NOTES ON QUANTITATIVE TOPOLOGY

Contents

Date: March 29, 2022.

0. Introduction

One of the basic questions of topology is: given topological spaces X and Y, when is there a continuous map $f: X \to Y$ with given properties. And this motivates a lot of what's done in topology: for instance, the fundamental group answers the question of when a closed curve in X extends to a disc.

In this class, we'll try to make some of these questions quantitative. Suppose we know that a map or a space with certain properties exists – what can we say about that map or space? How big is it? How complex is it?

I have a couple of goals here:

- Introduce some of the ideas and methods of quantitative geometry, like discretization, scaling, and limits
- Apply these ideas to geometric group theory and topology

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1. Lecture 1: 2022-01-25: Quantifying simple connectivity (notes by Robert Young)

Let's start with the question from the introduction: the fundamental group tells you when a closed curve in X extends to a disc. How do we quantify this?

Suppose X is a space, say a Riemannian manifold or simplicial complex. Let $\gamma: S^1 \to X$ be a null-homotopic curve in X. Then there is a homotopy from γ to a point; we can view this as a map $\beta: D^2 \to X$. How does the size of β depend on γ?

Specifically, we can define the *filling area* of a curve and the *Dehn function* of a space. Given a Lipschitz curve $\gamma: S^1 \to X$, the filling area of γ is

$$\delta_X(\gamma) = \inf_{\beta: D^2 \to X} \operatorname{area} \beta,$$

where the infimum is taken over the Lipschitz maps $\beta: D^2 \to X$ such that β agrees with γ on its boundary, i.e., $\beta|_S^1 = \gamma$. The *Dehn function* of X is the function

$$\delta_X(L) = \sup_{\gamma: S^1 \to X} \delta_X(\gamma),$$

where the infimum is taken over null-homotopic closed curves of length at most L.

(Instead of taking the infimum over null-homotopic closed curves, we can pass to the universal cover — any closed curve in the universal cover corresponds to a null-homotopic closed curve, so

$$\delta_X(L) = \sup_{\gamma: S^1 \to \tilde{X}} \delta_{\tilde{X}}(\gamma),$$

where the infimum is taken over *all* closed curves of length at most L.)

Remark (Lipschitz maps). Recall that a map $f\colon X \to Y$ is Lipschitz if there is some $C > 0$ such that $d_Y(f(p), f(q)) \le C d_X(p,q)$ for all $p, q \in X$. We use Lipschitz maps because we can define their length and area. By Rademacher's Theorem, if $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz, it is differentiable almost everywhere, in the sense that for almost every x there is a linear map $Df_x\colon \mathbb{R}^m \to \mathbb{R}^n$ such that when y is sufficiently close to x,

$$f(y) = f(x) + Df_x(y-x) + o(\|y-x\|).$$

(Recall that $o(\|y-x\|)$ denotes an error term that's strictly smaller than $\|x\|$, i.e.,

$$\lim_{y\to x} \frac{f(y) - (f(x) + Df_x(y-x))}{\|y-x\|} = 0.)$$

If $f\colon I \to \mathbb{R}^n$ is a Lipschitz curve, we define

$$\ell(f) = \int_I \|Df_x\|\, dx;$$

this is the same formula as the formula for the length of a C^1 curve. If $f\colon U \to \mathbb{R}^n$ is a Lipschitz map with $U \subset \mathbb{R}^m$ a measurable set, we define

$$\mathrm{vol}^m(f) = \int_U \sqrt{\det[(Df_x)^T Df_x]}\, dx;$$

this is likewise the same formula as the formula for the area of a surface, and these formulas generalize to Riemannian manifolds and (with some work) to simplicial complexes.

Note that if $f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ is C–Lipschitz, then $\mathrm{vol}^m(f) \le C^m \mathrm{vol}^m(U)$, where $\mathrm{vol}^m(U)$ is the euclidean volume of U.

For example, consider the case that $X = \mathbb{R}^2$ and γ is a simple closed curve. By the Jordan Curve Theorem, γ bounds a disc D, and the filling area of γ is equal to the area of D.

1.1. **Example:** $\mathbb{R}^n$.

Finding exact values of δ_X is generally difficult, but one can often prove asymptotics. For example, the following holds in $\mathbb{R}^n$:

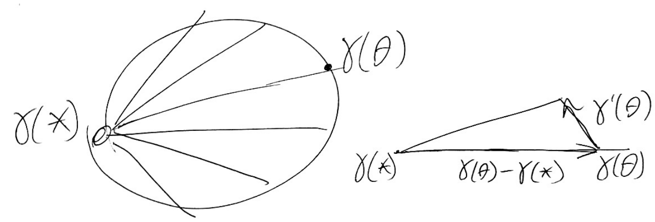
Proposition 1.1. *Let $\gamma\colon S^1 \to \mathbb{R}^n$ be a Lipschitz closed curve. Then*

$$\delta_{\mathbb{R}^n}(\gamma) \le \frac{1}{4}\ell(\gamma)^2.$$

Proof. We fill γ by a straight-line homotopy. Let $*$ be a basepoint in S^1, and let (r,θ) be polar coordinates on D^2. Let $\beta\colon D^2 \to \mathbb{R}^n$,

$$\beta(r,\theta) = \gamma(*) + r(\gamma(\theta) - \gamma(*)).$$

This is a Lipschitz disc filling γ. We can break the disc into wedges like so:



The area of the triangle on the right is at most $\frac{1}{2}\|\gamma(\theta)-\gamma(*)\|\|\gamma'(\theta)\|$, and, since γ has length L, $\|\gamma(\theta)-\gamma(*)\| \le \frac{L}{2}$. Therefore,

$$\begin{aligned}\operatorname{area}\beta &\le \frac{1}{2}\int_{S^1}\|\gamma(\theta)-\gamma(*)\|\|\gamma'(\theta)\|\,\mathrm{d}\theta\\ &\le \frac{L}{4}\int_{S^1}\|\gamma'(\theta)\|\,\mathrm{d}\theta \le \frac{L^2}{4},\end{aligned}$$

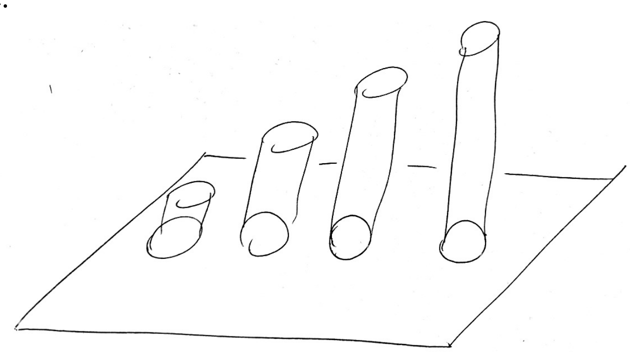
as desired. □

Thus $\delta_{\mathbb{R}^n}(L) \le \frac{L^2}{4}$. Conversely, the circle of radius r has length $2\pi r$ and area πr^2; it follows that

$$\delta_{\mathbb{R}^n}(L) \ge \pi(\frac{L}{2\pi})^2 = \frac{L^2}{4\pi}.$$

Thus $\frac{L^2}{4\pi} \le \delta_{\mathbb{R}^n}(L) \le \frac{L^2}{4}$ – we say that $\delta_{\mathbb{R}^n}(L) \approx L^2$.

1.2. **Example: $\delta_X(L) = \infty$.** Note that δ_X need not be finite. For example, consider a space X constructed by starting with the plane $\mathbb{R}^2$ and cutting a hole of radius $\frac{1}{4}$ around each point $(n,0)$. We glue a cylinder of height n to the hole around $(n,0)$, and cap it off with a disc. The resulting space X is homeomorphic to $\mathbb{R}^2$, but $\delta_X(1) = \infty$, because the boundary of each hole has length at most 1 and area $\approx n$.



We'll see later that this can't happen when X is more symmetric:

Proposition 1.2. *Suppose that X is a Riemannian manifold or simplicial complex with bounded degree that admits a cocompact action by isometries. (An action of G on X is cocompact if the quotient X/G is compact. Equivalently, there is a fundamental domain which is contained in a compact set.) Then $\delta_X(L) < \infty$ for all L.*

In particular, if K is compact and $\tilde{K}$ is its universal cover, then $\delta_{\tilde{K}}$ is finite.

1.3. **Example:** $\mathbb{H}^n$.

Another example: let $\mathbb{H}^n$ be hyperbolic space. We will not go into hyperbolic geometry too much, but a key feature of hyperbolic space is that geodesics in $\mathbb{H}^n$ diverge at an exponential rate. That is, a circular are in $\mathbb{H}^n$ with angle θ and radius r has length $\theta \sinh r$; in particular, the circle in the hyperbolic plane has circumference $2\pi \sinh r \approx e^r$ when r is large.

Another way to look at this: if γ and λ are two unit-speed geodesics that start at the same point x_0 and the angle $\theta = \angle(\gamma, \lambda)$ is small, then $d(\gamma(t), \lambda(t))$ can be small for a long time, i.e.,

$$d(\gamma(t), \lambda(t)) \approx \theta \sinh t \approx \theta e^t$$

for $1 < t < -\log\theta$. But then, around $t = -\log\theta$, we have $d(\gamma(t), \lambda(t)) \approx 1$, and the exponential growth kicks in. When $t > -\log\theta$, the length of the arc from γ to λ is growing quickly. Since λ and γ are unit-speed, $d(\gamma(t), \lambda(t))$ grows more slowly; in fact, $d(\gamma(t), \lambda(t)) = 2(t - |\log\theta|) + O(1)$ for $t > -\log\theta$.

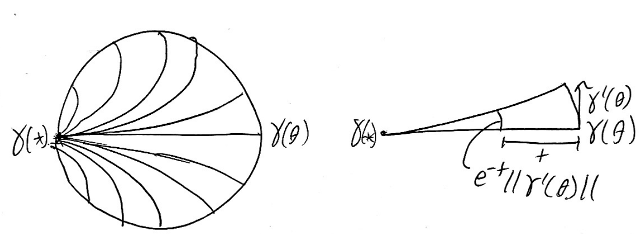
This affects the asymptotics of the Dehn function, as we see in the following proposition.

Proposition 1.3. $\delta_{\mathbb{H}^n}(L) \lesssim L$

Proof. We again use a straight-line homotopy. For $p, q \in \mathbb{H}^n$, let $\lambda_{p,q} \colon [0,1] \to \mathbb{H}^n$ be the geodesic from p to q. Using polar coordinates as before, let

$$\beta(r, \theta) = \lambda_{\gamma(*),\gamma(\theta)}(r).$$

This is a Lipschitz disc filling γ, and we can break the disc into wedges again, but the shape of the wedges is different:



Since the geodesics making up the sides of the wedge diverge exponentially, they also converge exponentially — the distance between the sides of the wedge is like $e^{-t}\|\gamma'(\theta)\|$, where t is the distance from $\gamma(\theta)$. The wedge then has area

$$\approx \int_0^{d(\gamma(*),\gamma(\theta))} e^{-t}\|\gamma'(\theta)\|\,\mathrm{d}t \lesssim \gamma(\theta),$$

and the disc has area

$$\operatorname{area}\beta \lesssim \int_{S^1} \|\gamma'(\theta)\|\,\mathrm{d}\theta = \ell(\gamma),$$

as desired. □

1.4. **Next time.** Next time, we'll see:

Proposition 1.4. *If K is a compact, simply-connected Riemannian manifold or simplicial complex, then $\delta_K(L) \lesssim_K L$.*

This gives us a variety of examples ($\mathbb{R}^n$, $\mathbb{H}^n$, and simply-connected compact spaces) where the Dehn function is small. Part of the reason for this is that these spaces are easy to navigate – in $\mathbb{R}^n$ and $\mathbb{H}^n$, there's a unique geodesic between any two points and that geodesic varies continuously. In a compact space, there need not be a unique geodesic, but the space itself can't be too complex. But in general, the Dehn function can be much larger.

