COMPOSING AND DECOMPOSING SURFACES AND FUNCTIONS

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Abstract. In mathematics, we are often drawn to the simple or elegant, but what lies at the other end of the spectrum? How can we build and study complex objects? How can we break them down? In this note, we will describe some tools for building functions and surfaces with structure at many different scales and, conversely, tools for decomposing complex objects into simple pieces. These methods are based on ideas from geometric measure theory and harmonic analysis, and we will give some applications to quantitative and metric geometry.

What makes one object more complex than another? For instance, what makes a high-genus surface more complex than a sphere, or what makes a random graph with n^2 vertices more complex than an $n \times n$ grid? A rough definition of complexity is that complex objects are hard to describe concisely. The *Kolmogorov complexity* of a string of 0's and 1's, for instance, measures the number of bits it takes to describe an algorithm that outputs that bit string. Then, on one hand, the 1000-bit sequence $0, 1, 0, 1, \ldots, 0, 1$ has low Kolmogorov complexity, since it is the output of a simple algorithm. On the other hand, there are at most $2^{k+1} - 1$ possible algorithms that can be described in at most k bits, so a generic string has large complexity: over 99% of the 1000-bit strings have complexity of at least 990 bits.

This highlights one of the properties of complex objects: while nearly all n-bit strings have complexity of at least $.99n$ bits, it is impossible to construct an example of such a string without using randomness — any explicit deterministic construction of an n-bit string is an algorithm, so its complexity is bounded by the length of the algorithm. Situations like this are not uncommon in combinatorics. Erdős [7], for instance, famously bounded the Ramsey numbers by showing that a random $2^{\frac{s}{2}}$-vertex graph is overwhelmingly likely to have no s-vertex cliques or s-vertex independent sets. Nevertheless, no specific graph is known to have this property, and there is no known way to construct such graphs for large s without using randomness.

This is one reason that random graphs, surfaces, and complexes can behave in strange and unexpected ways. All the familiar examples of graphs, surfaces, and complexes can be constructed algorithmically, so they are simple in the sense of Kolmogorov. Complex objects may be generic, but complex objects can be strange and unexpected to an intuition trained on familiar examples.

One may hope, however, that objects in $\mathbb{R}^n$ may behave in more familiar ways. In this note, we will confirm this intuition by constructing objects in $\mathbb{R}^n$ that are as complex as possible and bounding the complexity of such objects by decomposing them into simpler pieces. In Section 1, we will construct Lipschitz functions, and in Section 2, we will construct closed surfaces.

In both cases, we find that the complexity of these objects is bounded by geometric quantities. A Lipschitz function on the unit interval, for instance, has a graph which is a curve in $\mathbb{R}^n$. One can construct a Lipschitz function by starting with a linear function, then perturbing it repeatedly, but each perturbation increases the length of the graph, so the complexity of the function is ultimately bounded by the length of the graph. Likewise, a surface in $\mathbb{R}^n$ may be complex, but it can often be decomposed into a sum of several pieces. If the pieces of the decomposition are simple and their size is bounded, the decomposition gives an efficient description of the surface and bounds its complexity. Finally, in Section 3, we describe some applications of these techniques to geometric measure theory and metric geometry.

1. How to build a function

We start with a simple example. What does a generic 1-Lipschitz function of a single variable look like? (Most of these ideas can be generalized to higher dimensions, but we stick to one dimension for simplicity.) This question turns out to be surprisingly tricky. For example, one possible approach is to discretize; one can construct a Lipschitz function $f\colon [0,1] \to \mathbb{R}$ by choosing some $n \in \mathbb{N}$ and a sequence of bounded i.i.d. random variables $y_1, \ldots, y_n$ and defining

$$f\left(\frac{k}{n}\right) = \sum_{i=1}^{k} \frac{y_i}{n}.$$

We extend to all of $[0,1]$ by linear interpolation; as long as $|y_i| \leq 1$ for all i, this is 1-Lipschitz. For any finite n, this produces potentially interesting 1-Lipschitz functions, but the central limit theorem implies that as $n \to \infty$, these functions tend toward $g(x) = mx$, where m is the mean of the distribution that the y_i are drawn from.

The problem is that there are no nontrivial scale-invariant models of random 1-Lipschitz functions. By Rademacher's theorem, any Lipschitz function is differentiable almost everywhere, and the same is true for random 1-Lipschitz functions; if $f\colon [0,1] \to \mathbb{R}$ is a random 1-Lipschitz function drawn from some distribution, then for almost every $x \in [0,1]$, there exists a random variable $f'(x)$ with $|f'(x)| \leq 1$ such that

$$\mathbf{P}\left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)\right] = 1. \tag{1.1}$$

For any $r > 0$, we can rescale f around x by letting $f_r(h) = r^{-1}(f(x+rh) - f(x))$. Then (1.1) implies that f_r converges almost surely to the random linear function $h \mapsto f'(x)h$. It follows that any scale-invariant distribution on the space of 1-Lipschitz functions must be supported on the space of affine functions.

Instead, we can construct complex Lipschitz functions by combining functions with different scales. The simplest example is a Weierstrass–type construction; if $\phi_k(x) = 10^{-k}\sin(10^k x)$, then $f(x) = \frac{1}{L}\sum_{k=0}^{L-1} \phi_k(x)$ is a 1-Lipschitz function whose graph has bumps at the L different scales $1, 10^{-1}, \ldots, 10^{-L+1}$. The bumps at scale 10^{-k} have height roughly $\frac{1}{L}$ times their width, so we say that f is $\frac{1}{L}$-bumpy at L different scales. (One can construct a random 1-Lipschitz function with similar complexity by replacing the ϕ_k's by random 1-Lipschitz functions that oscillate with amplitude and wavelength roughly 10^{-k}.)

With a little more care, we can make this function more complex. The key is that

$$f'(x) = \sum_{k=0}^{L-1} \frac{1}{L} \cos(10^k x)$$

and there's little correlation between $\cos(10^k x)$ and $\cos(10^l x)$ when $k \neq l$. That is, $f'(x)$ is a sum of L values between $-\frac{1}{L}$ and $\frac{1}{L}$. At $x = 0$, all these values are $\frac{1}{L}$, so $f'(x) = 1$, but for a typical $x \in \mathbb{R}$, these values are close to independent, so $f'(x)$ is typically of order $\frac{\sqrt{L}}{L}$ — much smaller than 1. In fact, by the central limit theorem, if L is large and

$$g(x) = \frac{1}{L} \sum_{k=0}^{L^2} 10^{-k} \sin(10^k x),$$

then g is $\frac{1}{L}$-bumpy at L^2 different scales and the distribution of $g'(x)$ is close to a Gaussian with variance less than 1. While g is not 1-Lipschitz, it's almost Lipschitz in the sense that $|g'(x)| < 5$ for all but a tiny fraction of points, and we can make it 5-Lipschitz by changing it on a tiny fraction of its domain. This produces a Lipschitz function that is $\frac{1}{L}$-bumpy at L^2 different scales.

