procedure shows (3.6) is also true for ϕ in $W^{1,p}(\Omega, \mathbb{R})$. It follows from Lemma 3.4 and (3.6) that for $\mathcal{H}^m$ a.e. $\xi \in P$,

$$\begin{aligned} |f \circ H_\xi|^p_{L^\infty(D)} &\le c(p, q, D)\left(|d(f \circ H_\xi)|^q_{L^p(D)} |f \circ H_\xi|^{p-q}_{L^p(D)} + |f \circ H_\xi|^p_{L^p(D)}\right) \\ &\le c(p, q, D, c_0)\left(|(df)_{H_\xi(\cdot)}|^q_{L^p(D)} |f \circ H_\xi|^{p-q}_{L^p(D)} + |f \circ H_\xi||^p_{L^p(D)}\right). \end{aligned}$$

Integrating both sides with respect to ξ, and using Holder's inequality, we get

$$\begin{aligned} &\int_P |f \circ H_\xi|^p_{L^\infty(D)} d\mathcal{H}^m(\xi) \\ &\le c\left(\int_P |(df)_{H_\xi(\cdot)}|^p_{L^p(D)} d\mathcal{H}^m(\xi)\right)^{\frac{q}{p}} \left(\int_P |f \circ H_\xi|^p_{L^p(D)} d\mathcal{H}^m(\xi)\right)^{1-\frac{q}{p}} \\ &\quad + c\int_P |f \circ H_\xi|^p_{L^p(D)} d\mathcal{H}^m(\xi) \\ &\le c\left(|df|^q_{L^p(\Omega)} |f|^{p-q}_{L^p(\Omega)} + |f|^p_{L^p(\Omega)}\right). \end{aligned}$$

Here c depends on p, q, D, c_0, c_1 and c_2. In the last inequality above, we have used Lemma 3.3. □

4. Homotopy of Sobolev mappings

Let X and Y be topological spaces, we use $[X, Y]$ to denote the set of all homotopy classes of continuous maps from X to Y. Given any $f \in C(X, Y)$, we use $[f]_{X,Y}$ to denote the homotopy class corresponding to f as a map from X to Y. When it is clear what X and Y are, we simply write $[f]$ instead of $[f]_{X,Y}$.

For $\varepsilon_0 > 0$, denote

$$V_{2\varepsilon_0}(M) = \left\{y : y \in \mathbb{R}^l, \operatorname{dist}(y, M) < 2\varepsilon_0\right\}.$$

We assume ε_0 is small enough such that $V_{2\varepsilon_0}(M)$ is a tubular neighborhood of M and denote $\pi_M : V_{2\varepsilon_0}(M) \to M$ as the nearest point projection map, which is smooth because of the smallness of ε_0. Given any map $h : A \to M$, we define the corresponding $H : A \times B^l_{\varepsilon_0} \to M$ by $H(a, \xi) = \pi_M(h(a) + \xi)$. If Δ is a rectilinear cell, and $h : \Delta \to M$ is a Lipschitz map, then it is easy to see (H_1),(H_2) and (H_3)

in Section 3 are satisfied by H. For reader's convenience, we write down the proof of (H_3). Let $d = \dim(\Delta)$. Given any $y \in M$. Denote M_y as the tangent space of M at y. Define a map

$$\psi : \Delta \times \{\zeta \in \mathbb{R}^l : \zeta \perp M_y, |\zeta| \leq \varepsilon_0\} \to \Delta \times \mathbb{R}^l$$

by $\psi(x,\zeta) = (x, y + \zeta - h(x))$. Then clearly $H^{-1}(y) \subset \mathrm{im}(\psi)$. It follows from the area formula that $\mathcal{H}^{d+l-n}(H^{-1}(y)) \leq \mathcal{H}^{d+l-n}(\mathrm{im}(\psi)) \leq c(d,l,[h]_{Lip(\Delta)}, M)$. This verifies (H_3). Often we write h_ξ instead of H_ξ. The notations $V_{2\overline{\varepsilon}_0}(N)$ and π_N are defined similarly. When no confusions would occur, we write π instead of π_M and π_N. We start with a few simple facts.

Lemma 4.1. *Let X be any topological space, u_0 and u_1 be continuous maps from X to N. If $|u_0 - u_1|_{\infty,X} \leq \overline{\varepsilon}_0 = \overline{\varepsilon}_0(N)$, then $u_0 \sim u_1$ as maps from X to N.*

Proof. Simply take $H(x,t) = \pi_N((1-t)u_0(x) + tu_1(x))$ for $x \in X$, $0 \leq t \leq 1$ as the homotopy. □

Lemma 4.2. *If X is a compact metric space, then $[X,N]$ is countable.*

Proof. This follows from Lemma 4.1 and the fact $C(X,\mathbb{R})$ has a countable dense subset. □

The next lemma is concerned with certain topological classes introduced by a given Sobolev map when it is restricted to a lower dimensional set.

Lemma 4.3. *Assume $1 \leq p \leq n$, $u \in W^{1,p}(M,N)$, K is a finite rectilinear cell complex, the parameter space P is a m-dimensional Riemannian manifold, $H : |K| \times P \to M$ is a map such that $H|_{\Delta \times P}$ satisfies $(H_1), (H_2)$ and (H_3) for any $\Delta \in K$. Then there exists a Borel set $E \subset P$ such that $\mathcal{H}^m(E) = 0$ and $u \circ H_\xi \in \mathcal{W}^{1,p}(K,N)$ for any $\xi \in P \backslash E$. Assume either $k = 1$ or k is an integer with $0 \leq k < p$. Define a map $\chi = \chi_{k,H,u} : P \to \left[|K^k|, N\right]$ by setting $\chi(\xi) = \left[u \circ H_\xi|_{|K^k|}\right]$. Then χ is Lebesgue measurable, that is, $\chi^{-1}(\{\alpha\})$ is Lebesgue measurable for any $\alpha \in \left[|K^k|, N\right]$. Here K^k is the finite rectilinear cell complex defined by*

$$K^k = \{\Delta \in K : dim(\Delta) \leq k\}.$$

Proof. The existence of such an E follows from Lemma 3.3 and Corollary 3.1. Note that Lemma 3.3 is needed because we only know $u(x) \in N$ for $\mathcal{H}^n$ a.e. $x \in M$. But by the second half of Lemma 3.3, we conclude that for $\mathcal{H}^m$ a.e. $\xi \in P$, for each $\Delta \in K$ with $d = \dim(\Delta)$, $u \circ H_\xi$ takes values in N, $\mathcal{H}^d$ a.e. on Δ. The Sobolev embedding theorem implies that χ is pointwise well defined away from E. Note that $k = 1$ is special because a $W^{1,1}$ function on a closed interval is absolutely continuous after a modification on a measure zero set, but in general one does not have this for a $W^{1,k}$ function on a k dimensional disk for $k > 1$. Instead we will handle this issue in Lemma 4.6. Define $\tilde{u}(\xi) = u \circ H_\xi$ for $\xi \in P \backslash E$. It follows from Corollary 3.1 that $\tilde{u} \in L^p(P, \mathcal{W}^{1,p}(K,N))$. By the Lusin's theorem, we see the function $\tilde{u}$ is continuous on the whole parameter space P away from an arbitrary small measure set. Using Sobolev embedding theorem and Lemma 4.1, one concludes that the corresponding χ is locally constant away from such small measure sets. This along with Lemma 4.2 implies the measurability of χ. □

The next result is useful for the critical case $p \in \mathbb{N}$, $p \geq 2$, which is not covered by the previous Lemma 4.3 (see Lemma 4.6 below).

Lemma 4.4. *Assume m is a natural number, and $u \in W^{1,m}(B_1^m, N)$ such that the trace $T(u) = f \in W^{1,m}(\partial B_1, N) \subset C(\partial B_1, N)$. Then for any $\varepsilon > 0$, there exists a $v \in W^{1,m}(B_1, N) \cap C\left(\overline{B_1}, N\right)$ such that $|v-u|_{W^{1,m}(B_1)} \le \varepsilon$ and $v|_{\partial B_1} = f$. In addition, there exists an $\varepsilon = \varepsilon(m,u,N) > 0$ such that if $v_1, v_2 \in W^{1,m}(B_1,N) \cap C\left(\overline{B_1}, N\right)$ satisfy $v_i|_{\partial B_1} = f$ and $|v_i - u|_{W^{1,m}(B_1)} \le \varepsilon$ for $i=1,2$, then we have $v_1 \sim v_2$ relative to ∂B_1, that is, during the homotopy, the value on ∂B_1 is always fixed.*

Proof. As in the proof of Lemma 3.1, we may assume for some $\delta \in (0,1)$, $u(x) = f(x/|x|)$ for $1-\delta \le |x| \le 1$. Choose a $\eta \in C_c^\infty(B_1, \mathbb{R})$ such that $0 \le \eta \le 1$, $\eta|_{B_{1-\delta/2}} = 1$ and $\eta|_{B_1 \setminus B_{1-\delta/3}} = 0$. For $\varepsilon > 0$ small enough, we set $v_\varepsilon(x) = \fint_{B_\varepsilon(x)} u$ for $x \in B_{1-\delta/4}$. Then we define

$$w_\varepsilon(x) = (1-\eta(x)) f\left(\frac{x}{|x|}\right) + \eta(x) v_\varepsilon(x) \qquad \text{for } x \in B_1.$$

Clearly $w_\varepsilon \in W^{1,m}\left(B_1, \mathbb{R}^{\bar{l}}\right) \cap C\left(\overline{B_1}\right)$ and $w_\varepsilon \to u$ in $W^{1,m}(B_1)$. For $x \in B_{1-\delta/2}$, from Poincaré inequality we have

$$\fint_{B_\varepsilon(x)} |u(y) - \fint_{B_\varepsilon(x)} u| dy \le c\left(m, \bar{l}\right) \left(\int_{B_\varepsilon(x)} |\nabla u|^m\right)^{\frac{1}{m}},$$

hence $\operatorname{dist}(v_\varepsilon(x), N) \to 0$ uniformly for $x \in B_{1-\delta/2}$, this implies the same thing is true for w_ε on $B_{1-\delta/2}$ because $v_\varepsilon|_{B_{1-\delta/2}} = w_\varepsilon|_{B_{1-\delta/2}}$. On the other hand from uniform continuity of f we know $w_\varepsilon(x) - f(x/|x|) \to 0$ uniformly for $x \in \overline{B_1} \backslash B_{1-\delta/2}$ as $\varepsilon \to 0^+$. Hence $\operatorname{dist}(w_\varepsilon(x), N) \to 0$ uniformly for $x \in \overline{B_1}$ as $\varepsilon \to 0^+$, from which we deduce that $\pi \circ w_\varepsilon \to u$ in $W^{1,m}(B_1)$ as $\varepsilon \to 0^+$, $\pi \circ w_\varepsilon \in W^{1,m}(B_1, N) \cap C\left(\overline{B_1}, N\right)$ and $\pi \circ w_\varepsilon|_{\partial B_1} = f$. The first half of Lemma 4.4 follows. To prove the second half, clearly we may assume $u(x) = v_1(x) = v_2(x) = f(x/|x|)$ for $1/2 \le |x| \le 1$. Choose a $\eta \in C_c^\infty(B_1, \mathbb{R})$ such that $0 \le \eta \le 1$, $\eta|_{B_{2/3}} = 1$, $\eta|_{B_1 \setminus B_{3/4}} = 0$. For $\delta > 0$ small, we define

$$(v_i)_\delta(x) = (1-\eta(x)) f\left(\frac{x}{|x|}\right) + \eta(x) \fint_{B_\delta(x)} v_i, \tag{4.1}$$

for $x \in B_1$ and $i = 1,2$. From the continuity of f we know $(v_i)_\delta(x) - f(x/|x|) \to 0$ uniformly for $2/3 \le |x| \le 1$. On the other hand, for $|x| \le 2/3$, we have $(v_i)_\delta(x) = \fint_{B_\delta(x)} v_i$, hence

$$\begin{aligned} \operatorname{dist}\left((v_i)_\delta(x), N\right) &\le \fint_{B_\delta(x)} |v_i - \fint_{B_\delta(x)} v_i| \le c\left(m, \bar{l}\right) \left(\int_{B_\delta(x)} |\nabla v_i|^m\right)^{1/m} \\ &\le c\left(m, \bar{l}\right) \left(\varepsilon + \left(\int_{B_\delta(x)} |\nabla u|^m\right)^{1/m}\right) \le \bar{\varepsilon}_0/4 \end{aligned} \tag{4.2}$$

when $0 < \delta \le \delta_0\left(m, \bar{l}, u\right)$ and $c\left(m, \bar{l}\right)\varepsilon \le \bar{\varepsilon}_0/8$. In addition we may assume $\delta_0\left(m, \bar{l}, u\right)$ is small enough so that

$$|(v_i)_\delta(x) - f(x/|x|)| \le \bar{\varepsilon}_0/4 \quad \text{for } 2/3 \le |x| \le 1, 0 < \delta \le \delta_0\left(m, \bar{l}, u\right). \tag{4.3}$$

(4.2) and (4.3) tell us

$$\operatorname{dist}\left(\left(v_i\right)_\delta(x), N\right) \leq \bar{\varepsilon}_0/4 \quad \text{for } x \in B_1, 0<\delta \leq \delta_0\left(m, \bar{l}, u\right). \tag{4.4}$$

Note that for $1 \geq |x| \geq 2/3$, $\left(v_1\right)_{\delta_0}(x)=\left(v_2\right)_{\delta_0}(x)$. For $|x| \leq 2/3$, we have

(4.5)

$$\left|\left(v_1\right)_{\delta_0}(x)-\left(v_2\right)_{\delta_0}(x)\right| \leq \fint_{B_{\delta_0}(x)}\left|v_1-v_2\right| \leq\left(\fint_{B_{\delta_0}(x)}\left|v_1-v_2\right|^m\right)^{1/m} \leq c\left(m, \bar{l}\right) \frac{\varepsilon}{\delta_0}.$$

By taking $\varepsilon=\varepsilon(m, u, N)$ small enough, we have

$$\left|\left(v_1\right)_{\delta_0}(x)-\left(v_2\right)_{\delta_0}(x)\right| \leq \bar{\varepsilon}_0/4 \quad \text{for } x \in B_1. \tag{4.6}$$

From (4.4) and (4.6) we see easily that $\pi \circ\left(v_1\right)_{\delta_0} \sim \pi \circ\left(v_2\right)_{\delta_0}$ relative to ∂B_1, indeed the map $H(x, t)=\pi\left((1-t)\left(v_1\right)_{\delta_0}(x)+t\left(v_2\right)_{\delta_0}(x)\right)$ is the needed homotopy. On the other hand it is easy to see that $v_i \sim\left(v_i\right)_{\delta_0}$ relative to ∂B_1 for $i=1,2$, the second half of Lemma 4.4 follows. We should mention that for this part one may also use the so called VMO space theory by [BN]. □

Corollary 4.1. *Assume $m \in \mathbb{N}$, and $u \in W^{1,m}(B_1^m, N)$ such that the trace $T(u)=f \in W^{1,m}(\partial B_1, N) \subset C(\partial B_1, N)$. Then there exists an $\varepsilon_1=\varepsilon_1(m, u, N)>0$ such that, for any $v_0, v_1 \in C(\overline{B_1}, N) \cap W^{1,m}(B_1, N)$ with $f_0=v_0|_{\partial B_1}, f_1=v_1|_{\partial B_1} \in W^{1,m}(\partial B_1, N)$, if $|v_i-u|_{W^{1,m}(B_1)} \leq \varepsilon_1$ and $|f_i-f|_{W^{1,m}(\partial B_1)} \leq \varepsilon_1$ for $i=0,1$, then $|f_0(x)-f_1(x)| \leq \bar{\varepsilon}_0(N)$ for any $x \in \partial B_1$ and we may find a homotopy $v(\cdot) \in C([0,1], C(\overline{B_1}, N))$ such that, $v(0)=v_0, v(1)=v_1$ and $v(t)(x)=\pi_N((1-t)f_0(x)+tf_1(x))$ for $x \in \partial B_1$ and $0 \leq t \leq 1$.*

Proof. By Sobolev embedding theorem, we may take $\varepsilon_1(m, u, N)$ small enough such that $|f_i-f|_{\infty, \partial B_1} \leq \bar{\varepsilon}_0(N)/4$ for $i=0,1$. Let

$$\bar{u}(x)=\begin{cases} u(2x), & x \in B_{1/2}; \\ f(x/|x|), & x \in \overline{B_1} \backslash B_{1/2}. \end{cases}$$

Also for $i=0,1$ denote

$$\bar{v}_i(x)=\begin{cases} v_i(2x), & x \in B_{1/2}; \\ \pi_N((2-2|x|) f_i(x/|x|)+(2|x|-1) f(x/|x|)), & x \in \overline{B_1} \backslash B_{1/2}. \end{cases}$$

A simple computation shows

$$|\bar{v}_i-\bar{u}|_{W^{1,m}(B_1)} \leq c(m, u, N)\left(|v_i-u|_{W^{1,m}(B_1)}+|f_i-f|_{W^{1,m}(\partial B_1)}\right).$$