Proposition 1.5. *There is a compact simplicial complex K such that*

$$\delta_K(L) \geq e^{e^{e^{e^L}}}$$

for all sufficiently large L.

Proposition 1.6. *For all sufficiently large n, there is a compact simplicial 2–complex K_n with at most n vertices, edges, and faces such that*

$$\delta_{K_n}(3) \geq e^{e^{e^{e^n}}}.$$

In fact, in both cases, the Dehn function grows faster than any computable functions. To see this, we'll need to link filling area to computability, which we'll do next week.

2. Lecture 2: 2022-02-01 (notes by Yonghao Yu)

In the first lecture, we have seen some elementary example of $\delta_X(L)$. To go further, we need some general results, which are given by the following lemma.

Lemma 2.1. *Let X be a simplicial complex or Riemannian manifold. If X is equipped with an isometric G–action s.t. there exists a compact $K \subset X$ s.t. $GK = X$, (for instance, X can be a universal cover of a compact space), then*

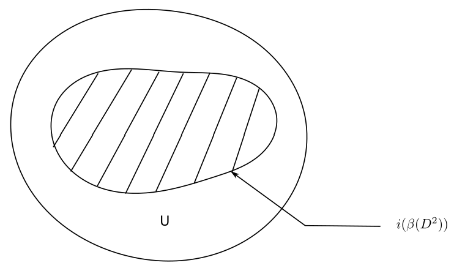
(1) *There exists $\epsilon > 0$ s.t. if $\gamma : S^1 \to X$ and $l(\gamma) < \epsilon$, then $\gamma \sim *$.*
(2) *If $\gamma : S^1 \to X$ is Lipshiitz and $\gamma \sim *$, then there exists a Lipschitz extension $\beta : D^2 \to X$ s.t. $\beta|_{S^1} = \gamma$.*
(3) *$\delta_X(L) < \infty$ for every $L > 0$.*

Proof. (1) This follows from the standard fact that the injectivity radius of a Riemannian manifold is a continuous positive function.

(2) We know that every continuous map $f : D^2 \to \mathbb{R}^n$ is Lipschitz on ∂D^2 can be approximated arbitrarily closely by a Lipschitz map.

One can generalize this result to a Riemannian manifold: Suppose $\gamma : S^1 \to X$ is Lipschitz and $\gamma \sim *$, then there exists $\beta : D^2 \to X$ s.t. $\beta_{S^1} = \gamma$, $\gamma(*) \in K$.

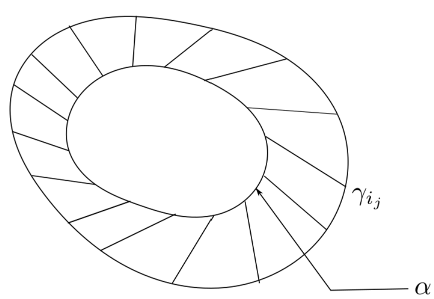
Since β is a continuous map with compact domain, $\beta(D^2)$ is compact. Then $\beta(D^2)$ is a subset of some bounded sub-manifold M. By Whitney embedding theorem, there exists an embedding $i : M \to \mathbb{R}^n$. By tubular neighborhood theorem, there exists a neighborhood U of $i(\beta(D^2))$ and a smooth retraction $r: U \to M$. Then one can approximate $i \circ \beta$ by a Lipschitz map $B: D^2 \to \mathbb{R}^n$. Then $r \circ B: D^2 \to i(M)$ is a Lipschitz map and $i^{-1} \circ r \circ B: D^2 \to M$ is a Lipschitz extension of γ.



(3) Because of the G–action, WLOG, we assume that $\gamma: S^1 \to X$ is a Lipschitz curve with $l(\gamma) \leq L$, $\gamma(*) \in K \subset X$.

Now we will apply the limit method for this problem. Let $\{\gamma_i\}_{i\in\mathbb{N}}$ be a sequence of closed curve in X s.t. $l(\gamma_i) \leq L$, $\delta_X(\gamma_i) \to \delta_X(L)$ and $\gamma_i(*) \in K$. Then $\gamma_i \subset N_L(K)$ where L is some neighborhood of K, and we can reparametrize the γ_i's as maps $\gamma_i[0, L] \to X$ with constant speed ≤ 1. By the Arzela–Ascoli theorem, there is a subsequence γ_{i_j} that converges uniformly to some curve α.

Hence when j is sufficiently large, $|\alpha(t) - \gamma_i(t)| < \frac{\epsilon}{10}$, where ϵ is the constant in part 1. Now subdivide the annulus into squares of length $\leq \epsilon$. Each square can be extend into a disc, which gives a homotopy between α and γ_{i_j}. In particular, we have $\alpha \sim \gamma_{i_j} \sim *$.



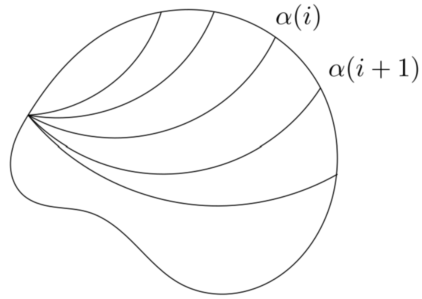
So $\delta_X(\alpha) < \infty$ and $\delta_X(\gamma_{i_j}) \leq \delta_X(\alpha) + \frac{10L}{\epsilon}\delta_X(\epsilon) < \infty$. Then $\delta_X(L) = \lim_{j\to\infty} \delta_X(\gamma_{i_j}) < \infty$, as desired.

□

Now we have enough tools to show Proposition 1.4, i.e., that the Dehn-function of a compact simply connected simplicial complex or Riemannian manifold can be bounded by a linear function.

Proof. For every $u, v \in X$, let $\gamma_{u,v}$ be a shortest path from u to v. Then $l(\gamma_{uv}) \leq dim(X)$. Let $D = dim(X)$. Given a curve $\alpha: S^1 \to X$ of length L, let n be the natural number s.t. $L \le n \le L+1$. Reparametrize α as a map $\alpha : [0,n] \to X$ with constant speed ≤ 1, so that $d(\alpha(i), \alpha(i+1)) \le 1$.

We can then decompose α into wedges $\Delta_i = \gamma_{\alpha(0)\alpha(i)}\alpha_{[i,i+1]}\gamma_{\alpha(i+1)\alpha(0)}$. Because $\pi_1(X) = 0$, $\Delta_i \sim *$, $l(\Delta_i) \le 2D+1$, so $\delta_X(\Delta_i) \le \delta_X(2D+1) < \infty$. Moreover, $\delta_X(\alpha) \le n\delta_X(2D+1) \le (L+1)\delta_X(2D+1)$.



□

2.1. **Computable functions.** Now we will show Proposition 1.5: There exists a compact simplicial complex K s.t. $\delta_K(L)$ is larger than any computable function. Before constructing such a K, we need some discussion on computable functions.

Definition 2.2. A computable function $f : \mathbb{N} \to \mathbb{N}$ is a function s.t. there exist some deterministic computer program (algorithm) to compute $f(n)$ for every n. For instance a Python program that does not include random number, internet and referencing outside source.

Note that there are only countably many computable functions, as a program is a finite string of bits. We identify the finite strings of bits with the natural numbers $\mathbb{N}$. We define $f_i(n)$ to be the function obtained by treating the ith finite string as a python program and running it on n. Note that $f_i(n)$ is only defined when the program f_i terminates on input n after outputting an integer. Then we have another definition for computable function:

Definition 2.3. f_i is a computable function iff $f_i(n) \in \mathbb{N}$ for all $n \in \mathbb{N}$.

Note also that there is a program (the Python interpreter) that takes i, n as inputs and outputs $f_i(n)$.

Now comes the question, can we find a computer program to determine whether or not $f_i(n)$ is defined for all i and n?

The answer is negative, as Alan Turing proved in 1936 that such computer program does not exists:

Theorem 2.4. *Let Halt(i,n) be the Halting function which determine whether $f_i(n)$ is well-defined or not. It outputs 1 if $f_i(n)$ exists and outputs 0 if $f_i(n)$ doesn't exists. Then the function Halt(i,n) is not computable.*

Proof. Suppose the Halting function is computable. Then one can define a program T which takes the natural number n as input.

Define T so that for any input n, if $\mathrm{Halt}(n,n)=1$, the program T loops infinitely. Else the program T outputs 0.

As program T is a finite string of bits, $T=f_N$ for some N. Now consider $T(N)=f_N(N)$. Then if $f_N(N)$ halts, $\mathrm{Halt}(n,n)=1$ and $T(N)$ does not halt, which is a contradiction. If $f_N(N)$ does not halt, then $\mathrm{Halt}(n,n)=0$ and $T(N)=0$, which is also a contradiction. Hence the halting function cannot be computable ☐

Corollary 2.5. *As a consequence, there is no computable function L s.t. if $f_n(n)$ halts, then it halts in at most $L(n)$ steps*

Proof. Suppose such an function L exists, then consider the following program H: given input n, run program f_n for $L(n)$ steps. If f_n terminated in $L(n)$ steps, output 1; else output 0. Then $H(n)$ computes $\mathrm{Halt}(n,n)$, one reach a contradiction.

Equivalently, one can define a function $L(n)$ as the longest number of steps that $f_m(m)$ takes before terminating for $m\le n$. Then we just shown that $L(n)$ is larger than any computable function $f(n)$. ☐

2.2. **Group presentations.** A *group presentation* is an expression $\langle g_1,\cdots,g_n|r_1,\cdots,r_s>$ where $g_1,\cdots g_n$ denotes the set of generators and $r_1,\cdots r_s$ denote the set of relations. Each r_i is a formal product (word) of g_i's and g_j's.

Let $F(g_1,\cdots,g_n)$ be the free group generated by the g_i's, i.e., the quotient of the set of words under the equivalence relations $wg_ig_i^{-1}w'\sim ww'$, and $wg_i^{-1}g_iw'\sim ww'$. Then a group presentation is the quotient

$$\begin{aligned}\langle g_1,\cdots,g_n\rangle &= F(g_1,\cdots,g_n)/\langle\langle r_1,\cdots r_s\rangle\rangle\\ &= \text{words}/\,wg_i^{\pm1}g_i^{\mp1}w'\sim ww', wr_i^{\pm1}w'\sim ww'\\ &= F(g_1,\cdots,g_n)/\Pi_{i=1}^d w_i r_{j_i}^{\pm1} w_i^{-1}\end{aligned}$$

2.3. **Example:** $\langle x,y|yx^{-1}y^{-1}\rangle=\mathbb{Z}^2$**.** Since $yx\sim(xyx^{-1}y^{-1})yx\sim xy$, and similarly one can shown that $xy^{-1}\sim y^{-1}x$, $x^{-1}y^{-1}\sim y^{-1}x^{-1}$, then every word is equivalent to x^ay^b for unique $a,b\in\mathbb{Z}$. Hence $\langle x,y|yx^{-1}y^{-1}\rangle=\mathbb{Z}^2$

Theorem 2.6 (Novikov–Boone)**.** *There is a group with unsolvable word problem. In other words, a group G s.t. there is no algorithm to determine whether two words w and l are equivalent or not.*

Indeed, simple groups can be computationally hard word problem. For instance, consider the group $BS_{1,2} = \langle a, b | aba^{-1}b^{-2} \rangle$. We have

$$\begin{aligned} a^n ba^{-n} &= a^{n-1} aba^{-1} a^{-n+1} \\ &= a^{n-1} b^2 a^{-n+1} \\ &= (a^{n-1} ba^{-n+1})(a^{n-1} ba^{-n+2}) \\ &= b^{2^n}, \end{aligned}$$

so words of length $2n+1$ in $BS_{1,2}$ "expand" to words of length 2^n.