This is just about the bumpiest a Lipschitz function can be. One way to see this is a theorem of Dorronsoro [5]. Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable. For $x \in \mathbb{R}$ and $r > 0$, we define a quantity $\beta_f(x, r)$ that measures how close f is to affine on $(x - r, x + r)$ by

$$\beta_f(x, r) = \frac{1}{r^2} \min_{\lambda} \int_{x-r}^{x+r} |f(y) - \lambda(y)| \, \mathrm{d}y,$$

where λ ranges over all affine functions. This is normalized to be scale invariant; if $g(x) = cf(c^{-1}x)$, then $\beta_g(x, r) = \beta_f(c^{-1}x, c^{-1}r)$, and Dorronsoro's Theorem implies that if f is supported on $[0, 1]$ and satisfies $\|f'\|_2 < \infty$, then

$$\sum_{n=0}^{\infty} \int_0^1 \beta_f(x, 2^{-n})^2 \, \mathrm{d}x \frac{\mathrm{d}r}{r} \lesssim \|f'\|_2^2. \tag{1.2}$$

It can help to interpret this inequality as an expectation. Let x be a uniformly distributed random point in $[0, 1]$. Then

$$\mathbf{E}_x \left[\sum_{n=0}^{\infty} \beta_f(x, 2^{-n})^2 \right] \lesssim \|f'\|_2^2. \tag{1.3}$$

In particular, for any $L > 0$, the expected number of n's such that $\beta_f(x, 2^{-n}) > \frac{1}{L}$ is at most a constant times L^2 — a 1-Lipschitz function is, for the most part, $\frac{1}{L}$-bumpy at a maximum of roughly L^2 different scales.

The exponent 2 in these bounds comes from the Pythagorean Theorem. By the Pythagorean Theorem, adding a bump of height $\frac{r}{L}$ to a segment of length r multiplies the length by roughly a factor of $\frac{1}{L^2}$, so covering a curve by such bumps (making it $\frac{1}{L}$-bumpy at scale r) increases its length by roughly a factor of $\frac{1}{L^2}$. If $f\colon [0, 1] \to \mathbb{R}$ is a 1-Lipschitz function, its graph is a curve of length between 1 and $\sqrt{2}$. The length is minimized when f is constant, and is larger when f is bumpier. Making f $\frac{1}{L}$-bumpy at a scale increases the length of the graph by roughly $\frac{1}{L^2}$, so the bound on the length of the graph implies that f can be $\frac{1}{L}$-bumpy at no more

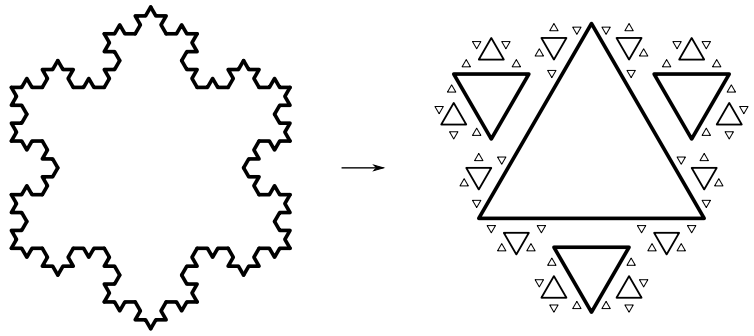

FIGURE 1. A stage in the construction of the Koch snowflake can be decomposed into triangles, and the total length of the triangles is bounded by the length of the original curve.

than roughly L^2 different scales. Going further in this direction leads to similar results for rectifiable curves, like Jones's Traveling Salesman Theorem [13].

2. How to build a surface

Now we turn our attention to surfaces. There are many ways to construct complicated closed surfaces embedded or immersed in $\mathbb{R}^n$. One can, for instance, construct codimension–1 surfaces by embedding a k-complex X in $\mathbb{R}^n$ and letting Σ be the boundary of a regular neighborhood of X; one can construct self-similar surfaces inductively, like the Koch snowflake or the Menger sponge; or one can use general position arguments or the Whitney Embedding Theorem to embed arbitrary k-manifolds in $\mathbb{R}^n$.

Although these surfaces can be complex, they can still be decomposed into simple pieces. For example, if $X \subset \mathbb{R}^n$ is an embedded simplicial complex, then its regular neighborhood R can be decomposed into neighborhoods of individual simplices R_δ. The fundamental class $[R]$ of R is the sum $[R] = \sum_\delta [R_\delta]$ of the fundamental classes of the pieces, and the boundary $A = [\partial R] = \partial [R]$ can be written $A = \sum_\delta \partial [R_\delta]$. Likewise, any step in the construction of the Koch snowflake or the Menger sponge can be written as a sum of the boundaries of equilateral triangles or cubes, as in Figure 1. These decompositions are efficient in the sense that the total area of the pieces is bounded by a multiple of the area of the original surface.

In this section, we will argue that arbitrary surfaces in $\mathbb{R}^N$ can't be too much more complex than Lipschitz functions. That is, given a surface $\Sigma \subset \mathbb{R}^N$, written as a k-cycle $M = [\Sigma] \in C_k(\mathbb{R}^n; \mathbb{Z}_2)$ with coefficients in $\mathbb{Z}_2$, we can write M as a sum $M = \sum_i A_i$ of k-cycles such that each of the A_i's can be approximated by graphs of Lipschitz functions (Lipschitz graphs) with bounded total volume. Furthermore, this decomposition is efficient, i.e., the total area of the A_i's is bounded by a multiple of the area of Σ.

In the following, we take $C_k(X; A)$ to be the set of singular Lipschitz k–chains in X with coefficient group A, i.e., formal sums $M = \sum_i a_i [\delta_i]$ where $a_i \in A \setminus \{0\}$ and $\delta_i \colon \Delta^k \to X$ are distinct Lipschitz k-simplices. When $A = \mathbb{Z}$ or $\mathbb{R}$, we define $\operatorname{mass} M = \sum_i |a_i| \operatorname{vol}^k \delta_i$, where $\operatorname{vol}^k \delta_i$ is the k-dimensional Hausdorff measure of δ_i, counted with multiplicity. When $A = \mathbb{Z}_2$, we define $\operatorname{mass} M = \sum_i \operatorname{vol}^k \delta_i$.

2.1. **Decomposing into cubes.** We first use cellular approximation and the Federer–Fleming Deformation Theorem to decompose cycles into sums of boundaries of cubes.

Let $0 < t < 1$ and let τ_t be the grid of side length t in $\mathbb{R}^n$. By cellular approximation, any chain in $\mathbb{R}^n$ can be approximated by a cellular chain in τ_t. The following special case of the Federer–Fleming Deformation Theorem makes this approximation quantitative.