Hence it follows from Lemma 4.4 that if we pick $\varepsilon_1(m, u, N)$ small enough, then we may find a map $\overline{H} \in C(\overline{B_1} \times[0,1], N)$ such that $\overline{H}(x, 0)=\bar{v}_0(x)$, $\overline{H}(x, 1)=\bar{v}_1(x)$ for $x \in \overline{B_1}$ and $\overline{H}(x, t)=f(x)$ for $x \in \partial B_1$, $0 \leq t \leq 1$. Let us define a map $\widetilde{H}$ on $\partial(\overline{B_1} \times[0,1])$ by

$$\widetilde{H}(x, t)=\begin{cases} v_0(x), & x \in \overline{B_1}, t=0; \\ \pi_N(3t f(x)+(1-3t) f_0(x)), & x \in \partial B_1, 0 \leq t \leq 1/3; \\ f(x), & x \in \partial B_1, 1/3 \leq t \leq 2/3; \\ \pi_N((3-3t) f(x)+(3t-2) f_1(x)), & x \in \partial B_1, 2/3 \leq t \leq 1; \\ v_1(x), & x \in \overline{B_1}, t=1. \end{cases}$$

Then it is clear that $\overline{H}|_{\partial(\overline{B_1}\times[0,1])}\sim\widetilde{H}$. On the other hand, if we set

$$H(x,t)=\begin{cases} v_0(x), & x\in\overline{B_1}, t=0;\\ \pi_N((1-t)f_0(x)+tf_1(x)), & x\in\partial B_1, 0\le t\le 1;\\ v_1(x), & x\in\overline{B_1}, t=1,\end{cases}$$

then, clearly $|H-\widetilde{H}|_{\infty,\partial(\overline{B_1}\times[0,1])}\le\overline{\varepsilon}_0(N)$. By Lemma 4.1, we know $H\sim\widetilde{H}$ on $\partial(\overline{B_1}\times[0,1])$. Hence $H\sim\overline{H}|_{\partial(\overline{B_1}\times[0,1])}$. It follows from Proposition 2.2 that H has a continuous extension to $\overline{B_1}\times[0,1]$ which takes value in N. The extension map is the needed homotopy. □

Lemma 4.5. *Let m be a natural number. K be a finite rectilinear cell complex with $dimK\le m$. Given any $u\in\mathcal{W}^{1,m}(K,N)$. Choose a $v\in C(|K|,N)\cap\mathcal{W}^{1,m}(K,N)$ such that $u|_{|K^{m-1}|}=v|_{|K^{m-1}|}$ and $|u-v|_{\mathcal{W}^{1,m}(K)}\le\varepsilon(m,K,N,u)$, a very small number. Define $\Theta(u)\in[|K|,N]$ by $\Theta(u)=[v]$. Then Θ is a well defined map from $\mathcal{W}^{1,m}(K,N)$ to $[|K|,N]$. In addition, Θ is a locally constant map.*

Proof. The existence of v and the well-definedness of $\Theta(u)$ follow from Lemma 4.4. Note that in Lemma 4.4, the homotopy between two approximation maps preserves the boundary value. This helps in patching the homotopy of all m dimensional cells into a global homotopy. The fact Θ is a locally constant map follows from Corollary 4.1. Again one just needs to apply Corollary 4.1 to m dimensional cells. □

The conclusion of Lemma 4.5 is in the same spirit as degree theory for VMO maps as studied in [BN]. By Lemma 4.5, Lemma 4.3 and its proof, one can easily deduce the following

Lemma 4.6. *Assume $p\in\mathbb{N}$, $2\le p\le n$, and K,P,M are the same as in Lemma 4.3. Then there exists a Borel set $E\subset P$ such that $\mathcal{H}^m(E)=0$ and for any $\xi\in P\backslash E$, we have $u\circ H_\xi\in\mathcal{W}^{1,p}(K,N)$. Define a map $\chi=\chi_{p,H,u}:P\to[|K^p|,N]$ by setting $\chi(\xi)=\Theta(u\circ H_\xi|_{|K^p|})$ (here Θ is the map defined in Lemma 4.5). Then χ is Lebesgue measurable.*

The next proposition is in the same spirit as Lemma 4.5. It says the homotopy classes we defined are stable under the weak and strong convergences of Sobolev mappings.

Proposition 4.1. *Assume $1\le p\le n$, $k\in\mathbb{Z}$, K,P,H are the same as in Lemma 4.3, and $u_i,u\in W^{1,p}(M,N)$. If either $0\le k\le p$ and $u_i\to u$ in $W^{1,p}(M,N)$ or $0\le k<p$ and $u_i\rightharpoonup u$ in $W^{1,p}(M,N)$, then after passing to subsequence we have $\chi_{k,H,u_{i'}}\to\chi_{k,H,u}$ $\mathcal{H}^m$ a.e. on P.*

Proof. It follows from Lemma 4.3 and Lemma 3.4 that we may find a Borel set $E_1\subset P$ such that $\mathcal{H}^m(E_1)=0$, for any $\xi\in P\backslash E_1$, $u\circ H_\xi$, $u_i\circ h_\xi\in\mathcal{W}^{1,p}(K,N)$ for every i. In addition, for $\xi\in P\backslash E_1$, $\Delta\in K$, $d=\dim(\Delta)$, we have u_i and u are approximately differentiable at $H_\xi(x)$ for $\mathcal{H}^d$ a.e. $x\in\Delta$, and

$$d^{ap}(u_i\circ H_\xi)_x=d^{ap}(u_i)_{H_\xi(x)}\circ(H_\xi)_{*,x},\quad d^{ap}(u\circ H_\xi)_x=d^{ap}u_{H_\xi(x)}\circ(H_\xi)_{*,x}$$

for $\mathcal{H}^d$ a.e. $x\in\Delta$.

First assume $0\le k\le p$ and $u_i\to u$ in $W^{1,p}(M,N)$. It follows from the proof of Lemma 3.4 that after passing to a subsequence $u_{i'}$, there exists a Borel set $E\subset P$, with $E_1\subset E$, $\mathcal{H}^m(E)=0$ such that for any $\xi\in P\backslash E$, $u_{i'}\circ H_\xi\to u\circ H_\xi$ in $\mathcal{W}^{1,p}(K,N)$. If $k<p$, it follows from Sobolev embedding theorem (applied to

every cell with dimension less than or equal to k) and Lemma 4.1 that $\chi_{k,H,u_{i'}}(\xi) \to \chi_{k,H,u}(\xi)$. If $k = p$, the same conclusion follows from Lemma 4.5.

Now assume $0 \leq k < p$ and $u_i \rightharpoonup u$ in $W^{1,p}(M, N)$. Fix a $q \in (k, p)$. Given any $\Delta \in K$ with $\dim(\Delta) \leq k$. It follows from Lemma 3.5 that

$$\int_P |u_i \circ H_\xi - u \circ H_\xi|^p_{L^\infty(\Delta)} d\mathcal{H}^m(\xi)$$
$$\leq c(p, q, \Delta, \bar{l}, c_0, c_1, c_2) \left(|du_i - du|^q_{L^p(M)} |u_i - u|^{p-q}_{L^p(M)} + |u_i - u|^p_{L^p(M)} \right).$$

Summing up, using the condition $u_i \rightharpoonup u$, we get

$$\int_P \sum_{\Delta \in K, dim(\Delta) \leq k} |u_i \circ H_\xi - u \circ H_\xi|^p_{L^\infty(\Delta)} d\mathcal{H}^m(\xi) \to 0,$$

as $i \to \infty$. After passing to subsequence $u_{i'}$, we may find a Borel set $E \subset P$ such that $E_1 \subset E$, $\mathcal{H}^m(E) = 0$ and for any $\xi \in P \backslash E$,

$$\sum_{\Delta \in K, dim(\Delta) \leq k} |u_{i'} \circ H_\xi - u \circ H_\xi|^p_{L^\infty(\Delta)} \to 0$$

as $i' \to \infty$. This together with Lemma 4.1 implies $\chi_{k,H,u_{i'}}(\xi) \to \chi_{k,H,u}(\xi)$. □

In the rest of this section, we want to present some results closely related to B. White's paper [Wh2]. These results will be needed later on. The following lemma says that $W^{1,p}$ maps have well defined $([p]-1)$-homotopy classes. The reader should compare it with Lemma 4.3 and Lemma 4.6.

Lemma 4.7. *Assume $1 \leq p \leq n$, $u \in W^{1,p}(M, N)$, K, P, H are the same as in Lemma 4.3 and P is connected, $k \in \mathbb{Z}$, $0 \leq k \leq [p]-1$, $\chi = \chi_{k,H,u}$. Then $\chi \equiv const$ $\mathcal{H}^m$ a.e. on P.*

Proof. By standard arguments, we only need to show that when $P = B^m_4$, one has $\chi \equiv const$ $\mathcal{H}^m$ a.e. on B^m_1.

Define a new rectilinear cell complex $\widetilde{K}$ by

$$\widetilde{K} = \{\Delta \times \{0\}, \Delta \times \{1\}, \Delta \times [0,1] : \Delta \in K\};$$

Then $|\widetilde{K}| = |K| \times [0,1]$.

We claim the following fact. For any $\zeta \in B^m_2$, there exists a Borel set $E_\zeta \subset B^m_2$ such that $\mathcal{H}^m(E_\zeta) = 0$ and for any $\xi \in B^m_2 \backslash E_\zeta$, we have $u \circ H_\xi, u \circ H_{\xi+\zeta} \in \mathcal{W}^{1,p}(K, N)$ and $u \circ H_\xi|_{|K^k|} \sim u \circ H_{\xi+\zeta}|_{|K^k|}$ as maps from $|K^k|$ to N. To show this fact, we define $\widetilde{H} : |\widetilde{K}| \times B_2 = |K| \times [0,1] \times B_2 \to M$ by $\widetilde{H}(x, t, \xi) = H(x, \xi + t\zeta)$. First assume $k + 1 < p$. Then by Lemma 4.3, we may find a Borel set $E_\zeta \subset B_2$ such that $\mathcal{H}^m(E_\zeta) = 0$ and $u \circ \widetilde{H}_\xi \in \mathcal{W}^{1,p}\left(\widetilde{K}, N\right)$ for any $\xi \in B_2 \backslash E_\zeta$. By Sobolev embedding theorem we may assume $u \circ \widetilde{H}_\xi$ is continuous on $|\widetilde{K}^{k+1}|$. Since $u\left(\widetilde{H}_\xi(x, 0)\right) = u(H_\xi(x))$, $u\left(\widetilde{H}_\xi(x, 1)\right) = u(H_{\xi+\zeta}(x))$, and $|K^k| \times [0,1] \subset |\widetilde{K}^{k+1}|$, we get $u \circ H_\xi|_{|K^k|} \sim u \circ H_{\xi+\zeta}|_{|K^k|}$. If $k + 1 = p$, then we only need to note that by Lemma 4.4, for the above chosen E_ζ, given any $\xi \in B_2 \backslash E_\zeta$, we may find a continuous map $\psi : |\widetilde{K}^{k+1}| \to N$ such that for any $\Delta \in \widetilde{K}$ with $d = \dim(\Delta) \leq k$, ψ and $u \circ \widetilde{H}_\xi$ are $\mathcal{H}^d$ a.e. equal on Δ. This clearly implies the needed homotopy.

Let E_0 be the measure zero set on which χ is not defined. If χ is not constant $\mathcal{H}^m$ a.e. on $B_1 \backslash E_0$, since $\left[|K^k|, N\right]$ is countable (by Lemma 4.2), we may find two different elements $\alpha_1, \alpha_2 \in \left[|K^k|, N\right]$ such that $\mathcal{H}^m(E_i) > 0$, where $E_i = \chi^{-1}(\{\alpha_i\}) \cap B_1$, $i = 1, 2$. Choose a density point $\xi_i \in E_i$, that is

$$\lim_{r\to 0^+} \frac{\mathcal{H}^m(B_r(\xi_i) \cap E_i)}{\mathcal{H}^m(B_r(\xi_i))} = 1.$$

Let $\zeta = \xi_1 - \xi_2 \in B_2$. Then $\chi(\xi) = \chi(\xi + \zeta)$ for $\xi \in B_2 \backslash E_3$, where $E_3 = E_\zeta \cup E_0 \cup (E_0 - \zeta)$, $\mathcal{H}^m(E_3) = 0$. Because ξ_1 is a density point for both E_1 and $\zeta + (E_2 \backslash E_3)$, we have $(\zeta + (E_2 \backslash E_3)) \cap E_1 \neq \emptyset$. Choose $\bar{\xi}_1 \in E_1$, $\bar{\xi}_2 \in E_2 \backslash E_3$ such that $\bar{\xi}_1 = \zeta + \bar{\xi}_2$. Then $\chi(\bar{\xi}_1) = \chi(\bar{\xi}_2)$, that is $\alpha_1 = \alpha_2$, contradiction. □

Remark 4.1. *Assume $1 \le p \le n$, $u \in W^{1,p}(M, N)$, K is a finite rectilinear cell complex, $h : |K| \to M$ is a Lipschitz map. Denote the corresponding $H : |K| \times B^l_{\varepsilon_0} \to M$ as $H(x, \xi) = \pi(h(x) + \xi)$. Then $\chi_{[p]-1,H,u} \equiv const$ a.e. on $B^l_{\varepsilon_0}$, we denote this constant as $u_{\#,p}(h)$.*

The next two lemmas say the object $u_{\#,p}(h)$ we defined in Remark 4.1 is indeed well behaved topologically.

Lemma 4.8. *Assume $1 \le p \le n$, $u \in W^{1,p}(M, N)$, K is a finite rectilinear cell complex, $h_0, h_1 : |K| \to M$ are Lipschitz maps and $h_0 \sim h_1$ as maps from $|K|$ to M. Then $u_{\#,p}(h_0) = u_{\#,p}(h_1)$.*

Proof. Let $\widetilde{K}$ be the same rectilinear cell complex as in the proof of Lemma 4.7. Then $|\widetilde{K}| = |K| \times [0, 1]$. We may find a $g \in Lip(|K| \times [0, 1], N)$ such that $g(x, 0) = h_0(x)$, $g(x, 1) = h_1(x)$ for any $x \in |K|$. Indeed the homotopy constructed in the proof of Proposition 2.3 (3) satisfies this requirement. It follows from Lemma 4.3 that there exists a Borel set $E \subset B^l_{\varepsilon_0}$ with $\mathcal{H}^l(E) = 0$ and for any $\xi \in B_{\varepsilon_0} \backslash E$, $u \circ g_\xi \in \mathcal{W}^{1,p}(\widetilde{K}, N)$. Observing $u \circ g_\xi(x, 0) = u \circ (h_0)_\xi(x)$ and $u \circ g_\xi(x, 1) = u \circ (h_1)_\xi(x)$ for $x \in |K|$, it follows from the proof of Lemma 4.7 that $u \circ (h_0)_\xi|_{|K^{[p]-1}|} \sim u \circ (h_1)_\xi|_{|K^{[p]-1}|}$. This clearly implies $u_{\#,p}(h_0) = u_{\#,p}(h_1)$. □

Remark 4.2. *Assume $1 \le p \le n$, $u \in W^{1,p}(M, N)$, and K is a finite rectilinear cell complex. Given any $\alpha \in [|K|, M]$, choose a $f \in Lip(|K|, M)$ with $[f] = \alpha$, then we write $u_{*,p}(\alpha) = u_{\#,p}(f)$. By Proposition 2.3 and Lemma 4.8, we see this gives us a well defined map from $[|K|, M]$ to $\left[|K^{[p]-1}|, N\right]$.*

Lemma 4.9. *Assume $1 \le p \le n$, $u, v \in W^{1,p}(M, N)$, K is a finite rectilinear cell complex, and $h : |K| \to M$ is a Lipschitz map. If h is a homeomorphism and $u_{\#,p}(h) = v_{\#,p}(h)$, then for any finite rectilinear cell complex L, any Lipschitz map $g : |L| \to M$, we have $u_{\#,p}(g) = v_{\#,p}(g)$.*

Proof. Without losing of generality, we may assume $\dim L \le [p] - 1$. By the cellular approximation theorem, we may find a $g_0 \in C(|L|, M)$ such that $g \sim g_0$ as maps from $|L|$ to M and $g_0(|L|) \subset h\left(|K^{[p]-1}|\right)$. Then $h^{-1} \circ g_0 \in C\left(|L|, |K^{[p]-1}|\right)$. Since $|K^{[p]-1}|$ is a Lipschitz neighborhood retractor in the corresponding Euclidean space, we may find a $\phi \in Lip\left(|L|, |K^{[p]-1}|\right)$ such that $\phi \sim h^{-1} \circ g_0$ as maps from $|L|$ to $|K^{[p]-1}|$. Hence $h \circ \phi \sim g_0$ as maps from $|L|$ to $h\left(|K^{[p]-1}|\right)$. It clearly follows from Remark 4.1 that $u_{\#,p}(h \circ \phi) = v_{\#,p}(h \circ \phi)$, this plus Lemma 4.8 tells us $u_{\#,p}(g) = v_{\#,p}(g)$. □

We note that Lemma 4.9 implies in particular that if $1 \le p \le n$, $u, v \in W^{1,p}(M, N)$, $h_i : K_i \to M$ are Lipschitz rectilinear cell decompositions for $i = 0, 1$, and $u_{\#,p}(h_0) = v_{\#,p}(h_0)$, then $u_{\#,p}(h_1) = v_{\#,p}(h_1)$. Hence we introduce the following

Definition 4.1. *Assume $1 \leq p \leq n$, $u, v \in W^{1,p}(M,N)$. If for any Lipschitz rectilinear cell decomposition $h : K \to M$, we have $u_{\#,p}(h) = v_{\#,p}(h)$, then we say u is $([p]-1)$-homotopic to v.*

It is easy to see the relation of $([p]-1)$-homotopy is an equivalence relation on $W^{1,p}(M,N)$ for the M, N, p in Definition 4.1. The following result, which was proved by B. White in [Wh2], plays important role in our future arguments. With the new concept $\mathcal{W}^{1,p}(K)$ and its properties in Section 3, we may use the classical Sobolev embedding theorem and Poincaré inequality on the unit ball instead of somewhat more complicated ones in section 2 of [Wh1] and section 1 in [Wh2]. This makes our proof technically simpler.

