(2021-11-04)

Last time: $\pi_1(S^n) \cong 0 \ \forall n \geq 2$. One quick consequence before moving on.

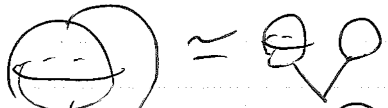
Thm (Borsuk–Ulam): Let $f: S^2 \to \mathbb{R}^2$. Then $\exists x \in S^2$ s.t. $f(x) = f(-x)$.

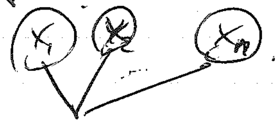
~~Pf:~~ (Note: Similar for $f: S^1 \to \mathbb{R}$ — let $g(x) = f(x) - f(-x)$
~~Then thm follows from intermediate value thm~~) Claim: $\exists x$ s.t. $g(x) = 0$.
Then g is cts, $g(x) = -g(-x)$ — so thm follows by IVT.

~~Here, need sthing higher dim~~ Now for the ~~sphe~~ 2-sphere.
Suppose $\nexists x$ s.t. $f(x) = f(-x)$. Let $g(x) = \dfrac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$ —
this is a cts map $g: S^2 \to S^1$ s.t. $g(x) = -g(-x)$.

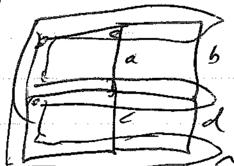
Consider the equator $\gamma(t) = (\cos(2\pi t), \sin(2\pi t), 0)$.
and the image $g \circ \gamma$. On one hand — $\gamma \simeq 0 \Rightarrow g\circ\gamma \simeq 0$.
OTOH, WLOG $g\circ\gamma(0) = x_0$, $g\circ\gamma(\frac{1}{2}) = -x_0$. Consider $\widetilde{g\circ\gamma}: [0,1] \to \mathbb{R}$.
Then $\widetilde{g\circ\gamma}(0) = 0$, $\widetilde{g\circ\gamma}(\frac{1}{2}) = k + \frac{1}{2}$. Further, $g\circ\gamma|_{[\frac{1}{2},1]}$ is a copy of $g\circ\gamma|_{[0,\frac{1}{2}]}$
so $\widetilde{g\circ\gamma}(\frac{1}{2}+t) = k + \frac{1}{2} + \widetilde{g\circ\gamma}(t)$
$\Rightarrow \widetilde{g\circ\gamma}(1) = 2k+1$ is odd. This contradicts $g\circ\gamma \simeq 0$.

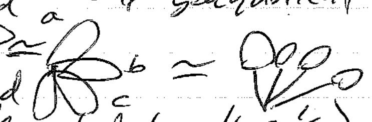
Today: Combining spaces.

Recall: Htpy equivs: 

lots of spaces are htpy equiv to a <u>bouquet</u>: 

Ex: Let X be a connected finite graph.
Then X has a <u>spanning tree</u> T (a tree that contains every vertex).
Spanning trees are maximal — if you add any edge, it forms a loop.
So if you quotient by the tree

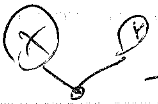




(Same is true for infinite graphs, but have to deal with ∞'s)

Q: π_1 of a bouquet?

Def: Let (X, x_0), (Y, y_0) be spaces. The <u>wedge sum</u> of X and Y is
$X \vee Y = \dfrac{X \sqcup Y}{x_0 \sim y_0}$. The bouquet of X and Y is

$W =$  If X, Y are complexes, then these are htpy equivalent.
~~We'll work with this because it's a little easier.~~

If X, Y are complexes then $\pi_1(X) * \pi_1(Y)$

Prop: $\pi_1(X \vee Y)$ is the <u>free product</u> of $\pi_1(X)$ and $\pi_1(Y)$.

Def: ~~$\pi_1(X) * \pi_1(Y)$ is the group consisting of words~~
~~A word in~~ A <u>word</u> in the alphabet $\cancel{\{\pi_1(X) \sqcup \pi_1(Y)\}}\ S$ is a formal product $w = \prod_{i=1}^{n} a_i$ where $a_i \in \cancel{\pi_1(X) \sqcup \pi_1(Y)}\ S$
$\in S^*$

The <u>free group</u> $\cancel{\pi_1(X) * \pi_1(Y)}\ G * H$ is the group consisting of words in $(G \sqcup H)^*$ under the equiv relation of "obvious relations":

$$w g g' w' \sim w(gg')w'$$
$$w h h' w' \sim w(hh')w'$$
$$w e_G w' \sim w w' \sim w e_H w'.$$

with ~~operation~~ concatenation as the group operation.

So $(a_1 \cdots a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \cdots a_2^{-1} a_1^{-1}$

$e_{G*H} = e_G = e_H = \emptyset$ (empty word).

