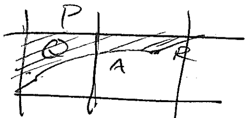
2022-02-22

Last time: Simplicial chains, Lipschitz singular chains.

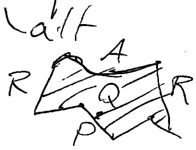
State: Thm (Federer–Fleming): Let X be a simplicial complex (or bilip equiv to a simplicial complex).

$\exists\, c>0$ s.t. $\forall A \in C_n^{Lip}(X)$, $\exists\, P(A) \in C_n^{\Delta}(X)$, $Q(A) \in C_{n+1}^{Lip}(X)$, $R(A) \in C_n^{Lip}(X)$ s.t.

- $A = P(A) + \partial Q(A) + R(A)$
- mass $P(A) \le$ ~~mass~~ c mass(A)
- mass $Q(A) \le c$ mass(A)
- mass $R(A) \le c$ mass(∂A).



and if $\partial A \in C_{n-1}^{\Delta}(X)$, we can take $R(A)=0$.



~~Sketch of proof: We want to construct a homotopy from A to something cellular.~~
~~Proceed by induction on~~

Pf Suppose $\partial A \in C_{n-1}^{\Delta}(X)$. (General is not much harder).

Proceed by induction:

Lemma: If $A \in C_n^{Lip}(X)$, and $\partial A \in C_{n-1}^{\Delta}(X)$, $\dim(X)=n$ then ~~A is thin-equivalent to a cellular chain~~,
A is homologous to ~~a cellular unique cellular chain~~.
~~ie~~, the chain $P = \sum_{\delta \in F^n(X)} \deg_\delta(A)\, \delta$.

where $\deg_\delta \sum a_i[\sigma] = \sum a_i \deg_{x_\delta}(\sigma)$ is well-defined for a.e. $x_\delta \in \delta$

~~(Prob state in terms~~ (Prob state "is homologous to a simplicial chain P".
i.e. $\exists\, Q$ s.t. $\partial Q = A - P$.) And since X is n-dim, mass(Q)

Let $A \in C_n^{\Delta}(X)$, $A = \sum a_i[\sigma_i]$

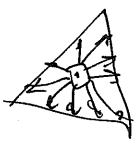
Now, ~~suppose~~ $\dim(X) = m > n$. We want to ~~show that A is~~ find $A_{m-1} \in C_n^{\Delta}(X^{(m-1)})$ homologous to A. — we find a series of htpies.

For every $\delta \in F^m(X)$, $\exists\, x_\delta \in \delta$ s.t. $x_\delta \notin \mathrm{supp}(A) = \bigcup \sigma_i(\Delta^n)$.
[In fact, $\exists\, \varepsilon > 0$ s.t. $B(x_\delta, \varepsilon) \cap \mathrm{supp} = \emptyset$.]

~~Let $\delta \in F^m$~~

Let $\rho_\delta : \delta \setminus \{x_\delta\} \to \partial\delta$ be radial projection from x_δ.

x_δ:



Repeat in every cell to get $\rho : X \setminus \{x_\delta\}_\delta \to X^{(m-1)}$

~~Then~~ Let $A_{m-1} = \rho_\#(A) \in C_n^{Lip}(X^{(m-1)})$

Let h_δ be straight-line htpy from id to ρ.

~~Let Q_{m-1}~~ $h_\delta : (X \setminus \{x_\delta\}_\delta) \times [0,1] \to X$

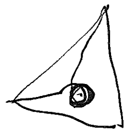
Let $Q_{m-1} = (h_\delta)_\#(A \times [0,1])$

Then $\partial Q_{m-1} = A - A_{m-1}$.

Repeat on A_{m-1} to get A_{m-2}, Q_{m-2}, etc.

~~Lef~~ Then $A_n \in C_n^{Lip}(X^{(n)})$ By lemma, hmlgs to $P \in C_n^{\Delta}(X^{(n)})$, ~~by say~~ say $A_n - P = \partial Q_\Delta$.

~~And~~ ~~Q~~. Then $A - P = \partial(\underbrace{Q_{m-1} + Q_{m+1} + \dots + Q_n + Q_\Delta}_{= Q})$.

Problem:  You could choose x_δ really badly.

Lemma: If x_δ are chosen randomly, then $E[\text{mass } R] \lesssim \text{mass}(A)$ and $E[\text{mass } Q] \lesssim \text{mass} A$ //.

Applications: $\forall\, k<n$, $\exists\, \alpha$ s.t. $FV_n^{k+1}(V) \le_\alpha V^{\frac{k+1}{k}}$.