Theorem 4.1. *If $1 \leq p \leq n$, $u, v \in W^{1,p}(M,N)$, and $A > 0$, then there exists a positive number $\varepsilon = \varepsilon(p, A, M, N)$ such that*

$$|du|_{L^p(M)}, |dv|_{L^p(M)} \leq A \text{ and } |u-v|_{L^p(M)} \leq \varepsilon \Longrightarrow u \text{ is } ([p]-1) \text{ homotopic to } v.$$

Proof. Indeed this theorem follows from Proposition 4.1 and a simple compactness arguments. Since the details of the proof below would be quite helpful for understanding the subsequent materials, we present it here. Fix a smooth triangulation of M, namely $h : K \to M$. By Remark 4.1 we may find a Borel set $E_1 \subset B^l_{\varepsilon_0}$ such that $\mathcal{H}^l(E_1) = 0$ and for any $\xi \in B_{\varepsilon_0} \backslash E_1$, we have $u \circ h_\xi, v \circ h_\xi \in \mathcal{W}^{1,p}(K,N)$ and $\left[u \circ h_\xi|_{|K^{[p]-1}|}\right] = u_{\#,p}(h)$, $\left[v \circ h_\xi|_{|K^{[p]-1}|}\right] = v_{\#,p}(h)$. Let m be a natural number which will be determined later. From Lemma 3.3 and Lemma 3.4 we know, for any $\Delta \in K$, $d = \dim(\Delta)$, we have

$$\fint_{B^l_{\varepsilon_0}} d\mathcal{H}^l(\xi) \int_\Delta |u(h_\xi(x)) - v(h_\xi(x))|^p d\mathcal{H}^d(x) \leq c(M) \int_M |u(y) - v(y)|^p d\mathcal{H}^n(y) \leq c(M)\varepsilon^p \tag{4.7}$$

and

$$\fint_{B^l_{\varepsilon_0}} d\mathcal{H}^l(\xi) \int_\Delta \left|d^{ap}(u \circ h_\xi|_\Delta)_x - d^{ap}(v \circ h_\xi|_\Delta)_x\right|^p d\mathcal{H}^d(x) \leq c(M) \int_M |d^{ap}u(y) - d^{ap}v(y)|^p d\mathcal{H}^n(y) \leq c(p, A, M). \tag{4.8}$$

This implies

$$\mathcal{H}^l\left(\left\{\xi \in B^l_{\varepsilon_0} : \int_\Delta |u(h_\xi(x)) - v(h_\xi(x))|^p d\mathcal{H}^d(x) \geq mc(M)\varepsilon^p\right\}\right) \leq \frac{\mathcal{H}^l\left(B^l_{\varepsilon_0}\right)}{m}, \tag{4.9}$$

$$\mathcal{H}^l\left(\left\{\xi \in B^l_{\varepsilon_0} : \int_\Delta \left|d^{ap}(u \circ h_\xi|_\Delta)_x - d^{ap}(v \circ h_\xi|_\Delta)_x\right|^p d\mathcal{H}^d(x) \geq mc(p, A, M)\right\}\right) \leq \frac{\mathcal{H}^l\left(B^l_{\varepsilon_0}\right)}{m}. \tag{4.10}$$

From (4.9), (4.10), Lemma 3.3, Lemma 3.4 and Corollary 3.1 and by taking m large enough (depends only on M), we may find a Borel set $E_2 \subset B^l_{\varepsilon_0}$ such that $\mathcal{H}^l(E_2) > 0$ and for any $\xi \in E_2$, the followings are true

- $u \circ h_\xi, v \circ h_\xi \in \mathcal{W}^{1,p}(K, N)$;
- For any $\Delta \in K$, denote $d = \dim(\Delta)$, we have u and v are approximately differentiable at $h_\xi(x)$ for $\mathcal{H}^d$ a.e. $x \in \Delta$; $d^{ap}(u \circ h_\xi|_\Delta)_x = d^{ap} u_{h_\xi(x)} \circ (h_\xi)_{*,x}$, $d^{ap}(v \circ h_\xi|_\Delta)_x = d^{ap} v_{h_\xi(x)} \circ (h_\xi)_{*,x}$ for $\mathcal{H}^d$ a.e. $x \in \Delta$;
- For any $\Delta \in K$, $d = \dim(\Delta)$, we have

$$\int_\Delta |u(h_\xi(x)) - v(h_\xi(x))|^p d\mathcal{H}^d(x) \le m \cdot c(M)\varepsilon^p = c(M)\varepsilon^p,$$

$$\int_\Delta |d^{ap}(u \circ h_\xi|_\Delta)_x - d^{ap}(v \circ h_\xi|_\Delta)_x|^p d\mathcal{H}^d(x) \le m \cdot c(p, A, M) = c(p, A, M).$$

Hence for any $\Delta \in K^{[p]-1}$, $d = \dim(\Delta)$,

$$|u \circ h_\xi|_\Delta - v \circ h_\xi|_\Delta|_{L^\infty(\Delta)} \le c(p, A, M)\varepsilon_1 + c(p, M)\varepsilon_1^{-\frac{pd}{p-d}}\varepsilon. \tag{4.11}$$

Choose $\varepsilon_1 = \varepsilon_1(p, A, M, N)$ such that $c(p, A, M)\varepsilon_1 \le \overline{\varepsilon}_0/2$, then choose $\varepsilon = \varepsilon(p, A, M, N)$ small enough, such that $c(p, M)\varepsilon_1^{-\frac{pd}{p-d}}\varepsilon \le \overline{\varepsilon}_0/2$, by (4.11) we easily see

$$|u \circ h_\xi|_{|K^{[p]-1}|} - v \circ h_\xi|_{|K^{[p]-1}|}|_\infty \le \overline{\varepsilon}_0. \tag{4.12}$$

By Lemma 4.1, (4.12) implies $u \circ h_\xi|_{|K^{[p]-1}|} \sim v \circ h_\xi|_{|K^{[p]-1}|}$ as maps from $|K^{[p]-1}|$ to N. Choosing a $\xi \in E_2 \backslash E_1$, we conclude Theorem 4.1. □

5. Path connectedness of spaces of Sobolev mappings

We use the same notations as in Section 4. Recall for $u, v \in W^{1,p}(M, N)$, if there exists a continuous path in $W^{1,p}(M, N)$ connecting them, then we write $u \sim_p v$. We have the following

Theorem 5.1. *Assume $1 \le p < n$, and $u, v \in W^{1,p}(M, N)$. Then $u \sim_p v$ if and only if u is $([p]-1)$-homotopic to v.*

Before we proceed, we note that if $p \ge n$, then by the Sobolev embedding theorem and Poincaré inequality (see [SU] or [BN]), one easily deduces that the path connected components of $W^{1,p}(M, N)$ corresponds bijectively to $[M, N]$ by a natural map.

We need some simple observations before proving Theorem 5.1.

Observation 5.1. *Assume $m \in \mathbb{N}$, $1 \le p < \infty$, and $u \in W^{1,p}(B_1^m, N)$ such that the trace $T(u) = f \in W^{1,p}(\partial B_1, N)$. For $0 < t \le 1$, define*

$$w(t)(x) = \begin{cases} u(x/t), & \text{for } |x| \le t; \\ f(x/|x|), & \text{for } t \le |x| \le 1. \end{cases}$$

Then $w \in C\left((0,1], W^{1,p}(B_1, N)\right)$ with $w(1) = u$. Note that usually we cannot extend w continuously to $t = 0$ if $p \ge m$.

Observation 5.2. *Assume $m \in \mathbb{N}$, $1 \le p < m$, and $u \in W^{1,p}(B_1^m, N)$ such that the trace $T(u) = f \in W^{1,p}(\partial B_1, N)$. For $0 \le t \le 1$, define*

$$w(t)(x) = \begin{cases} u(x/t), & \text{for } |x| \le t; \\ f(x/|x|), & \text{for } t \le |x| \le 1. \end{cases}$$

Then w is a continuous path in $W^{1,p}(B_1,N)$ with $w(0)(x)=f(x/|x|)$, $w(1)=u$ and $T(w(t))=f$ for any $0\le t\le 1$. Especially this gives us the following important boundary determination principle, for any $u,v\in W^{1,p}(B_1,N)$, if $T(u)=T(v)=f\in W^{1,p}(\partial B_1,N)$, then we may find a continuous path w in $W^{1,p}(B_1,N)$ connecting u and v with $T(w(t))=f$ for any $0\le t\le 1$.

Observation 5.3. *Assume $m\in\mathbb{N}$, $1\le p<m$ and f is a continuous path in $W^{1,p}(\partial B_1^m,N)$. Define $\tilde f$ by $\tilde f(t)(x)=f(t)(x/|x|)$ for $0\le t\le 1$ and $x\in B_1^m$. Then $\tilde f$ is a continuous path in $W^{1,p}(B_1^m,N)$.*

Proof of Theorem 5.1. Assume $u\sim_p v$. Then there exists a continuous path w in $W^{1,p}(M,N)$ with $w(0)=u$, $w(1)=v$. By compactness we may find an $A>0$ such that

$$\sup_{0\le t\le 1}|dw(t)|_{L^p(M)}\le A.$$

We may also find a $\delta>0$ such that for any $0\le t_1,t_2\le 1$ with

$$|t_1-t_2|\le\delta\Longrightarrow|w(t_1)-w(t_2)|_{L^p(M)}\le\varepsilon(p,A,M,N),$$

here $\varepsilon(p,A,M,N)$ is the number in Theorem 4.1. Choose a $m\in\mathbb{N}$ such that $\frac{1}{m}\le\delta$. Then for any $0\le i\le m-1$, $w\left(\frac{i}{m}\right)$ is $([p]-1)$-homotopic to $w\left(\frac{i+1}{m}\right)$, this implies $w(0)=u$ is $([p]-1)$-homotopic to $w(1)=v$.

On the other hand, suppose we are given two maps $u,v\in W^{1,p}(M,N)$ which are $([p]-1)$-homotopic. First let us assume $p\notin\mathbb{Z}$. For convenience we denote $k=[p]-1$. Choose a smooth triangulation of M, namely $h:K\to M$. From Section 3 and Section 4 we may find a $\xi\in B^l_{\varepsilon_0}$ such that $u\circ h_\xi, v\circ h_\xi\in\mathcal{W}^{1,p}(K,N)$ and $u\circ h_\xi|_{|K^k|}\sim v\circ h_\xi|_{|K^k|}$ as maps from $|K^k|$ to N. By Lemma 3.2 we may find a sequence $f_j\in Lip\left(|K^{k+1}|,\mathbb{R}^{\overline{l}}\right)$ such that $f_j\to u\circ h_\xi|_{|K^{k+1}|}$ in $\mathcal{W}^{1,p}\left(K^{k+1},\mathbb{R}^{\overline{l}}\right)$. By using Sobolev embedding theorem on each simplex we see for j large enough we have

$$\sup_{x\in|K^{k+1}|}|f_j(x)-u(h_\xi(x))|\le\overline{\varepsilon}_0.$$

It follows that the path $w(t)(x)=\pi((1-t)f_j(x)+tu(h_\xi(x)))$ is continuous in $\mathcal{W}^{1,p}\left(K^{k+1},N\right)$. We extend each $w(t)$ to a map $\tilde w(t)\in\mathcal{W}^{1,p}(K,N)$ in the following way, for each $(k+2)$-simplex Δ, in view that $\tilde w(t)$ has already been defined on $Bd(\Delta)$, we choose the barycenter of Δ as origin and do homogeneous degree zero extension to get $\tilde w(t)$ on Δ. Simply by induction we finish after working with n-simplex. It is easy to see that $\tilde w$ is a continuous path in $\mathcal{W}^{1,p}(K,N)$. In addition, from Observation 5.2 and Observation 5.3 we easily deduce that $\tilde w(1)$ can be connected to $u\circ h_\xi$ by a continuous path in $\mathcal{W}^{1,p}(K,N)$. Using h_ξ to go from $|K|$ to M, we may assume $u\circ h_\xi|_{|K^{k+1}|}$ is in $Lip\left(|K^{k+1}|,N\right)$ and $u\circ h_\xi$ is homogeneous degree zero extension on each simplex with dimension strictly higher than $k+1$. A similar procedure can also be applied to v. What we have shown so far is that we may assume both u and v has the additional properties that after composition with h_ξ, that are in $\mathcal{W}^{1,p}(K,N)$, Lipschitz on $|K^{k+1}|$ and homogeneous degree zero (in the sense just described above) on any $\Delta\in K$ with $\dim(\Delta)\ge k+2$. Indeed any $u,v\in\mathcal{W}^{1,p}(M,N)$ can be connected by a continuous path in $W^{1,p}(M,N)$ to maps with these additional properties. Since $u\circ h_\xi|_{|K^k|}\sim v\circ h_\xi|_{|K^k|}$ as maps from $|K^k|$ to N, from the proof of Proposition 2.2 (HEP), we may find a $f\in Lip\left(|K^{k+1}|,N\right)$ such that $f|_{|K^k|}=v\circ h_\xi|_{|K^k|}$ and $f\sim u\circ h_\xi|_{|K^{k+1}|}$ as maps from $|K^{k+1}|$ to N. From

Proposition 2.3, we may find a continuous path in $Lip\left(\left|K^{k+1}\right|, N\right)$ connecting f and $u \circ h_{\xi}|_{\left|K^{k+1}\right|}$, clearly it is also a continuous path in $\mathcal{W}^{1,p}\left(K^{k+1}, N\right)$. Any such f can be viewed as the restriction of a map in $\mathcal{W}^{1,p}(K, N)$, still denoted by f, to $\left|K^{k+1}\right|$. Indeed we simply define inductively for each $\Delta \in K$ with $\dim(\Delta) \geq k+2$, f to be the homogeneous degree zero extension (with respect to the barycenter of Δ) of its value on $Bd(\Delta)$. Then we see $u \circ h_{\xi}$ can be connected by a continuous path in $\mathcal{W}^{1,p}(K, N)$ to f by Observation 5.3. Therefore we only need to show f can be connected to $v \circ h_{\xi}$ by a continuous path in $\widetilde{W}^{1,p}(K, N)$. But now f and $v \circ h_{\xi}$ has one more additional property that $f = v \circ h_{\xi}$ on $\left|K^{k}\right|$. Applying Observation 5.1 to each $(k+1)$-simplex, we may assume for any $\Delta \in K$ with $\dim(\Delta) = k+1$, we have $f|_{\Delta \backslash B_{\delta}(c_{\Delta})} = v \circ h_{\xi}|_{\Delta \backslash B_{\delta}(c_{\Delta})}$. Here c_{Δ} is the barycenter of Δ and δ is a small number. Fix such a Δ, it must be the face of several $k+2$ simplices, namely $\Sigma_1, \cdots, \Sigma_r$, $r \geq 2$. Now consider $\Omega = \cup_{i=1}^{r} \Omega_i$, where $\Omega_i \subset \Sigma_i$ is formally equal to $\left(\overline{B_{2\delta}(c_{\Delta})} \cap \Delta\right) \times [0, \varepsilon]$, for which the product means we go in the Σ_i in the normal direction by ε length, ε is another small number. Define

$$\Omega_i' = \left(\overline{B_{2\delta}(c_{\Delta})} \cap \Delta\right) \times [0, \varepsilon/2], \ \Omega_i'' = \left(\overline{B_{2\delta}(c_{\Delta})} \cap \Delta\right) \times [\varepsilon/2, \varepsilon],$$

$$\Omega' = \bigcup_{i=1}^{r} \Omega_i', \ \Omega'' = \bigcup_{i=1}^{r} \Omega_i''.$$

Now consider a w defined on $\left|K^{k+2}\right|$ by setting $w|_{\Omega'} = v \circ h_{\xi}$, $w|_{\left|K^{k+2}\right| \backslash \Omega} = u \circ h_{\xi}|_{\left|K^{k+2}\right| \backslash \Omega}$. On each Ω_i'' we simply do homogeneous degree zero extension with respect to a point in $\operatorname{Int}(\Omega_i'')$. Clearly $w \in \mathcal{W}^{1,p}\left(K^{k+2}, N\right)$. We note that the set Ω is starshaped with respect to c_{Δ}, the barycenter of Δ. One may use simple radial (with the origin c_{Δ}) deformations as in Observation 5.2 to see the similar boundary determination principle is valid for Ω. In particular, w can be connected to $f|_{\left|K^{k+2}\right|}$ by a continuous path in $\widetilde{W}^{1,p}\left(K^{k+2}, N\right)$. Define $\tilde{w}$ inductively to be the homogeneous degree zero extension of w on each higher dimensional simplices Δ with $\dim(\Delta) \geq k+3$, from its value on $Bd(\Delta)$ as described before. Then for $\tilde{u} = \tilde{w} \circ h_{\xi}^{-1}$, one has $\tilde{u} \sim_p f \circ h_{\xi}^{-1} \sim_p u$. Moreover, since $\tilde{u} \circ h_{\xi}|_{\left|K^{k+1}\right|} = v \circ h_{\xi}|_{\left|K^{k+1}\right|}$, $\tilde{u} \sim_p v$ follows. Therefore we complete the proof of $u \sim_p v$.