3. Lecture 3: 2022-02-08 (notes by Mohammed Mannan)

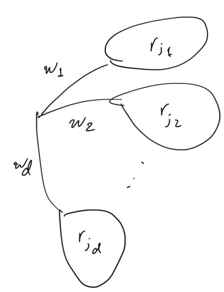
Definition 3.1. A van Kampen diagram D is a finite planar 2-complex embedded in $\mathbb{R}^2$ such that

- D is connected.
- D is simply-connected.
- Each edge is oriented and labeled by a generator.
- The boundary of each 2-cell is a relation.

Proposition 3.2. *Let $G = \langle g_1, \dots, g_n \mid r_1, \dots, r_s \rangle$ be a finitely presented group. Then $w =_G 1$ if and only if w is the boundary of a van Kampen diagram.*

Proof. ($\Leftarrow$) Deleting a cell adjacent to the boundary word creates a new boundary word that represents the same group element. By consecutive deletion, we see that the boundary word $=_G 1$.

($\Rightarrow$) If $w \in \mathscr{F}(g_1, \dots, g_n)$ is such that $w =_G 1$, then $w \in \langle\langle r_1, \dots, r_n \rangle\rangle$, so $w =_{\mathscr{F}} \prod_{i=1}^d w_i r_{j_i}^{\pm 1} w_i^{-1} = q$. There's a van Kampen diagram for q that looks like:

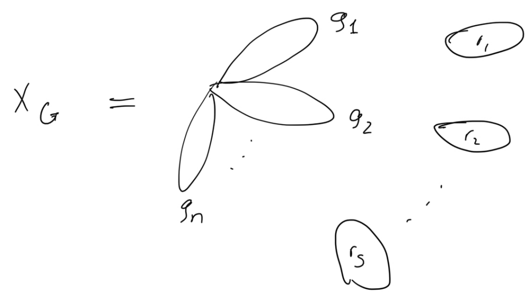


If q contains a substring $g^{\pm 1} g^{\mp 1}$, then two consecutive edges in the boundary have the same label but opposite orientations. These can be folded together to get a new van Kampen diagram whose boundary word is a free reduction of q. Since q can be freely reduced to w, there's a sequence of folds that turns the diagram for q into a diagram for w.

□

For a group $G = \langle g_1, \dots, g_n \mid r_1, \dots, r_s \rangle$, let X_G be the 2–complex given by the figure below, with one vertex, n edges, and s 2–cells, each glued according to one of the r_i's. An edge path in X_G is a path made up of edges. For each word $w \in$

$(g_1^{\pm 1}, \dots, g_n^{\pm 1})^*$, let λ_w be the corresponding edge path. Note that $\pi_1(X_G) = G$; indeed, the natural isomorphism identifies each element w with the homotopy class of λ_w.



A van Kampen diagram for G is naturally equipped with a map to X_G. This gives an alternative proof of the $(\Leftarrow)$ direction of the proposition, since if w is the boundary word of a van Kampen diagram, then the van Kampen diagram gives a null-homotopy of λ_w.

Conversely, suppose λ_w is null-homotopic. Then λ_w extends to a disc, but that disc need not be a van Kampen diagram. Regardless, we can always approximate the disc by a van Kampen diagram.

Lemma 3.3. *Let $w =_G 1$. Let $\beta : D^2 \to X_G$ be a Lipschitz map such that $\beta|_{S^1} = \lambda_w$. Then there is a van Kampen diagram D with boundary word w such that* $\operatorname{area}(D) \lesssim \operatorname{area}(\beta)$.

We define $\operatorname{area}(D)$ to be the number of 2-cells in D.

Proof. Without loss of generality we may suppose that β is smooth on the interior of every cell. By the co-area formula

$$\operatorname{area}(\beta) = \int_{X_G} \#\beta^{-1}(y)\,dy = \sum_{\sigma \in F^2(X)} \int_\sigma \#\beta^{-1}(y)\,dy$$

where $F^2(X)$ is the set of 2-cells in X. By Sard's theorem almost every $y \in \sigma$ is a regular point (that is, $D\beta_x$ is nonsingular for every $x \in \beta^{-1}(y)$). In particular, we can define the degree of β on σ. Pick a regular point $y \in \sigma$, then define

$$\deg_y(\beta) = \sum_{x \in \beta^{-1}(y)} \operatorname{sign}(\det D\beta_x).$$

Let y_σ be a regular point such that

$$\#\beta^{-1}(y_\sigma) \le \frac{1}{\operatorname{area}(\sigma)} \int_\sigma \#\beta^{-1}(y)\,dy < \infty.$$

Because y_σ is regular, the inverse function theorem applies. There is a neighborhood $U_\sigma \ni y_\sigma$ such that $\beta^{-1}(U_\sigma)$ consists of finitely many disjoint discs, each containing one element of $\beta^{-1}(y_\sigma)$, and each sent diffeomorphically to U_σ by β. Let $m = \min_\sigma \operatorname{area}(\sigma)$. Then

$$\#\beta^{-1}(y_\sigma) \le \frac{1}{m}\int_\sigma \#\beta^{-1}(y)\,dy,$$

so

$$\sum_\sigma \#\beta^{-1}(y_\sigma) \le \frac{1}{m}\operatorname{area}(\beta).$$

Let $r : X_G \to X_G$ be a map which sends each U_σ to σ and sends $\sigma \setminus U_\sigma$ to $\partial\sigma$. Consider $r\circ\beta$. Draw lollipops going around the U_σ. The image of the lollipops under $r\circ\beta$ is in $X_G^{(1)}$. The boundary curve is homotopic to $r\circ\beta|_{S^1} = \beta|_{S^1}$. Call the boundary curve $\gamma : S^1 \to X_G^{(1)}$. Then γ is homotopic to $\beta|_{S^1} \sim \lambda_w$ by a homotopy in $X_G^{(1)}$. Straighten out γ to be an edge path λ_q where q is of the form $q = \prod_{i=1}^d w_i r_{j_i}^{\pm 1} w_i^{-1}$ (where r_{j_i} is the relator bounding the cell that the ith lollipop is sent to).

Then q admits a van Kampen diagram with $\sum_\sigma \#\beta^{-1}(y_\sigma)$ 2–cells, and $\lambda_q \sim \gamma \sim \beta|_{S^1} \sim \lambda_w$ by a homotopy in $X_G^{(1)}$. Therefore, $q =_{\mathscr{F}} w$, so we a can fold the van Kampen diagram for q to get a van Kampen diagram for w. □

Proposition 3.4. *There is a 2-complex X such that $\delta_X(L) > e^{e^{\cdots^{e^L}}}$ for sufficiently large L.*

Theorem 3.5. *(Novikov–Boone) There is a finitely presented group G such that there is no algorithm to decide whether $w =_G 1$.*

Corollary 3.6. *With G as in Novikov–Boone, there is no computable function f such that if $w =_G 1$ and $\ell(w) \le L$, then there is a VKD for w with area $\le f(L)$.*

Proof. Suppose that such an f exists. Consider the algorithm, on input w, which attempts to construct a VKD for w with area $\le f(\ell(w))$. If one is found, then $w =_G 1$. Otherwise $w \ne_G 1$. □

Corollary 3.7. *There is no computable f such that $\delta_{X_G}(L) \le f(L))$ for all L (with G as in Novikov–Boone).*

Proof. We have seen that if $w_G = 1$, then there is a VKD for w with $\lesssim \delta_{X_G}(\lambda_w)$ 2-cells. Thus, for every computable f, there are words w of arbitrarily large length such that $w =_G 1$ and any van Kampen diagram for w has area $> f(\ell(w))$. It follows that $\delta_{X_G}(\lambda_G) \gtrsim f(\ell(w))$. □

4. Lecture 4: 2022-02-15: Filling problems in higher dimensions and Singular Lipschitz Homology (notes by Zhengjiang Lin)

An example of a group with unsolvable word problem can be found at `https://en.wikipedia.org/wiki/Word_problem_for_groups#Examples`; it has 10 generators and about 30 relators. Which invites the question:

Question. If the presentations of groups with unsolvable word problem are so complicated, why can't we just avoid them?

First, the unsolvability of the word problem implies the unsolvability of other problems through the following lemma (part of the Adian–Rabin theorem):

Lemma 4.1 (1-Embedding Lemma). *Given $G = \langle g_1, g_2, \dots, g_n \mid r_1, \dots, r_s \rangle$, $w \in \mathscr{F}(g_1, \dots, g_n)$, we can add generators and relations to G to get G_w, s.t. $G_w \cong \{1\}$ if and only if $w =_G 1$.*

This lemma implies that the triviality problem (given a group G, decide whether G is the trivial group) is unsolvable. Hence, for an arbitrary manifold X, deciding whether $\pi_1(X)$ is trivial is unsolvable. More generally, calculating $\pi_1(X)$ in any effective sense is unsolvable.

Moreover, this applies to manifolds, not just complexes.

Lemma 4.2. *Given a group presentation, there is a 4-dimensional closed manifold M_G such that $\pi_1(M_G) = G$.*

One can consturct such a manifold by embedding the presentation complex X_G of G into $\mathbb{R}^5$ and finding a neighborhood U of X_G that defomation retracts to X_G. Then $\pi_1(\partial U) \cong G$ and we can choose U such that ∂U is a manifold. Thus, classifying 4-dimensional manifolds in an effective sense (i.e., in a way that you can recognize whether a manifold is simply-connected) is impossible.

In higher dimensions, we can make a stronger statement:

Theorem 4.3 (Novikov). *The homeomorphism problem for n–manifolds is unsolvable if $n \geq 5$.*

This theorem implies that determining whether an n-complex is a manifold is unsolvable for $n \geq 6$. And if we can't recognize $\mathbb{S}^6$, then there must be Riemannian manifolds diffeomorphic to $\mathbb{S}^6$ but the diffeomorphism is uncomputably complicated.

Theorem 4.4 (Nabutovsky-Weinberger). *Let*

$$\mathscr{R}(\mathbb{S}^6) = \{\textit{Riemannian metrics on } \mathbb{S}^6 \textit{ with } |K| \leq 1\}.$$

Consider $\operatorname{diam} : \mathscr{R}(\mathbb{S}^6) \to \mathbb{R}$. *Then the function* diam *has infinitely many local minima. In fact, for any computable function F, there are infinitely many local minima M of depth $\geq F(\operatorname{diam}(M))$.*

4.1. **Filling problems in higher dimensions.** We return now to filling problems. The basic question in higher dimensions is the following:

Question. Given an n-dimensional surface in X, what is the smallest $(n+1)$-volume needed to fill it? That is, we want to find $FV^{n+1}(\alpha) \equiv \inf_{\partial\beta=\alpha} \operatorname{measure}(\beta)$.

