NOTES ON QUASI-ISOMETRIES AND COHOMOLOGY

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1. Introduction

These are some notes on Shalom's theorem that two quasi-isometric nilpotent groups have the same Betti numbers. Shalom's proof uses unitary representations of the two groups based on a measure equivalence, i.e., a pair of commuting actions. While Shalom simply uses the existence of such a measure equivalence, one can also construct an explicit pair of commuting actions on the space of quasi-isometries.

In this case, the cocycles that Shalom uses to describe the relationship between the two actions can be written in terms of a particular quasi-isometry $\phi\colon \Lambda \to \Gamma$, and the map from $H^*(\Gamma)$ to $H^*(\Lambda)$ can be written in terms of the pullback

$$\phi^*\omega(\lambda_0,\dots,\lambda_d) = \omega(\phi(\lambda_0),\dots,\phi(\lambda_d)),$$

where $\omega \in H^d(\Gamma)$ is an invariant cochain (a map $\omega\colon \Gamma^{d+1} \to \mathbb{C}$ which is invariant under the action of Γ) and $\phi^*\omega$ is a map $\phi^*\omega\colon \Lambda^{d+1} \to \mathbb{C}$ which is not necessarily invariant; we think of $\phi^*\omega$ as a d–cochain for $\Delta(\Lambda)$, where $\Delta(\Lambda)$ is the infinite-dimensional simplex whose vertex set is Λ. (When not otherwise specified, we take complex coefficients, so $H^*(\Gamma) = H^*(\Gamma;\mathbb{C})$.)

Since Λ is amenable, one can average $\phi^*\omega$ over a Følner sequence F_n to construct a cochain

$$\overline{\phi^*}\omega(\delta) := \lim_{n\to\infty} \frac{1}{|F_n|} \sum_{\lambda\in F_n} \phi^*\omega(\lambda\delta).$$

In general, this limit need not exist, but we will see that if ϕ satisfies an ergodicity property, then this limit always exists. In fact, the main goal of these notes is to prove the following proposition.

For any simplex $\delta = \langle g_0,\dots,g_d\rangle \in \Delta(G)$, any $g \in G$, and any $f\colon G \to H$, we write $g\delta = \langle gg_0,\dots,gg_d\rangle$ and $f(\delta) = \langle f(g_0),\dots,f(g_d)\rangle$.

Proposition 1.1. *Let Λ and Γ be finitely generated nilpotent groups and suppose that there is a surjective quasi-isometry from Λ to Γ. Then there are*

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a Følner sequence $G_n \subset \Lambda$ and quasi-isometries $\phi_0 \colon \Lambda \to \Gamma$ and $\psi_0 \colon \Gamma \to \Lambda$ such that $\phi_0 \circ \psi_0 = \mathrm{id}_\Gamma$ and

(1) *For $\omega \in C^*(\Gamma)$ and $\delta = \langle h_0, \dots, h_d \rangle \in \mathscr{F}^d(\Delta(\Lambda))$, the limit*

$$\overline{\phi_0^*}\omega(\delta) := \lim_{n\to\infty} \frac{1}{|G_n|} \sum_{\lambda \in G_n^{-1}} \omega\left(\phi_0(\lambda\delta)\right) \tag{1}$$

exists and $\overline{\phi_0^*}$ induces a homomorphism $\overline{\phi_0^*} \colon H^*(\Gamma) \to H^*(\Lambda)$.
For $d \geq 0$, $\zeta \in C^*(\Lambda)$ and $\delta \in \mathscr{F}^d(\Delta(\Gamma))$, the limit

$$\overline{\psi_0^*}\zeta(\delta) := \lim_{n\to\infty} \frac{1}{|G_n|} \sum_{\lambda \in G_n^{-1}} \zeta\left(\psi_0(\phi_0(\lambda)\delta)\right) \tag{2}$$

exists, and $\overline{\psi_0^*}$ induces a homomorphism $\overline{\psi_0^*} \colon H^*(\Lambda) \to H^*(\Gamma)$.

(2) *The composition $\overline{\psi_0^*} \circ \overline{\phi_0^*} \colon H^*(\Gamma) \to H^*(\Gamma)$ is the identity map.*

Consequently, $\beta_d(\Gamma) \leq \beta_d(\Lambda)$ for all $d \geq 0$.

