2022-02-08

Last time:
Group presentations:
$\langle g_1, \dots, g_n \mid r_1, \dots, r_s \rangle = \dfrac{F(g_1, \dots, g_n)}{\langle\langle r_1, \dots, r_s \rangle\rangle}$

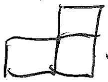
$\langle\langle r_1, \dots, r_s \rangle\rangle = \langle w_i r_{j_i}^{\pm 1} w_i^{-1} \mid w_i \in F(g_1, \dots, g_n) \rangle.$

Let's ~~you describe~~: $\mathbb{Z}^2 = \langle x, y \mid xy = yx \rangle$
$BS(1,2) = \langle a, b \mid aba^{-1} = b^2 \rangle$

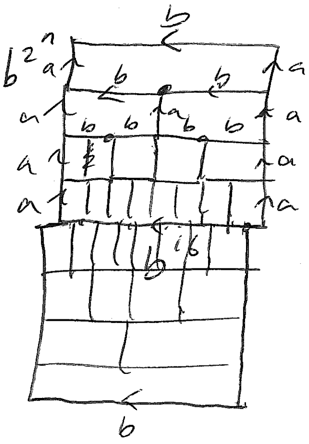
Last time, we discussed combinatorial interpretation, but there's also a geometric one: $G = \langle g_1, \dots, g_n \mid r_1, \dots, r_s \rangle$
Suppose ~~...~~ $w = w(g_1, \dots, g_n)$ (w is a word in the g_i's).
Prop: $w =_G 1 \iff w$ is the boundary word of a van Kampen diagram

Def: A van Kampen diagram D for G is a pic like ... finite planar 2-~~complex~~ complex embedded in $\mathbb{R}^2$ s.t.: - D is ~~simply~~ connected, simply-connected.
- Each edge of D is oriented and labeled with a generator.
- The boundary of each 2-cell is a relator.

The boundary word of D is the word reading around its outside.

Ex:  $\to x^2 y^2 x^{-1} y^{-1} x^{-1} y^{-1}$.

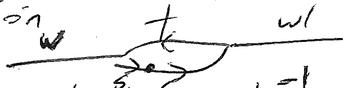
So: ~~BS(1,2)~~ ... Ex: $a^n b a^{-n} = b^{2^n}$

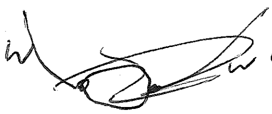


Ex: ~~$(a^n b a^{-n}) b (a^n b^{-1}$~~

Now take a copy of this, flip it, shift it over by 1:
$a^n b a^{-n} b a^n b^{-1} a^{-n} b^{-1}$
$= [a^n b a^{-n}, b]$
represents 1, but this is the smallest vKd.

Pf of Prop: ~~Deleting a~~ Consider the boundary curve - Moving it over a 2-cell is the same as applying a relation



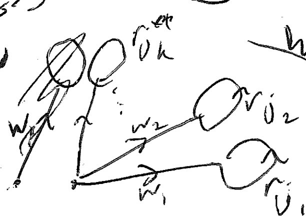


s.t. $r = s_2 t^{-1} s_1$
$r^{-1} = s_1^{-1} t s_2$,
$w s_1 s_2 w' \sim w s_1 r^{-1} s_2 w'$
$\sim w t w'$

Conversely, suppose $w =_G 1$
Then $w \in \langle\langle r_1, \dots, r_s \rangle\rangle$

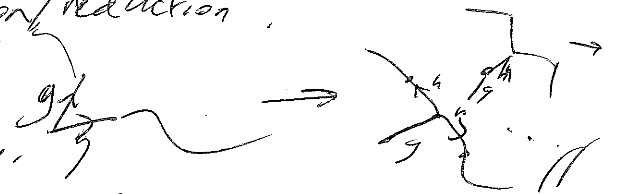
$w =_F \prod w_i r_{j_i}^{\pm 1} w_i^{-1}$

We can draw this:



This is a vkD with bdry word ~~freely equiv. to w,~~ equal to w in F.

— can get to w by free insertion/reduction.
But: if bdry word has gg^{-1}

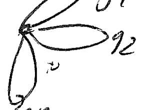


Repeating, this gives a vKD for w.

So: ~~Size of Area of vKd ⟺~~ vKd for w ⟺ reduction from w to 1.