If $p \in \mathbb{Z}$, we only need to use Lemma 3.2 and Lemma 4.4 to show the original maps u and v can be connected by continuous paths in $W^{1,p}(M, N)$ to maps with additional property $u \circ h_{\xi}, v \circ h_{\xi} \in \mathcal{W}^{1,p}(K, N)$, $u \circ h_{\xi}|_{\left|K^{p}\right|}$ and $v \circ h_{\xi}|_{\left|K^{p}\right|}$ are Lipschitz, $u \circ h_{\xi}|_{\left|K^{p}\right|} \sim v \circ h_{\xi}|_{\left|K^{p}\right|}$. The rest of the proof is the same as before. □

Now we will show how Theorem 5.1 reduces certain problems about Sobolev mappings, which are analytical problems, to pure topology problems.

Proposition 5.1. *Assume $1 \leq p < n$. For any Lipschitz rectilinear cell decomposition of M, namely $h : K \to M$, we set $M^j = h\left(\left|K^j\right|\right)$ for any j. Then the following two natural maps are bijections,*

$$C\left(M^{[p]}, N\right) / \sim_{M^{[p]-1}} \longleftarrow Lip\left(M^{[p]}, N\right) / \sim_{M^{[p]-1}, Lip} \longrightarrow W^{1,p}(M, N) / \sim_p .$$

Here for $f, g \in C\left(M^{[p]}, N\right)$, $f \sim_{M^{[p]-1}} g$ means $f|_{M^{[p]-1}}$ and $g|_{M^{[p]-1}}$ are homotopic as maps from $M^{[p]-1}$ to N. For $f, g \in Lip\left(M^{[p]}, N\right)$, $f \sim_{M^{[p]-1}, Lip} g$ means $f|_{M^{[p]-1}}$ can be connected to $g|_{M^{[p]-1}}$ by a continuous path in $Lip\left(M^{[p]-1}, N\right)$. The natural map for the left pointing arrow is the obvious one. The map for the right *pointing arrow is defined as follows, for any* $f \in Lip\left(M^{[p]}, N\right)$*, using* h *to pull* f *to* $K^{[p]}$*, after doing homogeneous degree zero extension on higher dimensional cells, we pull it to* M *by* h *and get* u*. Then we send the equivalence class corresponding to* f *to the equivalence class corresponding to* u*. This map is well defined by Theorem 5.1.*

Proof. It clearly follows from Proposition 2.3 that the left pointing arrow is a bijection. To prove the right pointing arrow is a bijection, first let us show it is one to one. Assume $f, g \in Lip\left(\left|K^{[p]}\right|, N\right)$, let $\tilde{f}$ and $\tilde{g}$ be homogeneous degree zero extension of f and g to $|K|$ (as we described in the proof of Theorem 5.1) respectively. Let $u = \tilde{f} \circ h^{-1}$, $v = \tilde{g} \circ h^{-1}$. It is clear that $u_{\#,p}(h) = \left[f|_{|K^{[p]-1}|}\right]$, $v_{\#,p}(h) = \left[g|_{|K^{[p]-1}|}\right]$. If $u \sim_p v$, then it follows from Theorem 5.1 that $f|_{|K^{[p]-1}|} \sim g|_{|K^{[p]-1}|}$. This shows the map is one to one. On the other hand, given any map $u \in W^{1,p}(M, N)$, we may find a $\xi \in B_{\varepsilon_0}$ such that $u \circ h_\xi \in \mathcal{W}^{1,p}(K, N)$. It follows from the proof of Theorem 5.1 that after going through a continuous path in $W^{1,p}(M, N)$ we may assume $u \circ h_\xi|_{|K^{[p]}|} \in Lip\left(\left|K^{[p]}\right|, N\right)$ and $u \circ h_\xi \in \mathcal{W}^{1,p}(K, N)$. Since $u \circ h_\xi \circ h^{-1} \sim_p u$, we may assume $u \circ h \in \mathcal{W}^{1,p}(K, \mathbb{N})$ and $u \circ h|_{|K^{[p]}|} \in Lip\left(\left|K^{[p]}\right|, N\right)$. Now it is easy to see that the equivalence class corresponding to $u|_{M^{[p]}}$ is mapped to the equivalence class corresponding to u. That is, the right pointing arrow is onto. □

Recall for any $1 \leq q < p < \infty$, we have a map $i_{p,q} : W^{1,p}(M, N) / \sim_p \rightarrow W^{1,q}(M, N) / \sim_q$ defined in the obvious way (see [BL]). An immediate consequence of the above proposition is the following

Corollary 5.1. *Assume* $k \in \mathbb{N}$*,* $k \leq q < p < k+1$*. Then* $i_{p,q}$ *is a bijection.*

Note that Corollary 5.1 gives a positive answer to conjecture 2 and 2′ in [BL].

Corollary 5.2. *Assume* $1 \leq p < n$*,* $\pi_i(N) = 0$ *for* $[p] \leq i \leq n$*. Then the two natural maps below are bijections,*

$$C(M, N) / \sim_M \longleftarrow Lip(M, N) / \sim_{M,Lip} \longrightarrow W^{1,p}(M, N) / \sim_p,$$

the notations are understood similarly as in Proposition 5.1.

Proof. By Proposition 5.1 we only need to verify the natural map $C(M, N) / \sim_M \rightarrow C\left(M^{[p]}, N\right) / \sim_{M^{[p]-1}}$ is a bijection for a smooth triangulation of M. But this clearly follows from cell by cell extension in view of the vanishing condition of homotopy groups of N. □

We note that Corollary 5.2 generalizes theorem 0.6 in [BL].

Corollary 5.3. *Assume* M *and* N *are connected,* $1 \leq p < n$*. If there exists a* $k \in \mathbb{Z}$*,* $0 \leq k \leq [p] - 1$ *such that* $\pi_i(M) = 0$ *for* $1 \leq i \leq k$*, and* $\pi_i(N) = 0$ *for* $k + 1 \leq i \leq [p] - 1$*, then* $W^{1,p}(M, N)$ *is path connected.*

Proof. By Proposition 5.1 we only need to verify that for a smooth triangulation of M, $C\left(M^{[p]}, N\right) / \sim_{M^{[p]-1}}$ has only one element, but this follows easily from theorem 3 and the proof of theorem 3′ in [Wh1]. □

Corollary 5.3 generalizes theorem 0.2, theorem 0.3 and proposition 0.1 in [BL].

We now turn to the question whether a given Sobolev map in $W^{1,p}(M, N)$ can be connected to a smooth map by a continuous path in $W^{1,p}(M, N)$. It turns out that there is a necessary and sufficient topological condition for that to be true.

Proposition 5.2. *Assume $1 \le p < n$, $u \in W^{1,p}(M,N)$, and $h : K \to M$ is a Lipschitz rectilinear cell decomposition. Then u can be connected to a smooth map by a continuous path in $W^{1,p}(M,N)$ if and only if $u_{\#,p}(h)$ is extendible to M with respect to N.*

Proof. Assume $u \sim_p v$ for some $v \in C^\infty(M,N)$. Then from Theorem 5.1 we have $u_{\#,p}(h) = v_{\#,p}(h)$, but clearly $v_{\#,p}(h)$ is extendible to M with respect to N.

On the other hand, if $u_{\#,p}(h)$ is extendible to M with respect to N, then we may find a $v \in C^\infty(M,N)$ such that $\left[v \circ h|_{|K^{[p]-1}|}\right] = u_{\#,p}(h)$. Thus u and v are $[p]-1$ homotopic, and hence $u \sim_p v$ by Theorem 5.1. □

Corollary 5.4. *Assume $1 \le p < n$. Then every map in $W^{1,p}(M,N)$ can be connected by a continuous path in $W^{1,p}(M,N)$ to a smooth map if and only if M satisfies $([p]-1)$-extension property with respect to N.*

Proof. Fix a smooth triangulation of M, namely $h : K \to M$.

Assume every map in $W^{1,p}(M,N)$ can be connected continuously to a smooth map. For any $f \in Lip\left(M^{[p]}, N\right)$, let g be the homogeneous degree zero extension of $f \circ h|_{|K^{[p]}|}$ to $|K|$. Then $u = g \circ h^{-1} \in W^{1,p}(M,N)$ and $u_{\#,p}(h) = \left[g|_{|K^{[p]-1}|}\right]$. Since u can be connected continuously to a smooth map, from Proposition 5.2 we know $f|_{M^{[p]-1}}$ has a continuous extension to M. By Proposition 2.2 and Proposition 2.3 we know M has the $([p]-1)$-extension property with respect to N.

On the other hand, assume M satisfies the $[p]-1$ extension property with respect to N, given any $u \in W^{1,p}(M,N)$, after going through a continuous path in $W^{1,p}(M,N)$, we may assume there exists a $\xi \in B^l_{\varepsilon_0}$ such that $u \circ h_\xi|_{|K^{[p]}|} \in Lip\left(|K^{[p]}|, N\right)$ and $u_{\#,p}(h) = \left[u \circ h_\xi|_{|K^{[p]-1}|}\right]$. Hence by Proposition 5.2 u may be connected continuously to a smooth map. □

Remark 5.1. *Corollary 5.4 covers theorem 0.5 of* [BL]*. It is a particular case that M satisfies the $([p]-1)$-extension property with respect to N. We also have the following statements. Assume M and N are connected, $1 \le p < n$. If either $[p]=1$ or $[p] \ge 2$ but $\pi_i(N) = 0$ for $[p] \le i \le n-1$, then every map in $W^{1,p}(M,N)$ can be connected to a smooth map. This, again, is because M has the $([p]-1)$-extension property with respect to N.*

Because of this necessary and sufficient topological condition for every map in $W^{1,p}(M,N)$ to be connected to some smooth maps by a continuous path in $W^{1,p}(M,N)$. We obtain the following corollary which provides a class of counterexamples to the conjecture 1 of [BL].

Corollary 5.5. *If $m_1, m_2 \in \mathbb{N}$, $m_2 < m_1$, and $3 \le p < 2m_2 + 2$, then in $W^{1,p}(\mathbb{CP}^{m_1}, \mathbb{CP}^{m_2})$, some maps cannot be connected to smooth maps by continuous paths.*

Proof. For any $m \in \mathbb{N}$, $\mathbb{CP}^m$ has a natural CW complex structure as

$$\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \cdots \subset \mathbb{CP}^m.$$

In addition by considering the fibration $\mathbb{CP}^m = \mathbb{S}^{2m+1}/\mathbb{S}^1$, we know $\pi_i(\mathbb{CP}^m) = 0$ for $0 \le i \le 2m-1$, $i \ne 2$.

We claim that there is no continuous map $f \in C(\mathbb{CP}^{m_1}, \mathbb{CP}^{m_2})$ such that $f|_{\mathbb{CP}^1} : \mathbb{CP}^1 \subset \mathbb{CP}^{m_1} \to \mathbb{CP}^1 \subset \mathbb{CP}^{m_2}$ is the identity map. To see that the claim is true, let α_i be the cohomology class in $H^2(\mathbb{CP}^{m_i})$ corresponding to $\mathbb{CP}^1$ for $i = 1, 2$.

We know the cohomology ring $H^*(\mathbb{CP}^{m_i})$ is isomorphic to $\mathbb{Z}[\alpha_i]/\{\alpha_i^{m_i+1}=0\}$ (see p174 and p175 of [Vi]). If such f exists, then $\alpha_1 = f^*(\alpha_2)$, which implies $\alpha_1^{m_2+1}=0$. The latter is impossible. Next we observe that the identity map from $\mathbb{CP}^{[p/2]} \subset \mathbb{CP}^{m_1}$ to $\mathbb{CP}^{[p/2]} \subset \mathbb{CP}^{m_2}$, when restricted to $\mathbb{CP}^{[(p-1)/2]}$, has no continuous extension by the claim above, using Corollary 5.4 we conclude the proof. □

Remark 5.2. *By considering cohomology ring with $\mathbb{Z}_2$ coefficients (see p175 of* [Vi]*), the same proof gives us the following statement. If $m_1, m_2 \in \mathbb{N}$, $m_2 < m_1$, and $2 \le p < m_2+1$, then in $W^{1,p}(\mathbb{RP}^{m_1}, \mathbb{RP}^{m_2})$, there are some maps which cannot be connected to smooth maps by continuous paths.*

6. Strong density problem for Sobolev mappings

An important technique in the study of approximation problems for Sobolev mappings is to use certain deformations with respect to the dual skeletons, which was used in the geometrical proof of the Poincaré duality theorem and in Federer-Fleming's theory of normal and integral currents. We present a version for finite rectilinear cell complex here. One should compare with section 1 of [Wh1], section 2 of [Ha] and pp143–146 of [Vi].

Let K be a finite rectilinear cell complex with $\dim K = m$. For each $\Delta \in K$, we pick up a point $y_\Delta \in \operatorname{Int}(\Delta)$. Denote $\mathcal{Y} = (y_\Delta)_{\Delta \in K}$. Given an integer $0 \le k \le m-1$. For $x \in |K^k|$, we set $|x|_k = 1$. For $k+1 \le i \le m$, if $|\cdot|_k$ has been defined on $|K^{i-1}|$, then for each $\Delta \in K$ with $\dim(\Delta) = i$, each $x \in \Delta$, we set

$$|x|_k = |x|_\Delta \cdot |y_\Delta + \frac{x - y_\Delta}{|x|_\Delta}|_k. \tag{6.1}$$

For the definition of $|x|_\Delta$, one should see (3.1). Hence by induction, we eventually get a function $|\cdot|_k$ on $|K|$. In fact the function $|\cdot|_k$ depends on K as well as the choice of $\mathcal{Y}$, but to avoid heavy notations, we don't explicitly write them out. Similar convention applies for many notations in this section, it will not cause confusions in the practice. For $0 \le \varepsilon \le 1$ we set $\Gamma_\varepsilon^k = \{x \in |K| : |x|_k = \varepsilon\}$. Then we may decompose $|K|$ as

$$|K| = \bigcup_{0 \le \varepsilon \le 1} \Gamma_\varepsilon^k, \quad \Gamma_1^k = |K^k|. \tag{6.2}$$

If we denote $L^{m-k-1} = \Gamma_0^k$, and set $L^m = |K|$, then we call L^i as the *dual i-skeleton* of K.

Now we want to define a map $\phi_1^k : \{0 < |x|_k \le 1\} \to \Gamma_1^k = |K^k|$. First look at $|K^{k+1}|$, for any $x \in |K^{k+1}|$, if $x \in |K^k|$, then we set $\phi_1^k(x) = x$, otherwise, there exists a unique $\Delta \in K$ with $\dim(\Delta) = k+1$ such that $x \in \operatorname{Int}(\Delta)$. Then we set

$$\phi_1^k(x) = y_\Delta + \frac{x - y_\Delta}{|x|_\Delta}. \tag{6.3}$$

Assume for some $k+2 \le i \le m$, $\phi_1^k : \{0 < |x|_k \le 1\} \cap |K^{i-1}| \to \Gamma_1^k$ has been defined. Then for $x \in |K^i|$, $0 < |x|_k \le 1$, if $x \in |K^{i-1}|$, then $\phi_1^k(x)$ has already been defined, otherwise, there exists a unique $\Delta \in K$ such that $\dim(\Delta) = i$ and $x \in \operatorname{Int}(\Delta)$. In the latter case we set

$$\phi_1^k(x) = \phi_1^k\left(y_\Delta + \frac{x - y_\Delta}{|x|_\Delta}\right). \tag{6.4}$$

By induction we eventually get a map ϕ_1^k from $\{0 < |x|_k \le 1\}$ to Γ_1^k.

Next we want to define a map $\phi^k : \{0 < |x|_k < 1\} \times (0,1) \to |K|$ with the property

$$|\phi^k(x,\varepsilon)|_k = \varepsilon \quad \text{for } 0 < |x|_k < 1, 0 < \varepsilon < 1. \tag{6.5}$$

For convenience we write $\phi_\varepsilon^k(x) = \phi^k(x,\varepsilon)$, hence $\phi^k(x,1)$ is also defined for $0 < |x|_k \le 1$. To define the needed ϕ^k, we first look at $|K^{k+1}|$. For any $x \in |K^{k+1}|$, $0 < |x|_k < 1$, there exists a unique $\Delta \in K$ such that $\dim(\Delta) = k+1$ and $x \in \operatorname{Int}(\Delta)$. Then we set

$$\phi^k(x,\varepsilon) = y_\Delta + \frac{\varepsilon}{|x|_\Delta}(x - y_\Delta) \quad \text{for } 0 < \varepsilon < 1. \tag{6.6}$$

Assume for some $k+2 \le i \le m$, $\phi^k(x,\varepsilon)$ has been defined for $x \in |K^{i-1}|$ with $0 < |x|_k < 1$, $0 < \varepsilon < 1$. Then for any $x \in |K^i|$ with $0 < |x|_k < 1$, if $x \in |K^i|$, then $\phi^k(x,\varepsilon)$ has already been defined for $0 < \varepsilon < 1$. Otherwise, there exists a unique $\Delta \in K$ such that $\dim(\Delta) = i$ and $x \in \operatorname{Int}(\Delta)$, then we set

$$\theta = 1 - (1-\varepsilon)\frac{1 - |x|_\Delta}{1 - |x|_k}; \tag{6.7}$$

$$\phi^k(x,\varepsilon) = y_\Delta + \theta \cdot \left(\phi^k\left(y_\Delta + \frac{x - y_\Delta}{|x|_\Delta}, \frac{\varepsilon}{\theta}\right) - y_\Delta\right). \tag{6.8}$$

By induction, we eventually get the needed map ϕ^k.