If Λ and Γ are quasi-isometric nilpotent groups and G is a sufficiently large finite group, then there is a surjective quasi-isometry from $G \times \Lambda \to \Gamma$. Taking the product with a finite group doesn't affect the cohomology (with coefficients in $\mathbb{C}$) of Λ, so

$$\beta_d(G \times \Lambda) = \beta_d(\Lambda) \leq \beta_d(\Gamma).$$

By symmetry, $\beta_d(\Lambda) = \beta_d(\Gamma)$. In particular, the maps $\overline{\psi_0^*}$ and $\overline{\phi_0^*}$ are vector space isomorphisms.

These notes are indebted to Shalom's original proof, and many of the constructions in these notes are simply translations of Shalom's constructions to this new context. Our goal is to make the proof more accessible and perhaps inspire extensions of Shalom's methods.

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2. Ergodicity and the space of quasi-isometries

The first step in the proof of Proposition 1.1 is to choose ϕ_0 and ψ_0. In this section, we construct a space Q of the possible choices of ϕ_0 and ψ_0, define actions of Λ and Γ on Q, and choose ϕ_0 and ψ_0 so that $(\phi_0, \psi_0) \in Q$ has an ergodic orbit. We then use Lindenstrauss's ergodic theorem for amenable groups to prove Proposition 1.1.(1).

Let Λ and Γ be as in Proposition 1.1. For $C > 0$, let $\mathrm{QI}_C(\Lambda, \Gamma)$ be the set of C–quasi-isometries from Λ to Γ. Let $l_\lambda \colon \Lambda \to \Lambda$ and $l_\gamma \colon \Gamma \to \Gamma$ be the left-multiplication maps. Let

$$X := \{(\phi, \psi) \in \mathrm{QI}_C(\Lambda, \Gamma) \times \mathrm{QI}_C(\Gamma, \Lambda) \mid \phi \circ \psi = \mathrm{id}_\Gamma\},$$

where C is chosen large enough that X is nonempty. Then Γ and Λ act on $\mathrm{QI}_C(\Lambda,\Gamma)$ by commuting actions; for $\gamma\in\Gamma$, $\lambda\in\Lambda$, and $(\phi,\psi)\in X$, we define the left action

$$\gamma\cdot(\phi,\psi)=(l_\gamma\circ\phi,\psi\circ l_{\gamma^{-1}})$$

and the right action

$$(\phi,\psi)\cdot\lambda=(\phi\circ l_\lambda, l_{\lambda^{-1}}\circ\psi).$$

Let $Q:=\{(\phi,\psi)\in X\mid \phi(0)=0\}$. This is a compact fundamental domain for the Γ–action on X, and, since ϕ is surjective, $Q\cdot\Lambda=X$. We can use the actions on X to define two actions on Q. For every $\lambda\in\Lambda$ and (ϕ,ψ), there is a unique $\gamma\in\Gamma$ such that $\gamma(\phi,\psi)\lambda\in Q$, namely $\gamma=\phi(\lambda)^{-1}$. We thus obtain a right action of Λ on Q by letting $\tau_\lambda\colon Q\to Q$,

$$\begin{aligned}\tau_\lambda(\phi,\psi)&=\phi(\lambda)^{-1}\cdot(\phi,\psi)\cdot\lambda\\ &=(l^{-1}_{\phi(\lambda)}\circ\phi\circ l_\lambda, l^{-1}_\lambda\circ\psi\circ l_{\phi(\lambda)}).\end{aligned}\tag{3}$$

(Equivalently, since the actions of Λ and Γ commute, Λ acts on $\Gamma\backslash X$, and there is a bijection between $\Gamma\backslash X$ and Q.)

Likewise, for every $\gamma\in\Gamma$, we have $\gamma(\phi,\psi)\psi(\gamma^{-1})\in Q$, and we define a right action of Γ by setting $\sigma_\gamma\colon Q\to Q$,

$$\sigma_\gamma(\phi,\psi)=\gamma^{-1}\cdot(\phi,\psi)\cdot\psi(\gamma).$$

It is straightforward to check that this is an action.

