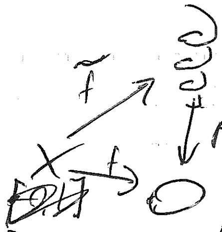
2021-10-21

Last time: Goal: $\pi_1(S^1) = \mathbb{Z}$.
~~Lemma:~~ Technique: lifting:
$p(t) = (\cos 2\pi t, \sin 2\pi t)$



Q's about quotient spaces, complexes

Thm: If $q: X \to X/\sim$ is quotient map, then $f: X/\sim \to Y$ is cts $\iff$ $f \circ q$ is cts.

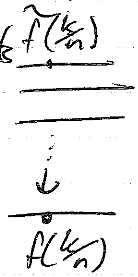
Cor: For every cell e^n_α of a CW complex X, $\exists$ a ~~characteristic~~ map $\Phi_\alpha: e^n_\alpha \to X$. f cts $\iff$ comp w/ chars are all cts.

Thm: Let ~~X be~~ Y be metrizable. ~~Let X~~ If $f: X \to Y$ is a proper cts bijection (preimage of cpt sets is cpt) then f is a homeo.

Lemma: Let $x_0 = (1, 0)$. If $f: [0,1] \to S^1$, $f(0) = x_0$, then $\exists!$ lift $\tilde{f}: [0,1] \to \mathbb{R}$ s.t. $\tilde{f}(0) = 0$.

Pf: Let $n > 0$ be large ~~enough~~ enough that if $I_k = [\frac{k}{n}, \frac{k+1}{n}]$ then $f(I_k) \subset J_k$, where J_k is an interval of length $\frac{1}{2}$.

Construct the lift inductively. Suppose a unique lift exists ~~constructed~~ on $[0, \frac{k}{n}]$ - extend to I_k.



Identify $p^{-1}(J_k)$ with $J_k \times \mathbb{Z}$.

$$\begin{array}{c} J_k \times \mathbb{Z} \\ \downarrow p \\ I_k \longrightarrow J_k \end{array}$$

Then $\tilde{f}(\frac{k}{n}) = (f(\frac{k}{n}), m)$
~~Define $\tilde{f}(x) = (f(x), m)$ for $x \in I_k$.~~
~~Thus Any continuous extension to I_k must have~~
Define $\tilde{f}(x) = (f(x), m)$, $x \in I_k$ — this ~~is the unique extension to~~ extension is the unique lift.

By induction $\exists$ a unique lift on $[0,1]$. //

This lets us define a map ~~$\pi_1(S^1, x_0) \to \mathbb{Z}$~~ $\{$loops based at $x_0\} \to \mathbb{Z}$:
$w(f) = \tilde{f}(1)$. To show this is well-defined:

Lemma: Let $h: [0,1]^2 \to S^1$, $h(0,0) = x_0$. Then $\exists!$ lift $\tilde{h}: [0,1]^2 \to \mathbb{R}$ s.t. $\tilde{h}(0,0) = 0$.

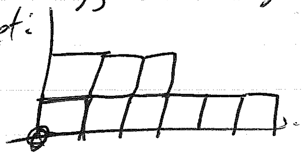
Pf: Same idea: let n s.t. $D_{j,k} = [\frac{j}{n}, \frac{j+1}{n}] \times [\frac{k}{n}, \frac{k+1}{n}]$ has $h(D_{j,k}) \subset J_{j,k}$ for some interval $J_{j,k}$.

How to induct?

$$\begin{tikzcd} & & J_{j,k} \times \mathbb{Z} \arrow[d] \\ A \arrow[urr, "\tilde{h}|_A"] \arrow[r, phantom, "\subset"] & D_{j,k} \arrow[ur, "\tilde{h}"] \arrow[r, "h"] & J_{j,k} \end{tikzcd}$$

If $A \subset D_{j,k}$ is nonempty and connected, then $\tilde{h}$ can be extended uniquely from A to $D_{j,k}$.

So: induct, ~~make sure that we always exten~~ by always extending from a nonempty connected set:



Thm: $\pi_1(S^1, x_0) \cong \mathbb{Z}$.