Theorem 2.1 ([8, 6])**.** *There is a $c_n > 0$ with the following property. Let $t > 0$. Let $T \in C_k(\mathbb{R}^n; A)$ be a singular Lipschitz k-chain over a coefficient group $A = \mathbb{Z}, \mathbb{R}$, or $\mathbb{Z}_\nu$ (i.e., a formal sum of Lipschitz maps $\Delta^k \to \mathbb{R}^n$ such that $\partial T = 0$). Suppose that $\partial T \in C_{k-1}(\tau_t; A)$ is a cellular chain. Then there are a cellular k-chain $P \in C_k(\tau_t; A)$ and a singular Lipschitz $(k+1)$-chain Q such that:*

1. $\operatorname{mass} P \le c_n \operatorname{mass} T$,
2. $\operatorname{mass} Q \le c_n t \operatorname{mass} T$, *and*
3. $\partial Q = P - T$.

In particular, $\partial P - \partial T = \partial^2 Q = 0$, so P and T have the same boundary.

Furthermore, if T is supported in a subcomplex $K \subset \tau_t$, then P is supported in the same subcomplex.

Let $T \in C_k(\tau_1)$ be a cellular k-cycle and let $M = \operatorname{mass} T$. We will decompose T by constructing a sequence of approximations of T. Let $P_0 = T$, and for each $i \ge 1$, let $P_i \in C_k(\tau_{2^i})$ be a cellular approximation of T in τ_{2^i} as in Theorem 2.1.

On one hand, P_i is a sum of k-cells of τ_{2^i}, so $\operatorname{mass} P_i$ is a multiple of 2^{ki}. On the other hand, $\operatorname{mass} P_i \le c_n \operatorname{mass} T$ for all i. Therefore, if $2^{ki} > c_n \operatorname{mass} T$, then $P_i = 0$. Let i_0 be the smallest integer such that $2^{ki_0} > c_n \operatorname{mass} T$. Then $P_{i_0} = 0$, so we can decompose T as

$$T = \sum_{i=0}^{i_0-1} (P_i - P_{i+1}).$$

For each i, $P_i - P_{i+1}$ is a cellular cycle in τ_{2^i}, so there is some cellular chain $R_i \in C_{k+1}(\tau_{2^i})$ such that $\partial R_i = P_i - P_{i+1}$. We can use Theorem 2.1 to find R_i. Let Q_i be a $(k+1)$-chain as in Theorem 2.1 so that $\partial Q_i = P_i - T$ and $\operatorname{mass} Q_i \le 2^i c_n \operatorname{mass} T$. Then $\partial(Q_i - Q_{i+1}) = P_i - P_{i+1} \in C_k(\tau_{2^i})$. That is, $Q_i - Q_{i+1}$ has cellular boundary, so we can apply Theorem 2.1 again to approximate it by a cellular chain $R_i \in C_{k+1}(\tau_{2^i})$ such that

$$\operatorname{mass} R_i \le c_n(\operatorname{mass} Q_i + \operatorname{mass} Q_{i+1}) \le c_n(2^i c_n \operatorname{mass} T + 2^{i+1} c_n \operatorname{mass} T) \lesssim 2^i \operatorname{mass} T$$

and

$$\partial R_i = \partial(Q_i - Q_{i+1}) = P_i - P_{i+1}.$$

We write R_i as a sum $R_i = \sum_j a_{i,j} R_{i,j}$, where the $R_{i,j}$'s are $(k+1)$-cells of τ_{2^i} and $\operatorname{mass} R_i = \sum_j |a_{i,j}| 2^{(k+1)i}$. Then

$$T = \sum_{i=0}^{i_0-1} \partial R_i = \sum_{i=0}^{i_0-1} \sum_j a_{i,j} \partial R_{i,j},$$

decomposes T as a sum of boundaries of cubes.

The number and size of the pieces of the decomposition is bounded in terms of the mass of T. For each $0 \le i \le i_0$, the total mass of the boundaries of the cubes is bounded by

$$\sum_j |a_{i,j}| \operatorname{mass} \partial R_{i,j} \approx \sum_j |a_{i,j}| 2^{ik} = 2^{-i} \operatorname{mass} R_i \lesssim \operatorname{mass} T,$$

so, since $2^{ki_0} \approx c_n \operatorname{mass} T$, we have

$$\sum_{i=0}^{i_0-1} \sum_j |a_{i,j}| \operatorname{mass}(\partial R_{i,j}) \lesssim i_0 \operatorname{mass}(T) \lesssim \operatorname{mass}(T) \log \operatorname{mass}(T).$$

This decomposition and similar decompositions are useful for studying isoperimetric inequalities. Recall that the isoperimetric inequality in $\mathbb{R}^n$ implies that for any k-cycle T, there is a $(k+1)$-chain S such that $\partial S = T$ and $\operatorname{mass} S \lesssim (\operatorname{mass} T)^{\frac{k+1}{k}}$. If R_i are as above, then $S = \sum_{i=0}^{i_0-1} R_i$ satisfies $\partial S = T$, and since $2^{ki_0} \approx c_n \operatorname{mass} T$,

$$\operatorname{mass}(S) \le \sum_{i=0}^{i_0-1} \operatorname{mass} R_i \lesssim \sum_{i=0}^{i_0-1} 2^i \operatorname{mass} T \le 2^{i_0} \operatorname{mass}(T) \lesssim \operatorname{mass}(T)^{\frac{k+1}{k}}.$$

More generally, this decomposition is useful for studying higher-dimensional versions of the *Dehn function* of a group or space, which measure the difficulty of filling a k-cycle in a space by a $(k+1)$-chain. In many cases (see for instance [20]), one can use a version of the Federer–Fleming Deformation Theorem to decompose an arbitrary k-cycle T into a sum of scalings of simple pieces $T = \sum_i S_i$ and construct a filling of T by adding together fillings of the S_i's.

2.2. **An inductive strategy.** One difficulty with this decomposition is that the total volume of the pieces grows when τ_1 is replaced by a finer grid. That is, the decomposition above writes a cycle $T \in C_k(\tau_1)$ as a sum of boundaries of cubes $T = \sum_{i=0}^{M-1} \sum_j \partial R_{i,j}$, where $2^{kM} \approx \operatorname{mass} T$ and $\sum_j \operatorname{mass}(\partial R_{i,j}) \approx \operatorname{mass} T$ for all i; the total mass of the $\partial R_{i,j}$'s is at most $M \operatorname{mass} T$.

Let $m < 0$ and let $T \in C_k(\tau_{2^m})$. By applying the same decomposition to a rescaling of T, we can write $T = \sum_{i=m}^{M-1} \sum_j \partial R_{i,j}$, where each $R_{i,j}$ is a cube of side length 2^i and $2^{kM} \approx \operatorname{mass} T$. Unfortunately, the total mass now satisfies

$$\sum_{i=m}^{M-1} \sum_j \partial R_{i,j} \lesssim (M-m) \operatorname{mass} T,$$

and even if $\operatorname{mass} T$ is fixed, this will get larger as $m \to -\infty$.