Let's formulize this question in simplicial topology first. One can see Hatcher's *Algebraic Topology* as a reference.

Let X be a simplicial complex.

$$F^n(X) = \{n\text{-dimensional simplices}\} = \{\langle v_0, \dots, v_n \rangle \subseteq X\}.$$

We fix a total order on the vertex set of X and we write simplices with vertices in ascending order so that there's a canonical way to write any simplex. Let

$$C_n(X) = \{\text{formal sums of } n\text{- simplices}\} = \Big\{\sum_{i=1}^{k} a_i\delta_i \mid a_i \in \mathbb{Z},\ \delta_i \in F^n(X)\Big\}.$$

And we define $\partial : C_n(X) \to C_{n-1}(X)$ to be the linear map such that

$$\partial(\langle v_0, \dots, v_n\rangle) = \sum_{i=0}^{n} (-1)^i \langle v_0, \dots, \widehat{v}_i, \dots, v_n\rangle.$$

We also set $B_n(X) \equiv \partial C_{n+1}(X) = n$-boundaries and $Z_n(X) \equiv \{T \in C_n(X) \mid \partial T = 0\} = n$-cycles $\supseteq$ sums of oriented simplicial n-surfaces. A direct calculation shows that $\partial^2(\langle v_0, \dots, v_n\rangle) = 0$. Hence, $B_n \subseteq Z_n$. But generally, $B_n \neq Z_n$ and $H_n(X) \equiv Z_n(X)/B_n(X)$ is not a trivial group. For example, a two dimensional torus has the H_1 equaling to $\mathbb{Z}^2$.

Now, let X be a simplicial complex. We define

$$\text{mass}(\sum_{i=1}^{k} a_i\delta_i) \equiv \sum_{i=1}^{k} |a_i|.$$

And for an $\alpha \in B_n(X)$, we define $FV^{n+1}(\alpha) \equiv \inf_{\partial\beta=\alpha} \text{mass}(\beta)$ and $FV^{n+1}(V) \equiv \sup_{\alpha\in B_n(X), \text{mass}(\alpha)\le V} FV^{n+1}(\alpha)$. We are interested in how to calculate $FV^{k+1}_{\mathbb{R}^n}$ for $k < n$.

First, we expect that $FV^{k+1}_{\mathbb{R}^n}(V) \sim V^{(k+1)/k}$. The reason is direct. Say, in $\mathbb{R}^n$, we have a 2-dimensional surface α with $\text{vol}(\alpha) = V$. We rescale α by $V^{-1/2}$ and get $\hat{\alpha}$ with volume equaling to 1. We fill $\hat{\alpha}$ with a 3-chain $\hat{\beta}$ with volume $\le FV^3(1)$. Then we scale β back by $V^{1/2}$ to get β such that $\partial\beta = \alpha$ and $\text{mass}(\beta) \le FV^3(1) \cdot V^{3/2}$. Therefore, $FV^3_{\mathbb{R}^n}(V) \le FV^3_{\mathbb{R}^n}(1) \cdot V^{3/2}$. But the problem here is that we do not know whether $FV^3_{\mathbb{R}^n}(1)$ is finite. Hence, we need more tools.

4.2. **Singular Lipschitz Homology.** We first define the set of singular Lipschitz chains as the following:

$$C_n^{\text{Lip}} \equiv \Big\{\sum_{i=1}^{k} a_i[\sigma_i] \ \Big|\ a_i \in \mathbb{Z}, \sigma_i : \Delta^n \to X \text{ is Lipschitz}\Big\}.$$

(We use square brackets to distinguish the map σ_i from the chain $[\sigma_i]$.) Here, $\Delta^n = \langle e_0, \dots, e_n\rangle$, and we define the boundary map as

$$\partial[\sigma] = \sum_{i=0}^{n} (-1)^i [\sigma|_{\langle e_0, \dots, \widehat{e}_i, \dots, e_n\rangle}].$$

If X is a simplicial complex and $\delta \in F^n(X)$, then there is a canonical map $\sigma_\delta : \Delta^n \to \delta$. This will induce an inclusion map $C_n(X) \hookrightarrow C_n^{\text{Lip}}(X)$. For any Lipschitz map $f : X \to Y$, there is a push-forward map $f_\# : C_n^{\text{Lip}}(X) \to C_n^{\text{Lip}}(Y)$, $f_\#(\sum_{i=1}^{k} a_i[\sigma_i]) = \sum_{i=1}^{k} a_i[f \circ \sigma_i]$. A standard theorem in topology guarantees the following:

Theorem 4.5. *Let $H_n^{\text{Lip}}(X) \equiv Z_n^{\text{Lip}}(X)/B_n^{\text{Lip}}(X)$, then $H_n^{\text{Lip}}(X) \cong H_n(X)$.*

We then define the mass on singular Lipschitz chains by $\mathrm{mass}(\sum_{i=1}^k a_i[\sigma_i]) \equiv \sum_{i=1}^k |a_i|\mathrm{vol}(\sigma_i)$. As before, we define

$$FV^{n+1}_{X,\mathrm{Lip}}(\alpha) \equiv \inf_{\partial\beta=\alpha} \mathrm{mass}(\beta)$$

and

$$FV^{n+1}_{X,\mathrm{Lip}}(V) \equiv \sup_{\alpha\in B_n^{Lip}(X),\ \mathrm{mass}(\alpha)\le V} FV^{n+1}_{X,\mathrm{Lip}}(\alpha).$$

Then the following approximation theorem holds:

Theorem 4.6 (Deformation Theorem, Federer-Fleming)**.** *Let X be a simplicial complex with standard metric (each simplex is isometric to unit simplex) or with a metric which is bi-Lipschitz equivalent to the standard metric. Then, for any $n > 0$, there is a $C > 0$, such that for any $A \in C_n^{\mathrm{Lip}}(X)$, there are $P(A) \in C_n(X)$, $Q(A) \in C_{n+1}^{\mathrm{Lip}}(X)$, and $R(A) \in C_n^{\mathrm{Lip}}(X)$, such that $A = P(A) + \partial Q(A) + R(A)$ and*

$$mass(P(A)) \le C \cdot mass(A),$$

$$mass(Q(A)) \le C \cdot mass(A),$$

$$mass(R(A)) \le C \cdot mass(\partial A).$$

Further, if $\partial A = 0$, then $R(A) = 0$ and $A = P(A) + \partial Q(A)$.

Here, $P(A)$ is a simplicial chain approximating A, $R(A)$ connects ∂A to the simplicial chain $\partial P(A)$, i.e.,

$$\partial R(A) = \partial A - \partial P(A) - \partial^2 Q(A) = \partial A - \partial P(A),$$

and $Q(A)$ is like a homotopy from A to $P(A)$. In particular, if A is a cycle (i.e., $\partial A = 0$), then $\partial P(A) = \partial A - \partial^2 Q(A) = 0$, so $P(A)$ is a cycle too.

5. LECTURE 5: 2022-02-22: ISOPERIMETRIC INEQUALITY IN EUCLIDEAN SPACE (NOTES BY DAN SIMON)

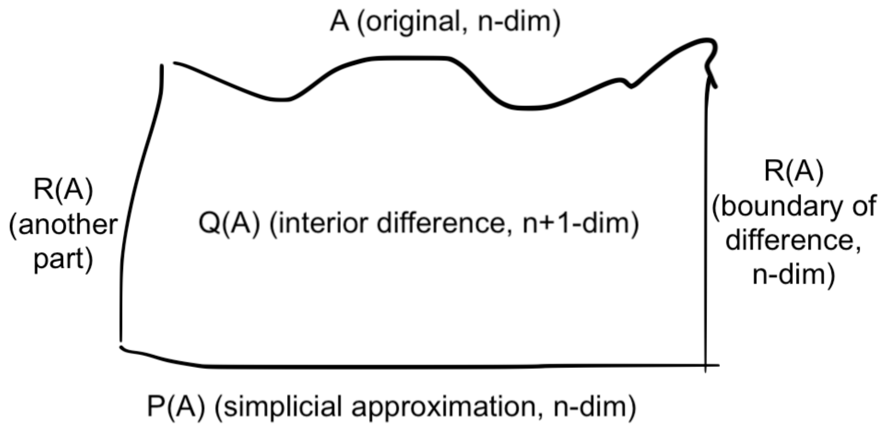
5.1. **The Federer–Fleming theorem.** Last time, we talked about simplicial chains and singular chains. It's a classic theorem in topology that these give you the same homology (we get homology by modding cycles (the kernel of the boundary map) by boundaries (the image of the one-dimension-higher boundary map)). We can give a quantitative version of this. Let C_n^{Lip} be the space of Lipschitz n–chains and let C_n^{Δ} be the space of simplicial n–chains.

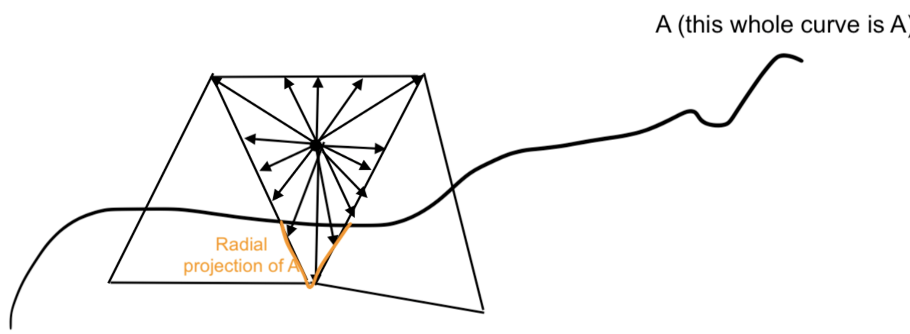
Theorem 5.1 (Federer-Fleming)**.** *Let X be a finite-dimensional simplicial complex, or bi-Lipschitz to a simplicial complex. There exists $c > 0$ (depending on the dimension of X, or on the bi-Lipschitz constant if we're in the "bi-Lipschitz to a simplicial complex" case) such that for all $A \in C_n^{\mathrm{Lip}}(X)$, there exist $P(A) \in C_n^{\Delta}(X)$, $Q(A) \in C_{n+1}^{\mathrm{Lip}}(X)$, and $R(A) \in C_n^{\mathrm{Lip}}(X)$ such that:*

(1) $A = P(A) + \partial Q(A) + R(A)$,
(2) $mass(P(A)) \le c\,mass(A)$,
(3) $mass(Q(A)) \le c\,mass(A)$,
(4) $mass(R(A)) \le c\,mass(\partial A)$.

If $\partial A \in C^{\Delta}_{n-1}(X)$*, then we can choose* $R(A) = 0$.