In the future, we shall need a map $F_{\delta,\varepsilon}^k : |K| \to |K|$ for $0 < \delta \le \varepsilon \le 1$, which is defined by

$$F_{\delta,\varepsilon}^k(x) = \begin{cases} x, & \text{when } \varepsilon \le |x|_k \le 1; \\ \phi^k(x,\varepsilon), & \text{when } \delta \le |x|_k \le \varepsilon; \\ \phi^k\left(x, \delta^{-1}\varepsilon|x|_k\right), & \text{when } 0 < |x|_k \le \delta; \\ x, & \text{when } |x|_k = 0. \end{cases} \tag{6.9}$$

Let $1 \le p < n$. Then we denote

$$\begin{aligned} R^{p,\infty}(M,N) = \big\{ & u : u \in W^{1,p}(M,N), \text{ there exists a smooth rectilinear} \\ & \text{cell decomposition of } M, \text{ say } h : K \to M, \text{ and a dual } (n-[p]-1)\text{-skeleton} \\ & L^{n-[p]-1} \text{ such that } u \text{ is } C^\infty \text{ on } M \backslash h\left(L^{n-[p]-1}\right)\big\}. \end{aligned} \tag{6.10}$$

The following statement was due to F. Bethuel (see [Be2], p154, Theorem 2). But for reasons explained in the introduction we need to give a somewhat different proof.

Theorem 6.1. *Assume $1 \le p < n$. Then $R^{p,\infty}(M,N)$ is dense in $W^{1,p}(M,N)$ under the strong topology.*

We need some preparations before proving this theorem.

Lemma 6.1. *Let Ω be any separable Riemannian manifold without boundary (possibly noncompact, incomplete and nonconnected), and $1 \le p < \infty$. If $\mathcal{E}$ is the Banach space $C(\Omega,\mathbb{R}) \cap L^\infty(\Omega) \cap W^{1,p}(\Omega)$ with norm $|u|_{\mathcal{E}} = |u|_{L^\infty(\Omega)} + |u|_{W^{1,p}(\Omega)}$, then $C^\infty(\Omega) \cap \mathcal{E}$ is dense in $\mathcal{E}$.*

Proof. Fix a $u \in \mathcal{E}$ and an $\varepsilon > 0$, choose a locally finite open cover of Ω, say $\{U_j\}_{j=1}^{\infty}$ such that $\overline{U_j} \subset \Omega$ is compact. Choose the corresponding decomposition of unit $\{\zeta_j\}_{j=1}^{\infty}$ (see p10 of [Wa]). Let $u_j = \zeta_j u$, choose $v_j \in C_c^{\infty}(U_j)$ such that $|v_j - u_j|_{\mathcal{E}} \le \varepsilon/2^j$. Let $v = \sum_{j=1}^{\infty} v_j$. Then $v \in C^{\infty}(\Omega)$. For any $V \subset \Omega$ open with $\overline{V}$ compact, we may find a $m > 0$ such that $\overline{V} \cap U_j = \emptyset$ for $j > m$. Hence $u|_V = \sum_{j=1}^{m} u_j$, $v|_V = \sum_{j=1}^{m} v_j$, we easily see $|u - v|_{\mathcal{E}} \le \varepsilon$. This implies the conclusion. □

Lemma 6.1 along with the nearest point projection π_N imply in particular that, if Ω is the same as in the lemma, then for any $1 \le p < \infty$, any $u \in W^{1,p}(\Omega, N) \cap C(\Omega, N)$, we may find $u_j \in C^{\infty}(\Omega, N) \cap W^{1,p}(\Omega, N)$ such that $\sup_{x\in\Omega} |u_j(x) - u(x)| \to 0$ and $u_j \to u$ in $W^{1,p}(\Omega, N)$.

To facilitate the proof of Theorem 6.1, we need to introduce various notions. Given two rectilinear cell complexes K_1 and K_2 such that $|K_1| = |K_2|$. Let $K = \{\Delta_1 \cap \Delta_2 : \Delta_1 \in K_1, \Delta_2 \in K_2, \Delta_1 \cap \Delta_2 \ne \emptyset\}$. Then K is a rectilinear cell complex which is a subdivision of both K_1 and K_2, we say K is the *rectilinear cell complex generated by K_1 and K_2*.

For any cube Q, we use K_Q to denote the rectilinear cell complex defined by $K_Q = \{\text{all faces of } Q\}$. We note that Q is a face of itself.

Assume $d \in \mathbb{N}$. If a cube in $\mathbb{R}^d$ is of the form $\prod_{i=1}^{d} [a^i, b^i]$, $a^i, b^i \in \mathbb{R}$, $a^i \le b^i$, then we say it is a *normal cube*. If K is a finite rectilinear cell complex such that each cell in K is a normal cube, then we say K is a *normal complex*. If K_1 and K_2 are two normal complexes such that $|K_1| = |K_2|$, then clearly the rectilinear cell complex generated by K_1 and K_2 is a normal complex too.

For $k \in \mathbb{Z}$, $1 \le k \le d$, we write $H_{k,t} = \{x : x \in \mathbb{R}^d, x^k = t\}$, here x^k is the k-th coordinate of x. For $a \in (\mathbb{R}^+)^d$, we denote $I_a = \prod_{i=1}^{d} [0, a^i]$. For any $0 \le t \le a^k$, let $Q_1 = \{x \in I_a : 0 \le x^k \le t\}$, $Q_2 = \{x \in I_a : t \le x^k \le a^k\}$. Then we denote $K_{a,k,t} = K_{Q_1} \cup K_{Q_2}$.

The following lemma is an easy consequence of Fubini type theorem (see also Corollary 3.1).

Lemma 6.2. *Assume $a \in (\mathbb{R}^+)^d$, K is a normal complex such that the polytope $|K| = I_a$, $1 \le p < \infty$, $i \in \mathbb{Z}$, $1 \le i \le d$. For any $t \in (0, a^i)$, we may use $H_{i,t}$ to slice K to form another normal complex, namely L_t, that is L_t is the normal complex generated by K and $K_{a,k,t}$. Assume $u \in \mathcal{W}^{1,p}(K, \mathbb{R})$. Then for $\mathcal{H}^1$ a.e. $t \in (0, a^i)$, we have $u \in \mathcal{W}^{1,p}(L_t)$.*

We remark that Lemma 6.2 says almost every slice is nice, hence when we choose generic slices in the future, we may always assume we are choosing those slices among the nice ones.

Let $a \in (\mathbb{R}^+)^d$. If we are given $m_i \in \mathbb{N}$, $0 = t_{i,0} < t_{i,1} < \cdots < t_{i,m_i} = a^i$ for $1 \le i \le d$, then we say $\{H_{i,t_{ij}} \cap I_a : 1 \le i \le d, 0 \le j \le m_i\}$ is a *net* on I_a, denote it as $\mathcal{N}$. Given $0 < \delta \le \min_{1\le i\le d} a^i$, set $m_i = [a^i/\delta]$. If for some $A \ge 1$, we have $\frac{\delta}{A} \le t_{i,j+1} - t_{i,j} \le A\delta$ for $1 \le i \le d$, $0 \le j \le m_i - 1$, then we say $\mathcal{N}$ is a (δ, A)-net. $\mathcal{N}$ divides I_a in to $m_1 \cdots m_d$ small cubes. That it is an (δ, A)-net simply means every small cube is $[0, \delta]^d$ after a translation and an inhomogeneous dilation. Also the Lipschitz constants of this transformation and its inverse are dominated by A.

We note that for any net $\mathcal{N}$ on I_a, we have a natural normal complex $K_{\mathcal{N}}$ such that $|K_{\mathcal{N}}| = I_a$. Indeed we just take it as the normal complex generated by

$\{K_{a,i,t_{ij}} : 1 \le i \le d, 0 \le j \le m_i\}$. Given any face Q of I_a and any net $\mathcal{N}$ on Q. $\mathcal{N}$ generates a normal complex $K_{Q,\mathcal{N}}$ such that $|K_{Q,\mathcal{N}}| = Q$, then we define a normal complex

$$K_{\mathcal{N}} = K_{Q,\mathcal{N}} \bigcup \{\Delta : \Delta \in K_{I_a} \text{ such that } \Delta \not\subset Q\}.$$

Clearly $|K_{\mathcal{N}}| = I_a$. If we are given m faces of I_a, namely $Q_1, \cdots, Q_m$, and for each i a net $\mathcal{N}_i$ on Q_i, then we call the normal complex generated by $K_{\mathcal{N}_1}, \cdots, K_{\mathcal{N}_m}$ as the *normal complex generated by* $\mathcal{N}_1, \cdots, \mathcal{N}_m$.

For any Riemannian manifold Ω, given a k-rectifiable subset S of Ω and a suitable differentiable function u on S, $1 \le p < \infty$, we denote $E_p(u,k,S) = \int_S |d_S u|^p d\mathcal{H}^k$, here $\mathcal{H}^k$ is the k-dimensional Hausdorff measure. We simply write $E(u,k,S)$ when it is clear what p is.

The next lemma contains one of the key analytic estimates that are needed in our proof of Theorem 6.1. We postpone the proof of it to the Appendix A.

Lemma 6.3 (Generic slicing lemma)**.** *Assume $a \in (\mathbb{R}^+)^d$, for each face of I_a, we pick up a net on it, all these nets together generate a normal complex K such that $|K| = I_a$, $1 \le p < \infty$, $u \in \mathcal{W}^{1,p}(K,\mathbb{R})$. Then there exists an absolute constant $A \ge 1$ such that for any $0 < \delta \le \min_{1\le i\le d} a^i$, there exists a (δ, A)-net $\mathcal{N}$ on I_a such that $u \in \mathcal{W}^{1,p}\left(\widetilde{K}\right)$, here $\widetilde{K}$ is the normal complex created from K and $\mathcal{N}$, and we have*

$$E\left(u,i,|\widetilde{K}^i| \cap \left(|K^j| \backslash |K^{j-1}|\right)\right) \le c(d)\left(\frac{1}{\delta}\right)^{j-i} E\left(u,j,|K^j|\right) \quad \textit{for } 1 \le i < j \le d.$$

The above inequalities imply in particular that

$$E\left(u,i,|\widetilde{K}^i| \cap |K^j|\right) \le c(d)\sum_{k=i+1}^{j}\left(\frac{1}{\delta}\right)^{k-i} E\left(u,k,|K^k|\right) + E\left(u,i,|K^i|\right)$$

for $1 \le i < j \le d$.

We also introduce the following map $\varphi_N : N \times \mathbb{R}^{\bar{l}} \to N$, which is defined by

$$\varphi_N(x,y) = \begin{cases} \pi\left(x + \frac{\bar{\varepsilon}_0 (y-x)}{|y-x|}\right) & \text{for } |y-x| \ge \bar{\varepsilon}_0; \\ \pi(y) & \text{for } |y-x| \le \bar{\varepsilon}_0. \end{cases}$$

We have $Lip(\varphi_N|_{N\times N}) \le c(N)$.

Finally we observe the following fact. Assume K is a finite rectilinear cell complex, $1 \le p < \infty$, $k \in \mathbb{Z}$, $k \ge 0$, and $u \in \mathcal{W}^{1,p}(K,\mathbb{R})$ with $u|_{|K^k|} \in C\left(|K^k|\right)$. Then there exists a sequence $u_i \in \mathcal{W}^{1,p}(K,\mathbb{R}) \cap C(|K|)$ such that $u_i|_{|K^k|} = u|_{|K^k|}$ and $u_i \to u$ in $\mathcal{W}^{1,p}(K)$. This fact follows from proofs of Lemma 3.1 and Lemma 3.2. As a consequence, we have the following

Corollary 6.1. *Assume K is a finite rectilinear cell complex, $1 \le p < \infty$, $k \in \mathbb{Z}$, $k \ge 0$, $u \in \mathcal{W}^{1,p}(K,N)$, $u|_{|K^k|} \in C\left(|K^k|, N\right)$, and there exists a $y_0 \in N$ such that $u(|K|) \subset \overline{B^{\bar{l}}_{\bar{\varepsilon}_0}(y_0)}$. Then there exists a sequence $u_i \in \mathcal{W}^{1,p}(K,N) \cap C(|K|,N)$ such that $u_i \to u$ in $\mathcal{W}^{1,p}(K)$, $u_i|_{|K^k|} = u|_{|K^k|}$ and $u_i(|K|) \subset \overline{B^{\bar{l}}_{3\bar{\varepsilon}_0/2}(y_0)}$.*

Proof. By the above observed fact, we may find a sequence $v_i \in \mathcal{W}^{1,p}\left(K, \mathbb{R}^{\bar{l}}\right) \cap C\left(|K|, \mathbb{R}^{\bar{l}}\right)$ such that $v_i|_{|K^k|} = u|_{|K^k|}$ and $v_i \to u$ in $\mathcal{W}^{1,p}(K)$. Then $u_i(x) = \varphi_N(y_0, v_i(x))$ is the needed sequence of maps. ∎

With all these preparations, we can proceed now with the proof of Theorem 6.1.

Proof of Theorem 6.1. Define $R^p(M,N)$ as the set similar to $R^{p,\infty}(M,N)$ but replace C^∞ by C^0. By the fact we stated after the proof of Lemma 6.1, it suffices to show $\overline{R^p(M,N)} = W^{1,p}(M,N)$. For convenience, we assume $p \notin \mathbb{Z}$ at first. Fix a smooth cubeulation of M, namely $h : K \to M$ such that each cube in K is normal. Given $u \in W^{1,p}(M,N)$, by Lemma 4.3 we may assume $f = u \circ h \in \mathcal{W}^{1,p}(K,N)$. Applying Lemma 6.3 on the n-cells in an arbitrary order, we get a (δ, A)-net on each of them. These nets together with the original K create a normal complex, called K_n, we have $f \in \mathcal{W}^{1,p}(K_n, N)$ and

$$E\left(f,i,|K_n^i|\right) \le c(M) \sum_{j=i}^{n} \left(\frac{1}{\delta}\right)^{j-i} E\left(f,j,|K^j|\right) \tag{6.11}$$

$$\le c(M)\left(\frac{1}{\delta}\right)^{n-i} E\left(f,n,|K^n|\right),$$

for $1 \le i \le n$ and all sufficiently small δ.

Fix a $\nu \in (0,p)$, for each n-cube Q in K_n, if for every $1 \le i \le n$, we have the normalized energy

$$\delta^{p-i} E\left(f,i,|K_Q^i|\right) \le \delta^\nu, \tag{6.12}$$

then we say Q is a *good cube*, otherwise we call it a *bad cube*. Denote $\mathcal{G}$ as the union of all good cubes and $\mathcal{B}$ as the union of all bad cubes. Clearly we have

$$\mathcal{H}^n(\mathcal{B}) \le c(M)\,\delta^{p-\nu} E\left(f,n,|K|\right), \tag{6.13}$$

hence $\mathcal{H}^n(\mathcal{B}) \to 0$ as $\delta \to 0^+$.

