2021-09-09

Last time:
In analysis, you learned about limits and continuity in terms of ε's and δ's - all these can be stated in terms of open sets, and we can use ~~Def~~ that to generalize to topological spaces.

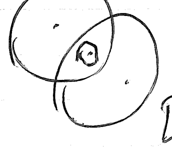
Def: A topological space is a set X equipped with a topology $\mathcal{T}$.
A topology on X is a set of subsets $\mathcal{T} \subset P(X)$ such that:
- $\phi, X \in \mathcal{T}$
- $\mathcal{T}$ is closed under finite intersections
- $\mathcal{T}$ is closed under arbitrary unions.

Ex: Metric ~~space topology~~. Let (X,d) be a metric space. ~~$U \subset X$ is open in the metric space topology if $\forall u \in U\ \exists r>0$ s.t.~~
Let $\mathcal{T} = \{U \mid \forall u \in U, \exists r>0 \text{ s.t. } B(u,r) \subset U\}$
This is a topology called the metric space topology on X.
(And in the problem set, you showed that ~~if you choose~~ limits in this topology are ~~the~~ limits in the ε-δ def.)

Today: Construct ~~more topologies~~, examples of topological spaces.

Problem: It's generally hard to list all open sets in a topology - we need a criterion (like the one for metric spaces). That characterizes metric space top. in terms of balls. Is there a generalization?

Def: A basis is a set $\mathcal{B} \subset P(X)$ such that:
(1) $\forall x \in X, \exists B \in \mathcal{B}$ s.t. $x \in B$.
(2) If $B_1, B_2 \in \mathcal{B}$, then $\forall x \in B_1 \cap B_2$, $\exists$ ~~B_3~~ $C \in \mathcal{B}$ s.t. $x \in C \subset B_1 \cap B_2$.



Like balls.

Def: The topology generated by $\mathcal{B}$ is the set
$\mathcal{T} = \{U \subset X \mid \forall u \in U, \exists B \in \mathcal{B} \text{ s.t. } x \in B \subset U\}$
Ex: $\mathcal{B} = \{B(x,r) \mid x \in X, r>0\}$ generates the metric space topology on $\mathcal{B}$.
Prop: $\mathcal{T}$ is a topology.
Pf: $\phi \in \mathcal{T}$ vacuously. $X \in \mathcal{T}$ because of (1). ~~because $\forall x \in X, \exists B \in \mathcal{B}$ s.t.~~

Let ~~$\mathcal{U}_\alpha \in \mathcal{B}$~~ $B_\alpha \in \mathcal{B}\ \forall \alpha \in A$, let $U = \bigcup_\alpha B_\alpha$.
Let $x \in U$. Then $x \in B_\alpha$ for some $\alpha \in A$.
Then $x \in B_\alpha \subset U$. ~~Since~~ This is true for all $x \in U$, so U is open.

Let $U, V \in \mathcal{T}$. ~~Let~~ Claim: $U \cap V \in \mathcal{T}$.
Let $x \in U \cap V$. Then $\exists B_1, B_2$ s.t. $x \in B_1 \subset U$, $x \in B_2 \subset V$.
Therefore $x \in B_1 \cap B_2$, so $\exists C$ s.t. $x \in C \subseteq B_1 \cap B_2 \subset U \cap V$.
By induction, if $U_1, \dots, U_k \in \mathcal{T}$, then $\bigcap_{i=1}^{k} U_i \in \mathcal{T}$ //.

In fact,
Lemma: $\mathcal{T} = \{$unions of elements of $\mathcal{B}\}$
Pf: ~~On one hand, if~~ Note that.
~~First, $\mathcal{B} \subset \mathcal{T}$~~ If $B \in \mathcal{B}$, then $B \in \mathcal{T}$ ~~by definition~~
Therefore, if $B_\alpha \in \mathcal{B}$ $\forall \alpha \in A$, then
$\bigcup_\alpha B_\alpha$ is a union of open sets
$\Rightarrow \bigcup_\alpha B_\alpha \in \mathcal{T}$.
So $\{$unions$\} \subset \mathcal{T}$. On the other hand, suppose $U \in \mathcal{T}$.
~~Th~~ $\forall x \in U$, let $B_x \in \mathcal{B}$ be s.t. $x \in B_x \subset U$.
Then $\bigcup_{x \in U} B_x = U$ //

~~Lemma: $\mathcal{T}$ is the Let $\mathcal{B}$ and $\mathcal{B}'$ be bases s.t. $\mathcal{B}$ generates $\mathcal{T}$, $\mathcal{B}'$ generates $\mathcal{T}'$. If $\mathcal{B} \subset \mathcal{T}'$ then $\mathcal{T} \subset \mathcal{T}'$.~~
Pf: ~~Exercise.~~ Lemma: If $\mathcal{B}$ generates $\mathcal{T}$ and $\mathcal{T}' \supset \mathcal{B}$, then $\mathcal{T}' \supset \mathcal{T}$.
Pf: Exercise.

