2018-10-25

Finish something from last time:

Thm (Gromov, Pansu): Suppose that $\Sigma \subset (\mathbb{H}, d_c)$ and that $\gamma: [0,1] \to \mathbb{H}$ is a curve ~~such~~ that intersects Σ transversally in the following sense: $\exists \varepsilon > 0$ s.t. if $\gamma': [0,1] \to \mathbb{H}$, $d(\gamma'(t), \gamma(t)) < \varepsilon$ then γ' also intersects Σ. Then $\dim_H(\Sigma) \geq 3$

Pf: We can approximate γ by a curve $\lambda: [0, L] \to (\mathbb{H}, d_c)$ that has unit speed wrt d_c. This is still transverse to Σ.

For $x \in \mathbb{H}$, $r > 0$, let $W(x,r) = \{h \in \mathbb{H} \mid h\lambda \text{ intersects } B_r(x)\}$.

Let's bound the ~~volume~~ measure of $W(x,r)$:

Let $v_i = \lambda(ir)$ for $i = 0, \dots, \frac{L}{r}$. If $h \in W(x,r)$, then, say, $h\lambda(t) \in B_r(x)$, then $\exists i$ s.t. $|ir - t| \leq r \Rightarrow hv_i \in B_{2r}(x)$,

$h \in B_{2r}(x) v_i^{-1}$.

Then $\mu(W(x,r)) \leq \sum_i \mu(B_{2r}(x) v_i^{-1}) \leq \frac{L}{r} \cdot \mu(B_{2r}(x)) \lesssim \frac{L}{r} r^4 = L r^3$

By transversality, $\exists$ a nbhd U of 1 s.t. $u\lambda$ intersects Σ $\forall u \in U$. If $\Sigma \subset \bigcup_i B_{r_i}(x_i)$, then $U \subset \bigcup_i W(x_i, r_i)$

$$\mu(U) \lesssim \sum L r_i^3 \Rightarrow \mathcal{H}^3(\Sigma) \gtrsim \frac{\mu(U)}{L} > 0. \quad //$$

So, what do we know?: Let γ be the curve $[X^n, [X^n, Y^n]]$.

- Any filling of γ has area $\gtrsim n^3$ wrt Riem metric
- Any filling of γ has Haus. dim ≥ 3 wrt sub-R metric

Q: How are these facts related?

(~~Though~~ I'll confess: I don't have a really rigorous answer to this, and I suspect that there may not be a single satisfying answer in the general case. Nevertheless, ...)

Idea: $(\mathbb{H}, d_c)$ is a limit of scalings of $(\mathbb{H}, d_0)$.

What does this mean? Need a defi

Def: If X is a metric space, $A, B \subset X$ are compact sets, the Hausdorff distance from A to B is

$$d_H(A,B) = \min\{r \mid B \subset N_r(A) \text{ and } A \subset N_r(B)\}.$$

Def: Let X, Y be compact metric spaces. The Gromov–Hausdorff distance is $d_{GH}(A,B) = \inf_M d_H(X, Y; M)$, where M ranges over metric spaces containing X, Y isometrically.

Ex: Let $D = \max\{\operatorname{diam} X, \operatorname{diam} Y\}$, let $M = (X \sqcup Y, d_M)$, where $d_M = d_X, d_Y$, ...

~~Since they're compact there's at least one such space~~

~~Ex: Def~~ For instance, $M = X \times Y$ contains both. Let $x \in X$, $y \in$

$X \times Y$

$\{x\} \times Y \quad \cup \quad X \times \{y\}$

Hausdorff dist? roughly ~~max~~ $\{\operatorname{diam} X, \operatorname{diam} Y\}$

Another way to define dist between metric spaces — ~~how close to an~~ can we find a map that's ε-close to an isometry

This is actually the same notion:

Lemma: Let $D = d_{GH}(X, Y)$. There is a map $\varphi : X \to Y$ s.t. $\forall x, x' \in X$, $|d(\varphi(x), \varphi(x')) - d(x, x')| \leq 2D$