~~Special ex~~ Ex: $\mathbb{Z} * \mathbb{Z} = F_2$ = free group of rank 2.
$\langle a \rangle * \langle b \rangle$

$F_2 = \{e, a, b, a^{-1}, b^{-1}, aa, ab, ab^{-1}, ba, bb, ba^{-1}, \ldots\}$

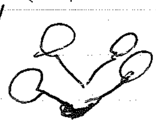
Prop: ~~Every element of~~ $G * H$ has a unique "<u>reduced</u>" form, $w = \prod a_i$ where $a_i \notin \{e_G, e_H\}\ \forall i$ and elements of G and H alternate

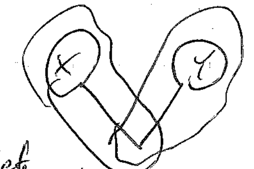
— If K is a group, $a: G \to K$, $b: H \to K$, then there is a unique homomorphism $f: G * H \to K$ s.t.
$f(g) = a(g)\ \forall g \in G$
$f(h) = b(h)\ \forall h \in H$.

(break): $\pi_1(W) = \pi_1(G) * \pi_1(H)$

Let X, Y be path-connected

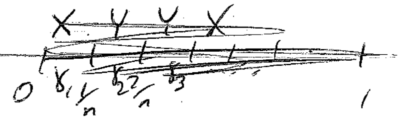
Prop: Let (W, w_0) be a bouquet of (X, x_0) and (Y, y_0). Then
$\pi_1(X, x_0) * \pi_1(Y, y_0) \longrightarrow \pi_1(W)$ is a surjection
(in fact, iso ~~but surj~~, but we'll see that later.)

Pf: Let γ be a loop in W. We want to write γ as a product of loops in X, loops in Y. But we need a ~~little work to~~ to avoid 



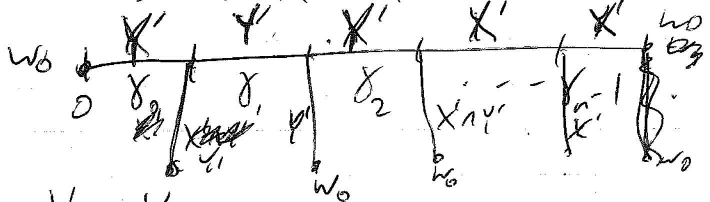
Let X' be a nbhd of left side ~~&~~
Let Y' be a nbhd of right side
so that ~~$X' \cap Y'$ is contractible~~ X' def retracts to X, Y' ~~def~~ $\to Y$
Let $\gamma: [0,1] \to W$.
Let $U_1 = \gamma^{-1}(X)$, $U_2 = \gamma^{-1}(Y)$. This is an open cover of $[0,1]$,
so $\exists\ n > 0$ s.t. every $\frac{1}{n}$-ball in $[0,1]$ lies in U_1 or U_2.

~~So cut up $[0,1]$:~~



Let $\gamma_i = \gamma|_{[\frac{i}{n}, \frac{i+1}{n}]}$ so $\gamma = \gamma_0 \gamma_1 \cdots \gamma_{n-1}$ where each γ_i is a path in X' or Y'.

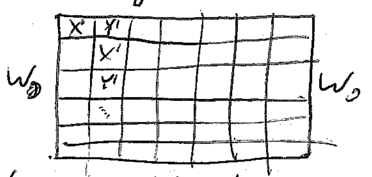
$\forall i = 0 \ldots n$, let λ_i be a path from w_0 to $\gamma(\frac{i}{n})$ s.t.
if $\gamma(\frac{i}{n}) \in X'$, then $\lambda_i \subset X'$
if $\gamma(\frac{i}{n}) \in Y'$, then $\lambda_i \subset Y'$
if $\gamma(\frac{i}{n}) \in X' \cap Y'$, then $\lambda_i \subset X' \cap Y'$.



Then $\forall i$, $w_i = \lambda_i \gamma_i \overline{\lambda_{i+1}}$ is a loop based at w_0 which lies in X' or Y'.
Also $\gamma \simeq$ ~~$(\lambda_0 \gamma_0 \bar{\lambda}_1)$~~ $w_0 w_1 \cdots w_{n-1} \in \pi_1(X) * \pi_1(Y)$. //

What if we want to prove that $\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y)$?

Suppose that $\gamma = \alpha_1 \cdots \alpha_m$, $\alpha_i \in \pi_1(X) \sqcup \pi_1(Y)$ and $\gamma \simeq 0$.
Need to show that $\gamma = e$ by using a htpy h.
Idea:



but implementation is quite tricky.

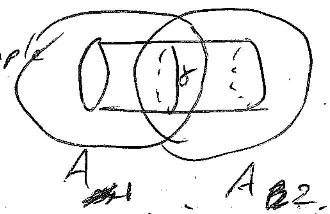
Instead, lets state the general version:

Thm (van Kampen): Let $X = \bigcup_\alpha A_\alpha$ is a union of open sets that share a basepoint x_0, ~~then:~~
~~If each intersection $A_\alpha \cap A_\beta$ is pathconnected,~~
~~then~~ Let $\Phi: *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ be the map ~~extending the maps induced by incl~~ that agrees with $(i_\alpha)_*$ on $\pi_1(A_\alpha)$
where $i_\alpha: A_\alpha \hookrightarrow X$.