Note: Only $P(A)$ is simplicial here; $Q(A)$ and $R(A)$ generally aren't.
Here is a picture with a one-dimensional A:

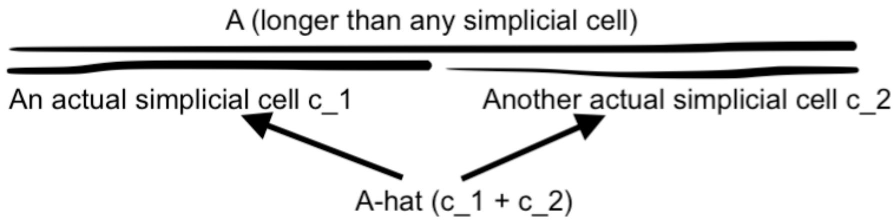




Proof. Suppose $\partial A \in C^{\Delta}_{n-1}(X)$ (the general case uses similar tools). Proceed by induction. Base case: $\text{supp}(A) \in X^{(n)}$. To handle the base case, we want to show:

Lemma 5.2. *If* $A \in C^{\text{Lip}}_n(X^{(n)})$ *(that is,* $A \in C^{\text{Lip}}_n(X)$ *and supp*$(A) \subset X^{(n)}$*) and* $\partial A \in C^{\Delta}_{n-1}(X)$*, then there exists* $B \in C^{\text{Lip}}_{n+1}(X^{(n)})$ *such that* $\partial B = A - \hat{A}$ *where* $\hat{A} = C^{\Delta}_n(X)$.

In general A and $\hat{A}$ may not be the same – since A is a Lipschitz chain, it can have cells that are bigger or smaller than the simplices of X.



In the above picture (which should be thought of as part of a line), A is a line segment, but in the cell complex on X there's no single line segment A; it's divided into two parts. So $\hat{A}$ needs to be the combination of those parts, and we can find $B \in C^{\mathrm{Lip}}_{n+1}(X^{(n)})$ with $\partial B = A - \hat{A}$ and with $\mathrm{mass}(B) = 0$ (since B is an $(n+1)$–chain in an n–dimensional complex).

In fact, $\hat{A}$ can be written in terms of degree:

$$\hat{A} = \sum_{\delta \in F^n(X)} \deg_\delta(A) \cdot \delta,$$

where $\deg_\delta(\sum a_i \sigma i) = \sum a_i \deg_{x_\delta} \sigma_i$ for generic x_δ. (Since ∂A lies in $X^{(n-1)}$, the degree is independent of the choice of x_δ.) As a consequence of this formula, $\mathrm{mass}(\hat{A}) \le \mathrm{mass}(\hat{A})$. (Note that backtracking along reused parts of simplices, such as curves, can occur in A but not in $\hat{A}$, so this is not guaranteed to be an equality.)

We omit the proof of this lemma for lack of time.

Having dealt with the base case, we will move on to the inductive case. Suppose $\mathrm{supp}(A) \subset X^{(k)}$ where $k > \dim(A)$. For every cell complex simplex $\delta \in F^k(X)$, choose a point $x_\delta \in \int(\delta)$. Define $\rho(\delta) : \delta \setminus \{x_\delta\} \to \partial\delta$ by radial projection. This is continuous on $\delta \setminus \{x_\delta\}$ and fixes $\partial\delta$ pointwise. Define $R : X^{(k)} \setminus \{x_\delta\}_\delta \to X^{(k-1)}$ so that R fixes $X^{(k-1)}$ and $R|_\delta = \rho_\delta$ for all $\delta \in F^k(X)$. If $x_\delta \not\subset \mathrm{supp}(A)$ for all δ then we let $A_{k-1} = R_k(A)$. Further, R is homotopic to the identity, say by $L_n : X^{(k)} \setminus \{x_\delta\}_\delta \times I \to X^{(k)}$.

(We need linearity to do radial projection. There are sort of three options here. One is to pull back along maps from simplices in Euclidean space, do radial projection there, and push forward. Another is to embed into high-dimensional Euclidean space where all cells are linear, and do it there. A third is to embed our simplicial complex in the infinite-dimensional simplex, and do it there.)

Let $Q_{k-1} = h_\#(A \times I) \in C^{\mathrm{Lip}}_{n+1}(X^{(k)})$. $\partial Q_{k-1} = A - A_{k-1}$. Let $d = \dim X$ and let A_d be original A. Radially project to get a sequence $A = A_d$, $A = A_{d-1}, \ldots, A_n$ and a sequence Q_k such that $\partial Q_k = A_k - A_{k-1}$. Then by the lemma, there is a simplicial $\hat{A}$ and a $B \in C^{\mathrm{Lip}}_{n+1}(X^{(n)})$ such that $\partial(B) = A_n - \hat{A}$. Let $P(A) = \hat{A}$ and let $Q(A) = Q_d + Q_{d-1} + \ldots + Q_{n+1} + B$. Then $P(A)$ is simplicial and

$$\partial Q(A) = (A - A_{d-1}) + (A_{d-1} - A_{d-2}) + \ldots + (A_{n+1} - A_n) + (A_n - \hat{A}) = A - P(A).$$

This handles the qualitative issues, but what about the quantitative issues? The main quantitative issue is the part of each simplex with large Lipschitz constant for the projection R, which is the part near each x_δ. If we choose x_δ randomly from a region in $\operatorname{int}(\delta)$, then the expected mass of $r_\delta(A)$ is $\mathbb{E}[\operatorname{mass} r_\delta(A)] \leq c\operatorname{mass}(A)$. Then there is some x_δ with $\operatorname{mass} r_\delta(A) \leq c\operatorname{mass}(A)$. Choose one of these in each simplex. Then $\operatorname{mass}(A_{k-1}) \leq c\operatorname{mass}(A_k)$. So $\operatorname{mass}(P(A)) \leq c^d \operatorname{mass}(A)$. (There are at most d steps involved.) This completes the proof.

(Note: the existence of such a c uses the bi-Lipschitz equivalence of our metric to the standard isometric simplicial metric. We can do the same thing for simplices that aren't equilateral, but the constant depends on the shape of the simplex, so if the simplices degenerate, the constant can blow up.) $\square$

5.2. **The isoperimetric inequality for Euclidean space.** How can we use this? First, we can bound the filling volume function for $\mathbb{R}^n$ and show that

$$\mathrm{FV}^{k+1}_{\mathbb{R}^n}(C) \lessapprox V^{(k+1)/k}\mathrm{FV}^{k+1}_{\mathbb{R}^n}(1).$$

Give $\mathbb{R}^n$ the structure of a subdivision of the unit grid, so that each simplex is bi-Lipschitz equivalent to the standard simplex. Let

$$m = \min_{\delta\in F^k(\mathbb{R}^n)} \operatorname{vol}(\delta).$$

(Minimal cell volume.) Let c be as in Federer–Fleming. Let $A \in C^{\mathrm{Lip}}_k(\mathbb{R}^n)$, $\partial A = 0$. Let $\operatorname{mass}(A) = V$.

We rescale A to have small volume. Define $s: \mathbb{R}^n \to \mathbb{R}^n$, $s(x) = \left((m/2cV)^{1/k}\right)x$. Then

$$\operatorname{mass} s(A) = \frac{m}{2cV}\operatorname{mass}(A) \leq \frac{m}{2c}$$

Let $\hat{A} = s(A)$. Then $\operatorname{mass} P(\hat{A}) \leq \frac{m}{2}$. But $P(\hat{A})$ is a sum of simplices, and each simplex has volume at least m, so $P(\hat{A}) = 0$. Since $\partial\hat{A} = 0$, Federer–Fleming implies that

$$\hat{A} = P(\hat{A}) + \partial Q(\hat{A}) + R(\hat{A}) = \partial Q(\hat{A})$$

and $\operatorname{mass} Q(\hat{A}) \leq c\operatorname{mass}(\hat{A}) \leq \frac{m}{2}$.

Scaling back, we have $\partial(s^{-1}(Q(\hat{A}))) = A$, and

$$\operatorname{mass}(s^{-1}(Q(\hat{A}))) \leq (2cV/m)^{(k+1)/k}\left(\frac{m}{2}\right) \lessapprox V^{(k+1)/k}.$$

This is the bound we wanted on the mass of A.

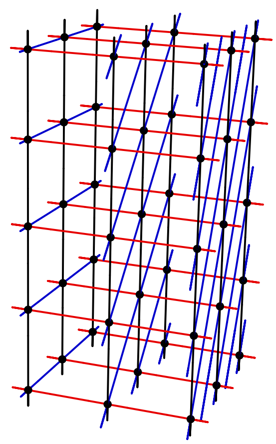
To summarize the idea of this proof, we rescale to make everything smaller so that when we apply Federer-Fleming, everything will be smaller than the cell size. This means that our rescaled A is entirely the Q term, which increases by roughly a factor of $V^{(k+1)/k}$ when we scale it back.

5.3. **The Heisenberg group.** We define the Heisenberg group as

$$\mathbb{H} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

Recall that the commutator of a and b is defined as $[a,b] = aba^{-1}b^{-1}$. Let $\mathbb{H}_{\mathbb{Z}}$ be the subgroup of $\mathbb{H}$ with entries in $\mathbb{Z}$ rather than in $\mathbb{R}$. It can also be written as $\langle X,Y,Z | [X,Y] = Z, [X,Z] = 1, [Y,Z] = 1\rangle$. By matrix calculations, $[\mathbb{H},\mathbb{H}]$ is matrices of the form $\begin{bmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $[[\mathbb{H},\mathbb{H}],\mathbb{H}] = 1$.

We can identify $\mathbb{H}$ with $\mathbb{R}^3$ so that (x, y, z) is identified with $\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$. The elements of $\mathbb{H}_{\mathbb{Z}}$ are the lattice points and the Cayley graph of $\mathbb{H}$ looks like this:



The way that the lines tilt as you go from left to right reflects how the multiplication works; if you multiply (x, y, z) by X (on the right), the x coordinate increases by 1. If you multiply by Z, the z coordinate increases by 1. If you multiply by Y, the y coordinate increases by 1, but the z coordinate also increases by x. Note that this is noncommutative: $[X,Y] = Z$.

How can we write elements of this group? We can swap Z and Z^{-1} with other stuff to put them at the end of the word. We can swap powers of X with powers of Y, but $[X,Y] = Z$, so this creates powers of Z, which we can push to the end. So we can write any element in the form $X^iY^jZ^k$ for integer i, j, k. It's not that hard to see that these i, j, k are unique for a given element of the Heisenberg groups.

So how does the Dehn function of the Heisenberg group behave?

On one hand, there are curves in $\mathbb{H}$ that are hard to fill. For instance, we can see that $X^nY^nX^{-n}Y^{-n} = Z^{n^2}$. So an isomorphism of this group will have to scale different axes differently, in a sense. Also, since Z commutes with anything, $[X^n, Z^{n^2}] = 1$, so the length-$10n$ word $[X^n,[X^n,Y^n]]$ is equal to 1. But reducing it to 1 in a naive way requires resolving things like $[X^n, Z^{n^2}] = 1$, which requires pushing n things past n^2 things and so takes n^3 steps. Is this within a constant factor of being optimal? (Yes; we'll see a quick argument today.)

On the other hand, $\mathbb{H}_k$ is the group of matrixes with ones on the diagonal, and all other entries 0 except for the topmost row and rightmost column. It can be written $H = \langle X_1, \dots, X_k, Y, \dots, Y_k, Z | [X_i, Y_i] = Z$, all other pairs commute$\rangle$. The only pairs that don't commute are X_i and Y_i, where the subscripts match. We have $[X_i^n, Y_i^n] = Z^{n^2}$ and $[X_i^n, [X_i^n, Y_i^n]] = 1$. But when $k \geq 2$, there's a reduction with roughly n^2 steps.

So why do these differ? Today, we'll see a quick argument for why the Dehn function of $\mathbb{H}$ is cubic; next time, we'll see why the Dehn function of $\mathbb{H}_n$ is quadratic when $n \geq 2$.

