(2018-11-29

Systolic geometry

Let M be a Riemannian manifold. The systole $sys_1(M)$ is the length of the shortest closed curve in M that is homotopically nontrivial.

Q: How is $sys_1(M)$ related to M? Especially $vol(M)$?

Ex: Suppose $M = \mathbb{T}^2$, with a flat metric. ~~If $sys_1(M)$~~
Then $M = \mathbb{R}^2/\Gamma$ for some lattice $\Gamma \subset \mathbb{R}^2$,
$sys_1(M) = \min\{\|v\| \mid v \in \Gamma\}$. Furthermore, $\forall x \in M$,
$injrad(x) \geq \frac{sys_1(M)}{2}$,

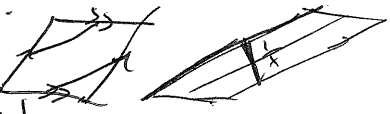
so, $Area(M) \geq Area(B_{\frac{sys}{2}}(x)) = \pi\left(\frac{sys_1(M)}{2}\right)^2 = \frac{\pi}{4} sys_1(M)^2$

In fact ~~optimum~~ $Area(M) \geq \frac{\sqrt{3}}{2} sys_1(M)^2$, which is sharp when Γ is hexagonal lattice (~~exercise~~). Pf: WLOG, take $sys_1(M) = 1$.

The remarkable thing is that the same inequality holds without any assumption on curvature!

Thm (Loewner) Let $M = (\mathbb{T}^2, g)$. Then
$$Area(M) \geq \frac{2}{\sqrt{3}} sys_1(M)^2$$
which is sharp when ~~the hexagonal lattice~~. $M = \mathbb{R}^2/\text{hex lattice}$.

Pf: ~~WLOG, take~~ ~~First~~ Reduce to flat case. WLOG, $Area(M) = 1$. By Uniformization thm, there is a flat ~~$(\mathbb{T}^2, h)$~~ metric $N = (\mathbb{T}^2, h)$ and a conformal factor $\rho: \mathbb{T}^2 \to \mathbb{R}^{+}$ s.t. $(\mathbb{T}^2, \rho h) \cong (\mathbb{T}^2, g)$.
Then $Area(M) = \int_{\mathbb{T}^2} \rho^2 \, d\mu_h = 1$.



Let γ be a simple closed geodesic in N of length λ.
Then $N \setminus \gamma$ is a cylinder of ~~length~~ width $\frac{1}{\lambda}$.
For $s \in [0, \frac{1}{\lambda}]$, let γ_s be the geodesic ~~parallel to γ, separated by s~~ at constant distance s from γ.

Then $\ell(\gamma_s, g) = \int_0^\lambda \rho(\gamma_s(t))\, dt$
$$\frac{\int_0^{1/\lambda} \ell(\gamma_s, g)\, ds}{1} = \frac{\int_0^{1/\lambda} \int_0^\lambda \rho(\gamma_s(t))\, dt\, ds}{1} = \frac{\int_{\mathbb{T}^2} \rho \, d\mu}{1}$$

By Cauchy-Schwarz, $\leq \sqrt{\int_{\mathbb{T}^2} \rho^2 \, d\mu} \sqrt{\int_{\mathbb{T}^2} 1 \, d\mu} = 1$.

So $\frac{\int_0^{1/\lambda} \ell(\gamma_s, g)\, ds}{\frac{1}{\lambda}} \leq \lambda \Rightarrow \exists s$ s.t. $\ell(\gamma_s, g) \leq \lambda$.

Therefore, $sys_1(M) \le sys_1(N) \le \frac{2}{\sqrt{3}}$. And if $sys_1(M) = \frac{2}{\sqrt{3}}$, then N is hex, C–S is sharp $\Rightarrow p \equiv 1$ and $M \cong N$. //

Other surfaces: Thm (Pu): If $M = (\mathbb{RP}^2, g)$, then
$$Area(M) \ge \frac{2}{\pi} sys_1(M),$$
with equality when ~~g~~ is the round metric.

~~But higher genus is not known precisely, and not expected to be so symmetric (eg. best known~~

Def: The systolic ratio of a surface Σ is
$$SR(\Sigma) = \min_g \frac{Area(\Sigma, g)}{sys_1(\Sigma, g)^2} = \text{area of smallest metric with systole } 1.$$

$SR(\mathbb{T}^2) = \frac{\sqrt{3}}{2}$, $SR(\mathbb{RP}^2) = \frac{2}{\pi}$, but higher genus is generally not known precisely — and the surfaces that achieve it aren't expected to be so symmetric. Eg. the best known example for genus 2 is six regular octagons glued in a certain pattern.