Ideally, given $m < 0$ and a cycle $T \in C_k(\tau_{2^m})$, we would like a decomposition $T = \sum_i K_i$ such that $\sum_i \operatorname{mass} K_i \lesssim \operatorname{mass} T$, with constant independent of m; such a decomposition opens up applications in geometric measure theory. In the rest of this section, we will pursue such decompositions.

One such decomposition appears in Wenger's proof of Gromov's Filling Inequality [19]. Gromov's Filling Inequality states the following.

Theorem 2.2. [12] *Let $k > 0$. There is a $c > 0$ such that for any Banach space X and any k-cycle $A \in C_k(X)$, there is a $(k+1)$-chain $D \in C_{k+1}(X)$ such that $\partial D = A$ and $\operatorname{mass} D \le c(\operatorname{mass} A)^{\frac{k+1}{k}}$.*

The methods used in the previous section cannot be used to prove Theorem 2.2, because the constants in the Federer–Fleming Deformation Theorem depend on the dimension of the ambient space. Nevertheless, it is straightforward to prove Theorem 2.2 when A is *round*, i.e., when $\operatorname{diam}\operatorname{supp} A \lesssim (\operatorname{mass} A)^{\frac{1}{k}}$.

Lemma 2.3 (Cone-type inequality)**.** *Let $k > 0$. There is a $c > 0$ such that for any Banach space X and any k-cycle $A \in C_k(X)$, there is a $(k+1)$-chain $D \in C_{k+1}(X)$ such that $\partial D = A$ and $\operatorname{mass} D \le c(\operatorname{mass} A)(\operatorname{diam}\operatorname{supp} A)$.*

Proof. We translate so that $0 \in \operatorname{supp} A$. Suppose that $A = \sum_i a_i[\delta_i]$ for some $a_i \in \mathbb{Z}$ and some Lipschitz simplices $\delta_i \colon \Delta^k \to X$. Let $\overline{\delta_i} \colon \Delta^k \times [0,1] \to X$, $\overline{\delta_i}(x,t) = t\delta_i(x)$. Then $D = \sum_i a_i[\overline{\delta_i}]$ satisfies the desired properties. □

Difficulties arise, however, when A is not round, for example when A is a long skinny cylinder. Wenger proves Theorem 2.2 by decomposing an arbitrary cycle A into a sum of round cycles and applying the cone-type inequality to each piece, and in this section, we will describe a version of his strategy, with some details simplified for the sake of brevity. Any inaccuracies and oversimplifications are entirely our fault.

We say that a k-chain T is *c-round* if $\operatorname{mass} T \ge c(\operatorname{diam}\operatorname{supp} T)^k$. Let $c > 0$ be small. The key idea of Wenger's decomposition is that if A is not c-round, then we can "cut" an open ball B out of A so that the cut-off piece is round. That is, we can decompose $A = A' + M$ so that M is a round cycle with $\operatorname{supp} M \subset \overline{B}$ and $\operatorname{supp} A' \subset X \setminus B$.

We find B by noting that since A is not c-round, there are $x \in \operatorname{supp} A$ and $0 < r < \operatorname{diam}\operatorname{supp} A$ such that $\operatorname{mass}(A \cap B(x,r)) \in [\frac{c}{2}r^k, cr^k]$. We let $B = B(x,r)$ and let $K = A \setminus B$ be the restriction of A to $X \setminus B$, so that $\operatorname{supp} K = (\operatorname{supp} A) \setminus B$ and $\partial K \in C_{k-1}(\partial B)$. Let $L \in C_k(\partial B)$ be a chain such that $\partial L = \partial K$ and let $A' = K - L$ so that A' is a cycle.

If c is sufficiently small and if x and r are chosen carefully, we can arrange that $\operatorname{mass} L \le \frac{c}{4}r^k$,

$$\operatorname{mass} A' = \operatorname{mass} A - \operatorname{mass}(A \cap B) + \operatorname{mass} L \le \operatorname{mass} A - \frac{c}{4}r^k,$$

and

$$\operatorname{mass}(A - A') = \operatorname{mass}(A \cap B) + \operatorname{mass} L \le 2cr^k.$$

Let $M = A - A'$. Then

$$\operatorname{diam}\operatorname{supp}(M) \le \operatorname{diam} B \le 2r \le 2c^{-\frac{1}{k}}(\operatorname{mass} M)^{\frac{1}{k}},$$

so M is $(2^{-k}c)$-round. Furthermore,

$$\operatorname{mass} M \lesssim \operatorname{mass} A - \operatorname{mass} A'. \tag{2.1}$$

Equation (2.1) is important because it lets us use this construction in an inductive argument. Let $c > 0$ be as in the argument above and let $A_0 = A$. If we have constructed A_i and A_i is not c-round, then there is a ball B_i and a decomposition $A_i = A_{i+1} + M_i$ where M_i is a round cycle with $\operatorname{supp} M_i \subset \overline{B}_i$ and $\operatorname{mass} M_i \lesssim \operatorname{mass} A_i - \operatorname{mass} A_{i+1}$. Otherwise, if A_i is c-round, we terminate the construction, letting $A_{i+1} = 0$ and $M_i = A_i$.

If this construction terminates with some $A_n = 0$, we have $A = \sum_{i=0}^{n-1} M_i$ and

$$\sum_{i=0}^{n-1} \operatorname{mass} M_i \lesssim \sum_{i=0}^{n-1} (\operatorname{mass} A_i - \operatorname{mass} A_{i+1}) = \operatorname{mass} A_0 - \operatorname{mass} A_n = \operatorname{mass} A.$$

Otherwise, for any $n > 0$, $A = A_n + \sum_{i=0}^{n-1} M_i$, and

$$\operatorname{mass} A_n + \sum_{i=0}^{n-1} \operatorname{mass} M_i \lesssim \operatorname{mass} A_n + \sum_{i=0}^{n-1} (\operatorname{mass} A_i - \operatorname{mass} A_{i+1}) = \operatorname{mass} A.$$

In general, we cannot guarantee termination, but we can choose the B_i's so that $\lim_i \operatorname{mass} A_i = 0$ and $\operatorname{diam} \operatorname{supp} A_i < 2 \operatorname{diam} \operatorname{supp} A$ for all i. Consequently, for any $\epsilon > 0$, there is an efficient decomposition $A = \sum_{i=0}^{n-1} M_i$ or $A = A_n + \sum_{i=0}^{n-1} M_i$ such that each of the M_i's is round and $\operatorname{mass} A_n < \epsilon$ (if the decomposition has an A_n term). By applying Lemma 2.3 to each summand, one constructs $D_0, \dots, D_n$ such that $\partial D_i = M_i$ and $\partial D_n = A_n$. Let $D = \sum D_i$. Then

$$\operatorname{mass} D \lesssim \epsilon \operatorname{diam} \operatorname{supp} A + \sum_{i=0}^{n-1} (\operatorname{mass} M_i)^{\frac{k+1}{k}},$$

and if ϵ is sufficiently small,

$$\operatorname{mass} D \lesssim (\operatorname{mass} A)^{\frac{k+1}{k}},$$

as desired.

