2022-03-08

Last time: $\mathbb{H}_2$ has quadratic Dehn function.
(And there are a lot of proofs of this: ~~Sapir~~ Ol'shanskii–Sapir gave a combinatorial proof, Allcock gave a proof based on symplectic geometry, ~~the one~~ I like this proof because it generalizes nicely to higher dims)
& Gromov gave a proof based on h-principle

~~Let~~ Recall: ~~$\delta_{\mathbb{H}_2}([X_1^n, Y_2^n])$~~ ~~$\delta_{\mathbb{H}^2}([X_1^n, Y_1^n][X_2^n, Y_2^n]^{-1}) \sim n^2$.~~
~~How can we use this more generally? to fill arbitrary curves?~~

$[X_1, Y_1][X_2, Y_2]^{-1}$ bounds a horizontal disc — how can we use this to fill arbitrary curves?

Lemma: $\mathbb{H}_2 \cong \langle X_1, \dots, Y_2, Z \mid [X_1, Y_1] = [X_2, Y_2],$
$[X_1, X_2] = [Y_1, Y_2]$
$= [X_1, Y_2] = [Y_1, X_2] = 1 \rangle.$

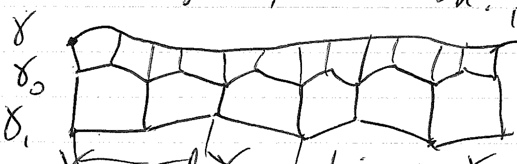
All of these relations bound horizontal discs, so any ~~wo~~ word representing 1 bounds a horizontal disc.
(~~But~~ We ~~can't~~ don't have any good bound the size of the disc) Say every edge path of length 100 can be filled by a horiz disc of area A
but we can have a trivial one – ~~only finitely~~

~~Prop~~ Thm: $\delta_{\mathbb{H}_2}(L) \lesssim L^2$. Enough to consider $L = 2^n$

Pf: Let $\gamma : [0, L] \to \mathbb{H}_2$ be a closed curve with unit speed.
We construct a seq. of approx of γ as follows: ~~Let $k \geq 0$~~
Let X be the Cayley graph of $\mathbb{H}_2$ wrt X_1, X_2, Y_1, Y_2 (just the horizontal generators). ~~$\mathbb{H}_2$ this is Every edge is horizontal~~ and let $X_k = s_{2^k}(X)$.
Since each edge of X is horizontal, each edge of X_k has length 2^k.
~~$\forall 0 \leq k \leq n$~~ we approx γ in X_k as follows:
Let $p_k : \mathbb{H}_2 \to X_k^{(0)}$ be nearest-point projection. ~~Approximate γ as follow~~
~~$\forall k$, we approximate~~
Let γ_k be the closed edge path in X_k connecting
$p_k(\gamma(0)), p_k(\gamma(2^k)), p_k(\gamma(2 \cdot 2^k)), \dots, p_k(\gamma(\frac{L}{2^k} \cdot 2^k))$

~~Now we construct htpies from $\gamma \to \gamma_0 \to \gamma_{1/2} \to \dots$~~

Note: – $\gamma_n = p_n(\gamma(0))$ is constant. So we construct htpies
– $\forall k$, ~~$X_{k+1} \subset X_k$~~ each edge of X_{k+1} is two edges of X_k.
$\gamma \to \gamma_0 \to \gamma_1 \to \dots \to \gamma_n$.



1 – annulus from γ to γ_0: breaks into L curves of length $\lesssim 1$.
~~2 connect γ_i to γ_{i+1} by edge the vertices of γ_i and γ_{i+1}~~
2 – γ_i and γ_{i+1} lie in X_i – so connect them in X_i to get $L/2^{i+1}$ edge paths of length $\lesssim 2^i$. Each consists of bddly many edges in X_i, so we can ~~fill it / a horizontal disc~~ ~~consists~~ scale down by 2^i, fill by a disc, scale back up.

Result: $\gamma \to \gamma_0$: L discs of area $\lesssim 1$. $L = 2^{2n}$
$\gamma_i \to \gamma_{i+1}$: $L/2^i$ discs of area $\sim (2^i)^2$ $L2^i$
$\vdots$ 2^{n+i}
$\gamma_{n-1} \to \gamma_n$: 1 disc of area $\sim (2^n)^2$. $= 2^{2n}$
$L^2 \sim 2^{2n}$!!