Pf: Embed X, Y in M s.t. $d_H(X, Y) = D$

$\forall x \in X$, let $\varphi(x) \in Y$ be the pt closest to x

~~Then~~ Then $d(x, \varphi(x)) \leq D$ $\forall x$, so lemma follows by Δ ineq

Exercise: Prove the converse: $\forall \varphi : X \to Y$,

$$d_{GH}(X, \varphi(X)) \lesssim \max |d(\varphi(x), \varphi(x')) - d(x, x')|$$

So, small GH distance $\Leftrightarrow$ "almost isometric".

~~We say~~ We define GH-convergence for compact spaces in usual way, and we define convergence for pointed non-cpt ~~for non-compact spaces,~~ spaces as follows:

Def: If $(X_1, x_1), (X_2, x_2), \ldots$ and (Y, y) are proper metric spaces, we say that $(X_i, x_i) \xrightarrow{GH} (Y, y)$ if $\forall R > 0$,

$$B_R(x_n; X_n) \xrightarrow{GH} B_R(y; Y).$$

Ex: grids in $\mathbb{R}^2$

But also: $(\mathbb{H}, \frac{d_0}{n}) \xrightarrow{GH} (\mathbb{H}, d_c)$..

In fact, we can construct almost-isoms: Let $s_t(x, y, z) = (tx, ty, t^2 z)$

Claim: $\forall u, v \in \mathbb{H}$, $\lim_{n \to \infty} \frac{d(s_n(u), s_n(v))}{n} = d_c(u, v)$.

Why? Let's undo the metric. Recall, d_0 is left-inv't Riem, ~~the left inv't fields~~

$X = (1, 0, 0)$

$Y = (0, 1, x)$ are left inv't fields so let's define

$Z = (0, 0, 1)$

$g = dx^2 + dy^2 + (dz - x\,dy)^2$. If we pull back under scaling

$$\frac{s_n^* g}{n^2} = \frac{n^2 dx^2 + n^2 dy + (n^2 dz - nx\,n\,dy)^2}{n^2}$$

$= dx^2 + dy^2 + n^2(dz - x\,dy)^2$ — i.e., (kernel $dz - x\,dy$)

horizontal vectors have odd length, Z has length going to ∞.

So, in an infinitesimal sense, convergence. Distance fn takes work.

Ex: Prove the claim about distances in $d_{G/n}$ vs. distances in d_C.

~~There's a remarkable generalization of this idea.~~

Okay: These are limits of metric spaces. ~~But~~ There's a remarkable generalization of this idea: ultralimits.

Ultralimits:

Thm: $\exists$ a linear functional $L : l_\infty \to \mathbb{R}$ s.t. $\forall (x_i) \in l_\infty$, $L((x_i))$ is a point of accumulation of x.

In particular, if (x_i) is convergent, then $L((x_i)) = \lim x_i$.

This is tricky even just for 0-1 sequences: $L(0,1,0,1,\dots) = 0$ or 1

$L(1,0,1,0\dots) = 0$ or 1,

but they can't be the same because $L(1,1,\dots) = 1 = L(0,1,\dots) + L(1,0,\dots)$.

Once you define L on 0-1 seqs, done. And we can do that w/ AoC.

Def: A filter is a nonempty subset $\omega \subset 2^{\mathbb{N}}$ s.t.:

- ω is closed under finite intersections.
- If $A \in \omega$ and $B \supset A$ then $B \in \omega$.
- $\phi \notin \omega$ and $\mathbb{N} \in \omega$. ~~(finitely additive measure w/ values in $\{0,1\}$)~~

ω is an ultrafilter if $\forall A \subset \mathbb{N}$, either $A \in \omega$ or $\mathbb{N} \setminus A \in \omega$.