Pf (scale down, approx, fill, scale up):
Give $\mathbb{R}^n$ the structure of a simplicial complex by subdividing the unit grid ~~Let m be the vol of the smallest k-simplex.~~
Let c as in Fed-Flem. Let $A \in C_k^{Lip}(\mathbb{R}^n)$ be a cycle. ($\partial A = 0$), mass$(A) = V$. Let $s: \mathbb{R}^n \to \mathbb{R}^n$, ~~$s(x) = V^{-1/k}(2c)^{-1/k}$~~

$$s(x) = \left(\frac{1}{2cV}\right)^{\frac{1}{k}} x \quad \text{so} \quad \text{mass}(s_\#(A)) = \frac{1}{2c}$$

Let $\hat{A} = s_\#(A)$. Then $\text{mass}(P(\hat{A})) \le \frac{1}{2}$. But $P(\hat{A})$ is a simplicial chain — ~~so~~ & each simplex has mass 1. — so $P(\hat{A}) = 0$.

Therefore, $\hat{A} = P(\hat{A}) + \partial Q(\hat{A}) + \underbrace{R(\hat{A})}_{0}$

$$\hat{A} = \partial Q(\hat{A}) \qquad \text{mass } Q(\hat{A}) \le c\, \text{mass}(\hat{A}) = \tfrac{1}{2}.$$

Let $B = (s^{-1})_\#(Q(\hat{A}))$; $A = \partial B$.

$$\text{mass}(B) \le \frac{1}{2} \cdot (2cV)^{\frac{k+1}{k}} \lesssim V^{\frac{k+1}{k}} //.$$

(break?)

Now we're getting closer to where I wanted to go w/ this course: ~~see the power of scaling~~. ~~scaling, limits, discretize~~
— ~~see the power of so how these all con~~
— see the power of scaling arguments. So plan:
1 – scaling in nilpotent groups. – many similarities to $\mathbb{R}^n$, but also a lot of new spaces/phenomena to share.
2 – work of Gromov, work of Fedor Manin on quantitative homotopy theory — ~~more get more into topology~~.
e.g. suppose $S^n \to K$ $\frac{L}{15}$ Lip — how does the map behave as L goes to ∞?

Nilpotent groups: The Heisenberg group: $\mathbb{H}_\mathbb{Z} = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x,y,z \in \mathbb{Z} \right\} = \langle X, Y, Z \mid [X,Z] = [Y,Z] = 1 \rangle$

$$\mathbb{H}_\mathbb{R} = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \,\middle|\, x,y,z \in \mathbb{R} \right\}.$$

$[X,Y]Z = XYX^{-1}Y^{-1}Z = 1$

~~XZ~~ $YX = XYZ^{-1}$ (and Z commutes w/ X,Y).

So: $Y^nX^n = X^nY^nZ^{-n^2}$. $[X^n,Y^n] = Z^{n^2}$.

Want to show you two things, poss today poss next week:

First ~~Further~~, $[X,Z]=1$, so $[X^n,[X^n,Y^n]]=1$ — and the obvious reduction to 1 has area n^3. — in fact, $\delta_{\mathbb{H}}(l)\sim l^3$.

~~(we'll see this)~~ How to show this?

~~OTOH~~ Second: $\mathbb{H}_n = \langle X_1,\dots,X_n, Y_1,\dots,Y_n, Z \mid [X_i,Y_i] = Z$, all other pairs commute $\rangle$.

is v. similar: $[X_i^n, Y_i^n] = Z^{n^2}$ but $\delta_{\mathbb{H}_n}(l)\sim l^2 \ \forall n\geq 2$

Why?

~~H: We~~ Identify $\mathbb{H}$ w/ $\mathbb{R}^3$ by $\begin{pmatrix}1&x&z\\&1&y\\&&1\end{pmatrix} \leftrightarrow (x,y,z)$.

Upper bound: Given a word $w(X,Y,Z) =_{\mathbb{H}} 1$. ~~Cancel~~ shuffle Z's right: Move Y's right one at a time. After each, move Z's right. Cancel.

Then $(x,y,z)(x',y',z') = (x+x', y+y', z+z'+xy')$

$(x,y,z)^{-1} = (-x,-y,-z+xy)$.

We give this the ~~metric~~ left-invt metric

$$ds^2 = dx^2+dy^2+(dz-x\,dy)^2$$

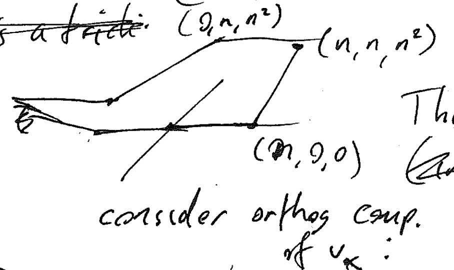
~~so at~~ (how to read: $\underset{v_X}{(1,0,0)}$, $\underset{v_Y}{(0,1,x)}$, $\underset{v_Z}{(0,0,1)}$ is orthogonal basis at (x,y,z)).