We think of these actions as "recentering" ϕ and ψ; given $q=(\phi,\psi)\in Q$, the graphs of ϕ and ψ are subsets of $\Lambda\times\Gamma$. The graph of ϕ goes through $(0,0)$ and the graph of ψ comes close to $(0,0)$; the action τ_λ corresponds to left-translating these graphs by $(\phi(\lambda),\lambda)^{-1}$, and the action of σ_γ corresponds to left-translating them by $(\gamma,\psi(\gamma))^{-1}$. Note that since $\phi(\psi(\gamma))=\gamma$,

$$\sigma_\gamma(\phi,\psi)=\phi(\psi(\gamma))\cdot(\phi,\psi)\cdot\psi(\gamma)=\tau_{\psi(\gamma)}.\tag{4}$$

To construct ϕ_0 and ψ_0, we use an ergodic theorem due to Lindenstrauss [Lin01]. A *Følner sequence* in G is a sequence of sets $F_n\subset G$ such that there is a finite generating set $S\subset G$ satisfying $|F_n|\to\infty$ and

$$\lim_{n\to\infty}|F_n\triangle sF_n|=0$$

for all $s\in S$. A *tempered Følner sequence* F_n is a Følner sequence such that there is some $C>0$ such that for all n,

$$\left|\bigcup_{k\le n}F_k^{-1}F_{n+1}\right|\le C|F_{n+1}|.$$

For instance, the sequence $F_n=[-n,n]\subset\mathbb{Z}$ is tempered, but $G_n=[2^n,2^n+n]$ is not. Lindenstrauss proved the following theorem.

Theorem 2.1 (Lindenstrauss)**.** *Let G be an amenable group acting ergodically on a measure space (X,η) and let F_n be a tempered Følner sequence. Then for any $f \in L_1(\eta)$ and for η–a.e. $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} f(gx) = \int_X f \, d\eta.$$

We use this to choose ϕ_0 and ψ_0 so that the closure of the orbit of (ϕ_0,ψ_0) supports an ergodic measure. For $p \in Q$, let δ_p be the point measure supported at p.

Lemma 2.2. *There are a Λ–invariant Radon measure μ on Q, a point $q = (\phi_0,\psi_0) \in \operatorname{supp}(\mu)$, and a symmetric Følner sequence $G_n = G_n^{-1} \subset \Lambda$ such that Λ acts ergodically on (Q,μ) and the averages*

$$\mu_n := \frac{1}{|G_n|} \sum_{\lambda\in G_n} \delta_{\tau_{\lambda^{-1}}(q)} \tag{5}$$

converge weakly to μ.

Proof. We first construct an invariant measure η on Q. Take a symmetric generating set for Λ and let $G_n \subset \Lambda$ be the ball of radius n in the word metric. This is a tempered Følner sequence and $G_n^{-1} = G_n$.

Let $C(Q)$ be the set of continuous complex-valued functions on Q. Let $p \in Q$, let $\overline{\lim}$ be an ultralimit, and for any continuous function $\omega\colon Q \to \mathbb{C}$, let

$$\alpha(\omega) = \overline{\lim_{n\to\infty}} \frac{1}{|G_n|} \sum_{g\in G_n} \omega(\tau_{g^{-1}}(p)).$$

Since G_n is a Følner sequence, for any $h \in \Lambda$,

$$\alpha(\omega\circ\tau_h) = \overline{\lim_{n\to\infty}} \frac{1}{|G_n|} \sum_{g\in G_n} \omega(\tau_h(\tau_{g^{-1}}(p))) = \overline{\lim_{n\to\infty}} \frac{1}{|G_n|} \sum_{g\in h^{-1}G_n} \omega(\tau_{g^{-1}}(p)) = \alpha(\omega),$$

i.e., α is Λ–invariant.

By the Riesz representation theorem, since Q is compact and Hausdorff, there is a Radon measure η on Q such that $\alpha(\omega) = \int_Q \omega \, d\eta$ and η is a Λ–invariant probability measure. That is, the set P of Λ–invariant probability measures on Q is nonempty, closed, and convex. Let μ be an extreme point of P. Then Λ acts ergodically on (Q,μ).

Let $f\colon Q \to \mathbb{C}$ be continuous. Since μ is a Radon measure, we have $\mu(Q \setminus \operatorname{supp}(\mu)) = 0$, and by Theorem 2.1, for μ–almost every $q \in \operatorname{supp}(\mu)$,

$$\lim_{n\to\infty} \frac{1}{|G_n|} \sum_{g\in G_n} f(\tau_{g^{-1}}(q)) = \int_Q f \, d\mu. \tag{6}$$

Since Q is compact and metrizable, there is a countable dense set of continuous functions. We can therefore find a $q \in \operatorname{supp}(\mu)$ such that (6) holds for every continuous function. Let μ_n be as in (5); then μ_n converges weakly to μ, as desired. □

We use this lemma to prove the first part of Proposition 1.1.