First, why does it take n^3 steps to reduce $[X^n, [X^n, Y^n]]$ in $\mathbb{H}$? The multiplication formula for $\mathbb{H}$ is

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy').$$

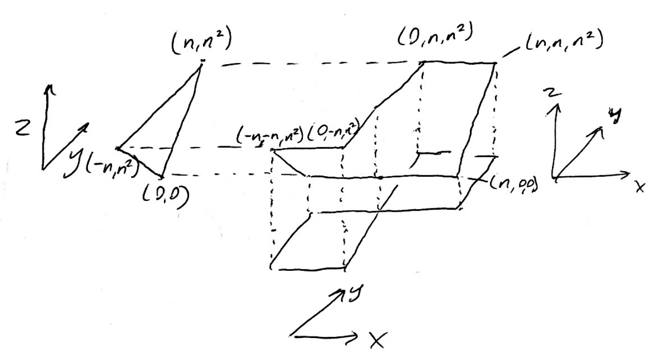
There are three left-invariant vector fields on $\mathbb{H}$, as follows: $X_{(x,y,z)} = (1, 0, 0)$, $Y_{(x,y,z)} = (0, 1, x)$, $Z_{(x,y,z)} = (0, 0, 1)$ (corresponding to the different colors of edges in the figure). We define a Riemannian metric by $dg^2 = dx^2 + dy^2 + (dz - x dy)^2$ so that these fields are orthogonal.

Consider $w_n = X^n Y^n X^{-n} Y^{-2n} X^{-n} Y^n X^n$ as a path in the Cayley graph. This is a non-intersecting closed curve in $\mathbb{H}$. It has filling area n^3. Indeed, if we project it into the yz-plane, it looks like a triangle with area n^3. But the map that projects surfaces into the yz-plane is, crucially, area-decreasing (because, with respect to the orthogonal basis X, Y, Z, the projection is an orthogonal projection followed by a map with determinant 1). So any filling has area at least n^3 in the projection and so actually has area at least n^3.

Note that we only get this area-decreasing property when projecting to the YZ-plane, not to the XY-plane. So we'd like to find some more general way to do this, which we'll discuss next time.

6. Lecture 6: 2022-03-01: Isoperimetric inequalities in the Heisenberg groups (notes by Hari Nathan)

Last time, we discussed the Heisenberg group $\mathbb{H}$ with multiplication $(x, y, z)(x', y', z') = (x + x', y + y', z + z')$ and the Dehn function of this group. We saw that the word $\gamma_n = X^n Y^n X^{-n} Y^{-2n} X^{-n} Y^n X^n$ makes a loop in the Heisenberg group:



This loop projects to a figure-8 in the xy–plane and a triangle in the yz–plane. Since the projection $\pi(x, y, z) = (y, z)$ is area decreasing and $\pi(\gamma_n)$ is a triangle with area n^3, any filling of the original loop γ_n has area at least n^3. Today, we will look at this using differential forms.

6.1. **Differential forms.** Let $\Omega^k(\mathbb{R}^n)$ the the set of k–forms, i.e., alternating multilinear functions that take k tangent vectors at a point to a real number. Let ω be the area form in the yz–plane:

$$\omega = dy \wedge dz \in \Omega^2(\mathbb{R}^2)$$

$$\omega((y_1, z_1), (y_2, z_2)) = y_1 z_2 - y_2 z_1 = \det \begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix}$$

Note that ω is closed (i.e., $d\omega = 0$), since $d\omega$ is a 3–form, and $\Omega^3(\mathbb{R}^2) = 0$.
Recall

Theorem 6.1 (Stokes)**.** *If $M \subset \mathbb{R}^n$ is an oriented manifold with boundary and $\omega \in \Omega^{k-1}(\mathbb{R}^n)$ then:*

$$\int_M dw = \int_{\partial M} w$$

(This generalizes the usual fundamental theorem of calculus: when M is a curve from p to q (and so $dM = q - p$) and $f \in \Omega^0(\mathbb{R}^n)$ is a real valued function of a point), we have

$$\int_M df = \int_M \nabla f \cdot dx = f(q) - f(p) = \int_{\partial M} f = f(q) - f(p)$$

Likewise, the curl theorem; if $f \colon D^2 \to M$ parametrizes M, then

$$\int_M \operatorname{curl}(V) \cdot dA = \int_{D^2} \operatorname{curl}(V) \cdot \frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v} \, du \, dv = \int_{\partial M} V \cdot dx.$$

Here, the 2–form is $(X, Y) \mapsto \operatorname{curl}(V) \cdot X \times Y$. The divergence theorem is left as an exercise.)

Stokes' theorem generalizes to Lipschitz singular chains: if $A = \sum_i a_i \sigma_i$ ($a_i \in \mathbb{R}, \sigma_i : \Delta^2 \to \mathbb{R}^2$) is a Lipschitz 2–chain we define:

$$\int_A \omega = \sum_i \left(a_i \int_{\sigma_i} \omega \right) = \sum_i \int_{\Delta^2} \omega \left(\frac{\partial \sigma_i}{\partial s}, \frac{\partial \sigma_i}{\partial t} \right) ds \, dt$$

Then, by Stokes' theorem, for any $\omega \in \Omega^k(\mathbb{R}^n)$ and $B \in C_{k+1}^{\text{Lip}}(\mathbb{R}^n)$:

$$\int_{\partial B} \omega = \int_B d\omega.$$

And we can use this to define the signed area of $A \in C_2^{\text{Lip}}(\mathbb{R}^2)$ as

$$\operatorname{sarea}(A) = \int_A dy \wedge dz.$$

If $\partial A = \partial B$ then $\partial(A - B) = 0$. Since $H_*(\mathbb{R}^2) = 0$, $\exists C$ s.t. $\partial C = A - B$. By Stokes' Theorem:

$$\int_C d(dy \wedge dz) = \int_{A-B} dy \wedge dz \Rightarrow \int_C d^2 y \wedge dz - dy \wedge d^2 z = \int_A dy \wedge dz - \int_B dy \wedge dz$$

$$\Rightarrow \int_C 0 = \text{sarea}(A) - \text{sarea}(B) \Rightarrow \text{sarea}(A) = \text{sarea}(B)$$

So if $\partial A = \partial B$, then sarea(A) = sarea(B). That is. the signed area depends only on the border, not the filling. More generally, if $\omega \in \Omega^K(\mathbb{R}^n)$ is closed (i.e. $dw = 0$) and if $A, B \in C_k^{Lip}(\mathbb{R}^n)$ s.t. $\partial A = \partial B$ then:

$$\int_A \omega = \int_B \omega$$

Now we apply this to the Heisenberg group.

6.2. **The Heisenberg Group.** Recall that we define three left invariant vector fields:

$$X_{(x,y,z)} = (1,0,0) \qquad Y_{(x,y,z)} = (0,1,x) \qquad Z_{(x,y,z)} = (0,0,1)$$

and give $\mathbb{H}$ the Riemmanian metric such that these are orthonormal. Then, for any $U, V \in \{X, Y, Z\}$ and any vectors S, T,

$$|\pi^*(\omega)(U,V)| \le 1 \Rightarrow \exists c > 0\, s.t. |\pi^*(\omega)(S,T)| \le c \cdot \|S \wedge \|_g$$

$$\Rightarrow \left| \int_A \pi^*(\omega) \right| \le c \cdot mass(A)$$

i.e. $\pi_*(\omega)$ is bounded. Let $\gamma_n = X^n Y^n X^{-n} Y^{-2n} X^{-n} Y^n X^n$ be the curve from the beginning of the section. If $A \in C_2^{Lip}(\mathbb{H})$ s.t. $\partial A = \gamma_n$ then $\partial \pi_\#(A) = \pi \circ \gamma_n$. So:

$$\int_A \pi^*(\omega) = \int_{\pi_\#(A)} \omega = \text{sarea}(\pi_\#(A)) = n^3 = \text{area of } \pi \circ \gamma_n$$

Since $\pi^*(\omega)$ is bounded,

$$\text{mass}(A) \ge c^{-1} \int_A \pi^*(\omega) \ge c^{-1} n^3.$$

6.3. **Homological lower bounds on filling area.** This gives us a general approach to finding lower bounds on filling volume:

(1) find a bounded closed form $\mu \in \Omega^k(X)$
(2) find a $(k-1)$–cycle M and an A s.t. $\partial A = M$
(3) if $\int_A \mu$ is large, then for any B s.t. $\partial B = M$ we have (1) $\int_A \mu = \int_B \mu$; and (2) mass$(B) \gtrsim |\int_A \mu| = |\int_B \mu|$ is large.

If the space is a group equipped with a left-invariant metric, it's convenient to take left-invariant forms. We define left-invariant 1–forms dual to X, Y, and Z by

$$\omega_X = dx \qquad \omega_Y = dy \qquad \omega_Z = dz - x\,dy.$$

(Recall that df is the 1–form corresponding to the gradient of f. dx is the gradient of the first coordinate function $x\colon \mathbb{R}^3 \to \mathbb{R}$, i.e., $dx(v)$ is the x–coordinate of v.) And one can check that:

$$\pi^*(dy \wedge dz) = \omega_Y \wedge \omega_Z = dy \wedge (dz - xdy) = dy \wedge dz - xdy \wedge dy = dy \wedge dz$$

is a left-invariant closed 2-form, so it gives a lower bound on filling area. Likewise, $\omega_X \wedge \omega_Z = dx \wedge dz - x dx \wedge dy$ is a closed bounded 2-form, so it gives a lower bound on filling area.

6.3.1. *Scaling* $\mathbb{H}$. We can see what lower bounds $\omega_X \wedge \omega_Z$ and $\omega_Y \wedge \omega_Z$ produce by looking at scalings of $\mathbb{H}$. One can check that for all $t > 0$, the map $s_t : \mathbb{H} \to \mathbb{H}$, $s_t(x, y, z) = (tx, ty, t^2 z)$ is an automorphism. In addition:

$$\begin{aligned} (s_t)_*(X) &= tX & (s_t)^*(\omega_X) &= t\omega_X \\ (s_t)_*(Y) &= tY & (s_t)^*(\omega_Y) &= t\omega_Y \\ (s_t)_*(Z) &= tZ & (s_t)^*(\omega_Z) &= t\omega_Z \end{aligned}$$

So $\omega_Y \wedge \omega_Z$ and $\omega_X \wedge \omega_Z$ both grow cubically, i.e.:

$$(s_t)^*(\omega_Y \wedge \omega_Z) = t\omega_Y \wedge t^2\omega_Z = t^3 \omega_Y \wedge \omega_Z$$

Thus, if $A \in C_2^{Lip}(\mathbb{H})$:

$$\int_{(s_t)_\#} \omega_Y \wedge \omega_Z = \int_A (s_t)^*(\omega_Y \wedge \omega_Z) = t^3 \int_A \omega_Y \wedge \omega_Z.$$

So if $\int_A \omega_Y \wedge \omega_Z \neq 0$, then ∂A has cubic filling area. Thus the Dehn function of the Heisenberg group grows cubically.

