

# THE GEOMETRY OF SURFACES AND 3-MANIFOLDS

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*Note: Most of the illustrations in these notes are omitted. Please draw your own!*

## 3. THE HYPERBOLIC PLANE

Remember the Euler characteristic argument we used to show that high-genus surfaces don't have flat structures:

If we triangulate a manifold, then we have

$$E = 3F/2$$

and if that manifold is a surface of genus  $g$ , we have

$$F - E + V = 2g - 2$$

or

$$F = 4g - 4 + 2V$$

The average number of faces around a vertex is  $3F/V$  (three "points" for each vertex, divided by the number of vertices), so we get

$$\frac{3F}{V} = \frac{12g - 12}{V} + 6$$

So, a triangulation of a sphere will have fewer than 6 triangles around each vertex on average, a torus will have an average of 6, and a genus  $g$  surface, when  $g$  is large, will have more than 6.

If you glue equilateral triangles together with 5 to a vertex, you get an icosahedron, which approximates one of our model spaces, the sphere. If you glue 6 at a vertex, you get a plane. If you glue 7 at a vertex, you get an approximation of the hyperbolic plane.

One way of saying this is that the sphere is a surface with *positive curvature*; roughly, there's a little less stuff around each point than in the plane – circles are a little smaller, triangles are a little thicker, and geodesics from a point spread out slowly and eventually intersect again.

The hyperbolic plane is the opposite; there's a little more stuff around each point than in the plane. We'll see that circles are a lot bigger, triangles are a lot thinner, and geodesics spread much more quickly than in the plane.

There are a couple of ways to make models of hyperbolic planes; you can glue together triangles so that seven triangles surround each vertex, or you can crochet hyperbolic planes by continuously adding stitches. If you've ever tried this, though, you'll see the problem – exponential growth. Whenever you add on a new layer of triangles, the length of the boundary grows by at least half, so a disc of radius  $r$  uses exponentially many triangles. So it can't really be embedded in  $\mathbb{R}^3$  – a large hyperbolic disc is essentially going to be a solid ball of paper.

Instead, we'll construct a couple of models of the hyperbolic plane. Here's one (figure omitted, though see

- <http://www.geometrygames.org/KaleidoTile/index.html>
- <http://bulatov.org/math/1001/index.html>)

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Poincare disc model: a unit disc with metric scaled by  $\frac{2}{1-r^2}$ . Note that although this is based on the unit disc, it's infinite in all directions.

The problem with having to view the hyperbolic plane this way is that it's hard to see all the symmetries. This picture has some obvious symmetries; for instance, rotating around the center is a symmetry. But there are more subtle ones too. These are all based on *inversions*.

Inversion “reflects” through a circle or line. (figure omitted) If  $C$  is the circle of radius 1, then inversion through  $C$  is the map (in polar coordinates)

$$(r, \theta) \rightarrow (1/r, \theta)$$

This:

- fixes the circle and swaps the inside and outside
- sends circles and lines to circles and lines
- preserves angles

**Exercise:** Check any of these assertions that we didn't prove in class.

We can define inversion through any circle with the same properties, and an inversion through a circle which is orthogonal to the unit circle is an isometry of the hyperbolic plane. Then:

- A composition of inversions through two diameters of the plane is a rotation fixing the center.
- A composition of inversions through two non-intersecting circles is a “translation”
- The hyperbolic plane is *isotropic* – there is an isometry taking any point and direction to any other point and direction. (Proof: It's enough to show that there's an isometry taking any point to the center. There's a family of circles, centered on the  $x$ -axis, which are orthogonal to the unit disc. Inversions through these take the center to anywhere on a radius of the disc, and rotations take this radius to the entire thing.

We can use these symmetries to describe the geodesics in the hyperbolic plane. Remember the symmetry trick – any axis of a reflection is a geodesic. Here, the axis of an inversion has to be a geodesic, by symmetry, and for any point and direction, one can find a circle which is orthogonal to the unit circle. So, in this picture, the geodesics in the hyperbolic plane are the circles (and lines) orthogonal to the unit circle.

Knowing this, we can see that a lot of the properties of the hyperbolic plane are the opposite of properties of the sphere: instead of no geodesics being parallel, lots of geodesics are parallel. Instead of triangles having larger angles than usual, they have smaller angles. Instead of circles growing slowly and then shrinking, they grow exponentially quickly.

**3.1. Hyperbolic surfaces.** The surface of genus 2 is homeomorphic to an octagon with some edges glued. (figure omitted) All eight vertices of the octagon are glued into one vertex. That vertex is surrounded by all eight corners of the octagon, so we can get a hyperbolic structure by constructing a regular octagon all of whose angles are  $\pi/4$ . (figure omitted)

We can do the same thing with other surfaces too; a hyperbolic genus  $g$  surface can be formed by gluing the edges of a regular  $4g$ -gon. (Note: As you saw in the exercises from the last section, there are lots of different flat structures on the torus. There are also lots of different hyperbolic structures on genus  $g$  surfaces, but that's another course...)

In fact, this even lets us put a structure on surfaces with holes (for instance, the torus with a point removed (figure omitted): if we drag vertices all the way out to the unit circle, we get ideal triangles (figure omitted),

which glue to give complete structures on surfaces with holes – these are called cusped hyperbolic surfaces.

There's another picture of the hyperbolic plane which illustrates these cusps better: take the disc picture and invert around a circle centered at  $(0, 1)$ . This sends the unit disc to a half plane; if we translate this half-plane so that the boundary is the  $x$ -axis, this is the upper half-plane model. This inversion clearly distorts the metric, but the distortion is very simple: lengths of curves at height  $y$  are scaled by a factor of  $y$ .

Most of what we've seen still holds true – geodesics in the disc model were circles and lines, and inversion preserves circles and lines, so the geodesics in this model are circles and lines orthogonal to the  $x$ -axis (that is, semicircles centered on the  $x$ -axis and vertical lines). But new symmetries show up; for instance, translating left or right is a symmetry of the hyperbolic plane, as is scaling from the origin.

**Exercise:** Draw some ideal triangles in the upper half-plane model. Make sure that their edges are geodesics! Can you draw an ideal triangle with an edge which is a vertical line?

**Exercise:** A translation of the upper half plane to the left or right generates a discrete, fixed-point free group of symmetries. Describe the complete hyperbolic surface corresponding to this group? (Look up “pseudosphere” or “tractrix” to see a piece of this surface embedded in  $\mathbb{R}^3$ .)