Proof of Proposition 1.1.(1). Let $q = (\phi_0, \psi_0)$ be as in Lemma 2.2 and let μ_n be as in (5). Let $d \geq 0$, let $\omega \in C^d(\Gamma)$, and let $\delta = \langle h_0, \ldots, h_d \rangle \in \mathscr{F}^d(\Delta(\Lambda))$. Let $f_\omega \colon Q \to \mathbb{C}$, $f_\omega(\phi, \psi) = \omega(\phi(\delta))$. Then f_ω is continuous and

$$\begin{aligned}\int_Q f_\omega \, \mathrm{d}\mu_n &= \frac{1}{|G_n|} \sum_{\lambda \in G_n} f_\omega(\tau_{\lambda^{-1}}(q)) \\ &= \frac{1}{|G_n|} \sum_{\lambda \in G_n} \omega(\phi_0(\lambda^{-1})^{-1}(\phi_0(\lambda^{-1}\delta))) \\ &= \frac{1}{|G_n|} \sum_{\lambda \in G_n} \omega(\phi_0(\lambda^{-1}\delta)),\end{aligned}$$

where the last equality follows from the Γ–invariance of ω. Thus, if $\overline{\phi_0^*}\omega(\delta)$ is as in (1), then

$$\overline{\phi_0^*}\omega(\delta) = \lim_{n\to\infty} \int_Q f_\omega \, \mathrm{d}\mu_n = \int_Q f_\omega \, \mathrm{d}\mu$$

by the weak convergence of μ_n. It is straightforward to check that $\overline{\phi_0^*}(d\omega) = d(\overline{\phi_0^*}\omega)$, so $\overline{\phi_0^*}$ induces a map on homology.

Similarly, let $\zeta \in C^d(\Lambda)$ and let $\delta \in \mathscr{F}^d(\Delta(\Gamma))$. Let $g_\zeta \colon Q \to \mathbb{C}$, $g_\zeta(\phi, psi) = \zeta(\psi(\delta))$. Then g_ζ is continuous and

$$\begin{aligned}\int_Q g_\zeta \, \mathrm{d}\mu_n &= \frac{1}{|G_n|} \sum_{\lambda \in G_n} \zeta(\lambda^{-1}\psi(\phi(\lambda^{-1})\delta)) \\ &= \frac{1}{|G_n|} \sum_{\lambda \in G_n^{-1}} \zeta(\psi(\phi(\lambda)\delta)).\end{aligned}$$

Thus, by the weak convergence of μ_n,

$$\overline{\psi_0^*}\zeta(\delta) = \lim_{n\to\infty} \int_Q g_\zeta \, \mathrm{d}\mu_n = \int_Q g_\zeta \, \mathrm{d}\mu.$$

This induces a map from $H^*(\Lambda)$ to $H^*(\Gamma)$ as desired. □

The second part of Proposition 1.1 relies on Theorem 4.1.3 of [Sha04], which is based on Theorem 10.1 of Blanc [Bla79]. (See also [Del77].)

Theorem 2.3. *Let Γ be a nilpotent group and let Y be a unitary representation with no fixed points. Then, for any $n \geq 0$, the reduced cohomology $\overline{H}^n(\Gamma; Y)$ is trivial.*

Recall that an invariant cochain $h \in C^d(G;Y)$ with coefficients in a G–module Y is a function $h\colon G^{d+1} \to Y$ such that for any $\lambda \in \Lambda$ and $\delta \in \mathscr{F}^d(\Delta(G))$, $h(\lambda\delta) = \lambda h(\delta)$. The reduced cohomology $\overline{H}^n(\Gamma;Y)$ is the quotient

$$\overline{H}^n(\Gamma;Y) = \ker(d_n)/\operatorname{clos}(\operatorname{im}(d_{n-1})),$$

where $d_n\colon C^n(\Gamma;Y) \to C^{n+1}(\Gamma;Y)$ is the coboundary map. This is a quotient of $H^n(\Gamma;Y)$, and if $H^n(\Gamma;Y)$ is finite-dimensional, then $H^n(\Gamma;Y) = \overline{H}^n(\Gamma;Y)$.