Let us first look at good cubes. Fix two positive numbers δ_1 and δ_2 such that $0 < \delta_1 \ll \delta_2 < \frac{1}{A}$. If Q is a good cube, from Sobolev embedding theorem we know $f|_{|K_Q^{[p]}|}$ is continuous and

$$\operatorname{osc}\left(f, |K_Q^{[p]}|\right) \le c(p,M)\,\delta^{\nu/p}. \tag{6.14}$$

Choose a $y_Q \in f\left(|K_Q^{[p]}|\right)$. By Lemma 6.3 we may find a $(\delta_1\delta, A)$ net $\mathcal{N}$ such that $f|_Q \in \mathcal{W}^{1,p}\left(\widetilde{K}_Q, N\right)$, $\mathcal{N}$ induces a net on each $(n-1)$-face of Q and $\widetilde{K}_Q$ is the normal complex created from K_Q together with all these induced nets. Moreover, we have

$$E\left(f,i,|\widetilde{K}_Q^i| \cap \left(|K_Q^j| \backslash |K_Q^{j-1}|\right)\right) \le c(M)\left(\frac{1}{\delta_1\delta}\right)^{j-i} E\left(f,j,|K_Q^j|\right) \tag{6.15}$$

for $1 \le i < j \le n$. Here A is an absolute constant. This, combined with (6.12), implies

$$(\delta_1\delta)^{p-i} E\left(f,i,|\widetilde{K}_Q^i|\right) \le c(\delta_1,p,M)\,\delta^\nu \quad \text{for } 1 \le i \le n. \tag{6.16}$$

By Sobolev embedding theorem we have $f|_{|\widetilde{K}_Q^{[p]}|}$ is continuous and

$$\operatorname{osc}\left(f, |\widetilde{K}_Q^{[p]}|\right) \le c(\delta_1,p,M)\,\delta^{\nu/p}. \tag{6.17}$$

If we set δ to be small enough (depending on δ_1) and $\hat{f}(x)=\varphi_N(y_Q,f(x))$ for $x\in Q$, then we have $\hat{f}=f$ on $|\widetilde{K}_Q^{[p]}|$. From Corollary 6.1 we may find a sequence $\hat{f}_j\in\mathcal{W}^{1,p}\left(\widetilde{K}_Q,N\right)\cap C(Q,N)$ such that $\hat{f}_j\to\hat{f}$ in $\mathcal{W}^{1,p}\left(\widetilde{K}_Q,N\right)$ and $\hat{f}_j=\hat{f}=f$ on $|\widetilde{K}_Q^{[p]}|$. Set $\bar{f}$ to be $\hat{f}_j$ on Q for some j large enough, this j depends on Q. Let x_Q be the barycenter of Q. Then for any $\alpha\in(0,1)$, we denote $Q_\alpha=(x_Q+(1-\alpha)(Q-x_Q))$. For any $x\in Q$, we define $r(x)$ to be the unique nonnegative number such that $x\in(x_Q+r(x)(Bd(Q)-x_Q))$, that is $r(x)=|x|_{Q,x_Q}$. Then we define a map $\phi:Q_{\delta_1}\to Q$ by

$$\phi(x)=\begin{cases} x, & x\in Q_{\delta_2};\\ x_Q+\left(1-\delta_2+(r(x)-1+\delta_2)\frac{\delta_1}{\delta_2-\delta_1}\right)\frac{x}{r(x)}, & x\in Q_{\delta_1}\backslash Q_{\delta_2}.\end{cases}\tag{6.18}$$

For any $x\in Q_{\delta_1}$, we set $\tilde{f}(x)=\bar{f}(\phi(x))$. Now we want to define $\tilde{f}$ on $Q\backslash Q_{\delta_1}$. We observe

$$\tilde{f}(x)=f(\phi(x))\qquad\text{for } x\in\phi^{-1}\left(|\widetilde{K}_Q^{[p]}|\right).\tag{6.19}$$

This relation is important for the final construction of $\tilde{f}$. Assume $\tilde{f}$ has already been defined on $|K_n^{n-1}|$ such that for any good cube Q,

$$\tilde{f}|_{Bd(Q)}\in\mathcal{W}^{1,p}\left(\widetilde{K}_Q^{n-1},N\right),\tilde{f}(x)=f(x)\ \text{ for } x\in|\widetilde{K}_Q^{[p]}|\tag{6.20}$$

and

$$E\left(\tilde{f},i,|\widetilde{K}_Q^i|\right)\le c(p,M)\,E\left(f,i,|\widetilde{K}_Q^i|\right)\qquad\text{for } [p]+1\le i\le n-1.\tag{6.21}$$

Then we define $\tilde{f}$ on $Q\backslash Q_{\delta_1}$ as follows. First set $\psi:Q\backslash Q_{\delta_1}\to Bd(Q_{\delta_1})$ as

$$\psi(x)=x_Q+(1-\delta_1)\frac{x-x_Q}{r(x)}\qquad\text{for } x\in Q\backslash Q_{\delta_1}.\tag{6.22}$$

Let C be a $[p]$-cell in $\widetilde{K}_Q$, on $\psi^{-1}(C)$ we simply define $\tilde{f}(x)=f(\psi(x))$. Now for any $[p]+1$ cell C in $\widetilde{K}_Q$, we observe $\psi^{-1}(C)$ is Lipschitz equivalent to $[0,\delta_1\delta]^{[p]+1}$, where the Lipschitz constants are dominated by a constant depending only on n, we simply do homogeneous degree zero extension on $\psi^{-1}(C)$ for $\tilde{f}$ of its value on $Bd\left(\psi^{-1}(C)\right)$. Inductively, we finish after we do this for $(n-1)$-cell in $\widetilde{K}_Q$. We need to emphasize that we haven't fixed the choice of $\tilde{f}$ on $|K_n^{n-1}|$ yet, we just need it to satisfy (6.20) and (6.21) for good cubes up to now, so there are still lots of freedom in choosing such a $\tilde{f}$.

Next we look at bad cubes. If Q is a bad cube, for any $\alpha\in(0,1/A)$, we may find a $(\alpha\delta,A)$-net $\mathcal{N}_Q$ such that $f|_Q\in\mathcal{W}^{1,p}\left(\widetilde{K}_Q,N\right)$, here $\widetilde{K}_Q$ is the normal complex created by $\mathcal{N}_Q$, moreover

$$E\left(f,n-1,|\widetilde{K}_Q^{n-1}|\right)\le\frac{c(M)}{\alpha\delta}E\left(f,n,|K_Q^n|\right)\tag{6.23}$$

for α sufficiently small enough. Assume $\tilde{f}$ has already been defined on $|\widetilde{K}_Q^{n-1}|$ such that $\tilde{f}|_{|\widetilde{K}_Q^{n-1}|}\in\mathcal{W}^{1,p}\left(\widetilde{K}_Q^{n-1},N\right)$ and in addition $\tilde{f}$ satisfies

$$E\left(\tilde{f},n-1,|\widetilde{K}_Q^{n-1}|\right)\le c(p,M)\,E\left(f,n-1,|\widetilde{K}_Q^{n-1}|\right).\tag{6.24}$$

Then on Q, we simply set $\tilde{f}$ as the homogeneous degree zero extension for each n-cell in $\widetilde{K}_Q$.

We haven't finished defining $\tilde{f}$ yet because we still need to define $\tilde{f}$ on the union of $|\widetilde{K}_Q^{n-1}|$ for all n-cells Q in K_n. It needs to satisfies (6.20), (6.21) for good cubes and (6.24) for bad cubes. To find such a $\tilde{f}$, we introduce a new normal complex K_{n-1}. K_{n-1} is created from the union of $\widetilde{K}_Q^{n-1}$ for all n-cells in K_n. In view of Lemma 6.2 we know $f \in \mathcal{W}^{1,p}(K_{n-1}, N)$. For any $(n-1)$-cell $Q \in K_{n-1}$, let λ be the minimal side length of Q, for any $\alpha \in (0,1)$, we may find a $(\alpha\lambda, A)$-net, namely $\mathcal{N}_Q$, such that $f|_Q \in \mathcal{W}^{1,p}\left(\widetilde{K}_Q, N\right)$ and

$$E\left(f, n-2, |\widetilde{K}_Q^{n-2}|\right) \leq \frac{c(M)}{\alpha\lambda} E(f, n-1, Q) \tag{6.25}$$

for sufficiently small α. Again if $\tilde{f}$ has already been defined on the union of $|\widetilde{K}_Q^{n-2}|$ and

$$E\left(\tilde{f}, n-2, |\widetilde{K}_Q^{n-2}|\right) \leq c(p, M)\, E\left(f, n-2, |\widetilde{K}_Q^{n-2}|\right), \tag{6.26}$$

then on Q we simply put $\tilde{f}$ to be the homogeneous degree zero extension on each $n-1$ cell in $\widetilde{K}_Q$. We keep this procedure going until we reach $K_{[p]}$, on $|K_{[p]}^{[p]}|$, we simply put $\tilde{f} = f$. Going back we get the needed $\tilde{f}$.

Let $\tilde{u} = \tilde{f} \circ h^{-1}$. Then a careful computation shows (see also [Be2], pp170–173)

$$|\tilde{u} - u|_{W^{1,p}(M)} \leq \beta_1(\delta, \delta_1, \delta_2) + \beta_2(\delta_1, \delta_2) + \beta_3(\delta_2) + \varepsilon,$$

where $\beta_1(\delta, \delta_1, \delta_2) \to 0$ if we fix δ_1, δ_2 and let $\delta \to 0^+$. $\beta_2(\delta_1, \delta_2) \to 0$ if we fix δ_2 and let $\delta_1 \to 0^+$, $\beta_3(\delta_2) \to 0$ when $\delta_2 \to 0^+$. Thus in order to make $\tilde{u}$ close to u, we first choose ε to be very small, then choose δ_2 small so that $\beta_3(\delta_2)$ will be also small. Next for such fixed δ_2, we choose δ_1 even smaller so that the resulting $\beta_2(\delta_1, \delta_2)$ is also very small. Finally we choose δ to be so small such that $\beta_1(\delta, \delta_1, \delta_2)$ is small. In this way we will be able to find a sequence of maps in $R^p(M, N)$ converging to u strongly, hence we get the theorem. If $p = 1$, the same proof goes through. If $p \in \mathbb{Z}$ and $p \geq 2$, then we only need to add the Lemma 4.4 on the p skeleton. This completes the proof of Theorem 6.1. $\square$

Our next goal is to show that under certain topological condition, a map in $R^{p,\infty}(M, N)$ can be approximated by smooth maps. We need some more notations. Let X and Y be two topological spaces, A be a subset of X, $\alpha \in [X, Y]$. Then we may define $\alpha|_A \in [A, Y]$ by $\alpha|_A = [f|_A]$ for any $f \in \alpha$. It is clear that $[f|_A]$ does not depend on the specific choice of f in α.

Theorem 6.2. *Assume $1 \leq p < n$, $h : K \to M$ is a Lipschitz rectilinear cell decomposition, $M^i = h\left(|K^i|\right)$ for $i \geq 0$, $L^{n-[p]-1}$ is one of the dual $(n-[p]-1)$-skeleton, and $u \in W^{1,p}(M, N)$ such that u is continuous on $M \backslash h\left(L^{n-[p]-1}\right)$. Then $u \in H_S^{1,p}(M, N)$ if and only if $u|_{M^{[p]}}$ has a continuous extension to M. In addition, if for some $\alpha \in [M, N]$, we have $u|_{M^{[p]}} \in \alpha|_{M^{[p]}}$, then we may find a sequence $u_i \in C^\infty(M, N)$ such that $[u_i] = \alpha$ and $u_i \to u$ in $W^{1,p}(M, N)$.*

Proof. If $u \in H_S^{1,p}(M, N)$, then we may find a sequence $u_i \in C^\infty(M, N)$ such that $u_i \to u$ in $W^{1,p}(M, N)$. Let $\varepsilon_0 = \varepsilon_0(M)$ be a small positive number, $H(x, \xi) = \pi(h(x) + \xi)$ for $x \in |K|$, $\xi \in B_{\varepsilon_0}^l$. Then $\chi_{[p],H,u_i} = [u_i \circ h|_{|K^{[p]}|}]$ a.e. on $B_{\varepsilon_0}^l$. It is clear that for some $\varepsilon_1 > 0$ small, $\chi_{[p],H,u} = [u \circ h|_{|K^{[p]}|}]$ a.e on $B^l_{\varepsilon_1}$. By Proposition 4.1, we see after passing to a subsequence, we have $[u_{i'} \circ h|_{|K^{[p]}|}] = [u \circ h|_{|K^{[p]}|}]$ for i' large enough. This implies that $u \circ h|_{|K^{[p]}|}$ has a continuous extension to $|K|$, hence $u|_{M^{[p]}}$ has a continuous extension to M.

To prove the inverse, first we observe that we may assume u is smooth on $M \backslash h\left(L^{n-[p]-1}\right)$. Indeed if this has been proved, then the theorem follows from the fact after Lemma 6.1.

To proceed we use the idea of the proof of theorem 1 in [Wh1], but with the new deformations we constructed at the beginning of this section. Let $k = [p]$. Since k is fixed, we shall write Γ_ε, ϕ and $F_{\delta,\varepsilon}$ instead of Γ^k_ε, ϕ^k and $F^k_{\delta,\varepsilon}$ for convenience. For $0 < \varepsilon \le 1$, $|K^k|$ is a deformation retractor of $\{x \in |K| : |x|_k \ge \varepsilon\}$, indeed $F_{t,1}$ for $\varepsilon \le t \le 1$ is the needed deformation. Choose a $v \in C(M,N)$ such that $[v] = \alpha$. Let $g_0 = v \circ h$, $f = u \circ h$. Since $f|_{|K^{[p]}|} \sim g_0|_{|K^{[p]}|}$, it follows that $f \sim g_0$ on $\{x \in |K| : |x|_k \ge \varepsilon\}$ and from Proposition 2.2 (homotopy extension theorem) we conclude that there exists a $g \in Lip\,(|K|, N)$ such that $g = f$ on $\{x \in |K| : |x|_k \ge \varepsilon\}$ and $g \sim g_0$. For $0 < \delta < \varepsilon \le 1/2$, we set $f_{\delta,\varepsilon}(x) = g\left(F_{\delta,\varepsilon}(x)\right)$ for $x \in |K|$. Then $f_{\delta,\varepsilon} \in Lip\,(|K|, N)$ and $f_{\delta,\varepsilon} \sim g \sim g_0$. In fact, we only need to consider $g \circ F_{\delta,t}$ for $\varepsilon \le t \le 1$ and $g \circ F_{s,1}$ for $\delta \le s \le 1$ to see the homotopy relation. We have the following basic facts (see Lemma B.2, Corollary B.1, Corollary B.2 and Corollary B.3 in Appendix B),

(P_1) $\mathcal{H}^n\left(\{x \in |K| : |x|_k \le \varepsilon\}\right) \le c\,(K,\mathcal{Y})\,\varepsilon^{k+1}$ for $0 < \varepsilon \le 1/2$;
(P_2) $0 < c\,(K,\mathcal{Y})^{-1} \le |d\left(|\cdot|_k\right)| \le c\,(K,\mathcal{Y})$ $\mathcal{H}^n$ a.e. on $|K|$;
(P_3) $|dF_{\delta,\varepsilon}(x)| \le c\,(K,\mathcal{Y})\,\varepsilon/|x|_k$ for $\delta \le |x|_k \le \varepsilon \le 1/2$;
(P_4) $|dF_{\delta,\varepsilon}(x)| \le c\,(K,\mathcal{Y})\,\varepsilon\delta^{-1}$ for $|x|_k \le \delta \le \varepsilon \le 1/2$;
(P_5) For $0 < \delta \le \varepsilon \le 1/2$, $J_{(\phi_\delta|_{\Gamma_\varepsilon})} \le c\,(K,\mathcal{Y})\,(\delta/\varepsilon)^k$ $\mathcal{H}^{n-1}$ a.e. on Γ_ε.

It is clear that

$$\{x \in |K| : f_{\delta,\varepsilon}(x) \ne f(x)\} \subset \{x \in |K| : |x|_k \le \varepsilon\}.$$

Hence to estimate $|f_{\delta,\varepsilon} - f|_{\widetilde{W}^{1,p}(K)}$ we only need to control

$$\int_{|x|_k \le \varepsilon} |df_{\delta,\varepsilon}(x)|^p d\mathcal{H}^n(x).$$

First of all we have

$$\begin{aligned}(6.27)\qquad &\int_{|x|_k \le \delta} |df_{\delta,\varepsilon}(x)|^p d\mathcal{H}^n(x)\\ &\le c\,(p,K,\mathcal{Y})\,[g]^p_{Lip(|K|)} \int_{|x|_k \le \delta} |dF_{\delta,\varepsilon}(x)|^p d\mathcal{H}^n(x)\\ &\le c\,(p,K,\mathcal{Y})\,[g]^p_{Lip(|K|)}\varepsilon^p\delta^{k+1-p} \quad (\text{by } (P_1) \text{ and } (P_4)).\end{aligned}$$

Secondly we know

$$\begin{aligned}(6.28)\qquad &\int_{\delta \le |x|_k \le \varepsilon} |df_{\delta,\varepsilon}(x)|^p d\mathcal{H}^n(x)\\ &\le c\,(p,K,\mathcal{Y})\,\varepsilon^p \int_{\delta \le |x|_k \le \varepsilon} |(df)\left(F_{\delta,\varepsilon}(x)\right)|^p |x|_k^{-p} d\mathcal{H}^n(x) \quad (\text{by } (P_3))\\ &\le c\,(p,K,\mathcal{Y})\,\varepsilon^p \int_{\delta \le |x|_k \le \varepsilon} |(df)\left(F_{\delta,\varepsilon}(x)\right)|^p |x|_k^{-p} J_{|\cdot|_k}(x)\, d\mathcal{H}^n(x) \quad (\text{by } (P_2))\end{aligned}$$

$$= c(p,K,\mathcal{Y})\,\varepsilon^p \int_\delta^\varepsilon dr \int_{|x|_k=r} r^{-p} |(df)(\phi_\varepsilon(x))|^p d\mathcal{H}^{n-1}(x) \quad \text{(by coarea formula)}$$
$$= c(p,K,\mathcal{Y})\,\varepsilon^p \int_\delta^\varepsilon dr \int_{\Gamma_\varepsilon} r^{-p} |df|^p J_{(\phi_r|_{\Gamma_\varepsilon})} d\mathcal{H}^{n-1} \quad \text{(by change of variable formula)}$$
$$\leq c(p,K,\mathcal{Y})\,\varepsilon \int_{\Gamma_\varepsilon} |df|^p d\mathcal{H}^{n-1} \quad \text{(by } (P_5)\text{)}.$$

Next we observe that for any $0 < t \leq 1/2$,

$$\int_t^{2t} dr \int_{\Gamma_r} |df|^p d\mathcal{H}^{n-1} \tag{6.29}$$
$$= \int_{t \leq |x|_k \leq 2t} |df(x)|^p J_{|\cdot|_k}(x)\, d\mathcal{H}^n(x) \quad \text{(by coarea formula)}$$
$$\leq c(p,K,\mathcal{Y}) \int_{t \leq |x|_k \leq 2t} |df(x)|^p d\mathcal{H}^n(x) \quad \text{(by } (P_2)\text{)}.$$

Hence we may find a $\varepsilon_t \in [t, 2t]$ such that

$$\int_{\Gamma_{\varepsilon_t}} |df|^p d\mathcal{H}^{n-1} \leq \frac{c(p,K,\mathcal{Y})}{t} \int_{t \leq |x|_k \leq 2t} |df(x)|^p d\mathcal{H}^n(x). \tag{6.30}$$

The latter inequality implies

$$\varepsilon_t \int_{\Gamma_{\varepsilon_t}} |df|^p d\mathcal{H}^{n-1} \tag{6.31}$$
$$\leq c(p,K,\mathcal{Y}) \int_{t \leq |x|_k \leq 2t} |df(x)|^p d\mathcal{H}^n(x) \to 0 \quad \text{as } t \to 0^+.$$

Putting (6.26), (6.27) and (6.30) together we get

$$|f_{\delta,\varepsilon_t} - f|_{\widetilde{W}^{1,p}(K,N)} \leq \alpha_1(\delta,t) + \alpha_2(t), \tag{6.32}$$

where $\alpha_1(\delta,t) \to 0^+$ if we fix t and let $\delta \to 0^+$, $\alpha_2(t) \to 0$ as $t \to 0^+$. We conclude that u is a strong limit of a sequence of Lipschitz maps of the form $u_{\delta,\varepsilon} = f_{\delta,\varepsilon} \circ h^{-1}$ in the $W^{1,p}(M,N)$. Since $[u_{\delta,\varepsilon}] = \alpha$, Theorem 6.2 follows. □

Now we describe several interesting consequences of Theorem 6.2.