(Show picture?)

~~Consider~~ Let $w_n = X^nY^nX^{-n}Y^{-2n}X^{-n}Y^n$

$\pi(x,y,z) = (y,z)$

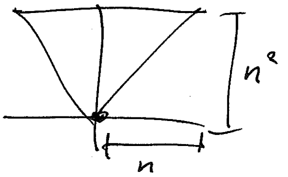
~~Now, here's a trick~~



Consider projection to yz-plane. This is area-~~preserving~~ decreasing: ~~As~~ $\pi_*(v_X)=0$, so it suffices to consider orthog comp. of v_X: $\pi_*(v_Y)=(1,x)$ ~~so~~ $\pi_*(v_Z)=(0,1)$

So if D is a disc w/ $\partial D = w_n$, then $\partial\pi(D) = \partial\pi(w_n)$ $\Rightarrow \text{area}(\pi(D)) \gtrsim n^3$

And:



~~$\pi(w_n)$ has area $\sim n^3$~~

$\Rightarrow \text{area}(D) \gtrsim n^3$.

But this is a little inelegant — relies on coords, doesn't work for $\pi:(x,y,z)\longmapsto(x,z)$. What's really going on?

Better to use differential forms;

~~Recall~~: Differential forms: ~~We work in $\mathbb{R}^4$ — we don't need a metric, so $\mathbb{H}$, $\mathbb{R}^3$ are simil~~ ~~Def in term~~
~~Def~~ ~~We won't need to use the metric at first, so we work in $\mathbb{R}^n$:~~

Def: Given a vector space V, define $\Lambda^k V^* = \{$multilinear alternating $f : V^k \to \mathbb{R}\}$.
Suppose $V = \mathbb{R}^n$. Then $V^* = \langle dx_1, \dots, dx_n \rangle = \Lambda^1 V^*$.
~~This is a vector space with basis~~
And $\Lambda^k V^*$ is a vector space w/ basis
$$\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} : i_1 < \cdots < i_k\}.$$
where $dx_{i_1} \wedge \cdots \wedge dx_{i_k}(v_1, \dots, v_k) =$ appropriate minor of $(v_1 \cdots v_k)$ (and a similar formula for arb. prodcts)
$$= \sum_{\sigma \in Sym(k)} sgn(\sigma) dx_{i_1}(v_{\sigma(1)}) \cdots dx_{i_k}(v_{\sigma(k)}).$$
In particular, $\Lambda^k \mathbb{R}^k \cong \mathbb{R}$ generated by the determinant ~~volume form ω~~.
~~What's the~~ ~~And we calculate~~: define ~~signed volumes by volume of a signed volume by~~ of $f : \Delta^k \to \mathbb{R}^k$

~~$\int_{\Delta^k}$~~ ~~Let Ω^k~~ For a smooth mfld M, ~~let $\Omega^k(M)$~~ $\Lambda^k(TM^*)$ is a vector bundle, and we define $\Omega^k(M) = \{$smooth sections of $\Lambda^k(TM^*)\}$
We can integrate these on surfaces: ~~if $K \subset M$ is a k-submfld~~, if $f : \Delta^k \to M$ is a Lipschitz map, $\omega \in \Omega^k(M)$

then $\int_{\Delta^k} \omega\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k}\right) dx$ generalizes a line integral, is independent of parametrization, subdivision, ~~etc.~~

In particular, if $K \subset M$ is an oriented submanifold (poss w/ bdry), X a simplicial complex, $f : X \to M$ an oriented Lipschitz homeo, then
$$\int_M \omega \overset{def}{=} \sum_{\substack{\delta \in F(X) \\ i_\delta : \Delta^k \to X}} \int_{\Delta^k} \omega\left(\frac{\partial f \circ i_\delta}{\partial x_1}, \dots, \frac{\partial f \circ i_\delta}{\partial x_k}\right) \text{ is well-defined.}$$

And you've seen this in multivariable — line integral of a v.field on a curve, is indep of param, flux through a surface can be written as a integral of a form, integral of curl over a surface is a form, etc.
(it's a little tricky because when you have a metric, there's an isomorphism between k-forms, $(n-k)$-forms, so ~~you're~~ you tend to think of grad and curl as v.fields, when they're really 1-forms and 2-forms. ~~Key fact:~~
~~Stokes' Theorem~~: ~~Let M be~~ an oriented k-manifold ~~with boundary~~.
Let ~~$\omega \in \Omega^k(M)$. Then $\int$~~

Further, div, grad, curl are instances of the exterior derivative:

$$d: \Omega^k(M) \to \Omega^{k+1}(M)$$

In any chart, df = differential of $f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j$.