~~Let~~ For $[f] \in \pi_1(A_\alpha)$, write $[f]_\alpha$ ~~$= \Phi([f])$~~

Then: ① If $A_\alpha \cap A_\beta$ is path-connected $\forall \alpha, \beta$, then Φ is surjective.

② Typically not injective: Trivial example:



$[\gamma] \in \pi_1(A_1 \cap A_2)$

So $\pi_1(A_1) * \pi_1(A_2)$ contains two copies of γ — $[\gamma]_1 \in \pi_1(A_1)$, $[\gamma]_2 \in \pi_1(A_2)$. And $\Phi([\gamma]_1) = (i_1)_*([\gamma]_1) = [\gamma]$, $\Phi([\gamma]_2) = (i_2)_*([\gamma]_2) = [\gamma]$

— Same curve. That is, $[\gamma]_1 [\gamma]_2^{-1} \in \ker \Phi$.

(and since the kernel is normal, $w[\gamma_1][\gamma]_2^{-1} w^{-1} \in \ker \Phi \ \forall w$.)

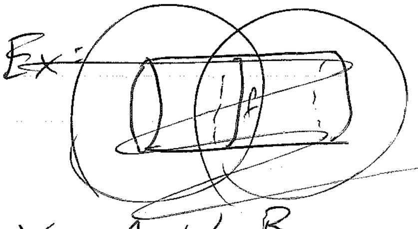
Second part of this is that these are the only things in the kernel

② If $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected for all α, β, γ, then

$$\ker \Phi = \langle w[\gamma]_\alpha [\gamma]_\beta^{-1} w^{-1} \mid \gamma \in \pi_1(A_\alpha \cap A_\beta, x_0), \ w \in *_\alpha \pi_1(A_\alpha) \rangle$$

$$= \langle\langle [f]_\alpha [f]_\beta^{-1} \mid f \in \pi_1(A_\alpha \cap A_\beta) \rangle\rangle$$

~~Ex:~~



And therefore, $\pi_1(X) \cong \dfrac{*_\alpha \pi_1(A_\alpha)}{\ker \Phi}$.

$\pi_1(A) = \langle a \rangle$ ~~$\pi_1(A \cap B) = \langle f \rangle$~~

~~$\pi_1(B) = \langle b \rangle$~~

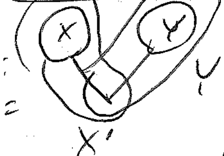
$\pi_1(A) * \pi_1(B) =$ ~~$\langle a, b \rangle$~~

$X = A \cup B$

~~$\pi_1(X) = \Phi(a) = \Phi(b)$~~

For ~~each element of $\pi_1(A \cap B)$, we have an element~~ ~~$\ker \Phi = \langle\langle ab^{-1} \rangle\rangle$~~

Ex:



$W =$ $\quad Y'$ $\quad X'$

$\Phi : \pi_1(X) * \pi_1(Y) \to \pi_1(W)$ is surjective.

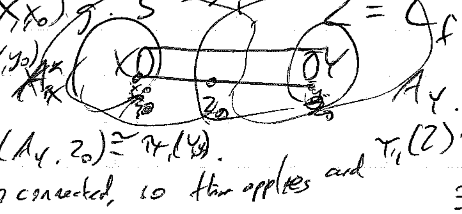
$$\ker \Phi = \langle\langle [f]_X [f]_Y^{-1} \mid f \in \pi_1(X' \cap Y') \rangle\rangle = 0.$$

Nontrivial is harder:

Def: Let X, Y be a space, f: ~~$S^1 \to$~~ $X \to Y$ a map. The <u>mapping cylinder</u> C_f is $C_f = Y \sqcup (X \times [0,1]) \,/\, (x,0) \sim f(x)$.

Then C_f def rets to Y.

Let ~~S^1~~ $f : (S^1, *) \to (X, x_0)$, $g : (S^1, *) \to (Y, y_0)$. $S^1 \to Y$ $\quad Z = C_f \sqcup C_g \,/\, (x,1) \sim (x,1)$



$\pi_1(Z)$?

$\pi_1(A_X, z_0) \cong \pi_1(X_0)$, $\pi_1(A_Y, z_0) \cong \pi_1(Y_0)$.

$A_X \cap A_Y$ is path connected, so this applies and $\pi_1(Z) \cong \dfrac{\pi_1(X) * \pi_1(Y)}{\ker \Phi} \cong \dfrac{\pi_1(X) * \pi_1(Y)}{\langle\langle fg^{-1} \rangle\rangle}$

[2021-11-11]

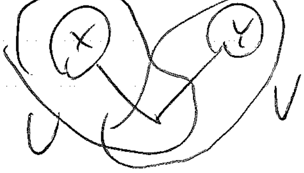
Last time:

Thm (Van Kampen): Let $X = \bigcup_\alpha A_\alpha$ be a union of open sets that share a base point x_0. Let $\Phi: \ast_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ be the canonical map.

① If $A_\alpha \cap A_\beta$ is path-connected $\forall \alpha, \beta$, then Φ is surjective.