~~Ex. $X = \mathbb{R}$, $\mathcal{B} = \{(a,b) \mid a<b\}$.~~
~~Generates the standard~~
~~Ex: (X,d) a metric space, $\mathcal{B} = \{B(x,r) \mid x \in X, r>0\}$~~
~~generates them~~

Ex: $X = \mathbb{R}$, $\mathcal{B} = \{[a,b) \mid a<b\}$. This is a basis.
Is: $[a,b)$ open? $[a,b]$? $(a,b]$? (a,b)?
How does this differ from standard topology? Finer
Does $\frac{1}{n} \to 0$? Does $-\frac{1}{n} \to 0$?
(lower limit topology).
~~(Lower limit topology - an unusual space. If you're~~ (break).

Order topology: ~~A set~~ Let S be a set. A total ordering on S is a relation $\preceq$ ~~s.t. with the~~
(a relation is a function $S \times S \to \{$true, false$\}$, eg: $=, <, \equiv$ (mod 3).)
that satisfies: reflexivity: $\forall x \in S$, $x \preceq x$.
antisymmetric: $\forall x, y \in S$, if $x \preceq y$ and $y \preceq x$, then $x = y$.
transitivity: $\forall x, y, z$ if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.
total: $\forall x, y$, $x \preceq y$ or $y \preceq x$.

We write $x \prec y$ if $x \preceq y$ and $x \neq y$. We write
$(x,y) = \{z \mid x \prec z \prec y\}$, $[x,y] = \{z \mid x \preceq z \preceq y\}$

~~Let $\mathcal{B} = \{(x,y) \mid x \leq$~~ Suppose X is totally ordered and has ≥ 2 elts.
Let ~~$\mathcal{B} = \{(x,y) \mid x < y\} \cup \{$~~
$\mathcal{B}$ be the set of :- Open intervals : (a,b) where $\forall a, b \in X$, $a<b$.
- half-open rays $\{z \mid x < z\}$ $\forall x \in X$.
- half-open rays $\{z \mid z < x\}$ $\forall x \in X$.

Then: $\mathcal{B}$ is a basis ~~and~~ the topology $\mathcal{T}$ generated by $\mathcal{B}$ is called the <u>order topology</u>.

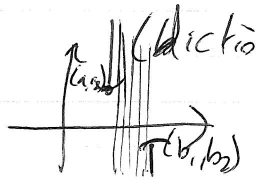
Ex: $X = (\mathbb{R}, \leq)$ $\mathcal{B}_{ord} = \{(a,b) \mid a<b\} \cup \{(a,\infty)\} \cup \{(-\infty,a)\}$
~~$\mathcal{T}$ = standard topology~~ generates $\mathcal{T}_{ord}$.

Claim : $\mathcal{T}_{ord} = \mathcal{T}_{met}$. (where $\mathcal{T}_{met}$ is gen. by $\mathcal{B}_{met} = \{(a,b) \mid a<b\}$)

Pf: Since $\mathcal{B}_{met} \subset \mathcal{B}_{ord}$, $\mathcal{T}_{met} \subset \mathcal{T}_{ord}$.
~~On the other hand every element of $\mathcal{B}$~~
OTOH, $\mathcal{B}_{ord} \subset \mathcal{T}_{met}$, so $\mathcal{T}_{ord} \subset \mathcal{T}_{met}$ //
We call this the '<u>standard topology on $\mathbb{R}$</u>' Unless otherwise spec.
But also: $X = \mathbb{R} \cup \{\pm\infty\}$. $\mathcal{B}_{ord} = \{(a,b) \mid a<b, a,b \in \mathbb{R}\}$ top. on $\mathbb{R}$
$\cup \{(a,\infty] \mid a \in \mathbb{R}\}$ is std.
$\cup \{[-\infty, a) \mid a \in \mathbb{R}\}$.

Does ~~(n) converge to ∞?~~
$n \to \infty$? If $U \ni \infty$, then $U \supset (a,\infty]$ for some $a \in \mathbb{R}$
$\Rightarrow$ $n \in U$ when n is sufficiently large.