Idea: A finitely additive ~~measure w/ values in~~ 0-1 valued measure

- if $A, B \subset \mathbb{N}$ and $A \cap B = \phi$, then

$\omega(A)$	$\omega(B)$	$\omega(A \cup B)$
1	1	impossible – $A \cap B = \phi \notin \omega$
1	0	1 – $A \cup B \supset A$
0	1	1 – $A \cup B \supset B$
0	0	0 – $(\mathbb{N} \setminus A) \cap (\mathbb{N} \setminus B) = \mathbb{N} \setminus (A \cup B) \in \omega \Rightarrow A \cup B \notin \omega$.

There's an obvious 0-1 valued measure – point measure.

A principal ultrafilter is a filter of the form

$$\omega_a = \{A \subset \mathbb{N} \mid a \in A\}.$$

Prop: $\exists$ a nonprincipal ultrafilter.

Pf: Axiom of Choice.

Let ω be a nonprincipal ultrafilter. Note that ω contains no finite sets ~~(or by)~~ (if $\omega \ni \{a_1, \dots, a_n\}$, then $\omega \ni \{a_i\}$ for some i, and then $\omega = \omega_{a_i}$.

~~Define $\lim_\omega x_i$~~ We can use this to define ~~one~~ ultralimits.

Roughly, we want to define $\lim_\omega x_i = \int x_i \, d\omega(i)$.
(And I think that's a sensible definition, but let me give the full version)

If $(x_i) \in \ell_\infty$, $y \in \mathbb{R}$, we say $\lim_\omega x_i = y$ if $\forall \varepsilon > 0$,
$\{i \in \mathbb{N} \mid |x_i - y| < \varepsilon\} \in \omega$.

Then:
- $\lim_\omega x_i$ is unique if it exists.
- $\lim_\omega x_i$ is an accumulation point.
- $\lim_\omega$ is linear.
- $\lim_\omega x_i$ exists $\forall (x_i) \in \ell_\infty$. Pf: Bifurcation.
- ~~Changes~~ ~~If $x_i = y_i$ for all but finitely many i then $\lim_\omega x_i = \lim_\omega y_i$~~

We also define ultralimits of metric spaces.

Let (X_i, x_i) be a sequence of ~~co~~ metric spaces.

Define $\ell_\infty(X_i) = \{\text{bounded sequences in } X_i\}$
$= \{(a_i \in X_i)_{i \in \mathbb{N}} \mid \| d(a_i, x_i) \|_\infty < \infty\}$

Given $(a_i), (b_i) \in \ell_\infty(X_i)$, $(d_{X_i}(a_i, b_i)) \in \ell_\infty(\mathbb{R})$.

Define $X_\omega = \left(\frac{\ell_\infty(X_i)}{\sim}, d_\omega \right)$

where $d_\omega((a_i), (b_i)) = \lim_\omega d_{X_i}(a_i, b_i)$.

and $(a_i) \sim (b_i) \iff d_\omega((a_i), (b_i)) = 0$.

~~Check~~ Then: X_ω is a metric space which we call the ultralimit of the X_i.

Ex: $X_i = (\mathbb{R}, 0) \xrightarrow{\omega} \mathbb{R}$. (obviously it should, but...)

~~$(x_i) \in \ell_\infty \mapsto \lim_\omega x_i$ is an isometry~~

$\ell_\infty / \sim \xrightarrow{\text{isom}} \mathbb{R}$

~~$(x_i) \sim (y_i) \iff \lim_\omega |x_i - y_i| = 0 \iff \lim_\omega x_i = \lim_\omega y_i$~~

So $d_\omega((x_i), (y_i)) = \lim_\omega |x_i - y_i| = |\lim_\omega x_i - \lim_\omega y_i|$

So $(x_i) \mapsto \lim_\omega x_i$ is a well-defined isom.