② If $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected $\forall \alpha, \beta, \gamma$, then
$\ker \Phi = \langle\langle [f]_\alpha [f]_\beta^{-1} \mid f \in \pi_1(A_\alpha \cap A_\beta, x_0) \rangle\rangle$.

Ex: ~~Easiest case:~~ ~~Let $X = U \cup V$ where $U \cap V$ is path-connected.~~



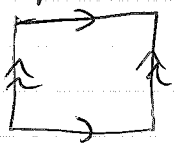
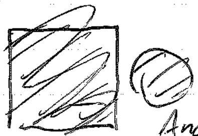
Simplest: $\pi_1(U \cap V) = 0$.

$X = U \cup V$, intersections are path-connected

so $\pi_1(X) \cong \dfrac{\pi_1(U) * \pi_1(V)}{\langle\langle 0 \rangle\rangle}$

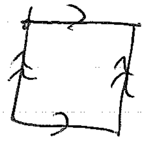
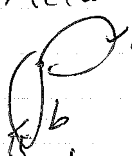
$\cong \pi_1(X) * \pi_1(Y)$

More complicated: $\pi_1(U \cap V) \cong \mathbb{Z}$.

$X =$  $U =$  $V =$  $U \cap V =$ 

~~Actually, let's be careful about~~ And let's be careful about base pts

x_0

U def retracts to  $\cong$ 

$\pi_1(U) = \langle a, b \rangle$

$\pi_1(V) = 0$

$\pi_1(X) = \dfrac{\langle a, b \rangle * \langle 0 \rangle}{\langle\langle [\gamma]_U [\gamma]_V^{-1} \rangle\rangle} = \dfrac{\langle a, b \rangle}{\langle\langle aba^{-1}b^{-1} \rangle\rangle}$

γ ~~retracts to~~ $\gamma \simeq aba^{-1}b^{-1}$

$[\gamma]_U [\gamma]_V^{-1} = aba^{-1}b^{-1}$

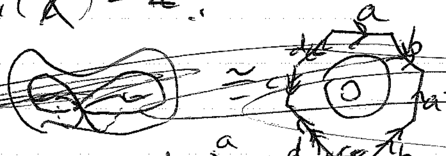
This quotient essentially sets $\gamma = 0$.

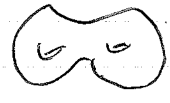
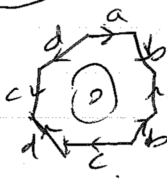
Written $\pi_1(X) \cong \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$

or $= \langle a, b \mid ab = ba \rangle$

i.e. $\pi_1(X) \cong \mathbb{Z}^2$.

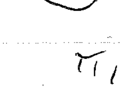
~~Likewise: X $\cong$ ... $\pi_1(X) = \langle a, b, c, d \rangle$~~

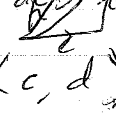


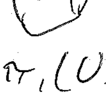
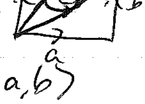
Ex:  $\cong$ 

~~$\pi_1(X) = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}$~~

$\pi_1(X) =$

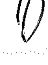
A couple maps:  or:  U  V

$V \cong$ 

 $\cong$ 

$\pi_1(V) = \langle c, d \rangle$

$\pi_1(U) = \langle a, b \rangle$

$U \cap V =$  γ $[\gamma]_U = aba^{-1}b^{-1}$

$[\gamma]_V = cdc^{-1}d^{-1}$. $\pi_1(X) = \dfrac{\pi_1(U) * \pi_1(V)}{\langle\langle [\gamma]_U [\gamma_V]^{-1} \rangle\rangle}$

$= \langle a,b,c,d \mid aba^{-1}b^{-1} = cdc^{-1}d^{-1} \rangle$

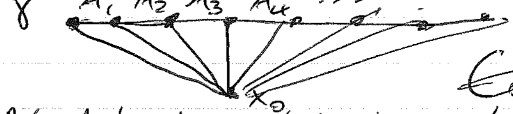
~~More~~ This is an example of an <u>amalgamated free product</u>:

~~$G*$~~ If $A \overset{i}{\hookrightarrow} G$, $A \overset{j}{\hookrightarrow} H$: $G *_A H = \dfrac{G * H}{i(a) \sim j(a)}$

$= \dfrac{G*H}{\langle\langle i(a) j(a)^{-1} \rangle\rangle}$

Pf:

Previously ① ~~Let~~ If $X = \bigcup_\alpha A_\alpha$ and $A_\alpha \cap A_\beta$ is path-conn $\forall \alpha, \beta$, then $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ is surjective.



$\gamma \sim \gamma_1 \gamma_2 \cdots \gamma_n$ where $\gamma_i \in \pi_1(A_i, x_0)$.

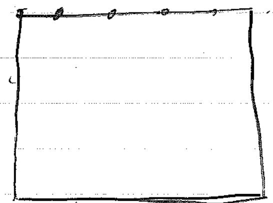
~~Call this a subdivision word for γ~~

② ~~Need to translate homotopies~~. Suppose ~~$x \simeq \gamma$~~.

