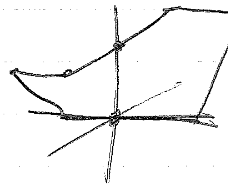
2022-03-01

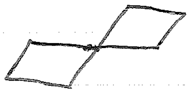
Last time: $\mathbb{R}$ $\mathbb{H}$

$(x,y,z)(x',y',z') = (x+x',\ y+y',\ z+z'+xy')$



γ_n $\quad \pi(x,y,z) = (y,z)$ is area-decreasing

$\Rightarrow \delta_{\mathbb{H}}(\gamma_n) \geq \delta_{\mathbb{R}^2}(\pi(\gamma_n)) = n^3$.



Differential forms in $\mathbb{H}$: $\Omega^k(\mathbb{R}^n) = \{k\text{-forms}\}$ = alternating multilinear fns taking k tangent vecs at a pt to a real #.

Let $\omega = dy \wedge dz$ = area form on yz-plane. Then ω is closed. $d\omega = 0$. Which is what lets us use it to define signed area. in this case $dy\wedge dz((y_1,z_1),(y_2,z_2)) = y_1 z_2 - y_2 z_1$

Thm (Stokes): If $M \subset \mathbb{R}^k$ is an oriented ~~closed~~ k-mfld w/ bdry, $\omega \in \Omega^{k-1}(\mathbb{R}^n)$, then $\int_M d\omega = \int_{\partial M} \omega$. Or if $A \in C_k^{Lip}(\mathbb{R})$, then $\int_A d\omega = \int_{\partial A} \omega$. (Special cases: FTC: $\omega = f \in \Omega^0$ $d\omega(v) = \nabla f \cdot v$

$\int_\gamma d\omega = \omega(\gamma(1)) - \omega(\gamma(0))$.

curl thm: $V \in \mathcal{V}(\mathbb{R}^3)$

$\omega_V(w) = V \cdot w \qquad \gamma = \partial D \qquad \int_\gamma V \cdot dx = \int_D (\operatorname{curl} V) \cdot du \times dv$

$d\omega(x,y) = (\operatorname{curl} \omega) \cdot x \times y$)

So, if $\omega = dy \wedge dz \in \Omega^2(\mathbb{R}^2)$, $A \in C_2^{Lip}(\mathbb{R}^2)$ then $\int_A$ = signed area of $A = \int_A \omega$. If $B \in C_2^{Lip}(\mathbb{R}^2)$ and $\partial A = \partial B$, then $\partial(A-B) = 0$, so $\exists\ C$ s.t. $\partial C = A - B$. By Stokes, $0 = \int_C d\omega = \int_{A-B} \omega \Rightarrow$

sarea(A) = sarea(B) i.e. signed area depends only on ∂A

(i.e., sarea$(A) = \int$ winding #.).

More generally, if $\omega \in \Omega^k(\mathbb{R}^n)$ is a closed form ($d\omega = 0$) and $A, B \in C_k^{Lip}(\mathbb{R}^n)$ satisfy $\partial A = \partial B$, then $\int_A \omega = \int_B \omega$. ~~$\exists\ C$ s.t. $\partial C = B - A$~~ In fact, $\exists\ \eta \in \Omega^{k-1}$ s.t. $d\eta = \omega$, $\int_A \omega = \int_{\partial A} \eta$.

So: Let $\pi(x,y,z) = (y,z)$, let $\mu = \pi^*(dy \wedge dz)$

$\mu(v,w) = (dy \wedge dz)(\pi_*(v), \pi_*(w))$.

And $d\mu = \pi^*(d(\underbrace{dy \wedge dz}_{\mu = dy \wedge dz})) = \pi^*(0) = 0$. So μ is closed.

Let $\gamma = \gamma_{w_n}$. If $\partial A = \gamma$, then $\int_A \nu = \int_{\pi_H(A)} dy \wedge dz = \text{sarea}(\pi_H(A))$
$= n^3$.

Further, $X_p = (1,0,0)$
$Y_p = (0,1,x)$ is orthogonal.
$Z_p = (0,0,1)$ Since $\max_{V_p, W_p \in \{X_p, Y_p, Z_p\}} \nu(V_p, W_p) = 1$,

$\Rightarrow \exists c$ s.t. $\nu(V,W) \le c\|V \wedge W\|_g$ (i.e., ν is a <u>bounded form</u>).
$\Rightarrow \underbrace{\left|\int_A \nu\right|}_{n^3} \le c\,\text{mass}(A)$.

So here's ~~an~~ general argument: ~~given a curve γ,~~ Homological lowerbounds:
Given a curve γ, find a bounded closed form ω s.t. $\int_A \omega$ is large on any A s.t. $\partial A = \gamma$.

~~In nilpotent groups,~~ In particular, <u>left-invariant forms</u>.