2.3. **Quasiminimizers and uniform rectifiability.** Wenger's proof of Gromov's Filling Inequality suggests a general strategy for constructing efficient decompositions inductively.

(1) Let $A \in C_k(\mathbb{R}^n)$ be a k-cycle. Let $A_0 = A$.
(2) Suppose by induction that we have constructed a cycle A_i. Find a region $U_i \subset \mathbb{R}^n$ and a cycle A_{i+1} such that A_i and A_{i+1} are the same outside U_i and
$$\operatorname{mass}(A_i - A_{i+1}) \lesssim \operatorname{mass} A_i - \operatorname{mass} A_{i+1}.$$
Let $M_i = A_i - A_{i+1}$.
(3) Repeat this process until there is some m such that $A_m = 0$ or $\operatorname{mass} A_m$ is as small as desired.
(4) Then $A = A_m + \sum_{i=0}^{m} M_i$, and

$$\operatorname{mass} A_m + \sum_{i=0}^{m} \operatorname{mass} M_i \lesssim \operatorname{mass} A_m + \sum_{i=0}^{m} (\operatorname{mass} A_i - \operatorname{mass} A_{i+1}) \lesssim \operatorname{mass} A.$$

In fact, this is the strategy behind the following theorem, which efficiently decomposes an arbitrary cellular mod-2 cycle as a sum of uniformly rectifiable pieces.

Theorem 2.4 ([21])**.** *Let $n > 0$. There is a $c > 0$ with the following property. Let $t > 0$ and let τ_t be the grid of side length t in $\mathbb{R}^n$. Any mod-2 cycle $A \in C_d(\tau_t; \mathbb{Z}_2)$ can be written as a sum $A = \sum_{i=0}^{m} M_i$ of mod-2 d-cycles $M_i \in C_d(E_i; \mathbb{Z}_2)$, where each E_i is a d-dimensional, c-uniformly rectifiable subcomplex of τ_t and $\sum |E_i| \lesssim \operatorname{mass} A$.*

Uniform rectifiability is a property defined and studied by David and Semmes [3]. It has many definitions; we will present a definition that uses a function $\beta_E(x,r)$ that measures the "bumpiness" of a set $E \subset \mathbb{R}^n$, similar to the function $\beta_f(x,r)$ in Section 1. Let $c > 1$, let $d > 0$ be an integer, and let $|\cdot|$ denote Hausdorff d-measure. A set $E \subset \mathbb{R}^n$ is *Ahlfors c-regular* if

$$c^{-1}r^d < |E \cap B(x,r)| < cr^d \text{ for all } x \in E \text{ and } 0 < r < \operatorname{diam} E. \tag{2.2}$$

For $x \in E$ and $r > 0$, let

$$\beta_E(x,r) = \frac{1}{r^{d+1}} \min_P \int_{E \cap B(x,r)} d(y,P)\, \mathrm{d}y,$$

where P ranges over all d-planes in $\mathbb{R}^n$ and $\mathrm{d}y$ represents Hausdorff d-measure. Then $\beta_E(x,r)$ measures how well $E \cap B(x,r)$ can be approximated by a plane. It is scale-invariant in the sense that if $cE = \{cy \mid y \in E\}$, then $\beta_E(x,r) = \beta_{cE}(cx, cr)$.

For any $c > 1$, we say that $E \subset \mathbb{R}^n$ is c-uniformly rectifiable if E is Ahlfors c-regular and if it satisfies the following inequality based on (1.2). For every $x \in E$ and $r > 0$,

$$\frac{1}{r^d} \int_{E \cap B(x,r)} \int_0^s \beta_E(y,s)^2 \frac{\mathrm{d}s}{s}\, \mathrm{d}y \le c. \tag{2.3}$$

That is, a uniformly rectifiable set is an Ahlfors regular set which is no bumpier than a Lipschitz function. The prototypical example of a uniformly rectifiable set is the graph of a Lipschitz function, or a *Lipschitz graph*.

The power of uniform rectifiability is that this condition is equivalent to a variety of other conditions on E. The work discussed in the next section, for instance, uses the fact that a uniformly rectifiable set admits a *corona decomposition*. Defining such a decomposition rigorously is rather technical, and we point interested readers to [3], but briefly, if E admits a corona decomposition, then there is a collection $\mathcal{C}$ of Lipschitz graphs with uniformly bounded Lipschitz constants that approximates E efficiently at most points and most scales. That is, on one hand, for almost every $x \in E$, there is a finite set of "bad scales" $S_x \subset \mathbb{Z}$ such that for all $i \in \mathbb{Z} \setminus S_x$, either $2^{-i} > \operatorname{diam} E$ or the intersection $B(x, 2^{-i}) \cap E$ can be approximated by an element of $\mathcal{C}$. Furthermore, as x ranges over E, the average size of S_x is bounded. On the other hand, the set $\mathcal{C}$ isn't too big; in particular, the total measure of the elements of $\mathcal{C}$ is comparable to the measure of E.

Conversely, if a set is not uniformly rectifiable, then it must be complex — it must be far from a plane at many scales. Fractals are typical examples; if E is self-similar but not a plane, then $\beta_E(x,r) > \epsilon$ for most x's and r's, so $\sum_{i=0}^{\infty} \beta_E(x, r2^{-i})^2$ is typically infinite.

Fractals also provide examples of sets that are rectifiable but not uniformly rectifiable. One such set is based on the four-corners Cantor set. For each $i \ge 0$, let K_i be the ith step in the construction of the four-corners Cantor set, so that $K_0 = \partial[0,1]^2$ is the boundary of the unit square, and each set K_{i+1} is obtained by replacing each square in K_i by four squares of one fourth the side length (Figure 2). Then for any $x \in K_i$ and any $j \le i$, the intersection $K_i \cap B(x, 4^{-j})$ isn't close to a line, so $\beta_{K_i}(x, 4^{-j+1}) \gtrsim 1$. Therefore, for any $y \in K_i$, we have

$$\int_{4^{-i}}^1 \beta_{K_1}(y,s)^2 \frac{\mathrm{d}s}{s} \approx i.$$

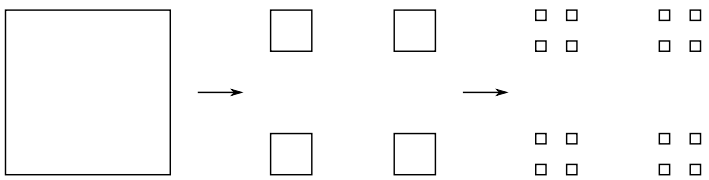

FIGURE 2. Three stages in the construction of the four-corners Cantor set. The stages of the construction are uniformly Ahlfors regular, but there is no c such that they are all c-uniformly rectifiable.

