2022-02-01

Last time: $\delta_X(\gamma) = \inf_{\substack{\beta: D^2 \to X \\ \beta|_{S^1} = \gamma}} \text{area } \beta$. $\delta_X(L) = \sup_{\ell(\gamma) \le L} \delta_X(\gamma)$.

Promised a couple of things: — let's prove them.

~~Prop: $\exists$ compact K s.t. $\delta_K(L) \geq e^{e^{e^L}}$~~

~~Prop: $\exists$ sequence K_n of s.c.'s with n vertices, edges~~

~~We'll need a lemma:~~

~~Lemma: Let K be a compact Riemannian manifold or simplicial complex~~

We'll need a lemma on how δ behaves for spaces that ~~have bounded geometry~~: ~~aren't~~ are sufficiently symmetric (~~complete~~ complete Riem mfld / simp. cplex)

Lemma: Let X be a space equipped with an ~~group action by~~ isometric G-action s.t. $\exists$ $K \subset X$ compact s.t. $GK = X$.

(e.g. X compact, X is ~~the~~ universal cover of a compact space, $\mathbb{R}^n$, $\mathbb{H}^n$ ~~etc~~)

then: ① - $\exists \varepsilon > 0$ s.t. if $\ell(\gamma) < \varepsilon$, then $\gamma \sim *$.

② - If $\gamma: S^1 \to X$ ~~is~~ a Lipschitz map and $\gamma \sim *$, then $\exists$ a Lipschitz extension ~~of~~ $\beta: D^2 \to X$ -

③ - $\delta_K(L) < \infty$ for all $L > 0$.

Pf: Key idea: Let $* \in S^1$. $\forall \gamma: S^1 \to X$, $\exists g$ s.t. $g\gamma(*) \in K$. Since G acts isometrically, $\gamma \simeq * \Leftrightarrow g\gamma \simeq *$ and $\delta_X(\gamma) = \delta_X(g\gamma)$ — so it suffices to consider curves based in K, ~~thus contained in $N_L(K) = \{y \in X \mid d(y,K) < L\}$~~ ~~(compact)~~ ~~also compact.~~

① - Standard exercise in Riemannian geometry

② - ~~By Whitney em Existence~~ Ex: Every continuous $f: D^2 \to \mathbb{R}^n$ which is Lipschitz on S^1 can be approximated arbitrarily closely by a Lipschitz map $\hat{f}$ s.t. $f = \hat{f}$ on S^1.

~~Let $f: N_L(K) \to \mathbb{R}^M$ be a smooth embedding (Whitney)~~.

~~Then~~ Let $\beta_0: D^2 \to X$ fill γ. Then $\beta_0(D^2)$ is compact. By Whitney Embedding Theorem, $\exists$ a smooth embedding ~~i~~ $i: U \to \mathbb{R}^M$ where $\beta_0(D^2) \subset U$

By Tubular Neighborhood Thm, there's ~~a retraction~~ a smooth retraction ~~from~~ $r: W \to i(U)$ where ~~$i(U)$~~ W is a nbhd of $i(U)$. Approximate $i \circ \beta_0$ by a Lipschitz map $\widehat{i \circ \beta_0}$, let $\beta = i^{-1} \circ r \circ \widehat{i \circ \beta_0}$

Picture: how would you do this on the sphere? 1) define convolution w/ spherical kernels in some way, 2) convolve, ~~to~~ gain some error, project back.

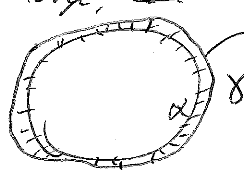
(3): ~~Suppose not. Then ∃ Cons~~ $\lim_i \delta_X(\gamma_i) = \delta_X(L)$,

Consider a seq. $\gamma_1, \gamma_2, \dots : S^1 \to X$ s.t. $\gamma_i(*) \in K$, $\ell(\gamma_i) < L$, and $\gamma_i \simeq *$ for all i. Parametrize them as maps
$\gamma_i : [0,L] \to X$ with speed ≤ 1, take $* = 0$.

~~By Arzela Asco~~ Then $\gamma_i([0,L]) \subset N_L(K) = \{y \in X \mid d(y,K) < L\}$, which is compact. By Arzela-Ascoli, ∃ a subseq s.t.

$$\gamma_{i_j} \xrightarrow{\text{unif}} \alpha.$$ Now discretize.

let ε be as in ①. If i_j is large, then $d(\alpha(t), \gamma_{i_j}(t)) < \frac{\varepsilon}{10}$, ~~we can draw~~
so there's a picture where each face has ~~area~~ perimeter $< \varepsilon$.
So we can fill each face with a disc $\Rightarrow \gamma_{i_N} \simeq *$.



and $$\delta_K(\gamma_{i_N}) \le \delta_K(\alpha) + \delta_K(\varepsilon)\,\frac{10L}{\varepsilon} < \infty.$$

So $$\delta_K(L) = \lim_{N\to\infty} \delta_K(\gamma_{i_N}) \le \delta_K(\alpha) + \delta_K(\varepsilon)\cdot\frac{10L}{\varepsilon} < \infty \quad //$$

~~Same discretization to prove~~ And we can use the same tech to prove:

Prop: Let X be compact, simply-connected. Then $\delta_X(L) \lesssim_X L$ for all $L \ge 1$.