More geometric: A vKd is a disc in an appropriate space:

Given $G=\langle g_1,\dots,g_n \mid r_1,\dots,r_s\rangle$, let X_G = 2-complex s.t.

$X_G^{(1)} =$  + a 2-cell for each relation.

Then $\pi_1(X_G)=G$. Each word $w\in F(g_1,\dots,g_n)$ corresp to a curve in X_G, which is null-homotopic ⟺ $w =_G 1$.

On one hand, a vKd is a null-homotopy.

Conversely, suppose $\gamma_w \simeq *$. Then what? The fact that $w =_G 1$ is a consequence of Seifert-van Kampen (thus, the vKd.)

~~But we a But if we want to quantify, better to~~

But let's quantify len: Suppose $\beta: D^2\to X_G$, $\beta|_{S^1}=\gamma_w$, β Lipschitz. Then there is a vKd for w with area $D \le$ area β.

WLOG, suppose β is smooth on the interior of each cell.

Pf: By the coarea formula, $\text{area}(\beta)=\int_{X_G}\#\beta^{-1}(y)\,dy$

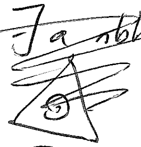
~~Therefore, for each 2 cell~~ $= \sum_{\delta\in X_G^{(2)}} \int_\delta \#\beta^{-1}(y)\,dy$.

For each δ, $\exists$ a pt $y_\delta\in\delta$ s.t. y_δ is regular and $\#\beta^{-1}(y_\delta) \le \frac{1}{\text{area}\,\delta}\int_\delta \#\beta^{-1}(y)\,dy$.

Let $m = \min_\delta \text{area}(\delta)$

Then $\sum_\delta \#\beta^{-1}(y_\delta) \le \sum_\delta \frac{1}{m}\int_\delta \#\beta^{-1}(y)\,dy \le \frac{1}{m}\,\text{area}(\beta)$.

~~So $\forall y_\delta$, $\exists$ a nbhd U_δ s.t. $\beta^{-1}(U_\delta)$~~

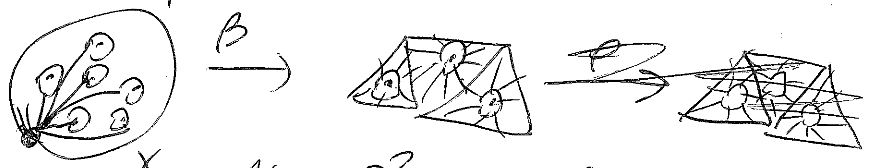


$\forall y_\delta$, if $x\in\beta^{-1}(y_\delta)$, then $D\beta_x$ is nonsingular.

Therefore, $\exists$ a nbhd U_δ of y_δ s.t. $\beta^{-1}(U_\delta)$ consists of $\#\beta^{-1}(y_\delta)$ discs, each sent homeomorphically to U_δ.

Let $\rho: X_G\to X_G$ be a map that sends each U_δ to δ and sends $\delta\setminus U_\delta$ to $\partial\delta$, & $\rho(x)=x$ $\forall x\in X_G^{(1)}$

Now, what does $\rho \circ \beta$ look like?



γ_w All of D^2 except for some disjoint discs is sent to $X^{(1)}_G$. The boundary of each disc is sent to ~~$\partial \delta$ or γ_{r_i} (or its reverse) for some~~ $\partial\delta = \gamma_{r_i}$ or $\gamma_{r_i}^{-1}$.

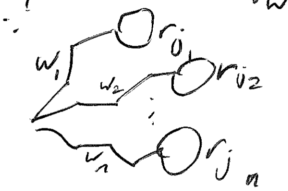
Fix a basepoint. Draw ~~loops around each~~ a bouquet of lollipops; ~~Then Then Then Let λ = boundary of lollipops.~~

~~Then $\beta \circ \lambda : S^1 \to X^{(1)}_G$ is homotopic in $X^{(1)}_G$ to γ_w.~~

~~Alter~~ We may suppose that each stem is a geodesic in $X^{(1)}_G$.

~~Let λ = boundary of lollipops. This is an edge path $\beta\circ\lambda = \gamma_w$.~~

~~Then $\beta \circ \lambda \simeq \gamma_w$ inside $X^{(1)}_G$ — so~~



This is a vKd for D_0. Let w' be its bdry.