Ex: Let ~~dx, dy, dz~~ $\omega_x, \omega_y, \omega_z$ be dual to X, Y, Z

i.e. $\omega_x^* = dx$ $\omega_z^* = dz - x\,dy$.
$\omega_y^* = dy$

Then: $d\omega_x^* = 0$ $d\omega_z^* = -dx \wedge dy - x\,d(dy)$
$d\omega_y^* = d(dy) = 0$ $= -dx \wedge dy = -\omega_x \wedge \omega_y$

(Why? $d\omega(V,W) = \lim_{r \to 0} \frac{1}{r^2} \int_{\lambda_r} \omega$ $\lambda_r =$

Then $d(Y^* \wedge Z^*) = Y^* \wedge dZ^* + dY^* \wedge Z^*$
$= 0 \Rightarrow Y^* \wedge Z^*$ is closed left-inv't form.

Likewise, $\omega_x^* \wedge \omega_z^*$ is closed, left-inv't.

~~Further~~ And the cubic growth comes from how they scale:

For $t > 0$, let $s(x,y,z) = (tx, ty, t^2 z)$. This is an automorphism.
$s_*(X) = tX$
$s_*(Y) = tY$ If γ is a <u>horizontal</u> curve
$s_*(Z) = t^2 Z$ (i.e. $\gamma'(t) = c_1(t) X + c_2(t) Y$ $\forall t$)

then $\|\gamma'\|_g = \sqrt{c_1^2 + c_2^2} = \|c_1 X + c_2 Y\|_g$
$\|(s_t \circ \gamma)'\|_g = \|t c_1 X + t c_2 Y\|_g = t\|\gamma'\|_g$.

i.e., $\ell(s_t(\gamma)) = t\,\ell(\gamma)$. And if $\partial A = \gamma$, then

$\int_{s_t(A)} \omega = \int_A s_t^*(\omega)$ $s_t^*(\omega_x) = t\omega_x$
$s_t^*(\omega_y) = t\omega_y$
$s_t^*(\omega_z) = t^2 \omega_z$.

$s_t^*(\omega_y \wedge \omega_z) = t^3 \omega_y \wedge \omega_z$, so $\int_{s_t(A)} \omega_y \wedge \omega_z = t^3 \int_A \omega_y \wedge \omega_z$

~~Why is $FV_{\mathbb{H}}$~~

grows cubically with scale.

So $FV(s_t(\gamma)) \gtrsim \int_{s_t(A)} \omega_y \wedge \omega_z \gtrsim t^3$.

(break)

Why is $FV_{\mathbb{H}_2}$ quadratic?

1) Previous argument doesn't work: ~~we can still define fields~~

$X_1 = (1,0,0,0,0)$ ~~and duals, but~~ X_1, X_2, Y_1, Y_2, Z,

$X_2 = (0,1,0,0,0)$ ~~and duals~~:

$Y_1 = (0,0,1,0,x_1)$ $\quad \omega_z = dz - x_1 dy_1 - x_2 dy_2$,

$Y_2 = (0,0,0,1,x_2)$ ~~but~~ - there are no closed bounded ~~cubic~~ left-inv forms of cubic growth.

$Z = (0,\dots,1)$

$$\omega_{y_1} \wedge \omega_z = dy_1 \wedge (dz - x_1 dy_1 - x_2 dy_2)$$
$$= dy_1 \wedge dz - x_2 dy_1 \wedge dy_2$$
$$d(\omega_{y_1} \wedge \omega_z) = 0 - dx_2 \wedge dy_1 \wedge dy_2 \neq 0.$$

~~2) There are surfaces with quadratic growth.~~

~~A surface~~

2) The scaling limit is bigger. We're touching on something called geometric group theory here — large-scale geom of discrete gps. And one tool is asymptotic cone or scaling limit: We want to study shape of space at large scales - say a ball of rad r. So let's take an r-ball, scale metric by r^{-1} to get a ~~new metric of~~ ~~rad~~ ball w/ radius 1. ~~How does that converge?~~ Do these approach a limit?

In order to talk about a limit of metric spaces, need a corresp. between the spaces. In this case, the scaling map works. So, let

$d_r(v,w) = \frac{1}{r} d(s_r(v), s_r(w))$ What ~~does~~ is the limit d_∞ as $r \to \infty$?