Theorem 6.3. *Assume N is connected and $1 \leq p < n$. Then $H_S^{1,p}(M,N) = W^{1,p}(M,N)$ if and only if $\pi_{[p]}(N) = 0$ and M satisfies $([p]-1)$-extension property with respect to N.*

We need the following topological lemma to prove this theorem.

Lemma 6.4. *Assume X and Y are two topological spaces, X can possess some CW complex structures, Y is path connected, $k \in \mathbb{N}$, and $\pi_k(Y) = 0$. Then X satisfies the $(k-1)$-extension property w.r.t. Y if and only if X satisfies the k-extension property w.r.t. Y.*

Proof. Fix a CW complex structure of X.

If X satisfies the $(k-1)$-extension property w.r.t. Y, then given any $f \in C(X^{k+1}, Y)$, there exists a $g \in C(X,Y)$ such that $f|_{X^{k-1}} = g|_{X^{k-1}}$. Because $\pi_k(Y) = 0$, we have $f|_{X^k} \sim g|_{X^k}$, hence $f|_{X^k}$ has a continuous extension to X by Proposition 2.1 (HEP). That is, X satisfies the k-extension property with respect to Y.

On the other hand, if X satisfies the k-extension property w.r.t. Y, then for any $f \in C\left(X^k, Y\right)$, there exists a $f_1 \in C\left(X^{k+1}, Y\right)$ such that $f_1|_{X^k} = f$. We may find a $g \in C(X,Y)$ such that $g|_{X^k} = f_1|_{X^k} = f$, hence g is a continuous extension of $f|_{X^{k-1}}$ to X, that is, X satisfies the $(k-1)$-extension property w.r.t. Y. Indeed what we have proved is any $f \in C\left(X^k, Y\right)$ has a continuous extension to X. □

Proof of Theorem 6.3. Assume we have $H_S^{1,p}(M,N) = W^{1,p}(M,N)$. Pick up a smooth triangulation of M, namely $h : K \to M$, denote $M^i = h\left(|K^i|\right)$ for $i \geq 0$. For each $\Delta \in K$, choose a $y_\Delta \in \mathrm{Int}\Delta$. Given any f in $Lip\left(M^{[p]}, N\right)$, let $f_0 = f \circ h$. Let $f_1 \in \mathcal{W}^{1,p}(K,N)$ be the map which we get from f_0 by doing homogeneous degree zero extension with respect to y_Δ on all simplices Δ with $\dim(\Delta) \geq [p]+1$. Let $u = f_1 \circ h^{-1}$. Then $u \in W^{1,p}(M,N)$. Hence $u \in H_S^{1,p}(M,N)$. It follows from Theorem 6.2 that $u|_{M^{[p]}} = f$ has a continuous extension to M. Now it follows from Proposition 2.3 and HEP that for any $f \in C\left(M^{[p]}, N\right)$, f has a continuous extension to M, this clearly implies $\pi_{[p]}(N) = 0$ and M satisfies the $([p]-1)$-extension property w.r.t. N.

On the other hand, assume $\pi_{[p]}(N) = 0$ and M satisfies $([p]-1)$-extension property w.r.t. N. Then if follows from the proof of Lemma 6.4 that for any CW complex of M, and $f \in C\left(M^{[p]}, N\right)$, f has a continuous extension to M. In view of Theorem 6.1, we only need to show $R^{p,\infty}(M,N) \subset \overline{C^\infty(M,N)}$, but this clearly follows from the topological condition and Theorem 6.2. □

An easy consequence of Theorem 6.3 and the proof of Corollary 5.3 is the following

Corollary 6.2. *Assume M and N are connected, $1 \leq p < n$, k is an integer such that $0 \leq k \leq [p]-1$ and $\pi_i(M) = 0$ for $1 \leq i \leq k$ and $\pi_i(N) = 0$ for $k+1 \leq i \leq [p]$. Then $H_S^{1,p}(M,N) = W^{1,p}(M,N)$.*

We note that Corollary 6.2 implies part (a) of theorem 1 of [Ha]. The next corollary gives another set of target manifolds N for which smooth maps from M into N are strongly dense in $W^{1,p}(M,N)$.

Corollary 6.3. *Assume N is connected, $1 \leq p < n$. If $\pi_i(N) = 0$ for $[p] \leq i \leq n-1$, then $H_S^{1,p}(M,N) = W^{1,p}(M,N)$.*

Proof. This follows from Theorem 6.3 and cell by cell extension. □

Remark 6.1. *It follows from Theorem 6.2 and the proof of Theorem 6.1 that for a map $u \in W^{1,p}(M,N)$, $1 \leq p < n$, $u \in H_S^{1,p}(M,N)$ if and only if for "generic" $[p]$-skeletons $M^{[p]}$, when $p \notin \mathbb{Z}$, $u|_{M^{[p]}}$ has a continuous extension to M, when $p \in \mathbb{Z}$, the homotopy class corresponding to $u|_{M^{[p]}}$ (because it is continuous on $M^{[p]-1}$ and in VMO on each $[p]$-cell, see Lemma 4.5) is extendible to M with respect to N. One needs to understand the word "generic" as in the way we create cell decompositions in the proof of Theorem 6.1.*

7. Weak sequential density problem for Sobolev mappings

The question whether smooth maps are sequentially weakly dense in the Sobolev space of mappings, $W^{1,p}(M,N)$, turns out to be much more subtle. It becomes important in finding minimizers of suitable energy functionals defined on the Sobolev space of mappings. Suppose $1 \leq p < n$ and p is not an integer, then it was shown in the earlier work of Bethuel [Be2] that $H_W^{1,p}(M,N) = H_S^{1,p}(M,N)$. Hence, in this case, the problem of the weak sequential density of smooth maps reduces to the strong density of smooth maps in $W^{1,p}(M,N)$, for which we have discussed in detail in the previous section. We also note that, in the special case $p = 1$, one always has $H_W^{1,1}(M,N) = H_S^{1,1}(M,N)$ due to analytical facts associated with L^1-weak convergence (see [Hn]). For general integer p's, $1 < p < n$, the space $H_W^{1,p}(M,N)$ is hard to characterize. We have the following

Theorem 7.1. *Assume $1 \le p < n$, $u \in W^{1,p}(M,N)$, $h : K \to M$ is a Lipschitz rectilinear cell decomposition of M. If $u \in H_W^{1,p}(M,N)$, then $u_{\#,p}(h)$ is extendible to M with respect to N, hence u may be connected to a smooth map by a continuous path in $W^{1,p}(M,N)$.*

Proof. This follows easily from Proposition 4.1 and Theorem 5.1. □

We also observe that, by Corollary 5.4 and Theorem 7.1, one has the following statements. If $H_W^{1,p}(M,N) = W^{1,p}(M,N)$ for some $1 \le p < n$, then M satisfies the $([p]-1)$-extension property with respect to N.

On the other hand, let $m_1, m_2 \in \mathbb{N}$, $m_2 < m_1$, then

- If $3 \le p < 2m_2 + 2$, then
$$H_W^{1,p}(\mathbb{CP}^{m_1}, \mathbb{CP}^{m_2}) \ne W^{1,p}(\mathbb{CP}^{m_1}, \mathbb{CP}^{m_2}).$$
- If $2 \le p < m_2 + 1$, then
$$H_W^{1,p}(\mathbb{RP}^{m_1}, \mathbb{RP}^{m_2}) \ne W^{1,p}(\mathbb{RP}^{m_1}, \mathbb{RP}^{m_2}).$$

These conclusions are direct consequences of Corollary 5.5 and Theorem 7.1.

Thus we have obtained a necessary topological condition for smooth maps to be weakly sequentially dense in $W^{1,p}(M,N)$. In view of this and earlier works [Be1, Be2, BBC, Ha, Hn], we make the following

Conjecture 7.1. *Assume $2 \le p < n$, $p \in \mathbb{Z}$, and $h : K \to M$ is a Lipschitz rectilinear cell decomposition of M. If $u \in W^{1,p}(M,N)$ such that $u_{\#,p}(h)$ is extendible to M with respect to N, then $u \in H_W^{1,p}(M,N)$.*

Conjecture 7.1 just says the topological obstruction we stated above is the only obstruction for the weak sequential approximability by smooth maps. In [HnL2], we shall prove Conjecture 7.1 under the additional assumption that $u \in R^p(M,N)$ (see the beginning of the proof of Theorem 6.1 for the definition). That is, at least for a dense subset of $W^{1,p}(M,N)$, the topological condition described in Theorem 7.1 is also sufficient for the map to be in $H_W^{1,p}(M,N)$.

Let $\widetilde{H}_W^{1,p}(M,N)$ be the smallest subset of $W^{1,p}(M,N)$ which is closed under the sequential weak convergence in $W^{1,p}(M,N)$ and contains $C^\infty(M,N)$. Then from [GMS], chapter 3, section 4.1 we know $\widetilde{H}_W^{1,p}(M,N)$ is equal to the successive sequential weak limits of $C^\infty(M,N)$ in $W^{1,p}(M,N)$ up to the first uncountable ordinal number. It follows from Theorem 6.1, Proposition 4.1 and the above result from [HnL2] that for any Lipschitz rectilinear cell decomposition of M, namely $h : K \to M$, and any $2 \le p < n$, $p \in \mathbb{Z}$,

$$\widetilde{H}_W^{1,p}(M,N)$$
$$= \{u : u \in W^{1,p}(M,N), u_{\#,p}(h) \text{ has a continuous extension to } M \text{ w.r.t. } N\}.$$

On the other hand, we also see easily that $\widetilde{H}_W^{1,p}(M,N) = \overline{H_W^{1,p}(M,N)}$. Here the closure is taken under the strong topology. This means one suffices to take a second time limit instead of taking limits to the first uncountable ordinal number to get $\widetilde{H}_W^{1,p}(M,N)$ from $C^\infty(M,N)$. Conjecture 7.1 just says we only need to take one time limits, that is $\widetilde{H}_W^{1,p}(M,N) = H_W^{1,p}(M,N)$ (see [HnL2] for further discussions). One may also conjecture that if $2 \leq p < n$, $p \in \mathbb{Z}$ and M satisfies the $(p-1)$-extension property with respect to N, then $H_W^{1,p}(M,N) = W^{1,p}(M,N)$.

In addition to Theorem 7.1, we have the following two statements.

Theorem 7.2. *Assume M and N are both connected, and $1 \leq p < n$. Then $H_S^{1,p}(M,N)$ is equal to $W^{1,p}(M,N)$ if and only if $\pi_{[p]}(N) = 0$ and $H_W^{1,p}(M,N)$ is equal to $W^{1,p}(M,N)$.*

If, in addition, we know $p \in \mathbb{N}$, $p > 1$ and $\pi_p(N) = 0$, then $H_S^{1,p}(M,N) = H_W^{1,p}(M,N)$.

Proof. The first fact follows from Theorem 6.3 and the statement after Theorem 7.1.

On the other hand, if we know p is an integer larger than 1, then given any $u \in H_W^{1,p}(M,N)$, it follows from Theorem 7.1 that for a generic skeleton M^{p-1}, $u|_{M^{p-1}}$ has a continuous extension to M. It follows then from the fact $\pi_p(N) = 0$ and the homotopy extension theorem that the homotopy class corresponding to $u|_{M^p}$ has a continuous extension to M (see the proof of Lemma 6.4). Thus by Remark 6.1 we have $u \in H_S^{1,p}(M,N)$. □

Appendix A. A proof of the generic slicing lemma

In this appendix, we shall give the detailed proof of Lemma 6.3, that is, the generic slicing lemma. For convenience, we first describe some notations.

Assume $a \in (\mathbb{R}^+)^d$. Let I_a be defined as $\prod_{i=1}^d [0, a^i]$. For each face of I_a, we pick up a net on it. All these nets together generate a normal complex K such that $|K| = I_a$. For $1 \leq i \leq k$, we denote by S_i the subset of $[0, a^i]$ of all points in the above nets in the i-th direction. S_i is a finite set. Let α be a subset of $\{1, \cdots, d\}$, we use $|\alpha|$ to denote the number of elements in α. If $\alpha = \emptyset$, then we set $K_\alpha = K$. Otherwise, if for any $i \in \alpha$, we have m_i numbers, namely $0 = t_{i,0} < t_{i,1} < \cdots < t_{i,m_i} = a^i$, then we denote K_α as the normal complex created from K together with $H_{i,t_{i,j}} \cap I_a$ for $i \in \alpha$, $0 \leq j \leq m_i$.

Proof of Lemma 6.3. We shall do slicing in each direction inductively. In view of Lemma 6.2, we don't need to worry about getting $u \in \mathcal{W}^{1,p}\left(\widetilde{K}, \mathbb{R}\right)$, hence for convenience we will not mention this point in the future proof.

Let us look at the first direction. For $1 \leq i \leq m_1 - 1$, let J_i be the closed interval $[(i - 1/8)\delta, (i + 1/8)\delta]$, $P_i = \{x : x \in I_a, x^1 \in J_i\}$. Fix a positive constant c_1, which will be determined later, we have

$$\begin{aligned}\text{(A.1)} \qquad & \int_{J_i} E\left(u, j-1, H_{1,t} \cap \left(|K^j| \backslash |K^{j-1}|\right)\right) dt \\ & \leq E\left(u, j, P_i \cap \left(|K^j| \backslash |K^{j-1}|\right)\right), \quad \text{for } 2 \leq j \leq d.\end{aligned}$$

If we set

$$\text{(A.2)} \qquad \mathcal{B}_j^1 = \left\{t : t \in J_i, E\left(u, j-1, H_{1,t} \cap \left(|K^j| \backslash |K^{j-1}|\right)\right)\right.$$

$$\geq c_1\delta^{-1}E\left(u,j,P_i\cap\left(|K^j|\backslash|K^{j-1}|\right)\right)\},\quad\text{for }2\leq j\leq d,$$

then it follows from (A.1) that

$$\mathcal{H}^1\left(\mathcal{B}_j^1\right)\leq\frac{\delta}{c_1}. \tag{A.3}$$

Let

$$\mathcal{B}=S_1\cup\bigcup_{j=2}^{d}\mathcal{B}_j^1.$$

Then from (A.3) we get

$$\mathcal{H}^1\left(\mathcal{B}\right)\leq\frac{d}{c_1}\delta. \tag{A.4}$$

In view of (A.4), if we take $c_1=c_1(d)$ large enough, we may find a point $t_{1,i}\in J_i\backslash\mathcal{B}$. By setting $t_{1,0}=0$, $t_{1,m_1}=a^1$, we get m_1 numbers in the first direction. In addition, we have

$$E\left(u,j-1,|K_{\{1\}}^{j-1}|\cap\left(|K^j|\backslash|K^{j-1}|\right)\right)\leq\frac{c\left(d\right)}{\delta}E\left(u,j,|K^j|\backslash|K^{j-1}|\right)\quad\text{for }2\leq j\leq d. \tag{A.5}$$

Indeed this follows from the way we choose $t_{1,i}$.