Pf: Discretize. $\forall u, v \in X$, let $\lambda_{u,v}$ be a shortest path from u to v. Then $\ell(\lambda_{u,v}) \le \operatorname{diam}(X) < \infty$. Let $D = \operatorname{diam} X$.
Let $\gamma : S^1 \to X$, let $n \in \mathbb{N}$, suppose $\ell(\gamma) \le n < \ell(\gamma)+1$. Parametrize ~~$\gamma$ with speed ≤ 1~~ $\gamma : [0,n] \to X$ with speed ≤ 1, so

$$d(\gamma(i), \gamma(i+1)) \le 1 \ \forall i.$$ We break γ into triangles:

$$\Delta_i = \lambda_{\gamma(0),\gamma(i)} \cdot \gamma|_{[i,i+1]} \cdot \gamma_{\gamma(i+1),\gamma(0)}.$$

Each triangle has length $\ell(\Delta_i) \le 2D+1$. Fill each triangle with a disc ~~set~~ to get

$$\delta_X(\gamma) \le n\,\delta_K(2D+1) \le (\ell(\gamma)+1)\,\delta_K(2D+1) \quad //$$

(break?)

Large Dehn functions:

Prop: $\exists$ compact K s.t. $\delta_K(L) \geq e^{e^{e^{L}}}$ for suff. large L.
(in fact, larger than any computable function)

~~Pf:~~ So we need to connect δ_K to computability.

Computable function: function ~~that can~~ where there's ~~an~~ a deterministic algorithm that terminates for all n to compute $f(n)$ from n. (Algorithm: say, a computer program) The ~~reason is~~ key is that the algorithm has to be representable as a finite string of bits — a number

Standard example of a non-computable fn:

Halt $(n) = \begin{cases} 1 & \text{if the } \sout{\text{program represen}} \text{ string of bits represented by } n \text{ } \sout{\text{represents}} \text{ is a program that halts on the input } n. \\ 0 & \text{otherwise} \end{cases}$

Thm (Turing): ~~The~~ There is no algorithm to compute Halt

Otherwise you could write a program T

```
input: n
  if Halt(n) = 1: loop infinitely.
  else:
    return 1.
```

Then $T = f_N$ for some N. — ~~what is~~ does T terminate on N? ~~$f_N(N)$?~~

if yes, then $Halt(N) = 1$, so $T(N)$ doesn't halt $\times$.
if no, then $Halt(N) = 0$, so $T(N)$ halts. $\times$ //

Cor: There is no computable function ~~that such that~~ L s.t. $\forall n$, if $f_n(n)$ halts, then it halts in at most $L(n)$ steps.

Pf: Otherwise, you could write an algorithm:

~~Halt(n) =~~ input n:
run $f_n(n)$ for $L(n)$ steps.
if it halted, return 1.
otherwise return 0.

— this algorithm computes $Halt(n)$ $\times$ //

To connect this to DFs, we need:

Group presentations:

A ~~group~~ group presentation is an expression of form

$F\langle g_1, \dots, g_n\rangle$

$\langle \underbrace{g_1, \dots, g_n}_{\text{generators}} \mid \underbrace{r_1, r_2, \dots, r_s}_{\text{relators}} \rangle$ where the r_i are elements of the free group generated by the g_i's (words in $\{g_1^{\pm 1}, \dots, g_n^{\pm 1}\}$)

~~This presents the group G generated by the g_i subject to relations that~~

~~This presents the group~~

This represents $G = \langle g_1, \dots, g_n\rangle / \langle\langle r_1, \dots, r_s\rangle\rangle$

the largest group ~~such~~ generated by the g_i's such that $r_i =_G 1$ for all i.

$\langle g_1, \dots, g_n \mid r_1, \dots, r_s\rangle = \langle g_1, \dots, g_n\rangle / \sim$ where $w g_i g_i^{-1} w' = ww'$ (free insertion/reduction)

$w r_i^{\pm 1} w' \sim ww'$ (applying a relator)

~~$ww' \to w$~~

$= \langle g_1, \dots, g_n\rangle / \langle\langle r_1, \dots, r_s\rangle\rangle$ = subgroup generated by conjugates of r_i.

because $w r_i w' = ww'(w'^{-1} r_i w')$ - applying a relator is ~~the~~ just multiplying by a conjugate of a relator.

Ex: ~~$\langle a_1, \dots, a_n \mid [a_i, a_j]$ for all $i, j\rangle \cong \mathbb{Z}^n$~~.

$\langle x, y \mid xyx^{-1}y^{-1}\rangle \cong \mathbb{Z}^2$. Why?

~~xy~~ $yx \sim xyx^{-1}y^{-1}yx \sim xy$.

(likewise, $xy^{-1} \sim y^{-1}x$, etc.)

So we can reduce $x^2yx^{-1}y \sim x^2x^{-1}yy \sim xy^2$

more generally, reduce any word to $x^a y^b$ for some a, b.

~~Recall~~ There's also a nice geometric interpretation:

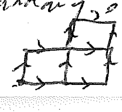
~~Recall: Let X be a 2-complex~~ Suppose $w \in \langle g_1, \dots, g_n\rangle$.

~~Then $w = 1$~~ Let $G = \langle g_1, \dots, g_n \mid r_1, \dots, r_s\rangle$. Then

$$w =_G 1 \iff w \text{ bounds a van Kampen diagram in } G$$

Def: A van Kampen diagram for G is a finite planar cell complex embedded in $\mathbb{R}^2$ s.t.:

- D is connected, simply connected
- Each edge is oriented, labeled with a generator.
- The boundary of each 2-cell is a relator.

Ex: ~~$x^2y^2x^{-1}$~~  Overflow: ~~can~~ describe equiv between comb. ~~gp~~ vKD.