6.4. **5-dimensional Heisenberg group ($\mathbb{H}_2$).** Similar to $\mathbb{H}$ (which we write as $\mathbb{H}_1$ from here on in to distinguish it from $\mathbb{H}_2$), we can construct $\mathbb{H}_2$ via matrices like:

$$\begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so:

$$(x_1, x_2, y_1, y_2, z) \cdot (x_1', x_2', y_1', y_2', z') = (x_1 + x_1', \cdots, y_2 + y_2', z + z' + x_1 y_1' + x_2 y_2').$$

As above we define fields and 1–forms:

$$\begin{aligned} X_1 &= (1,0,0,0,0) & \omega_{X_1} &= dx_1 \\ X_2 &= (0,1,0,0,0) & \omega_{X_2} &= dx_2 \\ Y_1 &= (0,0,1,0,X_1) & \omega_{Y_1} &= dy_1 \\ Y_2 &= (0,0,0,1,X_2) & \omega_{Y_2} &= dy_2 \\ Z &= (0,0,0,0,Z) & \omega_Z &= dz - x_1 dy_1 - x_2 dy_2 \end{aligned}$$

Here, $\omega_{Y_1} \wedge \omega_Z$ is still bounded and has cubic growth, but unlike with $\mathbb{H}_1$:

$$\begin{aligned} d(\omega_{Y_1} \wedge \omega_Z) &= d\omega_{Y_1} \wedge \omega Z - \omega_{Y_1} \wedge d\omega Z \\ &= 0 - dy_1 \wedge (-dx_1 \wedge dy_1 - dx_2 \wedge dy_2) \\ &= dy_1 \wedge dx_1 \wedge dy_1 + dx_1 \wedge dx_2 \wedge dy_2 \\ &= dy_1 \wedge dx_2 \wedge dy_2 \neq 0. \end{aligned}$$

So the argument we used before won't work.

So the question is, why does $\mathbb{H}_2$ have quadratic Dehn function? The reason is that the scaling limit has more surfaces. That is, let $d(v,w)$ be distance in $\mathbb{H}_2$ and for $r>0$ let:

$$d_r(v,w)=\frac{1}{r}\cdot d(s_r(v),s_r(w)).$$

As $r\to\infty$, this converges to a metric d_∞; $\mathbb{H}$ equipped with this metric is called the *scaling limit* of $\mathbb{H}$.

We can write d_r in terms of a Riemannian metric g_r. Recall that $dg^2=\omega_{X_1}^2+\cdots+\omega_{Y_2}^2+\omega_Z^2$. So, d_r corresponds to:

$$\begin{aligned} dg_r=s_r^*(dg^2)\cdot\frac{1}{r^2}&=\frac{1}{r^2}(r^2\cdot\omega_{X_1}^2+\cdots+r^2\cdot\omega_{Y_2}^2+r^4\cdot\omega_Z^2)\\ &=\omega_{X_1}^2+\cdots+\omega_{Y_2}^2+r^2\omega_Z^2. \end{aligned}$$

This is the Riemannian metric where the X's and Y's and Z are orthogonal and the X's and Y's have norm 1 but Z has norm r. In the limit, vectors in $ker(\omega_z)=\langle X_1,X_2,Y_1,Y_2\rangle$ have finite length and the other vectors have infinite length. This is a sub-riemannian metric.

If $\gamma[0,1]\to\mathbb{H}_2$ and $\gamma'(t)\in ker(\omega_z)$ we say γ is horizontal and $\ell(\gamma)=\int\|\gamma'\|=\int\|\gamma'\|_{g_r}=\ell_r(\gamma)$.

Theorem 6.2 (Chow)**.** *Any two points in $\mathbb{H}_1$ or $\mathbb{H}_2$ are connected by a horizontal curve.*

You can go from a point p to a point with the same first four coordinates as q by using lines parallel to $X_1,\dots,Y_2$. Then follow a commutator $[X_1,Y_1]$ to move up or down. In fact, if we let:

$$d_\infty=\lim_{r\to\infty}d_r,$$

then

$$d_\infty(0,(x_1,\dots,y_2,z))\approx|x_1|\cdots|y_2|+\sqrt{|z|}.$$

So $(\mathbb{H}_2,d_\infty)$ has Hausdorff dimension 6 (one for each of X_1, ..., Y_2 and two for Z).

The length of a horizontal curves scales linearly under s_r: $\ell(s_r\circ\gamma)=r\ell(\gamma)$, and we say that a surface $\sigma:\Delta^2\to\mathbb{H}_2$ is horizontal if $\sigma'(T\Delta^2)\subset ker(\omega_Z)$. Then the big difference between $\mathbb{H}_1$ and $\mathbb{H}_2$ is that $\mathbb{H}_1$ has no nondegenerate horizontal surfaces:

Proposition 6.3. *Horizontal surfaces in $\mathbb{H}$ are degenerate, i.e. σ' is never injective,*

but $\mathbb{H}_2$ has plenty, as we'll see in the next section.

6.5. **Horizontal Subgroups.** $\mathbb{H}_2$ has a large number of horizontal subgroups. To see this, we note that the Lie Algebra of $\mathbb{H}_2$ is:

$$\begin{aligned}\mathfrak{h}_2=\langle X_1,X_2,Y_1,Y_2,Z:&[X_1,Y_1]_L=[X_2,Y_2]_L=Z,\\ &[X_1,X_2]_L=[X_1,Y_2]_L=[Y_1,X_2]_L=[Y_1,Y_2]_L=0\rangle\end{aligned}$$

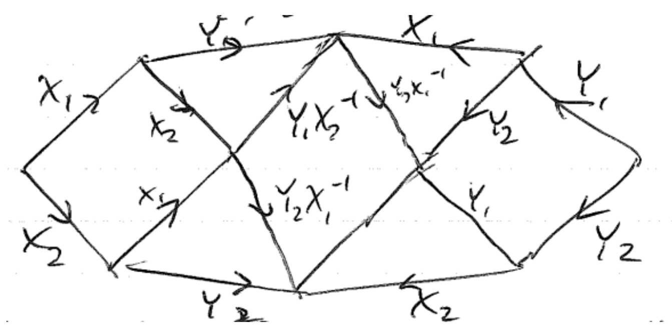
where $[\cdot,\cdot]_L$ refers to the Lie Bracket. Abelian subgroups of $\mathbb{H}_2$ correspond to abelian sub-algebras of η_2 e.g. $\langle Y_1, Y_2\rangle$ is an abelian subgroup so Y_1, Y_2 generate an abelian subgroup:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} = Y_1^a Y_2^b \right\} \cong \mathbb{R}^2$$

and this is horizontal (tangent to Y_1 and Y_2). The same is true for the other pairs of X's and Y's where $[\cdot,\cdot]_L$ is zero and even for some combinations:

$$[Y_1 - X_2, Y_2 - X_1]_L = [Y_1, Y_2]_L - [Y_1, X_2]_L - [X_2, Y_2]_L + [X_2, X_1]_L = Z - Z = 0.$$

And these fit together to fill horizontal curves. For example, the word $[X_1, Y_1][X_2, Y_2]^{-1}$ is a horizontal closed curve, and we can combine these subgroups to fill it as follows:



So the curve corresponding to $[X_1, Y_1][X_2, Y_2]^{-1}$ bounds a horizontal disc of area 5. By rescaling, $[X_1^n, Y_1^n][X_2^n, Y_2^n]^{-1}$ bounds a horizontal disc of area $5n^2$.

Thus, we can fill $[X_1^n, [X_1^n, Y_1^n]]$ quadratically as follows:

$$[X_1^n, [X_1^n, Y_1^n]] \underset{\sim n^2 \text{ steps}}{\longrightarrow} [X_1^n, [X_2^n, Y_2^n]] \underset{\sim n^2 \text{ steps}}{\longrightarrow} 1$$

7. Lecture 7: 2022-03-08: Heisenberg groups and quantitative homotopy theory (notes by Julian Cortes)

7.1. **Dehn function of $\mathbb{H}_2$.** Last time we showed that there are many horizontal discs in the five dimensional Heisenberg group. We'll start today by showing how to use these to bound the Dehn function of the Heisenberg group. [1]

Recall from last time that $[X_1, Y_1][X_2, Y_2]^{-1}$ bounds a horizontal disc of area $A \le 10$. When we scale the Heisenberg group, horizontal discs scale with it, i.e. $[X_1^n, Y_1^n][X_2^n, Y_2^n]^{-1}$ bounds a horizontal disc of area $A \le 10n^2$.

So: how can we use this to fill arbitrary curves? First, we note the following presentation for $\mathbb{H}_2$:

$$\mathbb{H}_2 = \left\langle X_1, Y_1, X_2, Y_2 \,\middle|\, [X_1, Y_1][X_2, Y_2]^{-1},\ [X_1, X_2],\ [Y_1, Y_2],\ [X_1, Y_2],\ [Y_1, X_2] \right\rangle \tag{1}$$

[1] There are many other proofs of this fact - Gromov [3] sketched the first, using microflexibility; Ol'shanskii–Sapir [5] gave a combinatorial proof; and Allcock [1] gave a proof using symplectic geometry.

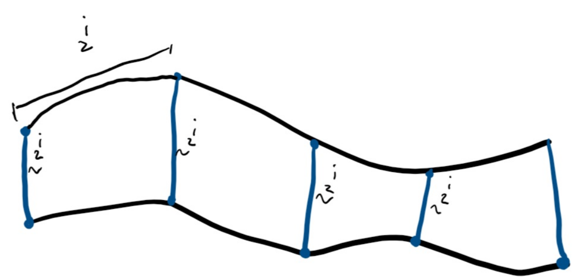
Every relation in (1) bounds a horizontal disc, so any word in X_1, Y_1, X_2, Y_2 that represents the identity bounds a horizontal disc. But note that the area of the disc depends on the length of the word – since there are only finitely many curves of length L, for any L, there is a $A > 0$ such that any word of length L bounds a horizontal disc of area at most A.

Theorem 7.1. $\delta_{\mathbb{H}^2(L)} \leq L^2$

Proof. It suffices to take $L = 2^k$, $k \geq 0$. Let $\gamma : [0, L] \to \mathbb{H}_2$ be a unit speed closed curve. We construct a sequence of horizontal approximation of γ as follows,

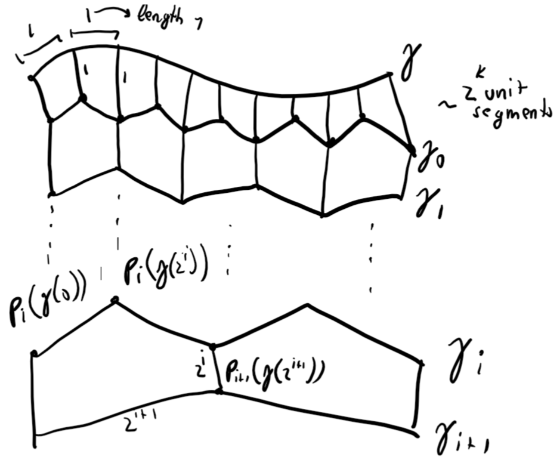
Let X be the Caley graph of $\mathbb{H}_2$ generated by X_1, X_2, Y_1, Y_2. Every edge is horizontal of length 1. On the other hand, take $i \geq 0$ and let $X_i = s_{2^i}(X)$, namely the scaling by a factor of 2^i. We still have that every edge is horizontal and of length 2^i and $X_{i+1} \subset X_i$.

Now let $p_i : \mathbb{H}_2 \to X_i^{(0)}$ be the nearest point projection and let γ_i be the edge path on X_i connecting $p_i(\gamma(0)), p_i(\gamma(2^i)) \ldots p_i(\gamma(2^{k-i} \cdot 2^i)$. This gives us a relatively nice curve.



Note that γ_i consists of 2^{k-i} edge paths of length $\sim 2^i$ each consisting of ~ 1 edges in X_i. When $i = k$, we have $p_k(\gamma(0)) = p_k(\gamma(2^k))$ so γ_k is constant.