Product rule: $d(f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \cancel{df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}}$

$$= \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \cdot dx_j\right) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$= df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Ex: 1-form in $\mathbb{R}^2$

$w = f\,dx + g\,dy$ ~~+h dz~~

$= \frac{\partial f}{\partial x} dx \wedge dx + \frac{\partial f}{\partial y} dy \wedge dx + \frac{\partial g}{\partial x} dx \wedge dy + \frac{\partial g}{\partial y} dy \wedge dy$

Stokes' Thm: Let $K \subset M$ be an oriented k-mfld w/ bdry. ~~Then~~ Let $w \in \Omega^{k-1}(M)$. Then

$$\int_K dw = \int_{\partial K} w.$$

Concise version:

What we're really using here is differential forms, de Rham cohomology.

~~Recall: diff form $w \in \Omega^k(\mathbb{R}^n)$ assigns~~

Let $w = dy \wedge dz$ be area form on y-z-plane. Then w is closed — $dw = 0$. Which is how we can use it to define signed area:

Stokes: if $M \subset \mathbb{R}^n$ is a k-mfld w/ bdry, $w \in \Omega^{k-1}(\mathbb{R}^n)$, then $\int_M dw = \int_{\partial M} w$. Or if $A \in C_k^{Lip}(\mathbb{R}^n)$, then $\int_A dw = \int_{\partial A} w$.

~~In particular~~ So if $w = dy \wedge dz \in \Omega^2(\mathbb{R}^2)$, and $A \in C_2^{Lip}(\mathbb{R}^2)$, signed area$(A) = \int_A w$. If $B \in C_2^{Lip}(\mathbb{R}^2)$, $\partial A = \partial B$, then $A - B$ is a cycle, $H_3(\mathbb{R}^2) = 0$, so $\exists\, C \in C_3^{Lip}(\mathbb{R}^2)$ s.t. $\partial C = A - B$, and

$$0 = \int_C dw = \int_{A-B} w = \text{sarea}(A) - \text{sarea}(B)$$

— signed area ~~is independent~~ of A depends only on ∂A.

— which is what we've seen already: signed area is the integral of the winding # of ∂A.

More generally, if $w \in \Omega^k(\mathbb{R}^n)$ is a closed form ($dw = 0$), ~~then~~ and $A, B \in C_k^{Lip}(\mathbb{R}^n)$, $\partial A = \partial B$, then $\exists\, C$ s.t. $\partial C = A - B$,

$$\int_C dw = \int_{A-B} w = 0. \Rightarrow \int_A w \text{ is determined by } \partial A.$$

In fact, $\exists$ a primitive η s.t. $d\eta = w$ and thus $\int_A w = \int_A d\eta = \int_{\partial A} \eta$.

In this case, let $\cancel{\omega}\ \mu \in \Omega^2(\mathbb{R}^3)$, $\pi(x,y,z) = (y,z) \in \mathbb{R}^2$

$\cancel{\omega}\ \mu = \pi^*(dy \wedge dz) = dy \wedge dz.$

~~Then $d\omega = \pi^*$~~

$$\pi^*(\eta)(v_1, \dots, v_k) = \eta(\pi_*(v_1), \pi_*(v_2), \dots, \pi_*(v_k))$$

$$d\mu = \pi^*(d(dy \wedge dz)) = 0. \quad \text{So } d\omega \text{ is closed.}$$

Let $\gamma = \gamma_{w_n}$ ~~$(w_n = [[x^n, y^n], x^n]$~~ $(w_n = x^n y^n x^{-n} \dots)$

If $\partial A = \gamma$, then $\int_A \mu = \int_{\pi_\#(A)} dy \wedge dz = \text{sarea}(\pi_\#(A)) = n^3.$

Furthermore, recall that our orthog basis was

$X_p = (1,0,0)$ ~~$\mu(X,Y) = 0$~~

$Y_p = (0,1,x)$ ~~$\mu(X,Z) = 0$~~

$Z_p = (0,0,1)$ ~~$\mu(Y,Z) = 1$~~

$\max_{\substack{V_p \in \mathbb{R}^3 \\ W_p \in \{X_p, Y_p, Z_p\}}} (\mu(V_p, W_p)) = 1 \Rightarrow \exists\, c$ (say 10) s.t. $|\mu(V,W)| \le c\, \|V \times W\|_g$

$\Rightarrow |\int_A \mu| \le c\, \text{mass}(A)$. (we say μ is a <u>bounded form</u>)

Thus any filling of γ has ~~area~~ mass $\ge n^3/c$ //

And this gives us ~~the generality~~ a general argument:

~~can we find closed forms to bounded closed~~

given a curve γ, find a bounded closed form ω s.t. $\int_A \omega$ is large on any ~~fillings~~ A s.t. $\partial A = \gamma$.

TBC.