Suppose $w = [f_1]_{\alpha_1} \cdots [f_n]_{\alpha_n} \sim$

~~Also~~ $w' = [f_1']_{\alpha_1'} \cdots [f_{n'}']_{\alpha_{n'}'}$

Claim: w ~~is equivalent to w' i.e.~~ can be transformed to w' ~~by~~ a series of ops:



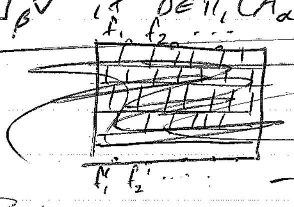
(A) $v[\gamma]_\alpha [\gamma']_\alpha v' \Leftrightarrow v[\gamma\gamma']_\alpha v'$ } same elt.

~~(B) $v[e]_\alpha v' \Leftrightarrow vv'$~~

(B) $v[\gamma]_\alpha v' \Leftrightarrow v[\gamma]_\beta v'$ if $\gamma \in \pi_1(A_\alpha \cap A_\beta)$.

Pf: ~~Take htp $\exists$ htpy~~

~~Pf~~



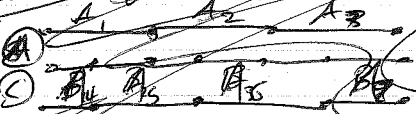
Subdivide into cells s.t. ~~Image of~~ each cell ~~lies in some~~ A_α.

– no more ~~than 3 edges~~ in a lapt.

~~– The vertices include the original verts~~

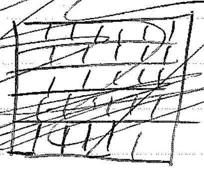
Lemma: ~~If γ is a curve in X~~ $\gamma \in \pi_1(X)$, ~~& and~~ ~~and if w, w'~~ and w, w' ~~are two different subdivision words for γ~~, then ~~w is equivalent to w'~~ ~~break to~~

Pf:



~~take a common subdivision~~ ~~comm~~ ops: ~~– change tails~~ ~~– add vertex~~ ~~– delete vertex~~ – change labeling.

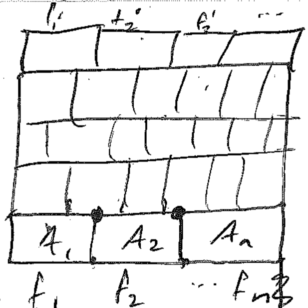
Pf of thm: Take a htpy:



Subdivide so that – each cell lies in some A_α – no more ~~that 3 edges at a vertex~~.

For

Pf: Subdivide a htpy:



- each cell lies in some A_α.
- no more than 3 edges at each vertex.
- The vertices include the original vertices

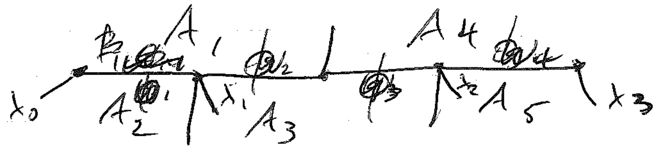
Add a tail to each vertex so that ~~λ~~ $\lambda_v \subset A_\alpha \cap A_\beta \cap A_\gamma$. $\in \pi_1(A_{\alpha_e} \cap A_{\beta_e})$

for each edge $e = (v_1, v_2)$ we have a loop $\lambda_e = \lambda_{v_1} g_e \bar{\lambda}_{v_2}$ word

Then every edge path ~~is a~~ can be read as a ~~word~~.

In fact, typically in multiple equivalent ways:

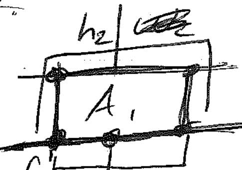
~~Top line is w'~~ ~~Bottom line w.~~



$[\lambda_0 g_1 \bar{\lambda}_1]_{2/1} [\lambda_1 g_2 \bar{\lambda}_2]_{3/1} [\lambda_2 g_3 \bar{\lambda}_3]_{3/4} [g_4]_{5/4} \sim ~~[g_1]_1 [g_2]~~$

but these are equivalent by move (B).

~~Top line is w', bottom is w.~~



~~And for each chunk,~~

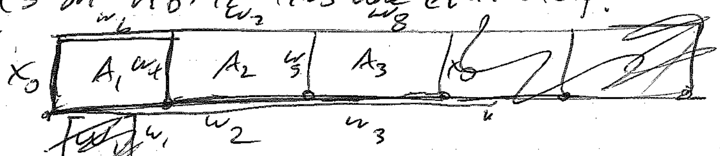
For each 2-cell,

the word reading around circumference

is ~~the ident~~ equiv to the identity by move (A).

So: ~~Each~~ The words on horiz lines are all equivalent.

~~Top line is~~



$w_1 w_2 w_3 \sim w_6 w_4 w_2 w_3 \sim w_6 w_7 w_5 w_3 \sim w_6 w_7 w_8$

top line is w', bottom is w. //

(break)

Overflow.

