Compactness:

~~Compactness is subtle. There's a story that in 1925~~
~~and for a long tim~~ ~~I warn you about th~~

Compactness is ~~possibly the~~ subtle. Most of what we've talked about so far, ~~to~~ ~~I can give~~ there are good reasons why it's defined that way, ~~and~~ we can be pretty clear about why connectedness is defined the way it is.

But compactness, not so much - the definition didn't really become set until 1930s, and before then, different people meant different things by "compact".

So: How many defs of compactness have you heard of?
- open-cover
- sequential.

These almost always coincide:

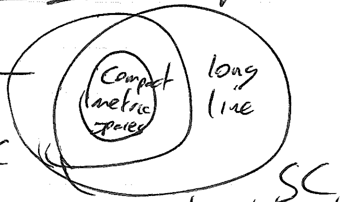
Thm: If X is a metric space, X is OCC $\Leftrightarrow$ X is SC.

But: ~~Ex~~ if X is not a metric space, there are ~~weird~~ counterexamples:

~~Ex: $[0,1]^{\mathbb{R}} = \{f:\mathbb{R}\to[0,1]\}$ with with product topology~~
~~with pointwise con~~ with ~~pointwise convergence~~
is ~~Comp OCC, not SC~~.

~~Ex~~

~~intfu~~ function spaces
$\{f:\mathbb{R}\to[0,1]\}$ OCC



Ult, decided that function spaces are cpt, long line is not.

Def: A space X is <u>compact</u> if every open cover of X has a finite subcover.

An <u>open cover</u> of X is a collection $\{S_\alpha\}_{\alpha\in A}$ of open sets s.t. $X = \bigcup S_\alpha$. It has a <u>finite subcover</u> if $\exists$ a finite set $B \subset A$ s.t. $\bigcup_{T\in B} T = X$.

This is an unusual condition: Let's see how to use it:

Main way to use it: Pick a clever cover.

Thm: If X is ~~a~~ compact, ~~metric space, then~~ $f: X\to\mathbb{R}$ is cts, then f is bounded.

Pf: Consider $S_n = f^{-1}((-n,n))$. These are open sets and $\bigcup_n S_n = X$. So $\exists$ a finite subcover,

$X = S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_k} \subset S_N$ where $N = \max\{n_1, \ldots, n_k\}$.
$\Rightarrow f(X) \subset [-N, N]$. //

~~EVT: If X compact, $f: X \to \mathbb{R}$, then f achieves a max on X.~~

Pf: ~~Suppose not. Let X Suppose not. Let $m = \sup f(X)$, and suppose that~~ Then ~~$g(x) = \frac{1}{f(x) - m}$~~ is continuous

~~$\Rightarrow g$ is bounded $\Rightarrow X$~~

Ex: Show that f achieves its max and min on X.

Can use cptness to approx ~~by s'thing~~ finite:

Thm: If X is compact, it is totally bounded $\Leftrightarrow$
($\forall \varepsilon > 0$, X can be covered by finitely many ε-balls)

Pf: Consider $\{B(x, \varepsilon)\}_{x \in X}$ //.

Thm: If X is a compact ~~metric space~~ and (x_n) a sequence in X, then (x_n) has a ~~convergent subsequence~~ point of accumulation.

~~Pf: (Note: Watch carefully)~~

Def: A sequence (x_n) has a point of accumulation c if $\forall$ nbhd U of c, $\exists$ infinitely many n s.t. $x_n \in U$.

If X is a metric space, and (x_n) accumulates to c, then $\exists$ a subseq (x_{n_i}) s.t. $x_{n_i} \to c$ (but not in general).

Pf: $\forall x \in X$, $\exists$ nbhd U_x s.t. $\{n \mid x_n \in U_x\}$ is finite. Take a cover. Finitely many finite sets. Note: in fact ctably compact.

Pf: Let $F_n = \overline{\{x_n, x_{n+1}, x_{n+2}, \ldots\}}$, ~~$E_n = X \setminus F_n$~~ $E_n = X \setminus F_n$. Then E_n is open.
~~Then E_n is open.~~

Claim: $\bigcap_{n=1}^{\infty} F_n = \{$points of accumulation of $(x_n)\}$.

If (x_n) accumulates to c, then $\forall$ nbhd U of c,
$\forall n$ $U \cap \{x_n, x_{n+1}, \ldots\} \neq \emptyset$. $\Rightarrow c \in F_n$ $\forall n \Rightarrow c \in \bigcap F_n$.

If (x_n) doesn't accumulate to c, then $\exists$ nbhd U of c
~~Let $c \in \bigcap F_n$, then $\forall$ nbhd U of c, $U \cap \{x_n, x_{n+1}, \ldots\} \neq \emptyset$. If $\{n \mid x_n \in U\}$ were finite~~

Let $c \in X$ s.t. ~~suppose~~ $\{n \mid x_n \in U\}$ is finite. Then this has a max N. ~~for some nbhd U of c.~~
$U \cap F_{N+1} = \emptyset \Rightarrow c \notin F_{N+1}$.

