(2018-11-08

~~X is a metric tree $\iff \forall x,y,z,w \in X$,~~

Asymptotic cones: Last time: definition

Today: How are cones related to the original spaces?

Two ideas: 1. A sequence of maps $X_i \to Y_i$ converges to a map $\lim_\omega X_i \to \lim_\omega Y_i$ — so a sequence of, e.g., longer and longer curves in X converges to a curve in $\mathrm{Cone}_\omega X$.

2. Finite configurations of points in $\mathrm{Cone}_\omega Y$ correspond to finite configurations in Y_i — so, properties that can be expressed in terms of finite configurations transfer.

(We saw this last time with polynomial growth: if a ball of radius r_i can ~~always~~ be covered by k balls of radius $\frac{r_i}{2}$, then the unit ball in $\mathrm{Cone}_\omega X$ can be covered by k balls of radius $\frac{1}{2}$.)

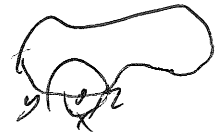
~~Ex:~~ For example, let's look at trees, which I tried last time but didn't quite manage:

Def: An <u>$\mathbb{R}$-tree</u> is a ~~connect~~ geodesic metric space ~~st~~ that contains no simple closed curves. (So there's a unique ~~geodesic between~~ path between any two pts: —o— .)

Thm: An ultralimit of $\mathbb{R}$-trees is an $\mathbb{R}$-tree.

Pf: ~~Let BWOC, let $\gamma: S^1 \to$~~

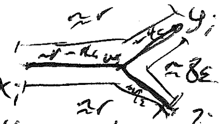
Let T_i be $\mathbb{R}$-trees, let $T_\omega = \lim_\omega T_i$. BWOC, suppose that $\gamma: S^1 \to T_\omega$ is a simple closed curve. Let $x \in \gamma$, choose a small $r>0$ s.t. $\exists\, y,z \in \gamma$ with $d(y,x) = d(z,x) = r$ and $[y,z]$ avoids $B_r(x)$. Let $\varepsilon = \frac{d(y,z)}{8}$.



After passing to a subsequence, we may suppose that $\exists\, x_i, y_i, z_i \in T_i$ s.t.

$$d(x_i,y_i) \to r$$
$$d(x_i,z_i) \to r$$
$$d(y_i,z_i) \to 8\varepsilon$$

So what's the config?



And every curve connecting y_i to z_i goes through the center w_i. That lets us discretize.

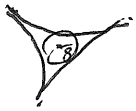
Choose $v^1, \dots, v^n \in [y,z]$ spaced $<\varepsilon$ apart. By assumption, $d(x, v^j) \geq r\ \forall j$. Pass to a subsequence again and choose $v_i^1, \dots, v_i^n \in T_i$ so pairwise distances converge. If i is suff. large, then

Connect y_i to z_i through the v_i^j's. This is a curve from y_i to z_i that never comes within 3ε of w_i ⚡.

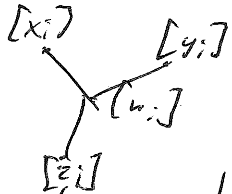
(And, as we saw last time, it's a big $\mathbb{R}$-tree — every pt is a vert, removing any pt produces unctbly many components.) (universal $\mathbb{R}$-tree)

Converse? No!

Thm (Gromov): A space X is δ-hyperbolic for some $\delta \geq 0$ if and only if ~~its~~ every asymptotic cone of X is an $\mathbb{R}$-tree.

Recall: δ-hyp $\iff$ every triangle has a δ-center: 

So ~~every triangle in~~ if $x, y, z \in \mathrm{Cone}_\omega X$, then there is a seq of $w_i \in (X, d_i)$ ~~of~~ of δ_i-centers, and $[w_i]$ is a 0-center — that implies it's an $\mathbb{R}$-tree



One more example, to demonstrate how complicated things can get.

Let $Sol_3 = \left\{ \begin{pmatrix} e^t & & x \\ & e^{-t} & y \\ & & 1 \end{pmatrix} \,\middle|\, x, y, t \in \mathbb{R} \right\}$ ~~This is Then~~ This is a Lie group structure

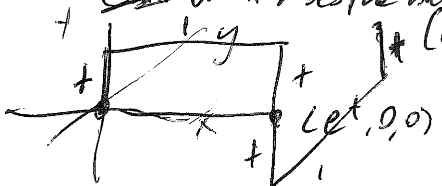
~~Let $g = e^{-2t}$~~

Let $g = e^{-2t}dx^2 + e^{2t}dy^2 + dt^2$ be left-inv metric.