~~The~~ Each d_r is Riemannian. Recall $ds^2 = \omega_{x_1}^2 + \omega_{x_2}^2 + \omega_{y_1}^2 + \omega_{y_2}^2 + \omega_z^2$

$$dg_r^2 = \frac{1}{r^2} s_r^*(ds^2) = \frac{1}{r^2}\left(r^2 \omega_{x_1}^2 + \dots + r^2 \omega_{y_2}^2 + r^4 \omega_z^2\right)$$
$$= \omega_{x_1}^2 + \dots + \omega_{y_2}^2 + r^2 \omega_z^2$$

As $r \to \infty$, this converges to a sub-Riemannian metric. - Like Riemannian, except only some vecs have finite length. Namely, $\ker(\omega_z) = \langle X_1, X_2, Y_1, Y_2 \rangle$

If $\gamma: \mathbb{R} \to \mathbb{H}$ or $\sigma: D \to \mathbb{H}$ satisfies $\operatorname{im} \gamma' \subset \ker \omega_z$ or $\operatorname{im} \sigma' \subset \ker \omega_z$, we say it's horizontal.

If γ ~~is horizontal~~, σ, then $\ell(s_r \circ \gamma) = r\ell(\gamma)$, $\text{area}(s_r \circ \sigma) = r^2 \text{area}(\sigma)$

Then: $\mathbb{H}_1$ has no nontrivial horizontal surfaces but $\mathbb{H}_2$ does: — in fact it has horizontal subgroups: ~~$\mathbb{H}_2$ is a Lie gp, w/ Lie algebra~~
$\mathbb{H}_2$ is a Lie gp w/ Lie algebra:

$$\langle X_1, X_2, Y_1, Y_2, Z \mid [X_1,Y_1]=[X_2,Y_2]=Z, \text{ all other gens commute} \rangle.$$

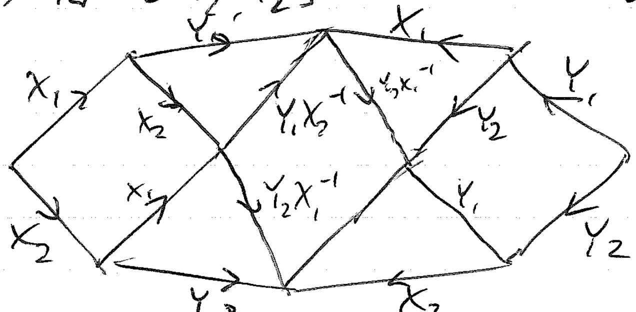
V, W generate an abelian subgp if $[V,W]=0$ — so, for instance $\langle X_1, Y_2\rangle \cong \mathbb{R}^2$. And this $\mathbb{R}^2$ is a horizontal surface (tangent to $\langle X_1, Y_2\rangle$ everywhere) ~~by left~~ Likewise, $\langle X_1, X_2\rangle$, etc.
But also $[Y_1 - X_2,\ Y_2 - X_1]$

$$= \cancel{[Y_1,Y_2]} - [Y_1,X_1] - [X_2,Y_2] + \cancel{[X_2,X_1]} = Z - Z = 0.$$

(combinatorial analogue: $Y_1X_2^{-1}$ and $Y_2X_1^{-1}$ generate abelian subgroup of $\mathbb{H}_2$) ~~which is a lattice in the~~

And we can glue these to construct interesting horizontal surfaces:

Ex: $[X_1,Y_1]=[X_2,Y_2]$ can be filled by a horizontal surface.



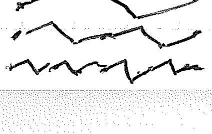
So $FV_{\mathbb{H}_2}([X_1^n,Y_1^n][X_2^n,Y_2^n]^{-1}) \lesssim n^2$.

Ex: $[X_1^n,[X_1^n,Y_1^n]] \xrightarrow[n^2]{} [X_1^n,[X_2^n,Y_2^n]] \xrightarrow[n^2]{} 1.$

~~Prop~~ In fact, Lemma: $\mathbb{H}_2^{\mathbb{Z}} \cong \langle X_1, X_2, Y_1, Y_2 \mid [X_1,Y_1]=[X_2,Y_2],$ ~~everything else commutes~~
$[X_1,X_2]=[X_1,Y_2]=[Y_1,Y_2]=[Y_1,X_2]=1\rangle$

~~So any~~
So, let $X \subset \mathbb{H}_2$ be Cayley graph of $\mathbb{H}_2^{\mathbb{Z}}$. Then any edge path in X bounds a horizontal disc.

(Almost enough to do a scaling arg — scale down, fill, scale up. But of course not every curve of length 1 is an edge path.)
So, discretize: let $\gamma:[0,1]\to\mathbb{H}_2$ ~~unit~~ sub-unit speed horizontal curve.
~~Let~~ For $k \geq 0$, let $X_k = s_{2^{-k}}(X)$, let $\pi_k: \mathbb{H}_2 \to X_k^{(0)}$ be closest pt. proj.
Note: $X_k \subset X_{k+1}$. Then we can construct a seq of approx of γ:
let γ_k be edge path ~~through~~ $\pi_k(\gamma(0)), \pi_k(\gamma(2^{-k})), \dots, \pi_k(\gamma(1))$.



γ_k has $\sim 2^k$ segs of length 2^{-k}
Connect.