Ex: ($\mathbb{R}^2$, lexicographic order). That is, $(a_1, a_2) \leq (b_1, b_2)$
if $a_1 < b_1$ or $a_1 = b_1$ and $a_2 < b_2$ ~~or~~



(dictionary order).

When does a sequence converge in order topology?

So we can construct std topology on $\mathbb{R}$ with only ordering. ~~What else? Same for $\mathbb{R}^n$!~~ Next: $\mathbb{R}^n$.

<u>Product topology</u>. Let X, Y be sets. Then

$X \times Y$ = ~~{ordered pairs}~~
$= \{(x,y) \mid x \in X, y \in Y\}$
ordered pairs

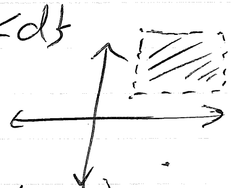
If X_1, X_2 are topological spaces, with topologies $\mathcal{T}_1, \mathcal{T}_2$, ~~$\mathcal{B}_X$ is a basis for topology of X, $\mathcal{C}$ is a basis for the topology of Y~~, the <u>product topology</u> $\mathcal{T}_{prod}$ on $X \times Y$ is the topology generated by

$$\mathcal{D} = \{ B_1 \times C_2 \mid B_1 \in \mathcal{T}_1, C_2 \in \mathcal{T}_2 \}.$$

If $\mathcal{B}_1$ is a basis for $\mathcal{T}_1$, $\mathcal{B}_2$ is a basis for $\mathcal{T}_2$, then $\mathcal{T}_{prod}$ is also gen by $\{B_1 \times B_2 \mid B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$.

Ex: $\mathbb{R} \times \mathbb{R}$ ~~with~~ where $\mathbb{R}$ has standard top.

Then $\mathcal{D} = \{(a,b) \times (c,d) \mid a<b, c<d\}$
$= \{$open rectangles$\}$



~~More generally~~
~~$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$. $\mathcal{D} = \{(a_1,b_1) \times (a_2,b_2) \times \cdots \times (a_n,b_n)\}$.~~

Claim: $\mathcal{T}_{prod}$ on $\mathbb{R}^2$ is equal to $\mathcal{T}_{met}$ on $\mathbb{R}^2$.

Pf: Let $\mathcal{B}_{met} = \{B(x,r) \mid x \in \mathbb{R}^2, r>0\}$.
$\mathcal{B}_{prod} = \{(a,b) \times (c,d)\}$

On one hand, if $(a,b) \times (c,d) \in \mathcal{B}_{prod}$, then $(a,b) \times (c,d) \in \mathcal{T}_{met}$.
$\Rightarrow \mathcal{T}_{prod} \subset \mathcal{T}_{met}$.

OTOH, if $B(x,r) \in \mathcal{B}_{met}$, then $B(x,r) \in \mathcal{T}_{prod}$
$\Rightarrow \mathcal{T}_{met} \subset \mathcal{T}_{prod}$. So $\mathcal{T}_{met} = \mathcal{T}_{prod}$. //

~~So we can construct~~
We call this the standard topology on $\mathbb{R}^2$.

More generally, for $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, let
$\mathcal{D} = \{(a_1,b_1) \times (a_2,b_2) \times \cdots \times (a_n,b_n)\}$.
~~and~~ the topology generated by $\mathcal{D}$ is equal to the metric space topology on $\mathbb{R}^n$. We call it the standard topology on $\mathbb{R}^n$. ~~Ex: What~~

<u>Sub-basis</u>: ~~Let~~ Let $\mathcal{S} \subset P(X)$. ~~It~~ is a <u>sub-basis</u> if $\forall x \in X$, $\exists C \in \mathcal{S}$ s.t. $x \in C$.

Then $\mathcal{B} = \{$finite intersections of elements of $\mathcal{S}\}$

~~The topology generated by $\mathcal{S}$ is the set of arbitrary unions of finite~~

$= \{C_1 \cap C_2 \cap \cdots \cap C_k \mid C_i \in \mathcal{S}\}$ is a basis

~~Let $\mathcal{T}$ = {arbitrary unions of finite intersections~~ of elements of $\mathcal{S}$}

~~This is the topology generated~~ by $\mathcal{S}$.

Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. I.e.
$\mathcal{T} = \{$arbitrary unions of finite intersections of ~~$\mathcal{S}$~~ els of $\mathcal{S}\}$

This is the <u>topology generated by $\mathcal{S}$</u>

Overflow: prod top as top gen by $\mathcal{S}$.