~~Fundamental groups of~~ complexes:

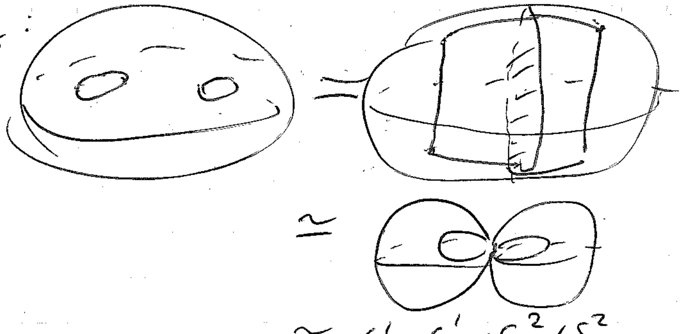
Couple examples w/ knot / link complements:

$\mathbb{R}^3 \setminus S^1$

$\simeq S^1 \vee S^2$

By vK1, $\pi_1(S^1 \vee S^2) \cong \mathbb{Z}$.

Sayme with two loops:

X 

$\simeq S^1 \vee S^1 \vee S^2 \vee S^2$

$\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$.

But: 

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Maybe: 

inside of torus is solid torus.
so is outside

~~Sometimes~~ Alt: $\mathbb{R}^3 \cup \{*\} \cong S^3$

~~Based on following decomp~~: $S^3 =$ two solid tori glued along their boundary.

~~One more~~: Torus knots: Recall: $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$

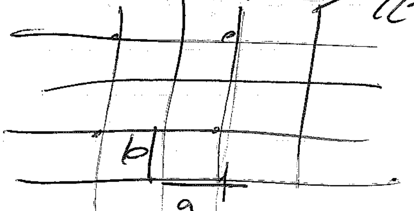
$S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$

~~Embed T in $\mathbb{R}^3$:~~  ~~Let m, n be~~ $K_1 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1, |w| \geq \frac{1}{2}\}$

~~relatively prime.~~ Thm: ~~Then Prop: $a^m b^n$ can be represented by a simple curve on T~~ $\Leftrightarrow$ ~~m, n~~ are relatively prime. K_2 of " " $|w| \leq \frac{1}{2}\}$

$D^2 \times S^1 \to K_1 \quad (z,w) \mapsto \left(\frac{z}{\sqrt{2}}, w\sqrt{1 - \frac{|z|^2}{2}}\right)$

Pf We can write $T = \mathbb{R}^2 / \mathbb{Z}^2$ where $\mathbb{Z}^2$ acts by translation:



$\forall\, m, n \in \mathbb{Z}$

$\gamma : [0,1] \to \mathbb{R}^2/\mathbb{Z}^2$

$\gamma(t) = (mt, nt)$

is a closed curve s.t. $\gamma \sim a^m b^n$. If $\gcd(m,n) = 1$,

this curve is simple (no self int).

Conversely, if $\gamma : S^1 \to \mathbb{R}^2/\mathbb{Z}^2$ is a simple closed curve, then

Thm: $\gamma \sim a^m b^n$ where $\gcd(m,n) = 1$.

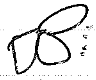
Embed T in $\mathbb{R}^3$ in standard way: 

The (m,n) torus knot is $\mathbb{R}^3 \setminus \gamma_{m,n}$, where $\gamma \subset T$ ~~repre~~ is simple htpic to $a^m b^n$.

$\pi_1(\mathbb{R}^3 \setminus K_{m,n})$ 

2021-11-18

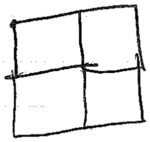
Fundamental groups of complexes

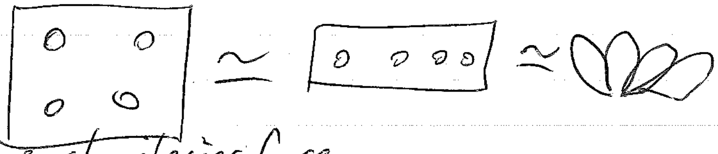
1-dim'l: Let X be a graph. It has a spanning tree $T \subset X$ which is contractible. Then $X \simeq X/T \cong$  – one loop for ~~edges~~ edge in $X \setminus T$.

$\pi_1(X) \cong \langle e_1, \dots, e_k \rangle$ ~~where $k = \#$ of edges in $X \setminus T$~~

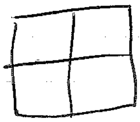
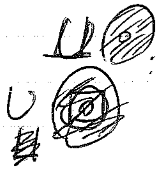
free group with one generator for each edge not in T.

$\operatorname{rank}(\pi_1(X)) = E - V + 1$.

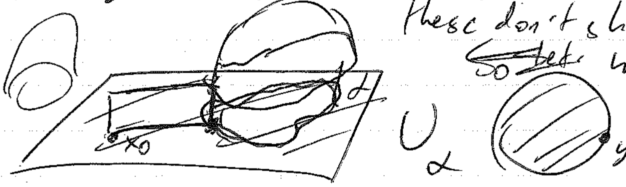
Alt:  ~~for~~ if X is planar, ~~...~~

$\simeq$ 

– one generator for each interior face.