We will prove Proposition 1.1.(2) by describing $\overline{\phi_0^*}$ and $\overline{\psi_0^*}$ in terms of cohomology with coefficients in $C(Q)$ and $L^2(\mu)$. As in the proof of Lemma 2.2, let $C(Q)$ be the set of continuous complex-valued functions on Q, equipped with left actions $\lambda \cdot f = f \circ \tau_\lambda$ and $\gamma \cdot f = f \circ \sigma_\gamma$ for $f \in C(Q)$, $\lambda \in \Lambda$, $\gamma \in \Gamma$.

First, we show that $\phi_0^*\colon C^*(\Gamma) \to C^*(\Delta(\Lambda))$ can be written $\phi_0^* = \Pi_\Lambda \circ T^\sharp$, where $T^\sharp\colon C^d(\Gamma) \to C^d(\Lambda; C(Q))$ and $\Pi_\Lambda\colon C^d(\Lambda; C(Q)) \to C^*(\Delta(\Lambda))$.

For $\omega \in C^d(\Gamma;\mathbb{R})$ and $\delta \in \mathscr{F}^d(\Delta(\Lambda))$, let

$$T^\sharp\omega(\delta)(\phi,\psi) := \omega(\phi(\delta)). \tag{7}$$

We check that $T^\sharp\omega$ is Γ–equivariant: for $\lambda \in \Lambda$,

$$(\lambda \cdot T^\sharp\omega(\delta))(\phi,\psi) = T^\sharp\omega(\delta)(\tau_\lambda(\phi,\psi)).$$

Since

$$\tau_\lambda(\phi,\psi) = (l_{\phi(\lambda)}^{-1} \circ \phi \circ l_\lambda, l_\lambda^{-1} \circ \psi \circ l_{\phi(\lambda)}),$$

we have

$$(\lambda \cdot T^\sharp\omega(\delta))(\phi,\psi) = \omega(\phi(\lambda)^{-1}\phi(\lambda\delta)) = \omega(\phi(\lambda\delta))$$

by the invariance of ω, and $\omega(\phi(\lambda\delta)) = T^\sharp\omega(\lambda\delta)(\phi,\psi)$. Furthermore, $T^\sharp$ commutes with the coboundary map.

Now, for $G = \Lambda$ or $G = \Gamma$ and every $h \in C^k(G;C(Q))$, we define a corresponding cochain $\Pi_G h \in C^k(\Delta(G))$ by

$$\Pi_G h(g_0,\dots,g_k) := h(g_0,\dots,g_k)(q)$$

for $g_0,\dots,g_k \in G$. This map likewise commutes with the coboundary map, and for $\delta \in \mathscr{F}^d(\Delta(\Gamma))$,

$$\Pi_\Lambda T^\sharp\omega(\delta) = T^\sharp\omega(\delta)(\phi_0,\psi_0) = \omega(\phi_0(\delta)) = \phi_0^*\omega(\delta),$$

i.e.,

$$\Pi_\Lambda T^\sharp\omega = \phi_0^*\omega. \tag{8}$$

In general, the image of Π_G consists of cochains which have symmetries like those of q. For example, if q is periodic, then $\Pi_G h$ is also periodic. That is, if there is some $\lambda \in \Lambda$ such that $\tau_\lambda(q) = q$, then for any $\delta \in \mathscr{F}^d(\Delta(\Lambda))$,

$$\Pi_\Lambda h(\lambda\delta) = h(\lambda\delta)(q) = h(\delta)(\tau_\lambda(q)) = h(\delta)(q) = \Pi_\Lambda h(\delta).$$

Likewise, if $\tau_{\lambda_1}(q)$ is sufficiently close to $\tau_{\lambda_2}(q)$, then $\Pi_\Lambda h(\lambda_1\delta)$ is close to $\Pi_\Lambda h(\lambda_2\delta)$.

We can define a map $T_\sharp\colon C^d(\Lambda;C(Q)) \to C^d(\Gamma;C(Q))$ in a similar way. This map roughly corresponds to the transfer maps constructed in [Sha04]. For $\delta \in \mathscr{F}^d(\Delta(\Gamma))$, $\zeta \in C^d(\Lambda;C(Q))$, and $(\phi,\psi)\in Q$, let

$$T_\sharp\zeta(\delta)(\phi,\psi) := \zeta(\psi(\delta))(\phi,\psi). \tag{9}$$

Then $T_\sharp$ commutes with coboundaries. We check that $T_\sharp\zeta$ is Γ–equivariant: for $\gamma\in\Gamma$, we note that

$$T_\sharp\zeta(\gamma\delta)(\phi,\psi) = \zeta(\psi(\gamma\delta))(\phi,\psi),$$

and

$$(\gamma\cdot T_\sharp\zeta(\delta))(\phi,\psi) = T_\sharp\zeta(\delta)(\sigma_\gamma(\phi,\psi)),$$

where

$$\sigma_\gamma(\phi,\psi) = (l_\gamma^{-1}\circ\phi\circ l_{\psi(\gamma)}, l_{\psi(\gamma)}^{-1}\circ\psi\circ l_\gamma).$$