But we can find asymptotics:

Thm (Buser–Sarnak). ~~$\forall g > 0$,~~ $\exists$ a sequence of arithmetic lattices $\Gamma_n \subset SL(2, \mathbb{R})$ s.t. $\mathbb{H}/\Gamma_n$ are hyperbolic surfaces with $genus(\mathbb{H}/\Gamma_n) \to \infty$ and $sys_1(\mathbb{H}/\Gamma_n) \sim \log(genus(\mathbb{H}/\Gamma_n))$.

Then $SR(\Sigma_g) \lesssim \frac{g}{(\log g)^2}$.

Thm (Gromov): $SR(\Sigma_g) \gtrsim \frac{g}{(\log g)^2}$ — i.e. if $M = (\Sigma_g, g)$ has area g, then it ~~has a close~~ $sys_1(M) \lesssim \log g$.

~~Pf (due to Katz and Sabourau):~~ ~~Want to prove this, but we need~~
~~Pf Before we prove Need some too~~ To prove this, we need entropy.

The volume entropy of a compact mfld (M, g) is
$$h(M, g) = \lim_{R \to \infty} \frac{\log vol\, B_R(\tilde{x}, \tilde{M})}{R}$$
where $\tilde{M}$ is univ cover, $\tilde{x} \in \tilde{M}$.

(Check: independent of $\tilde{x}$).

Sometimes it's easier to work with topological entropy. $\forall \gamma \in \pi_1(M, x)$, let $\|\gamma\| = \inf\{\ell(\lambda) \mid \lambda \sim \gamma\}$ by a htpy fixing endpts.

Then
$$h_{top}(M, g) = \lim_{R \to \infty} \frac{\log \#\{\gamma \in \pi_1(M, x) \mid \|\gamma\| < R\}}{R}$$

Lemma: $h(M, g) = h_{top}(M, g)$.

In particular, if $M = (S, g)$ is hyperbolic, then
$$h_{top}(M) = h(M) = \lim_{R\to\infty} \frac{\log \mathrm{vol}\, B_R(x;\mathbb{H})}{R} = 1.$$

Katok showed that this is ~~optimal~~ minimal:

Thm (Katok): If $\chi(M)<0$, then $\forall$ metric g,
$$h(M,g)^2 \ge \frac{2\pi|\chi(M)|}{\mathrm{Area}(g)}.$$
~~(check the scaling)~~

This is sharp $\Leftrightarrow$ g has constant curvature.
~~And note,~~ (Note: when g is hyp, $1=1$. Scaling by a const ~~preserves~~ scales both sides the same way.)

Pf: WLOG, scale so $\mathrm{Area}(g) = 2\pi|\chi(M)|$.
Then $\exists\, \rho: M \to \mathbb{R}^{>0}$ and a hyp metric σ so $(M, \rho\sigma) \cong (M,g)$. Let μ_σ be the measure corresp to σ — then, as before
$$\frac{\mathrm{Area}(g)}{\mathrm{Area}(\sigma)} = \frac{\int \rho^2\, d\mu_h}{\mathrm{Area}(h)} = 1 \Rightarrow ⨍ \rho^2\, d\mu_h = 1$$
$$\Rightarrow ⨍ \rho\, d\mu_h \le 1.$$

Write $⨍ \rho^2\, d\mu_h = 1$. The geodesic flow on (M, σ) is <u>ergodic</u>: if SM is the unit tangent bundle, $f: SM \to \mathbb{R}$ is continuous, then a.e. $v \in SM$, the hyp geod γ_v satisfies
$$\lim_{T\to\infty} \frac{\int_0^T f(\gamma(t))\,dt}{T} = ⨍_{SM} f.$$
— geods are evenly distrib in M.

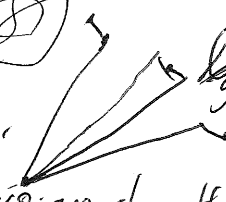
Choose $x \in M$ so that a.e. $v \in S_x M$, $\lim_{T\to\infty} \frac{\int_0^T \rho(\gamma(t))\,dt}{T} = ⨍_M \rho\, d\mu \le 1$
$$\Rightarrow \lim_{T\to\infty} \frac{\ell_g(\gamma_v([0,T]))}{T} \le 1.$$