2-dim'l:   : Idea: pop the 2-cells, remove a point in each cell, rest retracts to 1-skeleton, glue on a patch that covers the pt.

But ~~for vKT, we need basepoints. these basepoints are all different~~ these don't share a base point. ~~but~~ ~~So let~~ we can fix this, ~~add cells~~ change the sets, but it's easier to change the space.



U_α $e_\alpha : \partial D \to X$.

Let $Y = X \cup_{e_\alpha} \coprod D_\alpha$

X D

~~Th~~ Let γ_α be a path from $e_\alpha(y_0)$ to x_0

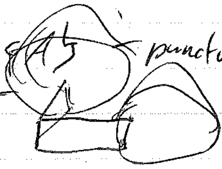
~~Glue a rectangle Let $X' = X \cup_\beta Z$~~
~~where $Z \cong$ and $\beta: Z_0 \to X$ along α and γ.~~

~~Then Let $\alpha' : \partial D \to Z_1$. Then $X \cup_\alpha D \cong X' \cup_1 D$~~

For each α, glue a ~~strip~~ rectangle to X along γ_α. one edge glues to γ_α, one to a segment in D_α.

Call this space Z. For each α, let c_α be a point in D_α.

Let $S = Z \setminus \{p_\alpha \mid \alpha \in A\}$ ← puncture every 2-cell

$T = Z \setminus X$ 

Then $\pi_1(S) = \pi_1(X)$ $\pi_1(S \cap T) = \langle g_\alpha, \alpha \in A \rangle$ ~~where~~

$\pi_1(T) = 0$

~~And~~ So: ~~vKT~~ vKT: $\pi_1(Z) \cong \frac{\pi_1(X) * \pi_1(D)}{\langle\langle [g_\alpha]_S [g_\alpha]_T^{-1} \rangle\rangle}$

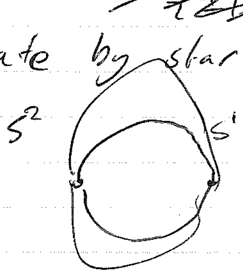
$\cong \frac{\pi_1(X)}{\langle\langle \gamma_\alpha e_\alpha \bar{\gamma}_\alpha \rangle\rangle}$

— ~~And~~ we can compute fund. gps of arbitrary 2-complexes.
~~(good to try this yourself, will put a problem on pset)~~

Ex: If X is a complex, $X^{(2)}$ its 2 skel, then $\pi_1(X) \cong \pi_1(X^{(2)})$.

Ex: Projective plane:
$\mathbb{RP}^n = S^n / \sim$ $\quad x \sim -x \ \forall x \in S^n$. Sometimes $S^n / \mathbb{Z}_2$

Cellulate by starting w/ a cellulation of S^n with action $x \mapsto -x$



w/ 2 cells in each dim from 0 to n.
Then $\mathbb{Z}_2$ ~~acts~~ action sends cells to cells.
And $S^n / \mathbb{Z}_2$ has one cell in each dimension.

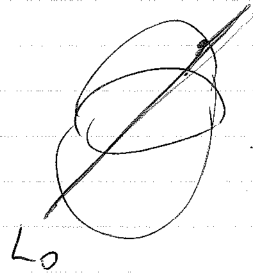
For $n \geq 2$, $\pi_1(\mathbb{RP}^n) \cong \pi_1(\mathbb{RP}^2)$

$\cong \frac{\langle a \rangle}{\langle\langle a^2 \rangle\rangle} \cong \mathbb{Z}_2$.

(And the same calc works with any cellulation - fewer cells make it easier)

Alternatively, for a geometric argument:

$\mathbb{RP}^n \cong \frac{\mathbb{R}^n \setminus \{0\}}{\mathbb{R}^\times} = \{\text{lines through the origin in } \mathbb{R}^n\}$.



Pick a "basepoint" L_0. Consider a loop based at L_0.
~~This is a path that starts and~~
There are two possibilities:
— ~~going around~~ the loop ~~flips~~ preserves the orientation of the line
— the loop flips orientation.

Then: - Loops that preserve orientation are null-homotopic (they correspond to a loop on the sphere).
— Loops that flip orientation are not null-homotopic (every htpic loop ~~also~~ flips too)
— ~~Compositions act~~ Concatenations of loops act ~~like~~ like a group. (flip, then flip = preserve, etc).
In fact, these classes are just $\pi_1(\mathbb{RP}^n)$.

And we can describe this in terms of lifting:

Prev:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[r, "\gamma"'] \arrow[ur, "\tilde{\gamma}"] & S^1 \end{tikzcd} \qquad \begin{tikzcd} & S^n = \text{oriented lines} \arrow[d, "p"] \\ {[0,1]} \arrow[r, "\gamma"'] \arrow[ur, "\tilde{\gamma}"] & \mathbb{RP}^n \cong \text{unoriented lines.} \end{tikzcd}$$

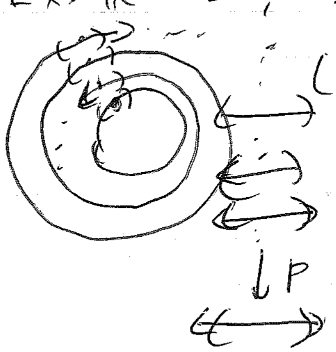
(paths lift, homotopy classes corresp to endpoint of lifting)

Can we generalize?