Pf: Let $w(f) = \tilde{f}(1)$ as above. If $f \simeq g$, then $\exists$ a htpy

$$\begin{array}{ccc} & f & \\ x_0 & \boxed{\;h\;} & x_0 \\ & g & \end{array} \quad \text{lifts to} \quad \begin{array}{ccc} & \tilde{f} & \\ 0 & \boxed{} & w(f) \\ & & \| \\ & \tilde{g} & w(g) \end{array}$$

So w is well-defined.

Homomorphism:

If $[f],[g] \in \pi_1(S^1, x)$, then ~~$[f\cdot g$~~ $w(f\cdot g) = w(f) + w(g)$.

~~Let~~ Let $\tau_n : \mathbb{R} \to \mathbb{R}$, $\tau_n(x) = x+n$. ~~If f~~ Then $\tau_n \circ \tilde{f}$ is the lift of f s.t. $\tau_n(\tilde{f}(0)) = n$. ~~So~~ So $\widetilde{f\cdot g} = \tilde{f} \cdot (\tau_{w(f)} \circ \tilde{g})$ and $\widetilde{f\cdot g}(1) = \tilde{f}(1) + \tilde{g}(1)$.

Surjective: Let $f_n(t) = (\cos(2\pi n t), \sin(2\pi n t))$ — then $\tilde{f}_n(t) = nt$, $w(f_n) = n$.

Injective: Suppose $w(f) = 0$. Then $\tilde{f}(0) = \tilde{f}(1) = 0$ so $\tilde{f}$ is a loop. Let $h_u(t) = p((1-u)\tilde{f}(t))$.

Then: $h_0(t) = p(\tilde{f}(t)) = f(t)$

$h_1(t) = h_u(0) = p((1-u)\cdot 0) = x_0$.

So h is a homotopy from f to a point ~~$[f] = 0$ in $\pi_1(S)$~~

f is null-homotopic //

Want to use this to prove some topological results — but first, one more property:

Functoriality:

Def: ~~If~~ For $A \subset X$, $B \subset Y$, a map of pairs $f : (X, A) \to (Y, B)$ is a map $f : X \to Y$ s.t. $f(A) \subset B$.

In particular, if (X, x_0) is a based space (i.e, $x_0 \in X$ is a basepoint), then

$\pi_1(X, x_0) = \{$htpy classes of paths $x_0 \to x_0\}$
$= \{$htpy classes of based maps $(S^1, s_0) \to (X, x_0)\}$ (wrt based homotopies).

Thm:

If $\alpha : (X, x_0) \to (Y, y_0)$, then there is a homomorphism

$$\alpha_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$
$$[\gamma] \longmapsto [\alpha \circ \gamma]$$

~~If~~ Further, if $\beta : (Y, y_0) \to (Z, z_0)$, then $\beta_* \circ \alpha_* = (\beta \circ \alpha)_*$.

and $(id_X)_* = id_{\pi_1(X, x_0)}$.

Well-defined: if h_t is a htpy from γ_1 to γ_2, then $\alpha \circ h_t$ is a htpy from $\alpha \circ \gamma_1$ to $\alpha \circ \gamma_2$.

Homomorphism: $\alpha \circ (\gamma_1 \cdot \gamma_2) = (\alpha \circ \gamma_1) \cdot (\alpha \circ \gamma_2)$.
Functorial: $\cancel{\alpha}(\beta \circ \alpha) \circ \gamma = \beta \circ (\alpha \circ \gamma)$
Identity: $id \circ \gamma = \gamma$. //

Further, this map is homotopy invariant: if $\alpha, \alpha' : (X, x_0) \to (Y, y_0)$ are homotopic as based maps, then $\alpha_* = \alpha'_*$.

This lets us turn topological problems into algebra.

Def: If $A \subset X$, a <u>retraction</u> of X to A is a map $r : X \to A$ s.t. $r(a) = a$ for all $a \in A$.

Thm: There is no retraction from $D^2 \to S^1$.
Pf: Suppose $r : D^2 \to S^1$ is a retraction. ~~Then~~ Let $i : S^1 \hookrightarrow D^2$ be the inclusion. Then

$$r \circ i = id_{S^1}$$
$$r_* \circ i_* = id_{\pi_1(S^1)}$$

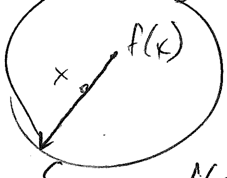
But $i_* : \pi_1(S^1) \to \pi_1(D^2) = 0$, so $r_* \circ i_* = 0$ ✗.