(And similar arguments work in other groups ~~if you can find the~~ — if you can fill enough horizontal curves ~~by~~ by horiz discs, quadratic filling.)

~~(Aside #1.~~ Open question: What happens in nilpotent groups where there aren't enough horizontal discs? Thm (Wenger): If G is ~~nilpotent~~ step 2 Carnot and $\delta_G \lesssim L^2$, then every horizontal curve in G_∞ can be filled by a limit of horizontal discs. So if there aren't enough horizontal discs, then $\delta_G \gtrsim L^2$ — ~~does this imply $\delta_G \gg$~~ can we say more? (G).

Quantitative Homotopy Theory: Homotopy theory studies homotopy classes of maps $X \to Y$. What can we say about the geometry of maps in a class/of the class itself? We've seen that when Y is sc, very dramatic phenomena. But when Y is s.c., things tend to be more subtle.

One reason: Computability issues tend not to arise in s.c spaces.
— ~~the word problem is difficult because~~ it's ~~uncomputable to recognize a group~~ - δ_X is large in spaces where it's hard/impossible to recognize the class of ~~an element~~ a ~~loop/~~ ~~knot in~~ ~~B~~. But π_n is abelian for $n \geq 2$, ~~and Abelian groups are easy to classify~~. ~~In fact~~
~~Thm~~ ~~and it turns out that π_n is well~~ ~~Abelian groups are~~
~~Thm (Sullivan)~~ - the word problem in abelian groups is easy! This doesn't make it easy to calculate π_n, but it makes ~~something~~ it ~~easy~~ possible to see certain aspects of it.
~~Ex~~ ~~Let's look at one question:~~ Let's look at ~~one~~ question:

~~Suppose~~ Let X be a space, let $\alpha \in \pi_n(X)$ ~~be~~ be an element of infinite order. Let $G_\alpha(L)$ = largest power of α that can be represented by an L-Lipschitz map.
How does this depend on L? Easy case first:

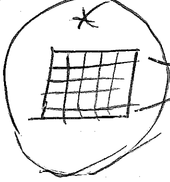
Ex: ~~Trivial~~ $\pi_n(S^n) \cong \mathbb{Z}$, let e be a generator. G_α?
Let $f: S^n \to S^n$ be a Lipschitz map. ~~Let $\omega \in \Omega^n(S^n)$ be area form on S^n~~
~~Then we can identify f by integrating ω: $f \simeq e^k$, where~~
~~$k = \int_{S^n} f^*(\omega) / \mathrm{vol}(S^n)$.~~

Then we can identify f by degree: Let ~~$p \in S^n$ be a generic point, consider $f^{-1}(p)$~~ $f = e^{ik}$ where $k = \deg(f)$. (count pre images)
But also by signed area: let $\omega \in \Omega^n(S^n)$ be volume form. Then

$$\mathrm{sarea}(f) = \int_{S^n} f^*(\omega) = \deg(f) \cdot \mathrm{vol}(S^n).$$

If f is L-Lip, then ~~Further,~~ $\|f^*(\omega)\| \lesssim L^n \Rightarrow \deg(f) \lesssim L^n$, so $G_e(L) \lesssim L^n$.

~~Conversely,~~ And this is sharp: Let $h : [0,1]^n \to S^n$ be a c-Lip map s.t. $h(\partial [0,1]^n) = *$ and h has degree 1.

$h_L = S^n$ 
$L \times \cdots \times L$ grid of copies of h
– ~~this is a~~ then $\mathrm{Lip}(h_L) \lesssim L$, $\deg(h_L) = L^n$.

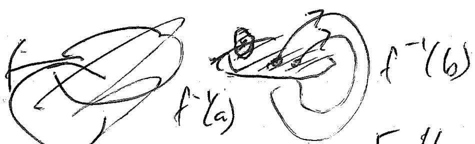
~~More complicated~~: Let's try something more complex: $\pi_3(S^2) \cong \mathbb{Z} = \langle h \rangle$,
where ~~h is the~~ $h : S^3 \to S^2$ is the Hopf fibration

$$S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}, \quad S^2 = \mathbb{C} \cup \{\infty\}$$

$h(z,w) = z/w$. Fibers are $h^{-1}(a) = S^3 \cap \{z = aw\}$ – circles lying in a complex plane. Why is this not null-htpic?