Properties:

~~Then~~ - Sol_3 has exponential growth: ~~the the volume form on Sol_3 is the usual form on $\mathbb{R}^3$, and~~ the b

~~Cone~~ What does the ball of radius r look like?



So $B_r(0) \approx (-e^r, e^r) \times (-e^r, e^r) \times (-r, r)$ has exponential volume.

- It also has exponential DF: ~~complete the~~ .

How does that show up in the cone? We know curves show up — a limit of longer curves converges in ~~Sol~~ the cone. But what about the discs?

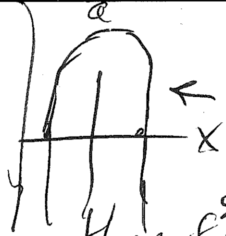
Let's calculate the cone. ~~Here's the trick:~~

First, we can write Sol_3 as a subset of a product:

$$Sol_3 = \left\{ \begin{pmatrix} e^t & & x \\ & e^{-t} & y \\ & & 1 \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} e^a & & x \\ & e^b & y \\ & & 1 \end{pmatrix} \right\} \cong \left\{ \begin{pmatrix} e^a & x \\ & 1 \end{pmatrix} \right\} \times \left\{ \begin{pmatrix} e^b & y \\ & 1 \end{pmatrix} \right\}$$

and $\left\{ \begin{pmatrix} e^a & x \\ & 1 \end{pmatrix} \right\}$ has metric $g = e^{-2a}dx^2 + da^2$. ~~Geodesics:~~

~~This~~



← Geodesics. Another way to understand this – this is a ~~surface it's complet~~ simply-connected, complete surface with transitive isometry group, and there are only three of those: sphere, plane, hyp. plane – ~~th~~ not sphere or plane, so it's hyp plane.

$$Sol_3 \subset \mathbb{H}^2 \times \mathbb{H}^2$$

$$Cone_\omega Sol_3 \subset Cone_\omega(\mathbb{H}^2 \times \mathbb{H}^2) = T_\omega \times T_\omega,$$

where T is universal $\mathbb{R}$-tree.

What subset? ~~Let's go back to the picture~~

~~The fn~~ $Sol_3 = \{(x, y, a, b) \in \mathbb{H}^2 \times \mathbb{H}^2 \mid a + b = 0\}$.

In the cone?

Consider $a : \mathbb{H}^2 \to \mathbb{R}$. This is 1-Lipschitz, ~~so we def~~

~~$a_i : (\mathbb{H}^2, d_i) \to \mathbb{R}$ ... so~~

~~a_ω ... $a_\omega([v_i]) = \lim_\omega a$~~

Let $a_\omega : Cone_\omega \mathbb{H}^2 \to \mathbb{R}$, $a_\omega([v_i]) = \lim_\omega \frac{a(v_i)}{i}$.

Then a_ω is 1-Lipschitz ~~What function is it?~~

~~A geodesic. Take a geodesic in $\mathbb{H}^2$ γ. unit speed~~

~~Take a_ω. If γ is a s $: \mathbb{R} \to \mathbb{H}^2$ is a geodesic,~~

~~then $\gamma_\omega : \mathbb{R} \to Cone_\omega$, $\gamma_\omega(t) = [\gamma(it)]$~~ is a unit-~~speed~~ geod.

~~If γ is vertical then~~ If you know the terminology, a is a Busemann function: we have a collection of vertical geodesics so that any pair of geods gets closer at larger times – we say these geodesics are all asymptotic to the same point at ∞, and a Busemann fn ~~measures~~ "distance" to that pt at ∞ – ~~its of its derivative~~ it's affine w/ slope 1 on each vertical geod.

Let's ~~take~~ look at a geod. Let $\gamma : \mathbb{R} \to \mathbb{H}^2$ be the vertical ~~geod~~ through the origin. Then $\gamma_\omega(t) = [\gamma_\omega(it)]$ is a geodesic in ~~Co~~ T_ω. and $a_\omega(\gamma_\omega(t)) = \lim_\omega \frac{a(\gamma(it))}{i} = t$.

Further, ~~we can~~ use γ_ω to calculate a_ω everywhere.

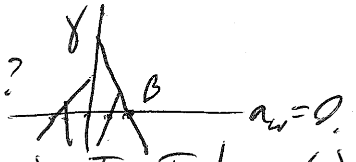
a has the property $a(v) = \lim_{t \to \infty} t - d(v, \gamma(t))$

In fact, a_ω measures "height".

Exercise: $\exists$ a ~~geod~~ unit speed geodesic $\gamma : \mathbb{R} \to T_\omega$ s.t. $a_\omega(\gamma(t)) = t$.