We often draw diagrams:

$$\begin{tikzcd} S^1 \arrow[rr, "id"] \arrow[dr, hook, "i"'] & & S^1 \\ & D^2 \arrow[ur, "r"'] & \end{tikzcd} \Longleftrightarrow \begin{tikzcd} \mathbb{Z} \arrow[rr, "id"] \arrow[dr, "i_*"'] & & \mathbb{Z} \\ & 0 \arrow[ur, "r_*"'] & \end{tikzcd}$$

In fact, Thm: if r is a retraction $X \to A$, then
Consequently: $i_* : \pi_1(A) \to \pi_1(X)$ is injective: diagram.

Thm (Brouwer Fixed Point Theorem): If $f : D^2 \to D^2$, then $\exists x \in D^2$ s.t. $f(x) = x$.
Pf: Suppose not. For all $x \in D^2$ let
$g(x)$ = end point of the ray from $f(x)$ to x:
This is continuous and $g(x) = x$ $\forall x \in S^1$. ✗



Thm (Fundamental Theorem of Algebra): Suppose $f(z) = a_n z^n + \dots + a_0$ where $n > 0$, $a_n \neq 0$, $a_i \in \mathbb{C}$. Then f has a root.

Suppose f has no roots.

Pf: ~~let f~~ $\forall r \geq 0$, let
$$\gamma_r(t) = \frac{f(re^{2\pi i t})/f(r)}{|f(re^{2\pi i t})/f(r)|}$$

Then $\gamma_r(0) = \gamma_r(1) = 1$ and $\gamma_r : [0,1] \to S^1$ and γ_r is a homotopy.

Since γ_0 is constant, $[\gamma_r] = 0 \in \pi_1(S^1)$ $\forall r$.

If R ~~is~~ sufficiently large, then $|a_n R^n| > |a_{n-1}R^{n-1}| + \cdots + |a_0|$.

Let $f_s(z) = a_n z^n + (1-s)(a_{n-1}z^{n-1} + \cdots + a_0)$. Then f_s has no zeroes on the circle $|z| = R$. Let
$$\lambda_s(t) = \frac{f_s(Re^{2\pi i t})/f_s(R)}{|f_s(Re^{2\pi i t})/f_s(R)|}$$
— this is a htpy from $\lambda_0 = \gamma_R$ to $\lambda_1(t) = e^{2\pi n i t}$ ~~$\neq 0$~~ ~~and $[\lambda_1] \neq 0 \in \pi_1(S^1)$~~. But $w(\lambda_1) \neq 0$, so ~~$\lambda_1 \not\simeq$~~
$$0 \simeq \gamma_R \simeq \lambda_1 \not\simeq 0 \quad \times$$

~~To go further, we Other spaces~~?

~~More functionality~~:

~~First, spaces that are "like" S^1.~~ Thm: ~~If r is a retraction, then r_*~~ is inj

Def: ~~Ade~~ If $A \subset X$ and $r : X \to A$ is a retraction, we say that r is a <u>deformation retraction</u> if $\exists$ a htpy $h_t : X \to X$ s.t. $h_0 = id$, $h_1 = r$, and $h_t(a) = a$ $\forall a \in A$, $t \in [0,1]$

— i.e., $r \simeq id$ through maps that fix A ptwise.

Thm: If r is a deformation retraction, then ~~$\pi_1(X)$~~ $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is an isomorphism.

Pf: ~~If r is a DR, then $i \circ r \simeq id_X$ by a htpy fixing a_0,~~ ~~As before,~~ Let $i : A \hookrightarrow X$. ~~As~~ Before, we saw that $r_* \circ i_* = id_{\pi_1(A)}$. Conversely, $i \circ r \simeq id_X$ by a htpy fixing a_0, so $(i \circ r)_* = i_* \circ r_* = id_{\pi_1(X)}$ //

And this broadens ~~the~~ our scope considerably:
$\mathbb{R}^2 \setminus \{0\}$, ~~graphs~~ cylinders, etc.