So this is a sequence of horizontal curves, all of about the same length, that approximate γ more and more coarsely. We can construct a null-homotopy of γ by constructing a homotopy from γ to γ_0 to γ_1 and so on until we reach γ_k which is constant.



We find a homotopy from γ_i to γ_{i+1} by connecting each point $\gamma_{i+1}(j2^{i+1})$ to $\gamma_i(j2^{i+1})$ by an edge path in X_i. Each of these paths has length $\approx 2^i$, and these paths subdivide the region between γ_i and γ_{i+1} into 2^{k-i-1} closed curves. Each curve is an edge path in X_i of length $\approx 2^i$, so each curve is a scaling of a word of length ≈ 1. Therefore, each curve bounds a horizontal disc of area $\approx 2^{2i}$.

So, in all, we use:

$$\gamma_0 \to \gamma_1 : \sim 2^k \text{ discs of area} \sim 1, \text{ total} \approx 2^k \cdot 1$$

$$\gamma_1 \to \gamma_2 : \sim 2^{k-1} \text{ discs of area} \sim 2^2, \text{ total} \approx 2^{k-1} \cdot 2^2 \qquad \vdots \qquad \vdots$$

$$\gamma_{k-1} \to \gamma_k : \sim 1 \text{ disc of area} \sim 2^{2k}, \text{ total} \approx 1 \cdot 2^{2k}$$

This adds up to $\approx 2^{2k} = L^2$. □

In general if G is Carnot and there are enough horizontal discs to fill arbitrary edge paths, the previous method can be applied to G.

The converse is trickier – if the Dehn function is quadratic, what does that imply about horizontal discs?

Theorem 7.2 (Wenger [6])**.** *If G is a Carnot group and $\delta_G(L) \le L^2$, then any horizontal curve can be filled by a limit of horizontal discs.*

Conversely, if there are not enough horizontal discs in G to fill all horizontal curves then $\delta_G(L)$ is strictly greater than L^2 (i.e., there is no C such that $\delta_G(L) < CL^2$ for all $L > 0$.)

Open question: Find a better lower bound!

7.2. **Quantitative homotopy theory.** Homotopy theory studies homotopy classes of maps $X \to Y$. We can try to quantify this in a couple of ways:

- We can ask what classes of maps from X to Y can be realized by L–Lipschitz maps?

- Suppose $f, g : X \to Y$ are homotopic. How big is the homotopy from f to g? (For example, if $\mathrm{Lip}(f)$, $\mathrm{Lip}(g) \le L$, what is the best Lipschitz constant of the homotopy?)

Let's try to address the first of these questions.

Definition 7.3. Let X be a space. Let $\alpha \in \pi_n(X)$, $n \ge 2$ For $L > 0$, let

$G_\alpha(L)$ = Growth of α = largest k such that α^k can be realized by an L-Lipschitz map.

7.2.1. $\pi_n(S^n)$ *and degree.* For example, take $\pi_n(S^n) \cong \mathbb{Z} = \langle e \rangle$. Let $e : S^n \to S^n$ be a generator; we can take e to be the identity map. Recall that the degree of a map f is the number of preimages of a regular point of f counted with orientation. Degree is well defined on homotopy classes; in fact, the degree map is an isomorphism from $\pi_n(S^n)$ to $\mathbb{Z}$.

But degree can also be computed using differential forms. Let $\omega \in \Omega^n(S^n)$ be a volume form. Then for $f\colon S^n \to S^n$,

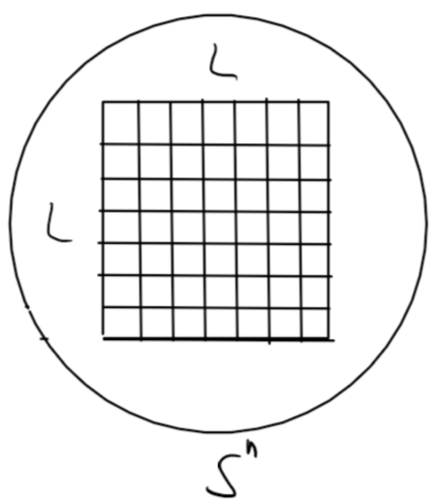
$$\int_{S^n} f^*(\omega) = \int_{S^n} \deg_x(f)\,\mathrm{d}x = \mathrm{vol}(S^n)\cdot\deg(f).$$

If f is L-Lipschitz, then $\|f^*(\omega)\|_\infty \le L^n$. This means that

$$\left|\int_{S^n} f^*(\omega)\right| \le L^n\,\mathrm{vol}(S^n),$$

so $\deg(f) \le L^n$.

This inequality is sharp – we can see this by drawing an n–dimensional cube D on the surface of an n–sphere and dividing it into an $L \times \cdots \times L$ grid.



Let $\beta : [0,1]^n \to S^n$ be a degree–1 map such that $\beta(\partial[0,1]^n) = *$. Let $f : S^n \to S^n$ send $S^n \setminus D$ to $*$ and let f restrict to β on each grid cell. Then $\mathrm{Lip}(f) \sim L$ and $\deg(f) = L^n$ so $G_e(L) \sim L^n$.

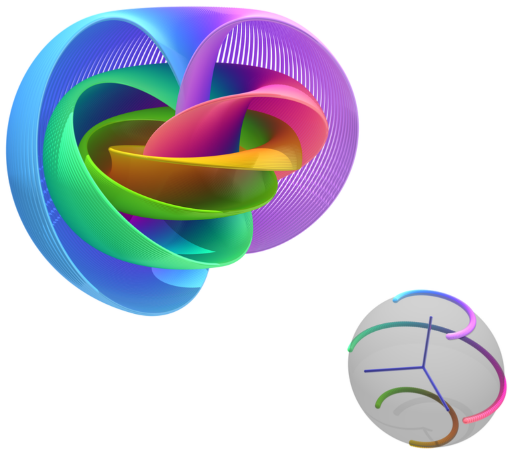


FIGURE 1. Hopf Fibration. *Source: Wikipedia*

7.2.2. *$\pi_3(S^2)$ and the Hopf fibration.* The *Hopf fibration* is a map from $S^3 \to S^2$ that generates $\pi_3(S^2)$. If we write

$$S^3 = \{(z, w) \in \mathbb{C}^2 \Big| |z|^2 + |w|^2 = 1\}$$

and

$$S^2 = \mathbb{C} \cup \{\infty\},$$

it is given by $h(z, w) = \frac{z}{w}$.

Note that: $\forall a \in \mathbb{C} \cup \{\infty\}$

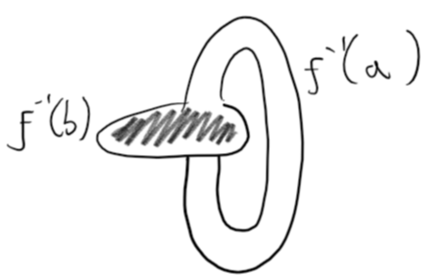
$$h^{-1}(a) = S^3 \cap \{z = wa\} \quad \text{is a circle.}$$

If S^1 is the unit circle in $\mathbb{C}$, then

$$h^{-1}(S^1) = \{|z| = |w| = \frac{1}{\sqrt{2}}.$$

The isomorphism from $\pi_3(S^2)$ to $\mathbb{Z}$ is given by the *Hopf invariant* $H : \pi_3(S^2) \to \mathbb{Z}$. Suppose $f : S^3 \to S^2$ is smooth and let $a, b \in S^2$ be regular points. We define

$$H(f) = \text{Linking \# of} f^{-1}(a) \text{ and } f^{-b}$$



That is, $f^{-1}(b)$ is a 1–cycle. Let B be a 2-chain such that $\partial B = f^{-1}(b)$. Then the linking number of $f^{-1}(a)$ and $f^{-1}(b)$ is the number of intersections of B and $f^{-1}(a)$, counted with multiplicity. Then, like the degree, H is constant on homotopy classes. One can show that H is an isomorphism from $\pi_3(S^2)$ to $\mathbb{Z}$ and that the circles $h^{-1}(a)$ and $h^{-1}(b)$ have linking number 1, so h generates $\pi_3(S^2)$.

Now we ask the same a question as before. How well can we realize the Hopf fibration and powers of the Hopf fibration as Lipschitz maps?

Whitehead gave a calculation of H via differential forms.

Theorem 7.4 (Whitehead). *Let $\omega_1, \omega_2 \in \Omega^2(S^2)$ such that $\int_{S^2} \omega_i = 1$. Consider the pullback $f^*(\omega_2) \in \Omega^2(S^3)$. Then $\mathrm{d}f^*(\omega_2) = f * (\mathrm{d}\omega_2) = 0$. Since $H^2(S^3) = 0$, there must be a primitive for $f^*(\omega_2)$, i.e, an $\alpha \in \Omega^1(S^3)$ such that $\mathrm{d}\alpha = f^*(\omega_2)$. Then*

$$H(f) = \int_{S^3} \alpha \wedge f^*(\omega_1)$$

Idea of proof. Consider the case ω_1 is supported on a neighborhood of a and ω_2 on a neighborhood of b. Then the primitive can be constructed to be supported on a neighborhood of B, where $\partial B = h^{-1}(b)$ as above. □

This gives us an upper bound on $G_L(L)$.

Let $f : S^3 \to S^3$ be L–Lipschitz. Suppose ω_1, ω_2 are bounded , $\|\omega_1\|_\infty, \|\omega_2\|_\infty \leq 1$. Then

$$\|f^*(\omega_2)\|_\infty \leq L^2$$
$$\|f^*(\omega_1)\|_\infty \leq L^2$$

We can take (this requires some work) $\alpha \in \Omega^1(S^3)$ such that $\|\alpha\|_\infty \lesssim \|f^*(\omega^2)\|_\infty$ and $\mathrm{d}\alpha = f^*(\omega_2)$. Then

$$H(f) = \int_{S^3} \alpha \wedge f * (\omega_1)$$

and since each of the terms in the integrand have norm L^2 we can conclude that

$$|H(f)| \leq L^4.$$

Now for the lower bound. Let $f_L : S^2 \to S^2$ be an L–Lipschitz map with degree L^2. Then $H(f_L \circ h) = L^4$ and $\mathrm{Lip}(f_L \circ h) \sim L$.

More generally: Sullivan's rational homotopy theory proves that:

Theorem 7.5 (Sullivan). *Let X be simply connected , let $n \geq 2$ and let $F : \pi_n(X) \to \mathbb{R}$ be a homomorphism. Then F can be computed by a formula involving differential forms on X, primitives, and wedge products.*

Gromov used this to show:

Theorem 7.6 (Gromov [4]). *Let X be simply connected, let $\alpha \in \pi_n(X)$ of infinite order. Then there is a d depending on α such that $G_\alpha(L) \leq L^d$.*

The question that arises is whether this bound is sharp. Recent results show that it need not be.

Theorem 7.7 (Berdnikov–Manin [2]). *Take the connected sum of* 4 *copies of* $\mathbb{C}P^2 \times S^2$ *and remove a point. This is a* 6*–dimensional manifold with non trivial* π_5. *Let* α *be the class of the puncture. Then rational homotopy theory says that* $G_\alpha \leq L^6$ *– but in fact this inequality is strict – for any* $c > 0$, $G_\alpha(L) < cL^6$ *whenever* L *is sufficiently large.*

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