~~More generally, if~~ ~~But also $\frac{1}{n}\mathbb{R} \xrightarrow{\omega} \mathbb{R}$,~~ More generally if $X_i \xrightarrow{GH} Y$ and Y is proper. But:

2018-11-01

Last time: Ultralimits of metric spaces:
- (X_i, x_i) a seq of metric spaces.
- $\ell_\infty(X_i) = \{(a_i \in X_i) \mid \exists C > 0 \text{ s.t. } d(a_i, x_i) < C \ \forall i\}$.
- $\lim_w X_i = (\ell_\infty(X_i)/\sim, d_w)$
  where $d_w((a_i),(b_i)) = \lim_w d(a_i, b_i)$ and $(a_i) \sim (b_i) \iff d_w((a_i),(b_i)) = 0$.

Write $[a_i]$ for the equivalence class of (a_i).

Today: Asymptotic cones: Let (X, d) be a metric space, $x_i \in X$ a seq. of basepoints, w an ultrafilter.

Let $\mathrm{Cone}_w(X, x_i) = \lim_w (X, \frac{d}{i}, x_i)$.

This depends on x_i and on w. Sometimes you'll see
$\mathrm{Cone}_w(X, x_i, d_i) = \lim_w (X, \frac{d}{d_i}, x_i)$ where $d_i \to \infty$.
but you can get the same effect by changing w.
~~If X has a G acts Further, dependence on basepoint~~ is often small – ~~the only place the basepoint appears~~ is in defining $\ell_\infty(X_i)$.
~~so if we move basepoints by a bdd dist, no change.~~
~~frequently, we write~~
~~Furthermore, the~~ This is particularly useful if ~~G X the~~ there's ~~Lemma Exercise~~: a group action on X:
- If G acts geometrically on X, then this is independent of basepoint
- If $f: X \simeq_{QI} Y$, then ~~[f]~~ $f_w([a_i]) = [f(a_i)]$ is a well-defined biLipschitz homeomorphism. (Exercise)

So $\mathrm{Cone}_w(G)$ is a group invariant.

~~In general, cone can be complicated.~~ Calculate:
~~Ex~~ First: ~~Exercise~~ a lemma:
Lemma: If $(X_i, x_i) \xrightarrow{GH} (Y, y)$ and Y is proper then $\lim_w X_i = Y$.
Pf: Since Y is proper, bounded seqs in Y have ultralimits
Pf: Let $\varphi_i: X_i \to Y$ be a sequence of almost-isometries so that $(\varphi_i(x_i)) \in \ell_\infty(Y)$. ~~Then $\forall (a_i) \in \ell_\infty(X_i)$.~~

~~Exercise Then $\Phi: \lim_w \varphi_i(X_i) \to \lim_w Y$~~
Then:
Exercise: $\Phi: \lim_w X_i \to Y$
$\Phi(a_i) = \lim_w \varphi_i(a_i)$ is an isometry ✓

So: $\mathrm{Cone}_w(\mathbb{R}^n) = \mathbb{R}^n$
$\mathrm{Cone}_w(\mathbb{H}, d_0) = (\mathbb{H}, d_0)$.

More generally,

Thm (Gromov): If G is torsion-free, nilpotent, then $(G, \frac{d}{i}) \xrightarrow{GH} X$ where X is a sub-Riemannian manifold.

In fact, the origin of asymptotic cones is a sort of converse to this statement – they originated in Gromov's proof of the polynomial growth theorem.

Suppose G is finitely-generated.
Growth of G is $\Gamma(r) = \# B_r(1)$ with respect to the word metric.

Thm (Gromov): If $\Gamma(r)$ is bounded by a polynomial, then G has a finite-index nilpotent subgroup.

Whole proof is too complicated for now, but let me show you one ingredient:

Prop: If $\Gamma(r) \sim r^k$, then $\exists$ s.t. $\mathrm{Cone}_\omega(G)$ is a proper metric space w/ transitive isom gp. and $(G, \frac{d}{i})$ subconverges in GH sense.