That is, the increasing complexity of the K_i's implies that there is no $c > 0$ such that all of the K_i's are c-uniformly rectifiable. Nevertheless, for each i, the fundamental class of K_i is a cellular cycle in $\tau_{4^{-i}}$, so Theorem 2.4 implies that it can be written as a sum of uniformly rectifiable pieces. In this case, K_i is a sum of the fundamental classes of 4^i disjoint squares, each of side length 4^{-i}.

The key tools for constructing the decomposition in Theorem 2.4 are the inductive strategy in Section 2.2 and a result of David and Semmes [4], which states that quasiminimizing sets are uniformly rectifiable. A set is quasiminimizing if compactly-supported deformations cannot decrease its area by too much. To state this rigorously, let $U \subset \mathbb{R}^n$ be a bounded open set. We say that a continuous map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a *deformation supported in* U if $f(U) \subset U$ and $f(x) = x$ for all $x \notin U$. Let $k > 1$. A set $S \subset \mathbb{R}^n$ such that $|S \cap B(0, \rho)| < \infty$ for all $\rho > 0$ is said to be (k, r)-*quasiminimizing* if for every open set $U \subset \mathbb{R}^n$ with $\operatorname{diam} U < r$, every deformation f supported in U satisfies

$$|f(S \cap U)| \geq \frac{1}{k}|S \cap U|.$$

For example, Lipschitz graphs are quasiminimizing sets. Let $\alpha\colon \mathbb{R}^{n-1} \to \mathbb{R}$ be a 1-Lipschitz function, let $\psi\colon \mathbb{R}^{n-1} \to \mathbb{R}^n$, $\psi(x) = (x, \alpha(x))$, and let $S = \psi(\mathbb{R}^n)$ be the graph of α. Let $\pi\colon \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1}$ be the projection $\pi(x, y) = x$. On one hand, π is distance-decreasing, so $|\pi(E)| \leq |E|$ for all $E \subset \mathbb{R}^n$. On the other hand, for any $E \subset S$, we have $E = \psi(\pi(E))$, and since ψ is 2-Lipschitz, we have

$$|\pi(E)| \leq |E| \leq 2^n |\pi(E)|.$$

For any deformation f supported in U, we have $\pi(f(U \cap S)) \supset \pi(U \cap S)$, so

$$|f(U \cap S)| \geq |\pi(f(U \cap S))| \geq |\pi(U \cap S)| \geq \frac{1}{2^n}|U \cap S|.$$

That is, S is $(2^n, \infty)$-quasiminimizing.

David and Semmes proved the following theorem.

Theorem 2.5 ([4])**.** *Let $n, d > 0$ and $k > 1$. There is a $c > 0$ such that if $S \subset \mathbb{R}^n$, $r \geq \frac{1}{10} \operatorname{diam} S$, and S is (k, r)-quasiminimizing, then S is c-uniformly rectifiable.*

The proof of Theorem 2.4 uses Theorem 2.5 to implement the inductive strategy. Let $k > 10$ and let c be as in Theorem 2.5. Let $A_0 = A \in C_d(\mathbb{R}^n; \mathbb{Z}_2)$ be a mod-2 cycle. If $i \geq 0$ and the support $S_i = \operatorname{supp} A_i$ is c-uniformly rectifiable, we let

$A_{i+1} = 0$ and terminate the construction. Otherwise, Theorem 2.5 implies that S_i is not a quasiminimizer. That is, there is a deformation $f_i\colon \mathbb{R}^n \to \mathbb{R}^n$, supported on some bounded set U_i with $\operatorname{diam} U_i < \frac{1}{10} \operatorname{diam} S_i$ such that $|f_i(S_i \cap U_i)| < \frac{1}{k}|S_i \cap U_i|$. If we choose the deformation carefully, we can ensure that $f_i(S_i)$ is a union of d-cells of τ_t, so that the push-forward $A_{i+1} = (f_i)_\sharp(A_i)$ is a cellular chain. We let $M_i = A_i - A_{i+1}$.

Since each A_i is a mod-2 cellular chain, we have $\operatorname{mass} A_i = |\operatorname{supp} A_i| = |S_i|$. Each step of this construction decreases the measure of S_i and thus decreases the mass of A_i. In fact,

$$\operatorname{mass} A_i - \operatorname{mass} A_{i+1} = |S_i \cap U_i| - |f_i(S_i \cap U_i)| \geq \frac{9}{10}|S_i \cap U_i| > 0$$

and

$$\operatorname{mass} M_i \leq |S_i \cap U_i| + |f_i(S_i \cap U_i)| \leq 2|S_i \cap U_i| \leq 3(\operatorname{mass} A_i - \operatorname{mass} A_{i+1}).$$

The number of cells in A_i is an integer and decreases with i, so this guarantees that the process terminates, i.e., $A_m = 0$ for some m. Then $A = \sum_{i=0}^{m-1} M_i$ and

$$\sum_{i=0}^{m-1} \operatorname{mass} M_i \leq \sum_{i=0}^{m-1} 3(\operatorname{mass} A_i - \operatorname{mass} A_{i+1}) \leq 3 \operatorname{mass} A.$$

That is, this is an efficient decomposition of A. Choosing the f_i so that the M_i's are supported on uniformly rectifiable sets, however, is more difficult, and we point interested readers to the full proof in [21].

Thus, while surfaces in $\mathbb{R}^n$ can be complex, their complexity is bounded by their volume, and complex surfaces can be efficiently decomposed into pieces that are not too much bumpier than Lipschitz graphs.

3. Applications

In this section, we will describe some ways to apply the decompositions described in the previous section to bound the topology of cycles and currents in $\mathbb{R}^n$ and to bound embeddings of nilpotent groups into Banach spaces.

3.1. Geometric measure theory and quantifying the topology of embedded submanifolds.

Theorem 2.4 can be used to bound the topological complexity of an arbitrary cycle in $\mathbb{R}^n$. Specifically, it can be used to quantify how difficult it is to orient a mod-2 cycle, that is, to lift it to a cycle with integer coefficients.

Given a cycle $A \in C_d(\mathbb{R}^n; \mathbb{Z}_2)$, a *pseudo-orientation* of A is a cycle $P \in C_d(\mathbb{R}^n)$ such that $P \equiv A \pmod 2$. In general, a cycle A has many pseudo-orientations. We define the *nonorientability* of A to be the infimal mass of a pseudo-orientation, i.e.,

$$\operatorname{NO}(A) = \inf\{\operatorname{mass} P \mid P \text{ is a pseudo-orientation of } A\}.$$

The nonorientability of a surface measures how difficult it is to cut that surface into orientable pieces. That is, let $\Sigma \subset \mathbb{R}^4$ be an arbitrary nonorientable surface and let $A = [\Sigma]$. Let Γ be a graph (a 1-dimensional simplicial complex) embedded in Σ such that each connected component of $\Sigma \setminus \Gamma$ is orientable. (For instance, let Σ be a Klein bottle and let Γ be a simple closed curve that cuts Σ into a cylinder.) Let $C_1, \ldots, C_k$ be these components. Orient them arbitrarily and let $C = \sum_{i=1}^k [C_i]$. Let $G = \partial C = \sum_{i=1}^k \partial[C_i]$.