Then

$$\begin{aligned}(\gamma\cdot T_\sharp\zeta(\delta))(\phi,\psi) &= \zeta(\psi(\gamma)^{-1}\psi(\gamma\delta))(\sigma_\gamma(\phi,\psi))\\ &\overset{(4)}{=} \zeta(\psi(\gamma)^{-1}\psi(\gamma\delta))(\tau_{\psi(\gamma)}(\phi,\psi))\\ &= \zeta(\psi(\gamma\delta))(\phi,\psi)\end{aligned}$$

by the Λ–equivariance of ζ, so

$$\gamma\cdot T_\sharp\zeta(\delta) = T_\sharp\zeta(\gamma\delta). \tag{10}$$

We can draw the following commutative diagram relating $T^\sharp$ and $T_\sharp$ to ψ_0^* and ϕ_0^*.

$$\begin{tikzcd} & C^d(\Lambda;C(Q)) \arrow[r, "T_\sharp"] \arrow[d, "\Pi_\Lambda"] & C^d(\Gamma;C(Q)) \arrow[d, "\Pi_\Gamma"] \\ C^d(\Gamma) \arrow[ur, "T^\sharp"] \arrow[r, "\phi_0^*"'] & C^d(\Delta(\Lambda)) \arrow[r, "\psi_0^*"'] & C^d(\Delta(\Gamma)) \end{tikzcd}$$

The left side of the diagram commutes by (8). For $\zeta \in C^d(\Lambda;C(Q))$ and $\delta\in\mathscr{F}^d(\Gamma)$,

$$\Pi_\Gamma T_\sharp\zeta(\delta) = T_\sharp\zeta(\delta)(\phi_0,\psi_0) = \zeta(\psi_0(\delta))(\phi_0,\psi_0),$$

and

$$\psi_0^*\Pi_\Lambda\zeta(\delta) = \Pi_\Lambda\zeta(\psi_0(\delta)) = \zeta(\psi_0(\delta))(\phi_0,\psi_0),$$

so the right side commutes.

Furthermore, since $\phi\circ\psi = \mathrm{id}_\Gamma$ for all $(\phi,\psi)\in Q$,

$$T_\sharp T^\sharp\omega(\delta)(\phi,\psi) = T^\sharp\omega(\psi(\delta))(\phi,\psi) = \omega(\phi(\psi(\delta))) = \omega(\delta) \tag{11}$$

and $\psi_0^*\phi_0^*\omega(\delta)=\omega(\delta)$. That is, $T_\sharp\circ T^\sharp$ is the inclusion of $C^d(\Gamma)$ into $C^d(\Gamma;C(Q))$ induced by the inclusion $\mathbb{C}\subset C(Q)$ and $\psi_0^*\circ\phi_0^*$ is the inclusion of $C^d(\Gamma)$ into $C^d(\Delta(\Gamma))$.

Next, we show that the maps above extend to chains with coefficients in $L^2(\mu)$. Every function in $C(Q)$ is L^2, so we can view $T^\sharp$ as a map from $C^d(\Gamma)$ to $C^d(\Lambda;L^2(\mu))$. Extending $T_\sharp$ takes a little more work.

Lemma 2.4. *The map $T_\sharp$ extends to a continuous map from $C^d(\Lambda;L^2(\mu))$ to $C^d(\Gamma;L^2(\mu))$.*

Proof. For $\zeta\in C^d(\Lambda;C(Q))$ and $\delta'\in\mathscr{F}^d(\Lambda)$, we have

$$\|\zeta(\delta')\|_{L^2(\mu)}=\int\zeta(\delta')(\phi,\psi)^2\,\mathrm{d}\mu(\phi,\psi)=\lim_{n\to\infty}\frac{1}{|G_n|}\sum_{\lambda\in G_n^{-1}}\zeta(\delta')(\tau_\lambda(q))^2.$$

