Quantitative geometry and topology I 2022-01-25

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Office hours by appt. I may set up recordings just for the heck of it

Collect emails.

What is this course?

One of the basic ideas of topology: when is there a continuous map $X \to Y$ with given properties?

QT: Suppose that such a map exists. What can you say about it? How big/complex is it?

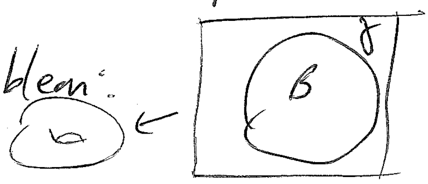
Goal – Introduce ideas/methods for solving problems like this (esp. discretization, scaling, limits)
– Apply these to geometric group theory, topology.

Today (start): Quantifying simple connectivity.

Q: Let X be a space (complex or manifold), $\gamma: S^1 \to X$ null-homotopic. Then $\exists \beta: D^2 \to X$ that extends α — how does the size of β depend on α?

(Equiv, ~~pass~~ ~~let~~ pass to univ. cover: $\gamma: S^1 \to \tilde{X}$ where $\tilde{X}$ is simply connected. How does the size of β depend on γ?)

This is kind of like the isoperimetric problem:



So let's do this for $\mathbb{R}^n$:

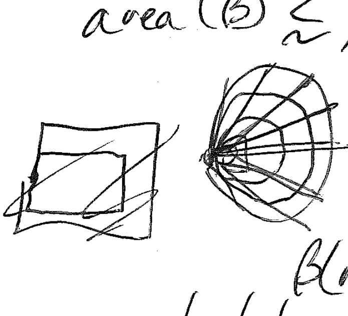
Prop: Let $\gamma: S^1 \to \mathbb{R}^n$ be a Lipschitz closed curve

[Sidebar: Lipschitz: $\exists C$ s.t. $d(\gamma(x), \gamma(y)) \leq C d(x,y)$ $\forall x,y \in S^1$
By Rademacher's thm, any Lip $f: \mathbb{R}^m \to \mathbb{R}^n$ is diff almost everywhere, so Lipschitz maps have lengths, areas, etc.]

Then $\exists \beta: D^2 \to \mathbb{R}^n$ with $\beta|_{S^1} = \gamma$ s.t.

$$\text{area}(\beta) \lesssim \ell(\gamma)^2.$$

Pf:



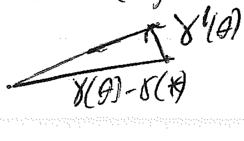
straight-line homotopy

Put polar coords (r,θ) on D^2.
Let $* \in S^1$ be a basepoint, let

$$\beta(r,\theta) = \gamma(*) + r(\gamma(\theta) - \gamma(*))$$

And we can calculate area:

$$\begin{aligned}\text{area}(\beta) &= \tfrac{1}{2}\int \|\gamma'(\theta) \times (\gamma(*) - \gamma(\theta))\| \, d\theta \\ &\leq \tfrac{1}{2}\int \|\gamma'(\theta)\| \cdot \|\gamma(*) - \gamma(\theta)\| \, d\theta \\ &\leq \tfrac{1}{2}\ell(\gamma) \cdot \tfrac{1}{2}\ell(\gamma) = \tfrac{1}{4}\ell(\gamma)^2. \quad //\end{aligned}$$

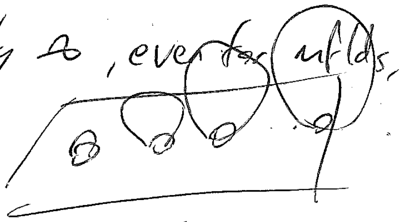


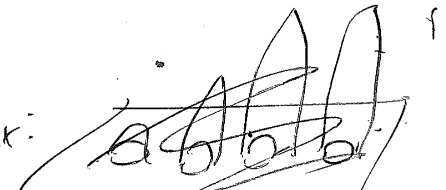
The bound on area(β) depends on the space in question: let's ~~define~~:

Def: Let X be a simply-connected metric space (Riemannian manifold or simplicial complex). For any Lipschitz $\alpha: S^1 \to X$, the __filling area__ $\delta_X(\alpha) = \inf\{\text{area}\,\beta \mid \beta|_{S^1} = \alpha\}$. For any $L>0$, the __Dehn function__ $\delta_X(L) = \sup_{\substack{\alpha: S^1 \to X \\ \ell(\alpha) < L}} \delta_X(\alpha)$.