Each edge e of Γ is in the boundary of two components C_i and C_j, so e occurs exactly twice in the sum $\sum_{i=1}^{k} \partial[C_i]$. If the orientations of C_i and C_j agree on e, then the occurences of e cancel out and it has coefficient 0 in G. If the orientations disagree, then e has coefficient ± 2. Since all coefficients of G are even, $G/2$ is a 1-cycle with integer coefficients, so there is some D such that $\partial D = G/2$. By the isoperimetric inequality, we can choose D such that $\operatorname{mass} D \leq (\operatorname{mass} G)^2 \leq \ell(\Gamma)^2$, where $\ell(\Gamma)$ is the total length of the edges of Γ. Let $P = C - 2D$; then $P \equiv C \equiv A \pmod 2$, so P is a pseudo-orientation of A and

$$\mathrm{NO}(A) \leq \operatorname{mass} P \leq \operatorname{mass} A + 2\ell(\Gamma)^2.$$

In [21], Theorem 2.4 was used to prove the following.

Theorem 3.1 ([21])**.** *Let $n > 0$ and $0 < d < n$. There is a $c > 0$ with the following property. Let $t > 0$ and let τ_t be the grid of side length t in $\mathbb{R}^n$. Then for any mod-2 cycle $A \in C_d(\tau_t; \mathbb{Z}_2)$, $\mathrm{NO}(A) \leq c \operatorname{mass} A$.*

By Theorem 2.4, $A = \sum_i M_i$ where each M_i is a cycle supported on a uniformly rectifiable set E_i. Nonorientability is subadditive, so it suffices to use the uniform rectifiability of the E_i's to bound $\mathrm{NO}(M_i)$.

The proof relies on approximating the uniformly rectifiable set E_i by Lipschitz graphs. To give a brief sketch, since E_i is uniformly rectifiable, it has a corona decomposition, i.e., a collection of Lipschitz graphs $\mathcal{C}$ such that for most points $x \in E_i$ and most scales $0 < r < \operatorname{diam} E_i$, the intersection $B(x, r) \cap E_i$ is close to one of the Lipschitz graphs $\Lambda \in \mathcal{C}$. Then the restriction of M_i to $B(x, r)$ can be approximated by a chain in Λ, and since Λ is orientable, that chain inherits an orientation from Λ. If every intersection $B(x, r) \cap E_i$ could be approximated by oriented surfaces and all of the orientations agreed, then it would be easy to lift M_i to a cycle with integer coefficients. Difficulties only arise where E_i is complex — from choices of x and r such that $B(x, r) \cap E_i$ can't be approximated by a Lipschitz graph or when elements of $\mathcal{C}$ don't have consistent orientations — but the uniform rectifiability of E_i bounds how much complexity it can have.

3.2. **Metric geometry and embeddings of nilpotent groups.** In more recent work, these tools have been applied to study embeddings into Banach spaces, especially the Heisenberg groups. The Heisenberg groups are the simplest family of nonabelian nilpotent groups, and their noncommutativity makes them difficult to embed into Banach spaces in a way that preserves their metrics.

An introduction to the Heisenberg group can be found, for instance, in [1], but briefly, the integer Heisenberg group $H_n^{\mathbb{Z}}$ is the group generated by $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, and Z such that $[X_i, Y_i] = X_i Y_i X_i^{-1} Y_i^{-1} = Z$ and all other pairs of generators commute. This is isomorphic to a group of upper-triangular $(n+2) \times (n+2)$ matrices of the form

$$\begin{pmatrix} 1 & x_1 & \cdots & x_n & z \\ & \ddots & & & y_1 \\ & & \ddots & & \vdots \\ & & & \ddots & y_n \\ & & & & 1 \end{pmatrix} \tag{3.1}$$

where the x_i's, y_i's, and z are all integers. Here, $X_i \in H_n$ is identified with the matrix of form (3.1) with all coefficients zero except that $x_i = 1$ and likewise for Y_i and Z. The integer Heisenberg group is a lattice in the nilpotent Lie group obtained by taking the matrices of the form (3.1) with real coefficients; we call this Lie group H_n and write elements of H_n as points $(x_1, \dots, x_n, y_1, \dots, y_n, z) \in \mathbb{R}^{2n+1}$.

A key feature of the Heisenberg group is the difference between the vertical direction Z and the horizontal directions X_i and Y_i. For any n, one can calculate that $[X_i^n, Y_i^n] = Z^{n^2}$. That is, quadratically large powers of Z can be written as products of linearly many X_i's and Y_i's. Let $\langle Z \rangle$ be the z-axis in H_n; in the terminology of geometric group theory, $\langle Z \rangle$ is a quadratically distorted subgroup. Left-invariant metrics on H_n inherit this quadratic distortion; we will equip H_n with the left-invariant metric such that

$$d(\mathbf{0}, (x_1, \dots, x_n, y_1, \dots, y_n, z)) = \max\left\{|x_i|, |y_i|, \sqrt{|z|}\right\}.$$

Then, for $t \neq 0$, the map

$$s_t(x_1, \dots, x_n, y_1, \dots, y_n, z) = (tx_1, \dots, tx_n, ty_1, \dots, ty_n, t^2 z),$$

which scales horizontal directions by t and scales the vertical direction by t^2, is an automorphism of H_n that scales the metric by a factor of t.

A natural question is how well H_n can be embedded into different spaces, especially Banach spaces. This has applications in theoretical computer science, where the accuracy of some algorithms depends on how well certain metric spaces embed in L_1 (see for instance [15, 11]). In this section, we will consider embeddings of H_n into L_p spaces.

For any $n > 1$ and any $p \in [1, \infty)$, H_n, equipped with its subriemannian metric, does not embed in L_p by a bilipschitz map. When $p > 1$, this follows from a version of Pansu's differentiability theorem [18], which states that Lipschitz maps from H_n to L_p can be locally approximated by homomorphisms from H_n to L_p. Since L_p is abelian and $Z = [X_1, Y_1]$, any such homomorphism must send each *vertical line* (coset of $\langle Z \rangle$) to a point; thus, Lipschitz maps from H_n to L_p cannot be bilipschitz.

This differentiability fails, however, for maps to L_1; Lipschitz maps need not be differentiable anywhere. Instead, Cheeger and Kleiner [2] proved that H_n does not embed in L_1 by showing that the behavior of maps from H_n to L_1 depends on the structure of surfaces in H_n.