Let $\delta\in\mathscr{F}^d(\Gamma)$. For $\lambda\in\Lambda$, note that

$$T_\sharp\zeta(\delta)(\tau_\lambda(q))=\zeta(\lambda^{-1}\psi_0(\phi_0(\lambda)\delta))(\tau_\lambda(q)).$$

Let

$$\delta_\lambda=\lambda^{-1}\psi_0(\phi_0(\lambda)\delta).$$

Let $D(\delta)=\{\delta_\lambda\mid\lambda\in\Lambda\}$. Since ϕ_0 and ψ_0 are quasi-inverses, $D(\delta)$ is a finite set.

Then

$$\int T_\sharp\zeta(\delta)(\phi,\psi)^2\,\mathrm{d}\mu(\phi,\psi)=\lim_{n\to\infty}\frac{1}{|G_n|}\sum_{\lambda\in G_n^{-1}}T_\sharp\zeta(\delta)(\tau_\lambda(q))^2.$$

By the above,

$$T_\sharp\zeta(\delta)(\tau_\lambda(q))^2=\zeta(\delta_\lambda)(\tau_\lambda(q))^2\le\sum_{x\in D(\delta)}\zeta(x)(\tau_\lambda(q))^2.$$

Therefore,

$$\|T_\sharp\zeta(\delta)\|_2\le\sum_{x\in D(\delta)}\|\zeta(x)\|_2,\tag{12}$$

so $T_\sharp$ is a continuous map from $C^d(\Lambda;C(Q))$ to $C^d(\Lambda;L^2(\mu))$. Since μ is a Radon measure, $C(Q)$ is dense in $L^2(\mu)$, so $T_\sharp$ can be extended to $C^d(\Lambda;L^2(\mu))$ by continuity. □

For $G=\Lambda,\Gamma$, let $M_G\colon C^d(G;L^2(\mu))\to C^d(G)$ be the integral

$$M_G\alpha(\delta)=\int_Q\alpha(\delta)\,\mathrm{d}\mu\tag{13}$$

for $\delta\in\mathscr{F}^d(\Delta(G))$. If $\alpha\in C^d(G;C(Q))$, then Lemma 2.2 implies

$$M_G\alpha(\delta)=\lim_{n\to\infty}\frac{1}{|G_n|}\sum_{\lambda\in G_n}\alpha(\delta)(\tau_{\lambda^{-1}}(q)).\tag{14}$$

Then M_G commutes with coboundaries and the following lemma holds.

Lemma 2.5. *For all $\omega \in C^d(\Gamma)$ and $\zeta \in C^d(\Lambda)$, we have $\overline{\phi_0^*}\omega = M_\Lambda T^\sharp\omega$ and $\overline{\psi_0^*}\zeta = M_\Gamma T_\sharp\zeta$, where we view ζ as an element of $C^d(\Lambda; C(Q))$ via the inclusion $\mathbb{C} \subset C(Q)$.*

Proof. Let $\delta \in \mathscr{F}^d(\Delta(\Lambda))$. By (14) and (7),

$$M_\Lambda T^\sharp\omega(\delta) = \lim_{n\to\infty} \frac{1}{|G_n|} \sum_{\lambda\in G_n} T^\sharp\omega(\tau_{\lambda^{-1}}(q))$$
$$= \lim_{n\to\infty} \frac{1}{|G_n|} \sum_{\lambda\in G_n} \omega(\lambda\phi_0(\lambda^{-1}\delta)).$$

By the invariance of ω,

$$M_\Lambda T^\sharp\omega(\delta) = \lim_{n\to\infty} \frac{1}{|G_n|} \sum_{\lambda\in G_n} \omega(\phi_0(\lambda^{-1}\delta)) = \overline{\phi_0^*}\omega(\delta).$$

Likewise, let $\delta \in \mathscr{F}^d(\Delta(\Gamma))$. Then, by (9),

$$M_\Gamma T_\sharp\zeta(\delta) = \lim_{n\to\infty} \frac{1}{|G_n|} \sum_{\lambda\in G_n} T_\sharp\zeta(\tau_{\lambda^{-1}}(q))$$
$$= \lim_{n\to\infty} \frac{1}{|G_n|} \sum_{\lambda\in G_n} \zeta(\lambda\psi_0(\phi_0(\lambda^{-1})\delta))(\tau_{\lambda^{-1}}(q)).$$

Since $\zeta \in C^d(\Lambda)$, $\zeta(\delta')$ is a constant function for any δ', and $\zeta(\lambda\delta') = \zeta(\delta')$. Therefore, by (2),

$$M_\Gamma T_\sharp\zeta(\delta) = \lim_{n\to\infty} \frac{1}{|G_n|} \sum_{\lambda\in G_n} \zeta(\psi_0(\phi_0(\lambda^{-1})\delta)) = \overline{\psi_0^*}\zeta(\delta).$$

□

Finally, we prove Proposition 1.1.(2).