If $\beta : \mathbb{R} \to T_\omega$ is a unit-speed geod s.t. $\beta(t) = \gamma(t)$ for all suff. large t, then $a_\omega(\beta(t)) = t$ $\forall t$.

What do I mean ~~what~~ by height?

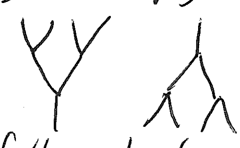


Then ~~Sol_3~~ $\mathrm{Cone}_\omega \mathrm{Sol}_3 = \{ (v,w) \in T_\omega \times T_\omega \mid a_\omega(v) + b_\omega(w) = 0 \}$.

~~Then: This is a 1-dimensional~~ — injective ~~...~~ discs ~~no embedded discs~~,
~~wild~~ space. ~~Wild~~

Thm: ~~...~~ No ball in $\mathrm{Cone}_\omega \mathrm{Sol}_3$ is simply connected.

Pf:



(In fact, $\pi_1(\mathrm{Cone}_\omega \mathrm{Sol}_3)$ is uncountable.

(following level sets).

~~But also folded flats~~: But also: This space, $T_\omega \times T_\omega$ is an example of a building — these are spaces, often coming from ~~algebraic~~ arithmetic groups, that have flats and where the geometry of the flats is important to the geom of the space. Here, a flat is a geod $\times$ a geod — in this picture, a piece of paper folded in quarters. The intersection with Sol_3 is this corner.

What's nice about this picture is that it generalizes —

$$\mathrm{Sol}_5 = \left\{ \begin{pmatrix} e^{t_1} & & & x \\ & e^{t_2} & & y \\ & & e^{t_3} & z \\ & & & 1 \end{pmatrix} \;\middle|\; t_1 + t_2 + t_3 = 0 \right\} \subset \left\{ \begin{pmatrix} e^{t_1} & & & x \\ & e^{t_2} & & y \\ & & e^{t_3} & z \\ & & & 1 \end{pmatrix} \right\} \cong \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$$

~~Then~~ Sol_5 has quadratic DF, ~~But~~ ~~Cone~~ $\pi_1(\mathrm{Cone}_\omega \mathrm{Sol}_5)$ is ?,

but $\pi_2(\mathrm{Cone}_\omega \mathrm{Sol}_5)$ is uncountable.

So: ~~If DF is large, the~~ the exponential discs in Sol_3 simply disappear in $\mathrm{Cone}_\omega \mathrm{Sol}_3$. ~~Which makes some sense.~~

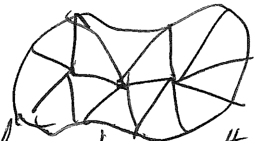
In general? A pair of theorems, maybe discuss more next time:

Thm (Gromov): ~~If every asymptotic cone of X is s.c., then~~ ~~Suppose~~ Let G be a f.p. group. If every asymp cone of G is s.c. then $\exists \alpha > 0$ s.t. $\delta_G(n) \lesssim n^\alpha$.

Thm (Gromov): If $\delta_G(n) \lesssim n^2$, then every asymp cone of G is s.c.

Pf: Remember the idea - to pass from cone to space, we need to express properties in terms of finitely many points.

So: let's say that X is k-triangulable if $\forall$ suff. large $L>0$, $\forall$ ~~curves γ: S~~ closed curves $\gamma : S^1 \to X$ of length $\leq L$, we can ~~decompose γ into~~ fill γ ~~with~~ a triangulation with $\leq k$ triangles, each of length $\leq L/2$.



~~If X is~~

If X is k-triangulable for some k, then we can repeat the process.

Every time, length goes down by 2. # of triangles up by k, so this eventually results in a disc with polynomial area.

Further, k-triangulable just relies on finitely many pts!

So let's do this! Suppose X is not k-triangulable $\forall k$. Then $\exists$ curves $\gamma_k : S^1 \to X$ with $\ell(\gamma_k) \to \infty$ s.t. γ_k has no k-triangulation. Consider $X_\omega = \lim_\omega (X, \frac{d}{\ell(\gamma_k)}, \gamma_k(0))$. This is a cone of X.

$\gamma_\omega = \lim_\omega \gamma_k : S^1 \to X_\omega$ is unit speed, length 1, and by assumption, $\exists$ $\beta : D^2 \to X_\omega$ filling γ. β is unif cts, so a suff fine triangulation of D^2 gives a triang of γ_ω into triangles with length $\leq 1/3$. The vertices of this triang form a triangulation in ~~some γ_k~~ all suff. large γ_k $\times$.