Let $a, b \in S^2$, $a \neq b$. Then $h^{-1}(a)$ and $h^{-1}(b)$ have nonzero linking number. (picture?) So we define:
Def: The Hopf invariant $H(f)$ of a map $f : S^3 \to S^2$ is the linking # of two generic fibers $f^{-1}(a)$ and $f^{-1}(b)$.

~~That is, if a, b are regular points,~~ Ex: Takes a little work to define:
- if a, b are regular points, then $f^{-1}(a)$, $f^{-1}(b)$ are 1-cycles.



The normals map to S^2, so the normals are oriented – this lets us orient $f^{-1}(a)$, $f^{-1}(b)$.
Fill $f^{-1}(a)$ by a disc A. This intersects $f^{-1}(b)$ – the linking # is the # of intersections, w/ ~~mult~~ sign.

Thm: $H(f)$ is well-defined, invariant under htpy, $H : \pi_3(S^2) \xrightarrow{\cong} \mathbb{Z}$.
In particular, h generates $\pi_3(S^2)$.

Growth of h?

Thm (Whitehead): We can calculate $H(f)$ using diff. forms. Let $\omega, \omega' \in \Omega^2(S^2)$ be forms s.t. $\int_{S^2} \omega = \int_{S^2} \omega' = 1$. Let $f: S^3 \to S^2$.
Then $d f^*(\omega') = 0$. ~~But~~ Since $H^2(S^3) = 0$, $\exists$ a primitive $\beta \in \Omega^1(S^3)$ s.t. $d\beta = f^*(\omega)$. ~~then $H(\beta) = \int_S$~~
For any such ω, ω', β, $H(f) = \int_{S^3} \beta \wedge f^*(\omega)$.

(Ex: ~~Show how~~ Take ω, ω' supported on a small nbhd of a, b. Show how this ~~calculates~~ agrees w/ ~~previous def~~ linking #.)

Thm (Gromov): $G_h(L) \sim L^4$. Pf: ~~Apply~~ Make this quantitative. Let $\omega, \omega' \in \Omega^2(S^2)$ as above. Let $f: S^3 \to S^2$ be L-Lip. Then $\|f^*(\omega)\|_\infty \leq L^2$. There is a primitive ~~β s.t. $\|$~~ $\beta \in \Omega^1(S^3)$ s.t. $\|\beta\|_\infty \lesssim \|f^*(\omega)\|_\infty \leq L^2$. Then

$$H(f) = \int_{S^3} \beta \wedge f^*(\omega)$$

$$\lesssim \|\beta\|_\infty \|f^*(\omega)\| \lesssim L^4$$

Conversely, if $h: S^3 \to S^2$ is ~~the~~ Hopf map, $g_L: S^2 \to S^2$ is a map of degree L^2, then ~~$g_L \circ h$ has~~ $H(g_L \circ h) = L^4$ //.

Thm (Sullivan ~~Gromov~~): Let ~~X be~~ X be s.c. Riemannian mfld. ~~$\forall \alpha \in \pi_n(X)$~~ ~~$G_\alpha(L)$ is at most polynomial in L~~ ~~there is a~~ ~~process~~ integral ~~of~~ ~~f~~ for let $S: \pi_n(X) \to \mathbb{R}$ a homomorphism
~~Pf~~ There is an integral of forms ~~as~~ as above ~~st~~ that calculates $S(f)$ ~~= $\int$~~ ...

Thm (Gromov): Let $\alpha \in \pi_n(Y)$ be ∞ order. Then $G_\alpha(L)$ is ~~at~~ ~~at most~~ bounded by a polynomial of degree depending on the Sullivan construction.

Q: Is this optimal? ~~Can~~ we always construct ~~maps~~ maps ~~that~~ ~~"maximize" of this sort?~~ with this growth? ~~Can see the~~ ~~In the case of Hopf, can we always~~ E.g. in Hopf, needed maps so $\|\beta \wedge f^*(\omega)\| \sim L^4$ — not obvious that these exist.

Development: There are examples where they do. But also, examples where they don't — ~~where they~~ ~~attach~~ ~~some rational invt, some algebra~~
(Berdnikov–Manin): This isn't sharp in $\pi_5([\#4\,\mathbb{C}P^2 \times S^2]^0)$
~~because the algebra of~~ — roughly, the algebra is too complex to be represented by ~~a~~ forms on S^5.