Note: Possibly ∞, even for mflds, complexes: ex:

Ex:





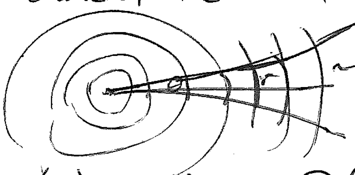
But we'll see that if ~~$X = \tilde{M}$ for some compact M~~, X has enough symmetry, then $\delta_X(L) < \infty$.

What other examples?

Ex: $\mathbb{H}^n$ — hyperbolic space

(Sidebar: $\mathbb{H}^n$ is the unique simply-connected complete n-manifold with constant sectional curvature -1.

~~Key pr~~ Because of the curvature, geodesics diverge quickly



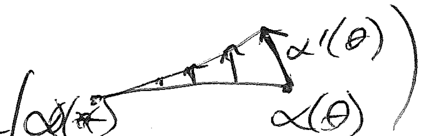
$\sim e^r A$ (technically, $\theta \sinh r$)

Prop: $\delta_{\mathbb{H}^n}(L) \lesssim L$. Pf: $\forall u, v \in \mathbb{H}^n$, let $\lambda_{u,v}$ be unique geodesic from u to v. Let ~~$\beta(r,\theta) = \lambda(r)$~~

$\alpha: S^1 \to \mathbb{H}^n$, let $\beta(r,\theta) = \lambda_{\alpha(*)\alpha(\theta)}(r)$.

~~Then Area(β)~~ (straight-line homotopy).

Then Area $(\beta) = \int$ area $\left(\vphantom{\int} \right)$ $d\theta \lesssim C\alpha'(\theta)$



But area because of the exponential decay. So area$(\beta) \lesssim \ell(\alpha)$. //

Next example needs some lemmas:

Lemma: Let K be a compact Riemannian manifold or simplicial complex.

Then: - $\exists \varepsilon > 0$ s.t. if $\ell(\gamma) < \varepsilon$, then $\gamma \sim *$ (standard exercise in ~~Riem~~ geometry).

- If $\gamma: S^1 \to K$ is a Lipschitz map and $\gamma \sim *$, then $\exists$ a Lipschitz extension $\beta: D^2 \to K$. (exercise: ~~use Whitney embedding theorem, tubular nbhds~~.)

- $\delta_{\tilde{K}}(L) < \infty \quad \forall L > 0$.

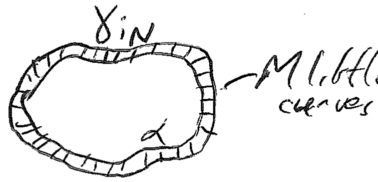
Pf of last ~~(because it uses limits)~~: __Pass to a limit.__

Consider a seq.

~~Suppose~~ not. ~~Then~~ $\exists\ \gamma_1, \gamma_2, \dots : S^1 \to K$ s.t. $\gamma_i \sim *$ and $\ell(\gamma_i) \leq L$. Parametrize ~~with constant speed~~ ~~and~~ $\delta_K(\gamma_i)$ ~~...~~ $\to \delta_K(L)$ these as maps $\gamma_i : [0, L] \to K$ with constant speed (< 1).

By Arzela-Ascoli, there is a subseq. $\gamma_{i_1}, \gamma_{i_2}, \dots$ s.t.

$$\gamma_{i_j} \xrightarrow{\text{uniformly}} \alpha.$$



Let ε be as in part ①. Consider:
Each little curve has length $< \varepsilon$. So we can fill in the annulus. So $\gamma_{i_N} \sim \alpha \sim *$ and $\delta_K(\gamma_{i_N}) < \delta_K(\alpha) + \delta_K(\varepsilon) \cdot M$.

$$\text{So } \delta_K(L) = \lim_{N \to \infty} \delta_K(\gamma_{i_N}) < \infty \quad //$$

Prop: Suppose X is compact (manifold or complex) and simply connected. Then $\exists\ C > 0$ s.t. $\delta_X(L) \leq CL$ for all $L > 1$.
(In fact for all L, but we're mostly interested in large case).