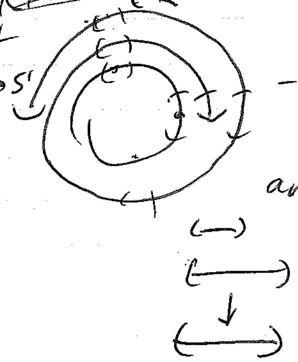
<u>Covering spaces</u>: A <u>covering space</u> $\tilde{X}$ of X is a space $\tilde{X}$ w/ a map $p: \tilde{X} \to X$ s.t. $\forall x \in X$, $\exists U \ni x$ s.t. $p^{-1}(U)$ is a disjoint union of copies of U, ~~&~~ s.t. each copy of U is sent homeomorphically to U by p.

~~So~~, So $p^{-1}(U) \cong U \times E$ where E is discrete.
~~These are called sheets of $\tilde{X}$.~~

Ex: $\mathbb{R} \to S^1$, ~~$S^n \to \mathbb{RP}^n$~~



But not $(0, 3\pi) \to S^1$



components of premiage ~~Premise~~ ~~aren't sent to~~ aren't sent homeomorphically

Ex: $S^1 \to S^1$ $\quad (S^1 = \{z \in \mathbb{C} \mid |z| = 1\})$
$z \longmapsto z^n$

Ex: $S^n \to \mathbb{RP}^n$ ~~If we take a line a collection of nearby lines~~ ~~then~~ Every unoriented line has two oriented lines. If we take a line L, nbhd $U \ni L$, $p^{-1}(U)$ is two copies of U, one oriented one way, one oriented the other way.

Key ~~fact~~ thm:
Lifting: If $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a cover and $f: [0,1]^n \to X$ is a map, s.t. $f(0) = x_0$,

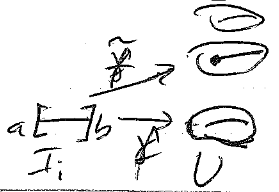
there is a unique lift $\tilde{f}: [0,1]^n \to \tilde{X}$ s.t. $\tilde{f}(0) = \tilde{x}_0$.

Pf: $(n=1)$ ~~Lift short segments~~ Subdivide and lift.

$\forall x \in X, \exists U_x$ s.t. $p^{-1}(U_x) \cong U_x \times E_x$, E_x discrete,
$p(u,e) = u$.

Then $f^{-1}(U_x)$ is an open cover of $[0,1]$. $\exists \varepsilon > 0$ s.t. each $f([a, a+\varepsilon])$ is contained in some U_x.

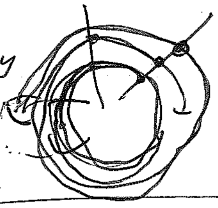
Subdivide $[0,1]$ into ε-intervals, $I_1, \dots, I_k$.



Lift over each interval.

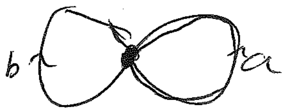
Argue for $n \geq 1$ as before.

This is why $(0, 3\pi) \to S^1$ doesn't work

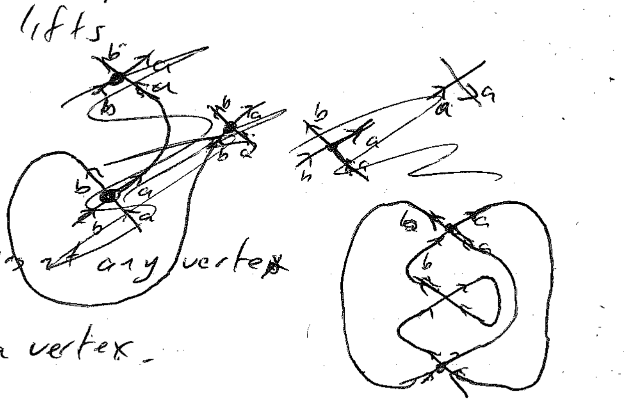


~~But~~ ~~And we ca~~ And we can use this point of view to construct more complex lifts

$X =$



$\tilde{X}$



a and b have lifts starting at any vertex

Follow those lifts, end at a vertex.

- Any graph with edges labeled a and b, with one a ~~edge~~ in and ~~out of each vertex, one b edge in and out~~ of each type in and out of any vertex is a cover of X.

Thm (Fundamental Theorem of Covering Spaces): If X is a connected ~~complex~~ manifold, then there is a bijection

$$\left\{ \begin{array}{c} \text{connected} \\ \text{based covers} \\ \text{of } (X, x_0) \end{array} \right\} \Big/ \text{isomorphism} \longleftrightarrow \left\{ \begin{array}{c} \text{subgroups of} \\ \pi_1(X, x_0) \end{array} \right\}.$$

$$(\tilde{X}, \tilde{x}_0) \longmapsto p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

Ex: circle, free groups