Pf: ~~Let $n>0$. Let $r>0$. Let $r>0$.~~
Say $c^{-1} r^k \le \Gamma(r) \le c r^k$. Claim: $\forall n, \exists M(n) > 0$ s.t. $\forall r$, $B_r(1)$ can be covered by $M(n)$ balls of radius $\frac{r}{n}$.

Pf: Let $\{a_1, \dots, a_m\}$ be a ~~maximal~~ $\frac{r}{2n}$-net in $B_r(1)$ (i.e. maximal set s.t. $d(a_i, a_j) \ge \frac{r}{2n}$ when $i \ne j$).
Then $B_{\frac{r}{n}}(a_i)$ cover $B_r(1)$ and $B_{\frac{r}{2n}}(a_i)$ are disjoint balls in $B_{r+\frac{r}{2n}}(1)$.

So $\# B_{r(1+\frac{1}{2n})}(1) = \Gamma(r(1+\frac{1}{2n})) \ge m\, \Gamma(\frac{r}{2n})$

$$\Rightarrow m \le \frac{c(r(1+\frac{1}{2n}))^k}{c^{-1}(\frac{r}{2n})^k}$$

$\forall r$, Let $\{a_{1,r}, \dots, a_{M,r}\}$ be pts s.t. $\bigcup B_{\frac{r}{n}}(a_{j,r})$ covers $B_r(1)$.

~~Then~~ $B_1([1], \mathrm{Cone}_\omega G) \subset \bigcup_{j=1}^{M} ~~B_{\frac{1}{n}}([a_{j,i}])~~$

Suppose $[b_i] \in B_1([1]) \subset \mathrm{Cone}_\omega(G)$.

WLOG, $b_i \in B_i(1)\ \forall i \Rightarrow \forall i, \exists\ j_i \in \{1, \dots, M\}$ s.t. $d(b_i, a_{j_i, i}) \le \frac{i}{n}$.

Let $k_\omega = \lim_\omega k_i$. Then $d_\omega([b_i], [a_{k_\omega, i}]) \le \frac{1}{n}$.

$\Rightarrow B_1([1]) \subset \bigcup_k B_{1/n}([a_{k,i}])$. $\Rightarrow B_1([1])$ is compact.

Similarly for any ball.

~~Furthermore~~ Subconvergence: Transitivity: Let $[g_i] \in Cone_\omega(G)$. ~~Then~~ Let $f : Cone_\omega(G) \to Cone_\omega(G)$, $f([x_i]) = [g_i x_i]$. This is an isometry s.t. $f([1]) = [g_i]$.

~~Once we have this~~, Montgomery–Zippin–Yamabe (Hilbert's 5th) implies that $Isom(Cone_\omega(G))$ is a finite-dimensional Lie group, and you can use poly growth to characterize the Lie group.

(break)

On the other hand, if G doesn't have poly growth, then $Cone_\omega(G)$ can be very complicated.

~~Lemma: If $\Gamma(r)$ has exponential growth is superpolynomial, then its asymptotic cone is not proper.~~ Lemma: If $\Gamma(r)$ has exponential growth, then its asymptotic cone is not proper.

Pf: If $e^{c'r} \le \Gamma(r) \le e^{cr}$, then ~~$\exists n$ s.t. the number of $\frac{r}{n}$-balls needed to cover an r-ball is unbounded $\to \infty$~~

~~# of pts in a maximal $\frac{r}{n}$-net in the r-ball $\to \infty$ as $r \to \infty$.~~

~~Let $\{a_{i,1}, \dots, a_{i,n_i}\}$ be a maximal $\frac{i}{n}$-net in B_i.~~

~~$\Rightarrow B_1 \subset Cone_\omega(G)$ cannot be covered by~~ finitely many $\frac{1}{n}$-balls.

~~Let~~ ~~Let N_i be a maximal $\frac{i}{n}$-net in the i-ball~~

~~Since~~ If $e^{c'r} \le \Gamma(r) \le e^{cr}$ $\forall r$, then ~~$\#N \to \infty$~~ $\exists n > 0$ s.t. if N_i is a ~~maximal~~ $\frac{i}{n}$-net in the i-ball, then $\#N_i \to \infty$.