Proof of Proposition 1.1.(2). It suffices to show that for any $\omega \in C^d(\Gamma)$, the difference $\omega - \overline{\psi_0^*} \circ \overline{\phi_0^*}(\omega)$ is a coboundary.

Let

$$\nu := T^\sharp\omega - M_\Lambda T^\sharp\omega \in C^d(\Lambda; C(Q)).$$

Then $\nu \in C^d(\Lambda; L^2_0(\mu))$, where $L^2_0(\mu) = \{f \in L^2(\mu) \mid \int f \, \mathrm{d}\mu = 0\}$. Since Λ acts ergodically on μ, $L^2_0(\mu)$ is a unitary representation of Λ with no fixed points, so by Theorem 2.3, ν is a reduced coboundary in $C^d(\Lambda; L^2_0(\mu))$, i.e., a limit of coboundaries.

Since $C(Q)$ is dense in $L^2(\mu)$, there are $\eta_i \in C^{d-1}(\Lambda; C(Q))$ such that $d\eta_i \to \nu$ and thus $dM_\Gamma T_\sharp\eta_i \to M_\Gamma T_\sharp\nu$. Therefore, $M_\Gamma T_\sharp\nu$ is a reduced coboundary in $C^d(\Gamma)$. Since $H^d(\Gamma)$ is finite-dimensional, the subspace of coboundaries has finite codimension in $C^d(\Gamma)$ and is thus closed. Thus

$$M_\Gamma T_\sharp \nu = M_\Gamma T_\sharp T^\sharp \omega - M_\Gamma T_\sharp M_\Lambda T^\sharp \omega$$

is a coboundary.

By (11),

$$M_\Gamma T_\sharp T^\sharp \omega = M_\Gamma \omega = \omega,$$

and by Lemma 2.5,

$$M_\Gamma T_\sharp M_\Lambda T^\sharp \omega = \overline{\psi_0^*} \circ \overline{\phi_0^*}(\omega),$$

so ω is cohomologous to $\overline{\psi_0^*} \circ \overline{\phi_0^*}(\omega)$, as desired. □

3. Questions

(1) Can we construct the cochain that arises from Theorem 2.3 explicitly? That is, a similar argument implies that if $\alpha \in C^d(\Delta(\Gamma))$ generates an ergodic measure μ on $C^d(\Delta(\Gamma))$ and

$$\lim_{n\to\infty} \frac{1}{|F_n^\Gamma|} \sum_{g \in F_n^\Gamma} g\alpha = 0,$$

then $\alpha = \Pi_\Gamma(\beta)$ for some $\beta \in C^d(\Gamma; C_0(Q))$, where $Q = C^d(\Delta(\Gamma))$ and $C_0(Q) = C(Q) \cap L_0^2(\mu)$. By Theorem 2.3, β is a reduced boundary, so there are $\eta_i \in C^{d-1}(\Gamma; C_0(Q))$ such that $\partial\eta_i \to \beta$. One can show that the images $\Pi_\Gamma \eta_i$ also generate ergodic measures, so α is the limit of the coboundaries of cochains that generate ergodic measures and average to zero along a Følner sequence.

Can we construct these cochains explicitly?

(2) Can we state a version of Theorem 2.3 for cocycles in $C^d(\Delta(\Gamma))$? For instance, suppose that $\alpha \in C^d(\Delta(\Gamma))$ is a cocycle which averages to zero along a Følner sequence. When is α the limit of coboundaries of cochains that also average to zero along a Følner sequence?

(3) Suppose $\phi\colon \Lambda \to \Gamma$ is a quasi-isometric embedding. When can we define a pullback $\overline{\phi^*}\colon H^*(\Gamma) \to H^*(\Lambda)$? When is this pullback functorial?

(4) Can we prove Sauer's result [Sau06] that the cohomology ring is quasi-isometry invariant this way?

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