Then we switch to the second direction. For $1\leq i\leq m_2-1$, let J_i be the closed interval $[(i-1/8)\delta,(i+1/8)\delta]$, $P_i=\{x:x\in I_a,x^2\in J_i\}$. Fix a positive constant c_2, which will be determined later, we have

$$\int_{J_i}E\left(u,j-1,H_{2,t}\cap\left(|K^j|\backslash|K^{j-1}|\right)\right)dt \tag{A.6}$$
$$\leq E\left(u,j,P_i\cap\left(|K^j|\backslash|K^{j-1}|\right)\right)\qquad\text{for }2\leq j\leq d,$$

$$\int_{J_i}E\left(u,j-2,H_{2,t}\cap|K_{\{1\}}^{j-1}|\cap\left(|K^j|\backslash|K^{j-1}|\right)\right)dt \tag{A.7}$$
$$\leq E\left(u,j-1,P_i\cap|K_{\{1\}}^{j-1}|\cap\left(|K^j|\backslash|K^{j-1}|\right)\right)\qquad\text{for }3\leq j\leq d.$$

Define

$$\mathcal{B}_j^2=\{t:t\in J_i,E\left(u,j-1,H_{2,t}\cap\left(|K^j|\backslash|K^{j-1}|\right)\right) \tag{A.8}$$
$$\geq c_2\delta^{-1}E\left(u,j,P_i\cap\left(|K^j|\backslash|K^{j-1}|\right)\right)\}\qquad\text{for }2\leq j\leq d,$$

$$\mathcal{B}_j^{1,2}=\left\{t:t\in J_i,E\left(u,j-2,H_{2,t}\cap|K_{\{1\}}^{j-1}|\cap\left(|K^j|\backslash|K^{j-1}|\right)\right)\right. \tag{A.9}$$
$$\left.\geq c_2\delta^{-1}E\left(u,j-1,|K_{\{1\}}^{j-1}|\cap\left(|K^j|\backslash|K^{j-1}|\right)\right)\right\}\quad\text{for }3\leq j\leq d.$$

Then it follows from (A.6) and (A.7) that

$$\mathcal{H}^1\left(\mathcal{B}_j^2\right)\leq\frac{\delta}{c_2},\quad\mathcal{H}^1\left(\mathcal{B}_j^{1,2}\right)\leq\frac{\delta}{c_2}. \tag{A.10}$$

Let

$$\mathcal{B}=S_2\cup\left(\bigcup_{j=2}^{d}\mathcal{B}_j^2\right)\cup\left(\bigcup_{j=3}^{d}\mathcal{B}_j^{1,2}\right).$$

Then

$$\mathcal{H}^1\left(\mathcal{B}\right)\leq\frac{c\left(d\right)}{c_2}\delta. \tag{A.11}$$

In view of (A.11), if we take $c_2 = c_2(d)$ large enough, we may find a point $t_{2,i} \in J_i \backslash \mathcal{B}$. By setting $t_{2,0} = 0$, $t_{2,m_2} = a^2$, we get m_2 numbers in the second direction. In addition, we have

$$\begin{aligned} & E\left(u, j-1, |K^{j-1}_{\{2\}}| \cap \left(|K^j| \backslash |K^{j-1}|\right)\right) \\ & \leq \frac{c(d)}{\delta} E\left(u, j, |K^j| \backslash |K^{j-1}|\right) \text{ for } 2 \leq j \leq d, \end{aligned} \tag{A.12}$$

$$\begin{aligned} & E\left(u, j-2, |K^{j-2}_{\{1,2\}}| \cap \left(|K^j| \backslash |K^{j-1}|\right)\right) \\ & \leq \frac{c(d)}{\delta} E\left(u, j-1, |K^{j-1}_{\{1\}}| \cap \left(|K^j| \backslash |K^{j-1}|\right)\right) \quad \text{for } 3 \leq j \leq d. \end{aligned} \tag{A.13}$$

This follows from our choices of $t_{2,i}$. In addition

$$\begin{aligned} & E\left(u, j-1, |K^{j-1}_{\{1,2\}}| \cap \left(|K^j| \backslash |K^{j-1}|\right)\right) \\ & \leq E\left(u, j-1, |K^{j-1}_{\{1\}}| \cap \left(|K^j| \backslash |K^{j-1}|\right)\right) + E\left(u, j-1, |K^{j-1}_{\{2\}}| \cap \left(|K^j| \backslash |K^{j-1}|\right)\right) \\ & \leq \frac{c(d)}{\delta} E\left(u, j, |K^j|\right). \end{aligned} \tag{A.14}$$

We used (A.5) and (A.12) in the last inequality.

Assume this process has been done for $(k-1)$-th direction for some $3 \leq k \leq d$, now let us look at the k-th direction. For $1 \leq i \leq m_k - 1$, let J_i be the closed interval $[(i - 1/8)\delta, (i + 1/8)\delta]$, $P_i = \{x : x \in I_a, x^k \in J_i\}$. Fix a positive constant c_k, which will be determined later, for any $\alpha \subset \{1, \cdots, k\}$ such that $k \in \alpha$, we have

$$\begin{aligned} & \int_{J_i} E\left(u, j-|\alpha|, H_{k,t} \cap |K^{j-|\alpha|}_{\alpha\backslash\{k\}}| \cap \left(|K^j| \backslash |K^{j-1}|\right)\right) dt \\ & \leq E\left(u, j-|\alpha|+1, |K^{j-|\alpha|+1}_{\alpha\backslash\{k\}}| \cap \left(|K^j| \backslash |K^{j-1}|\right)\right) \qquad \text{for } |\alpha|+1 \leq j \leq d. \end{aligned} \tag{A.15}$$

Define

$$\begin{aligned} \mathcal{B}^\alpha_j = \Big\{ t : t \in J_i, & E\left(u, j-|\alpha|, H_{k,t} \cap |K^{j-|\alpha|}_{\alpha\backslash\{k\}}| \cap \left(|K^j| \backslash |K^{j-1}|\right)\right) \\ & \geq c_k \delta^{-1} E\left(u, j-|\alpha|+1, |K^{j-|\alpha|+1}_{\alpha\backslash\{k\}}| \cap \left(|K^j| \backslash |K^{j-1}|\right)\right)\Big\} \end{aligned} \tag{A.16}$$

for $k \in \alpha, |\alpha| + 1 \leq j \leq d$. Then it follows from (A.15) that

$$\mathcal{H}^1\left(\mathcal{B}^\alpha_j\right) \leq \frac{\delta}{c_k}. \tag{A.17}$$

Let

$$\mathcal{B} = S_k \cup \bigcup_{k\in\alpha, |\alpha|+1 \leq j \leq d} \mathcal{B}^\alpha_j.$$

Then

$$\mathcal{H}^1(\mathcal{B}) \leq \frac{c(d)}{c_k}\delta. \tag{A.18}$$

In view of (A.18), if we take $c_k = c_k(d)$ large enough, we may find a point $t_{k,i} \in J_i \backslash \mathcal{B}$. By setting $t_{k,0} = 0$, $t_{k,m_k} = a^k$, we get m_k numbers in the k-th direction. In addition we have

$$E\left(u, j-|\alpha|, H_{k,t} \cap |K^{j-|\alpha|}_{\alpha\backslash\{k\}}| \cap \left(|K^j| \backslash |K^{j-1}|\right)\right) \tag{A.19}$$

$$\leq \frac{c\,(d)}{\delta} E\left(u, j-|\alpha|+1, |K^{j-|\alpha|+1}_{\alpha\backslash\{k\}}| \cap \left(|K^j|\backslash|K^{j-1}|\right)\right) \qquad \text{for } k \in \alpha, |\alpha|+1 \leq j \leq d.$$

Hence the induction gives us $K_{\{1,\cdots,k\}}$. If we set $\widetilde{K} = K_{\{1,\cdots,k\}}$, one then deduces

$$E\left(u, i, |\widetilde{K}^i| \cap \left(|K^j|\backslash|K^{j-1}|\right)\right) \leq c\,(d) \left(\frac{1}{\delta}\right)^{j-i} E\left(u, j, |K^j|\right) \quad \text{for } 1 \leq i < j \leq d. \tag{A.20}$$

This gives us the first estimate in Lemma 6.3. The second one follows easily from the first one. □

Appendix B. Deformations associated with the dual skeletons

In this appendix, we shall give detailed proofs for some basic properties of the deformations defined at the beginning of Section 6. Assume K is a finite rectilinear cell complex with $\dim K = m$. For each $\Delta \in K$, pick up a point $y_\Delta \in \mathrm{Int}\,(\Delta)$. Fix an integer $0 \leq k \leq m-1$. Then we have Γ^k_ε as the level set of the function $|\cdot|_k$ which is defined inductively by (6.3). For $\delta, \varepsilon \in (0,1)$, we have a natural map $\phi^k_\varepsilon|_{\Gamma^k_\delta}$ from Γ^k_δ to Γ^k_ε.

Lemma B.1. *For any $\delta, \varepsilon \in (0,1)$, $\phi^k_\varepsilon|_{\Gamma^k_\delta}$ is a bijection from Γ^k_δ to Γ^k_ε, its inverse is $\phi^k_\delta|_{\Gamma^k_\varepsilon}$.*

Proof. It follows from an induction argument that for any $\delta, \varepsilon \in (0,1)$, any $0 < |x|_k < 1$,

$$\phi^k_\delta\left(\phi^k_\varepsilon\,(x)\right) = \phi^k_\delta\,(x)\,. \tag{B.1}$$

Lemma B.1 follows because for any $\delta \in (0,1)$, any $x \in \Gamma^k_\delta$, $\phi^k_\delta\,(x) = x$. □

From now on we always assume K is a finite rectilinear cell complex with $\dim K = n$ and for any $x \in |K|$, there exists a $\Delta \in K$ with $\dim(\Delta) = n$ such that $x \in \Delta$. For each $\Delta \in K$, we pick up a point $y_\Delta \in \mathrm{Int}\,(\Delta)$. Let $\mathcal{Y} = (y_\Delta)_{\Delta \in K}$. Fix an integer $0 \leq k \leq n-1$.

Lemma B.2. *There exists a constant $c\,(K, \mathcal{Y}) > 0$ such that*

$$0 < c\,(K, \mathcal{Y})^{-1} \leq |d\left(|\cdot|_k\right)| \leq c\,(K, \mathcal{Y}) \quad \mathcal{H}^n \text{ a.e. on } |K|. \tag{B.2}$$

Proof. This follows from an easy induction if we observe the following two facts. First one, given any rectilinear cell Δ with $\dim(\Delta) = m \in \mathbb{N}$, pick up any point $y_\Delta \in \mathrm{Int}\,(\Delta)$, define a map $\psi : \Delta \to \overline{B^m_1}$ by

$$\psi(x) = |x|_\Delta \cdot \frac{x - y_\Delta}{|x - y_\Delta|} \qquad \text{for any } x \in \Delta. \tag{B.3}$$

Then ψ is a bi-Lipschitz map. Second one, given any suitably differentiable function f on ∂B_1, set $u\,(x) = |x| f\,(x/|x|)$ for $x \in \overline{B_1}$. Then we have

$$|du\,(x)\,|^2 = |f\,(x/|x|)\,|^2 + |df\,(x/|x|)\,|^2. \tag{B.4}$$

□

Lemma B.3. *The map* ϕ^k *satisfies*

$$|\partial_2 \phi^k(x,\varepsilon)| \le c(K,\mathcal{Y}) \quad \text{for } 0<|x|_k<1, 0<\varepsilon<1. \tag{B.5}$$

Here ∂_2 *means derivative with respect to* ε. *For derivatives with respect to* x, *we have*

$$|d_x \phi^k(x,\varepsilon)| \le c(K,\mathcal{Y}) \left(\frac{\varepsilon}{|x|_k} + \frac{1-\varepsilon}{1-|x|_k}\right). \tag{B.6}$$

Proof. This follows from induction along with the formulas (6.7) and (6.8). Note that for any $\Delta \in K$, $x \in \Delta$, we have $|x|_k \le |x|_\Delta$. □

Corollary B.1. *For* $0<\delta\le\varepsilon\le 1/2$, *we have*

$$|dF^k_{\delta,\varepsilon}(x)| \le c(K,\mathcal{Y})\,\varepsilon/|x|_k \quad \text{for } \delta \le |x|_k \le \varepsilon, \tag{B.7}$$

$$|dF^k_{\delta,\varepsilon}(x)| \le c(K,\mathcal{Y})\,\varepsilon\delta^{-1} \quad \text{for } |x|_k \le \delta. \tag{B.8}$$

Proof. This follows from Lemma B.3 and an easy computation. □

To understand more refined properties of the map ϕ^k, we need to introduce some notations. Given any $(n-k)$ number $\varepsilon_i \in [0,1]$ for $k+1 \le i \le n$, we want to define the set $\Upsilon^k_{\varepsilon_{k+1},\cdots,\varepsilon_n}$. This will be done inductively. For $\varepsilon_{k+1}\in[0,1]$, we set

$$\Upsilon^k_{\varepsilon_{k+1}} = \bigcup_{\Delta\in K,\ dim(\Delta)=k+1} \left(y_\Delta + \varepsilon_{k+1}\left(Bd(\Delta) - y_\Delta\right)\right). \tag{B.9}$$

Clearly $\Upsilon^k_{\varepsilon_1} \subset |K^{k+1}|$. Assume for some $k+2\le i\le n$, $\Upsilon^k_{\varepsilon_{k+1},\cdots,\varepsilon_{i-1}}$ has already been defined as a subset of $|K^{i-1}|$. Then we set

$$\Upsilon^k_{\varepsilon_{k+1},\cdots,\varepsilon_i} = \bigcup_{\Delta\in K,\ dim(\Delta)=i} \left(y_\Delta + \varepsilon_i\left(\left(\Delta \cap \Upsilon^k_{\varepsilon_{k+1},\cdots,\varepsilon_{i-1}}\right) - y_\Delta\right)\right). \tag{B.10}$$

Eventually we get $\Upsilon^k_{\varepsilon_{k+1},\cdots,\varepsilon_n}$ for $\varepsilon_i\in[0,1]$, $k+1\le i\le n$. Clearly we have

$$\Upsilon^k_{\varepsilon_{k+1},\cdots,\varepsilon_n} \subset \Gamma^k_{\varepsilon_{k+1}\cdots\varepsilon_n}. \tag{B.11}$$

The importance of $\Upsilon^k_{\varepsilon_{k+1},\cdots,\varepsilon_n}$ lies in the following

Lemma B.4. *Assume* $0<\varepsilon_i\le 1$ *for* $k+1\le i\le n$, $\varepsilon=\varepsilon_{k+1}\cdots\varepsilon_n<1$. *Then for any* $0<\delta\le 1$, *we have*

$$\left|d\left(\phi^k_\delta|_{\Upsilon^k_{\varepsilon_{k+1},\cdots,\varepsilon_n}}\right)\right| \le c(K,\mathcal{Y})\,\delta\varepsilon^{-1} \quad \mathcal{H}^k \text{ a.e. on } \Upsilon^k_{\varepsilon_{k+1},\cdots,\varepsilon_n}. \tag{B.12}$$

Proof. This follows easily from an induction argument in view of the definition of ϕ^k by (6.7) and (6.8). □

Corollary B.2. *For* $0<\delta\le\varepsilon\le 1/2$, *we have*

$$J_{\left(\phi^k_\delta|_{\Gamma^k_\varepsilon}\right)}(x) \le c(K,\mathcal{Y})\,(\delta/\varepsilon)^k \quad \text{for } \mathcal{H}^{n-1} \text{ a.e. } x\in\Gamma^k_\varepsilon. \tag{B.13}$$

Proof. It follows from Lemma B.3 that

$$\left|d\left(\phi^k_\delta|_{\Gamma^k_\varepsilon}\right)\right| \le c(K,\mathcal{Y})\left(\frac{\delta}{\varepsilon} + \frac{1-\delta}{1-\varepsilon}\right) \le c(K,\mathcal{Y}). \tag{B.14}$$

On the other hand for $x \in \Gamma_\varepsilon^k$, we may find $(n-k)$ numbers, namely $\varepsilon_i \in (0,1]$ for $k+1 \le i \le n$ such that $x \in \Upsilon^k_{\varepsilon_{k+1},\cdots,\varepsilon_n}$. Now it follows from Lemma B.4 that

$$|d\left(\phi_\delta^k|_{\Upsilon^k_{\varepsilon_{k+1},\cdots,\varepsilon_n}}\right)(x)| \le c(K,\mathcal{Y})\,\delta\varepsilon^{-1}, \tag{B.15}$$

which implies $d\left(\phi_\delta^k|_{\Gamma_\varepsilon^k}\right)(x)$ has operator norm bounded by $c(K,\mathcal{Y})\,\delta\varepsilon^{-1}$ on a k-dimensional subspace of the tangent space of Γ_ε^k at x. Combining this last estimate with (B.14), one concludes Corollary B.2. □

Corollary B.3. *For $0<\varepsilon\le 1/2$, we have*

$$\mathcal{H}^n\left(\{x \in |K| : |x|_k \le \varepsilon\}\right) \le c(K,\mathcal{Y})\,\varepsilon^{k+1}. \tag{B.16}$$

Proof. From Lemma B.1 we know for any $0<\delta\le 1/2$, $\phi_\delta^k|_{\Gamma^k_{1/2}}$ is a bijection from $\Gamma^k_{1/2}$ to Γ^k_δ, hence from area formula we have,

$$\mathcal{H}^{n-1}\left(\Gamma_\delta^k\right) = \int_{\Gamma^k_{1/2}} J_{\left(\phi_\delta^k|_{\Gamma^k_{1/2}}\right)}(x)\,d\mathcal{H}^{n-1}(x) \le c(K,\mathcal{Y})\,\delta^k. \tag{B.17}$$

Here we use Lemma B.4 in the last step. Now for any $0<\varepsilon\le 1/2$, we have

$$\mathcal{H}^n\left(\{x\in|K| : |x|_k\le\varepsilon\}\right) \le c(K,\mathcal{Y})\int_{|x|_k\le\varepsilon} J_{|\cdot|_k}(x)\,d\mathcal{H}^n(x) \quad \text{(by Lemma B.2)}$$

$$= c(K,\mathcal{Y})\int_0^\varepsilon \mathcal{H}^{n-1}\left(\Gamma_\delta^k\right)d\delta \quad \text{(by coarea formula)} \le c(K,\mathcal{Y})\,\varepsilon^{k+1} \quad \text{(by (B.17))}.$$

□

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