$\forall \varepsilon>0, \exists\, T_0$, ~~st. for all~~ $P \subset S_x M$ s.t. $\mu(P) \ge \frac{1}{2}\mu(S_x M)$ and $\forall T > T_0$,
$$\ell_g(\gamma_v([0,T])) \le (1+\varepsilon)T.$$

Now, we bound h_{top}. $\forall v \in S_x M$, let $\bar\gamma_{v,T}$ be the closed curve obtained by connecting $\gamma_v(T)$ to x by a shortest path:

~~Then~~ $\forall T > T_0$, $\forall v \in P$,
$$\ell_g(\bar\gamma_{v,T}) \le (1+\varepsilon)T + O(1).$$
~~Because we're in H, these geods diverge exp. quickly, so~~ e

In $(\tilde M, \sigma)$,



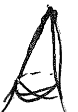
If two are close, they have same end pt, htpic. If not, then at

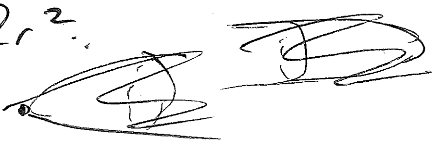
But because ~~of hyp.~~ "close" $\Rightarrow$ v. close: $\exists\, c > 0$ s.t. (depending on ϵ)
if $T > T_0$ and $\angle(v,w) > \frac{c}{e^T}$, then $\overline{\gamma_{v,T}} \not\simeq \overline{\gamma_{w,T}}$.

Therefore (M,g) has $\gtrsim e^T$ non-homotopic closed curves of length $(1+\varepsilon)T + O(1)$ $\Rightarrow h_{top}(M,g) \geq 1$ //

Thm (Gromov): $SR(\Sigma_g) \gtrsim \frac{g}{(\log g)^2}$.

Lemma: There is a ~~metric~~ surface ~~on~~ $M = (\Sigma_g, g)$ so that
$SR(\Sigma_g) = \frac{\text{Area}(M)}{\text{sys}_1(M)^2}$. This surface satisfies:
– There is a closed geodesic of length $\text{sys}_1(\Sigma_g, M)$ through every point of ~~Σ~~ M. (otherwise, we could shrink the area).
– $\forall\, x \in M$, $0 < r < \frac{\text{sys}_1(M)}{2}$,
$$\text{area}(B_r(x)) \geq 2r^2.$$





Pf: Let M be as in the lemma, ~~with~~ ~~area~~ $A = \text{Area}(M)$.
Let $\{x_1, \dots, x_n\}$ be a maximal $\frac{r}{6}$-net, $r = \text{sys}_1(M)$.
Since $B_{\frac{r}{12}}(x_i)$ are disjoint, $n \leq \frac{A}{2\frac{r^2}{144}} \leq \frac{72A}{r^2}$.

~~Let G be the graph whose vertices are~~ $x_1, \dots, x_n$.
Draw ~~arcs~~ a geodesic between each pair of (x_i, x_j) s.t.
$d(x_i, x_j) < \frac{r}{2}$. Claim: Every closed curve ~~based at x_0~~
Call this G. $\gamma \in \pi_1(M, x_1)$ is htpic to a curve in G.

Let $\gamma : [0,L] \to M$ be a unit speed closed curve, & let
$t_1 = 0 < t_2 < \dots < t_m = L$
be pts s.t. $|t_{i+1} - t_i| \leq \frac{r}{12}$, $m \lesssim \frac{12L}{r}$.

Let ~~g_i be the ver~~ a_i be s.t. x_{a_i} is the closest vertex to $\gamma(t_i)$
Connect x_{a_i} and $\gamma(t_i)$ by λ_i
of length $\leq \frac{r}{6}$. Then:
$$d(x_{a_i}, x_{a_{i+1}}) \leq \frac{r}{6} + \frac{r}{12} + \frac{r}{6} < \frac{r}{2},$$ so
x_{a_i} and $x_{a_{i+1}}$ are adj., and all these loops have length $\leq r \Rightarrow$ null htpic.

~~Further~~, So, $h_{top}(M) = \lim\limits_{R\to\infty} \frac{\log \#\{\gamma \mid \|\gamma\| < R\}}{R} \leq \lim\limits_{R\to\infty} \log \left(72\frac{A}{r^2}\right)^{\frac{12R}{r}}$

$\frac{12}{r} \log \frac{72 A}{r}$ $\Rightarrow \frac{A}{r^2} \gtrsim \frac{|\chi(M)|}{(\log|\chi(M)|)^2}$ $= \frac{12}{R}\log\left(72\frac{A}{r^2}\right) \geq \sqrt{\frac{2\pi|\chi(M)|}{A}}$