Let λ = boundary of lollipops. ~~The boundary of D_0~~

Then $\beta \circ \lambda = \gamma_{w'}$. ~~and $\beta\circ\lambda \simeq$~~

Since $\lambda \sim \partial D^2$, $\beta\circ\lambda \sim \beta|_{S^1} = \gamma_w$.

I.e. $\gamma_w \simeq \gamma_{w'} \Rightarrow w = w'$ in $\pi_1(X^{(1)}_G) = F(g_1, \dots g_n)$.

~~So there's~~ So there's a free reduction from ~~w~~ w to w'.

Fold the edges of D_0 appropriately to get a vKd for w. //

Now: Prop: There is a 2-complex X s.t. $\delta_X(L) \geq e^{e^{\cdot^{\cdot^{e^L}}}}$.

Thm (Novikov–Boone): There is a group pres. $G = \langle g_1, \dots g_n \mid r_1, \dots, r_s \rangle$ s.t. ~~there~~ is no algorithm to decide ~~whether a word is 1~~

~~(the wp~~ with unsolvable word problem

(no algorithm ~~to~~ that takes a word w as input and outputs

$$WP(w) = \begin{cases} 1 & w =_G 1 \\ 0 & \text{~~othe~~ } w \neq_G 1 \end{cases} \Big)$$

Pf of Prop: Suppose $\exists$ a computable fn D s.t. $\delta(L) < D(L)$ $\forall L$.

Then there's an alg to compute $WP(w)$:

input w ~~of length~~

consider all vKd's of area $\leq D(\ell(w))$

If any such vKd has bdry word w, ~~done~~ then $w =_G 1$

Otherwise, $w \neq_G 1$.

$\forall n$

P: There is a simply-connected ~~2~~ simplicial 2-complex K_n with ~~n~~ $\le n$ edges, vertices, tris, s.t. $\delta_{K_n}(10) > e^{e^{\cdot^{\cdot^{e}}}}{}^{n}$

Thm (Adyan – Rabin): Triviality (and many other properties) is undecidable. ~~In~~

In fact,

Embedding Lemma: Given a pres G and a word w, we can augment G with additional gens/relations to get G_w s.t.

$G_w \cong 1 \iff w =_G 1$. ~~Then $|X_{G_w}| \sim \ell(w)$~~

Proof of prop^n: Consider X_{G_w}. ~~By ad~~ & Barycentrically subdivide to get a simplicial cplex K_w. ~~Each generator is a curve of length~~ with

~~Suppose~~ Each gen is a curve of length ≤ 10. $|K_w| \lesssim \ell(w)$

~~So~~

Suppose there's a computable fn s.t. if $\ell(w) < n$, then $\delta_{K_w}(10) < f(n)$. Then there's an algorithm to ~~check~~ solve word problem.

input w:

Construct X_{G_w}.

Try to find a vKd for each generator of G_w of area $\le D(w)$. If we can, then $G_w \cong 1 \Rightarrow w =_G 1$.

Otherwise, $G_w \not\cong 1 \Rightarrow w \neq_G 1$ //

Further directions:

① Geometric Group Theory: ~~Study of~~ Suppose G acts ~~on X by isom~~ isometrically on X, ~~and such that the~~ cocompactly ($\exists$ compact K s.t. $GK = X$), and properly discontinuously ($\forall$ compact $K \subset X$, ~~$\#$~~ $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite)

How ~~does the~~ is the geometry of X ~~assoc to the gr~~ connected to the group theory of G?

Thm (Gromov): ~~If Let δ_G = the # applica~~ ~~δ_X grows~~

The growth rate of δ_X is an invariant of G.

In fact $\delta_X(L)$ grows at the same rate as ~~δ_G~~ $\delta_G(L)$, where

~~δ_G = # of applications of relators necessary to~~

$\delta_G(w)$ = ~~# of relators~~ min area of a vKd for w.

$\delta_G(L) = \max\limits_{\ell(w) < L} \delta_G(w)$

② Embedding uncomputability in other topological structures.

Thm (Nabutovsky): Let $S \subset \mathbb{R}^7$ be a closed 6-mfld. It is undecidable whether $S \cong S^6$