Pf: Discretize: $\forall u, v \in X$, let $\lambda_{u,v}$ be a shortest path from u to v. Then $\ell(\lambda_{u,v}) \leq \operatorname{diam} X < \infty$. Let $D = \operatorname{diam} X$.
Let $\gamma : S^1 \to X$, Let $n \in \mathbb{N}$, suppose $\ell(\gamma) \leq n$.
Parametrize γ with constant speed $\gamma : [0, n] \to X$ — then $d(\gamma(i), \gamma(i+1)) \leq 1\ \forall i$. We break γ into triangles

$$\Delta_i = \lambda_{\gamma(0), \gamma(i)} \cdot \gamma|_{[i, i+1]} \cdot \lambda_{\gamma(i+1), \gamma(0)} \text{ — then } \ell(\Delta_i) \leq 2D + 1.$$

Fill each Δ_i by a disc of area $\leq \delta_K(2D+1)$ — this fills γ with area $\leq n\, \delta_K(2D+1)$ //

~~Thm~~: But this makes things seem easy — draw a bunch of triangles, use them to fill the curve. In general, not so easy:

Thm: $\forall n$ suff. large, $\exists$ a simplicial 2-complex K_n
(Sidebar: you know what a simplicial complex is?)
with ~~n triangles~~ ~~n vertices~~ s.t. ~~n triangles~~ ~~s.t.~~ at most n vertices, n triangles, and edges s.t.

$$\delta_{K_n}(3) > e^{e^{e^{e^{\cdot}}}}$$

(in fact, $\delta_{K_n}(3) >$ any computable function of n).

Need to connect Dehn functions to computational complexity.

We look at <u>Group Presentations</u>:

A <u>group presentation</u> is an expression of the form

$$\langle \overbrace{g_1, \dots, g_n}^{\text{generators}} \mid \overbrace{r_1, \dots, r_s}^{\text{relators}} \rangle$$ where the r_i are <u>words</u> in the alphabet $g_1, \dots, g_n$ — i.e.,

$$r_i = g_{j_1}^{\pm 1} g_{j_2}^{\pm 1} \cdots g_{j_k}^{\pm 1}.$$

It <u>presents</u> the group G generated by $g_1, \dots, g_n$ subject to the relations that ~~all the r_i are the identity~~ $r_i = 1_G \; \forall i$.

i.e., $$G = \frac{\langle g_1, \dots, g_n \rangle}{\langle\langle r_1, \dots, r_s \rangle\rangle}$$

Equiv: $G = \{g_1, \dots, g_n\}^* / \sim$ (words)

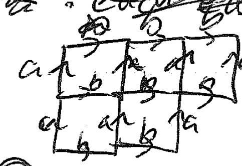
where $w g_i^{\mp 1} g_i^{\pm 1} w' \sim w w'$ (free insertion/reduction) and $w r_i^{\pm 1} w' \sim w w'$ (applying a relator)

Ex: $\mathbb{Z}^n = \langle g_1, \dots, g_n \mid [g_i, g_j] \text{ for all } i, j \rangle$.

These have a nice geometric interpretation:

Suppose $w \in \{g_1, \dots, g_n\}^*$. Then $w \sim$ empty word $(\varepsilon) \iff$ w bounds a <u>van Kampen diagram</u>.

Def: A <u>van Kampen diagram</u> D for a presentation P is a ~~singular cellulation of the disc~~ finite planar cell complex embedded in $\mathbb{R}^2$ s.t. D is connected, simply connected



– Each edge is oriented and labeled with a generator

– The boundary of each 2-cell is a relator in P.

$P = \langle a, b \mid [a, b] = 1 \rangle$

but also ∂D

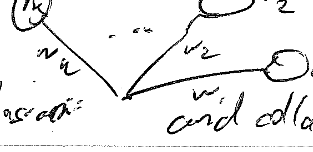
The <u>boundary word</u> is the word obtained by reading around the boundary.

Then: – $\partial D \sim \varepsilon$ (induction)

– If $w \sim \varepsilon$, then w is the boundary word of a vKd.

Pf: $w \in \langle\langle r_1, \dots, r_s \rangle\rangle$

$$w =_{F(g_1, \dots, g_n)} \prod_i w_i r_i^{\pm 1} w_i^{-1}$$



Lollipop diagram and collapse.