Suppose (x_n) has no accum pts. Then $\bigcup_{n=1}^{\infty} E_n = X \setminus \bigcap_{n=1}^{\infty} F_n = X$.
So $\{E_n\}$ is an open cover. $\Rightarrow \bigcup_{n=1}^{N} E_n = X$. But this doesn't contain x_{N+1}. $\times$

In particular, if X is a metric space, then compact $\Rightarrow$ sequentially cpt. (breaks)

~~More generally: Before the break, ~~we talked about~~ mostly metric spaces. Now let's work~~
~~more generally~~: Before the break, ways to use cpt ness. Now more general.
Def: X is compact if every open cover of X has a finite subcover.

Thm: Suppose X is a space, $Y \subset X$ is a subspace. Then:
- Y is compact $\iff$ every ~~covers~~ of Y by open subsets of X has a finite subcover
- If X is compact, Y is closed, then Y is compact.
- If X is Hausdorff, Y is compact, then Y is closed.

Pf: 1, 2: ~~easy~~ leave as exercise.
3: I want to show you something: ~~Arguments about compactness~~
Compact is v. close to finite. Often, ~~you can argue~~ ~~ab~~ you can take an argument about finite sets, make an arg about compact.

Ex: If X is Hausdorff, Y is finite then Y is closed.
Pf: Let $Y = \{y_1, \dots, y_k\}$, suppose $x \in X \setminus Y$.
ETS that $\exists$ a nbhd U of x s.t. $U \cap Y = \phi$.
Since X is Hausdorff, $\forall$ ~~x~~ i, $\exists$ B_i, C_i s.t. $B_i \ni y_i$, $C_i \ni x$
$B_i \cap C_i = \phi$. Then $\bigcap_{i=1}^k C_i$ is a nbhd of x s.t. $y_j \notin \bigcap_{i=1}^k C_i \; \forall j$.

And same for cpt:
Pf: $\forall y \in Y$, $\exists$ B_y, C_y open sets, $B_y \ni y$, $C_y \ni x$.
Can't take $\bigcap_{y \in Y} C_y$, but $\exists\, y_1, \dots, y_k$ s.t.
$\bigcup_{i=1}^k B_{y_k} \supset Y$. Then $\bigcap_{i=1}^k C_{y_k} \ni x$ and $Y \cap \bigcap_{i=1}^k C_{y_k} = \emptyset$ //

So: Thm (Heine-Borel): ~~Every compact~~ If $S \subset \mathbb{R}$ then
S is compact $\iff$ S is closed and bounded.

Pf: compact $\Rightarrow$ closed ✓ compact $\Rightarrow$ bounded ✓.
$\Leftarrow$: ~~ETS that~~ If S is closed, bounded then $S \subset [-N, N]$
for some N. ~~So ETS that $[-N,N]$ is compact.~~
A closed subset of a compact space is cpt, so ETS that $[-N, N]$ is compact.

Suppose $[-N, N] = \bigcup_{\alpha \in A} S_\alpha$.

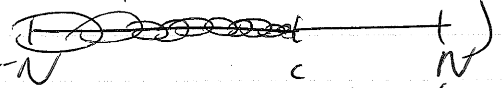
Finite sub. using LUB:

Let $c = \sup\{x \mid x \le N$ and $[-N, x])$ is covered by finitely many S_α's$\}$.

~~Then $c \ge N$.~~ ~~Then $c > -N$.~~

~~A priori:~~

Claim: $[-N, c]$ is covered by fin. many S_α's and $c = N$.

(Main issue: )

Let $S_\alpha \ni c$. Let $\varepsilon > 0$ s.t. $(c-\varepsilon, c+\varepsilon) \subset S_\alpha$.

Then there's a finite cover of $[-N, c-\frac{\varepsilon}{2}]$. Add S_α to this to get a finite cover of $[-N, c]$.

~~Suppose~~ So $[-N, c]$ has a finite cover. ~~Further, if $c < N$, then~~ $\times$. In fact, this covers $[-N, c+\frac{\varepsilon}{2})$, so if $c < N$, then $\times$ //

~~One last thing~~:

Finally: Cptness is preserved by cts maps:

Lemma: If X is cpt, $f : X \to Y$ cts, then $f(X)$ is cpt.

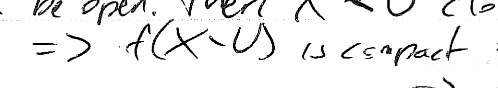
Pf: Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $f(X)$. Then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of $X \Rightarrow \{f^{-1}(U_\alpha)\}$ has a finite subcover $\Rightarrow \{U_\alpha\}$ has a finite subcover. //

Cor: If X is compact, Y is Hausdorff, and $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.

Pf: Need: f^{-1} is cts. I.e., if $U \subset X$ is open, then $f(U)$ is open.

Let $U \subset X$ be open. Then $X \setminus U$ closed $\Rightarrow X \setminus U$ compact $\Rightarrow f(X \setminus U)$ is compact $\Rightarrow f(X \setminus U)$ is closed $\Rightarrow f(U)$ is open. //

(Not true in general:



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