Consider $(a_i), (b_i) \in \ell_\infty(N_i)$. If a_i, b_i are eventually different, then $d_\omega([a_i],[b_i]) = \lim_\omega \frac{d(a_i, b_i)}{i} \ge \frac{1}{n}$.

Since $\#N_i \to \infty$, there are ∞ly many ~~different~~ different sequences that are eventually different. $\Rightarrow B_1 \subset Cone_\omega(G)$ is not compact.

In fact, it's worse than that.

Let ~~x~~ $n_i = \# N_i$. ~~Let~~ For $\alpha \in [0,1]$, let

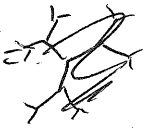
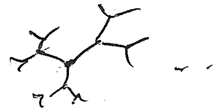
~~$x_\alpha = \{a$~~ $N_i = \{a_{i,1}, \dots, a_{i,n_i}\}$.

let

$$x_\alpha = [a_{i,\lfloor \alpha n_i \rfloor}].$$ If $\alpha \neq \beta$, then $d_\omega(x_\alpha, x_\beta) =$

$\Rightarrow \exists$ ~~uncountably many points in B_1~~ separated by $\frac{1}{n}$.

$\exists$ an uncountable $\frac{1}{n}$-net in B_1.

Ex: $X =$  3-regular tree. 

~~Exponential growth~~

~~then $Cone_\omega(X)$ is the universal $\mathbb{R}$-tree~~

~~- $Cone_\omega(X)$ is a metric tree: $\exists$ a unique simple path between any two points. there are no simple closed curves.~~

Lemma: $Cone_\omega(X)$ contains no simple closed curves. ~~That is, $Cone_\omega(X)$ is a metric tree.~~

Pf: ~~Suppose~~

Then:

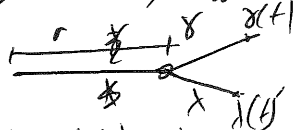
— ~~$\forall x \in Cone_\omega(X)$, Every point in $Cone_\omega(X)$ is a cut point~~

1 - Big:

~~Not~~ Exponential growth $\Rightarrow$ cone is not proper, uncountable $\frac{1}{2}$-net in 1-ball.

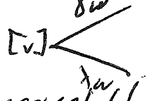
And we can see this: Let $\gamma, \lambda : [0,\infty) \to T$ be distinct unit-speed geodesic rays with $\gamma(0) = \lambda(0) = v_0$. ~~Then~~

Let $\gamma_\omega(t) = \lim_\omega \gamma(it)$, $\lambda_\omega(t) = \lim_\omega \lambda(it)$. These are geodesics in $Cone_\omega(T)$.

We have 

so $d(\gamma(t), \lambda(t)) = 2(t-r)$ when $t > r$.

$\Rightarrow d_\omega(\gamma_\omega(t), \lambda_\omega(t)) = \lim_\omega \frac{2(it-r)}{i} = 2t$, i.e., $\lambda_\omega, \gamma_\omega$ disjoint



There are uncountably many distinct geodesics based at v, so there are uncountably many disjoint geods from $[v]$.

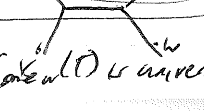
~~Further, $Isom(T)$ is transitive~~

~~In fact, Ken~~ ~~Properties~~: Further, $Isom(T)$ acts cocompactly on T

$\Rightarrow Isom(Cone_\omega(T))$ acts transitively on $Cone_\omega(T)$

~~Ex~~ $\Rightarrow$ uncountably many rays from every pt.

~~In fact~~

2 - ~~Treelike: You can characterize a tree by a 4 pt condition~~ 

treelike (Ex: $Cone_\omega(T)$ has no simple closed curves) $\Rightarrow Cone_\omega(T)$ is universal $\mathbb{R}$-tree