To give a rough idea of this argument, suppose that M is a measure space and that $f\colon H_n \to L_1(M)$ is a Lipschitz function. We claim that f is not bilipschitz. For almost every $m \in M$, the coordinate function $f_m\colon H_n \to \mathbb{R}$ has bounded variation [2]. Almost every sublevel set of a BV function has finite perimeter. (A finite perimeter set is a set that can be approximated by sets whose boundary has uniformly bounded Hausdorff measure.) If $E = f_m^{-1}((-\infty, t])$ is such a sublevel set, then an analogue of De Giorgi's theorem in H_n implies that the reduced boundary $\partial^* E$ has a approximate tangent plane at almost every $x \in \partial^* E$ [9]. Furthermore, this tangent plane is a *vertical plane* (a union of vertical lines); vertical planes are the only scale-invariant codimension-1 subgroups of H_n. Thus, when $r > 0$ is small, the intersection $B(x, r) \cap E$ is close to the intersection of $B(x, r)$ with a half-space bounded by a vertical plane.

In fact, for almost every $m \in M$ and $x \in H_n$, there is an $r > 0$ such that $f_m|_{B(x,r)}$ is close to a function $\bar{f}_m$ whose sublevel sets are half-spaces bounded by vertical planes. We call $\bar{f}_m$ a *vertical function*. By Fubini's theorem, there is some

$x \in H_n$ and some $r > 0$ such that $f_m|_{B(x,r)}$ is close to a vertical function $\bar{f}_m$ for all but a small fraction of $m \in M$; then $f|_{B(x,r)}$ is close to a map $\bar{f}$ such that every coordinate function of $\bar{f}$ is vertical. Then $\bar{f}$ sends vertical lines in $B(x,r)$ to a point, so f is not bilipschitz on $B(x,r)$.

This argument links the metric properties of f to the shape of the sublevel sets $E_{m,y} = f_m^{-1}((-\infty, y])$. For example, if the $E_{m,y}$'s are all smooth at some scale ρ (i.e., if $B(x,\rho) \cap E_{m,y}$ is close to a vertical half space for all m, t, and $x \in \partial E_{m,y}$), then f collapses vertical line segments at scale ρ. Conversely, if f is c-bilipschitz at scales between R and R' (i.e., there are c, R, and R' such that

$$cd(p,q) \leq \|f(p) - f(q)\|_1 \leq d(p,q)$$

for all $p, q \in H_n$ such that $d(p,q) \in [R, R'])$, then the $E_{m,y}$'s must be bumpy at scales between R and R'; in the language of Sections 1 and 2, the $E_{m,y}$'s must be roughly c-bumpy at roughly $\log \frac{R'}{R}$ different scales.

The techniques of Section 1 and Section 2, however, bound how bumpy a surface can be. Given a set $U \subset \mathbb{R}^n$ of finite perimeter, we can approximate U by a set U^t which is a union of cells of τ_t. As $t \to 0$, this approximation converges to U, and since U has finite perimeter, the $(n-1)$-volume of the boundary $|\partial U^t|$ stays bounded. The boundary ∂U^t can be viewed as an $(n-1)$-cycle, so by Theorem 2.4, ∂U^t is a sum of cycles supported on uniformly rectifiable sets. Each of these pieces is no bumpier than a Lipschitz function; they all satisfy (2.3).

The results of [16] extend this to the Heisenberg group H_n and prove that the boundary ∂U of a set $U \subset H_n$ of finite perimeter can be decomposed as a sum of cycles supported on sets $E \subset H_n$. Uniform rectifiability is not as well studied in H_n as in $\mathbb{R}^n$, and it is not known which definitions of uniform rectifiability generalize to H_n, but [16] shows that the E's admit intrinsic corona decompositions. These are collections of intrinsic Lipschitz graphs that approximate E at most points and scales. (Intrinsic Lipschitz graphs were introduced in [10] as an analogue of Lipschitz graphs.) When $n \geq 2$, intrinsic Lipschitz graphs in H_n are about as bumpy as Lipschitz graphs in $\mathbb{R}^n$. That is, like Lipschitz graphs, an intrinsic Lipschitz graph can be $\frac{1}{L}$-bumpy at no more than roughly L^2 scales. This corresponds to the following bound:

Theorem 3.2. [16] *For any $n \geq 2$ there is a $c > 0$ such that for sufficiently large $R > 1$, there are no 1-Lipschitz maps $f \colon H_n \to L_1$ that are $c(\log R)^{-\frac{1}{2}}$-bilipschitz at scales between 1 and R. Furthermore, this is sharp; there is a $c' > 0$ such that for every sufficiently large R, there is a 1-Lipschitz map $f \colon H_n \to L_1$ that is $c'(\log R)^{-\frac{1}{2}}$-bilipschitz at scales between 1 and R.*

Intrinsic Lipschitz graphs in H_1, however, can be much bumpier than Lipschitz graphs in $\mathbb{R}^n$. In [17], it is shown that intrinsic Lipschitz graphs in H_1 can be $\frac{1}{L}$-bumpy at up to roughly L^4 scales, but no more than that. This matches the bound for Lipschitz curves proved in [14] and corresponds to the following bound.

Theorem 3.3. [17] *There is a $c > 0$ such that for sufficiently large $R > 1$, there are no 1-Lipschitz maps $f \colon H_1 \to L_1$ that are $c(\log R)^{-\frac{1}{4}}$-bilipschitz at scales between 1 and R. Furthermore, there is a $c' > 0$ such that for every sufficiently large R, there is a 1-Lipschitz map $f \colon H_1 \to L_1$ that is $c'(\log R)^{-\frac{1}{4}}$-bilipschitz at scales between 1 and R.*

This leads to the following consequence:

Theorem 3.4. [17] *There is a metric space M that has a bilipschitz embedding into L_1 and L_4, but not L_p for $1 < p < 4$.*

In short, the extent to which f preserves the metric on H_n depends on the bumpiness/complexity of the sublevel sets $E_{m,y}$, and the maximum possible complexity of a finite-perimeter subset $E \subset H_n$ depends on the ambient dimension n.

4. Conclusion

Some possible next questions include:

- How can other topological properties of a manifold be quantified? How do they depend on the complexity of the manifold?
- These decompositions bound the complexity of manifolds embedded in $\mathbb{R}^n$, suggesting that manifolds embedded in $\mathbb{R}^n$ are less complex than abstract manifolds. How does that affect their geometry?
- How does the maximum possible complexity of a manifold embedded in a space X depend on the geometry of X?

More generally, these tools suggest a link between complexity and geometry. If a manifold is simple in the sense that it embeds in $\mathbb{R}^n$, then it is simple in the sense that it can be decomposed into simple pieces. In a way, any object that can be drawn on a piece of paper embeds, more or less, in a low-dimensional Euclidean space; conversely, a key property of very complex objects like random graphs and arithmetic manifolds is that they are hard to embed, hard to decompose, and hard to visualize. Perhaps the objects whose shapes we can imagine are the objects simple enough to fit in our imagination. How can we better understand the shape of objects on the far end of the complexity spectrum?

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