Embeddings of the Heisenberg group, uniform rectifiability, and the Sparsest Cut problem

Robert Young
New York University
(joint work with Assaf Naor)

June 2018

A.N. was supported by BSF grant 2010021, the Packard Foundation and the Simons Foundation. R.Y. was supported by NSF grant DMS 1612061, the Sloan Foundation, and the Fall 2016 program at MSRI. The research that is presented here was conducted under the auspices of the Simons Algorithms and Geometry (A&G) Think Tank.

Distortion

Let X be a metric space.

- Let $f : X \to Y$ and let $D \geq 1$. We say that f has distortion at most D if there is an $r > 0$ such that

$$\frac{d(f(a), f(b))}{d(a, b)} \in [r, Dr]$$

for all $a, b \in X$, $a \neq b$.

Distortion

Let X be a metric space.

- Let $f : X \to Y$ and let $D \geq 1$. We say that f has distortion at most D if there is an $r > 0$ such that

$$\frac{d(f(a), f(b))}{d(a, b)} \in [r, Dr]$$

  for all $a, b \in X$, $a \neq b$.
- For $p > 0$, the L_p*–distortion* of X is the infimal $D \in [1, \infty]$ such that there is an embedding $f : X \to L_p$ such that $d(a, b) \leq \|f(a) - f(b)\|_p \leq Dd(a, b)$ for every $a, b \in \mathcal{M}$.

Examples

- (Kuratowski) For any metric space X, $c_\infty(X) = 1$.

Examples

- (Kuratowski) For any metric space X, $c_\infty(X) = 1$.
- (Bourgain) If X is an n–point metric space, then $c_p(X) \lesssim \log n$ for any $1 \leq p \leq \infty$.

Examples

- (Kuratowski) For any metric space X, $c_\infty(X) = 1$.
- (Bourgain) If X is an n–point metric space, then $c_p(X) \lesssim \log n$ for any $1 \leq p \leq \infty$.
- (Matoušek) If X is an n–point expander graph, and $1 \leq p < \infty$, then $c_p(X) \gtrsim \log n$.

The main theorem

Theorem (Naor-Y.)

Let $k \geq 2$ *and let* $B_{\mathbb{Z}}^{2k+1}(n)$ *be the set of integer points in the ball of radius* n *in the Heisenberg group* H^{2k+1}*. Then*

$$c_1(B_{\mathbb{Z}}^{2k+1}(n)) \asymp \sqrt{\log n}.$$

The main theorem

Theorem (Naor-Y.)

Let $k \geq 2$ and let $B_{\mathbb{Z}}^{2k+1}(n)$ be the set of integer points in the ball of radius n in the Heisenberg group H^{2k+1}. Then

$$c_1(B_{\mathbb{Z}}^{2k+1}(n)) \asymp \sqrt{\log n}.$$

Theorem (Naor-Y.)

Let $B_{\mathbb{Z}}^{3}(n)$ be the set of integer points in the ball of radius n in the Heisenberg group H^3. Then

$$c_1(B_{\mathbb{Z}}^{3}(n)) \asymp (\log n)^{\frac{1}{4}}.$$

c_1 and the Sparsest Cut problem

For $n > 0$, let

$\alpha(n) = \max\{c_1(X) \mid X$ is an n–point metric space of negative type$\}$.

This is the *Goemans–Linial integrality gap*.

c_1 and the Sparsest Cut problem

For $n > 0$, let

$\alpha(n) = \max\{c_1(X) \mid X \text{ is an } n\text{–point metric space of negative type}\}.$

This is the *Goemans–Linial integrality gap*.

Theorem (Goemans–Linial)

There is a polynomial-time algorithm that approximates the Nonuniform Sparsest Cut Problem to within a factor of $\alpha(n)$.

c_1 and the Sparsest Cut problem

For $n > 0$, let

$\alpha(n) = \max\{c_1(X) \mid X \text{ is an } n\text{–point metric space of negative type}\}.$

This is the *Goemans–Linial integrality gap*.

Theorem (Goemans–Linial)

There is a polynomial-time algorithm that approximates the Nonuniform Sparsest Cut Problem to within a factor of $\alpha(n)$.

Theorem (Lee-Naor)

The Heisenberg group is bilipschitz equivalent to a metric of negative type.

The Goemans–Linial question

How does $\alpha(n)$ grow with n?

The Goemans–Linial question

How does $\alpha(n)$ grow with n?

- $\alpha(n) \lesssim (\log n)^{\frac{1}{2}+o(1)}$ (Arora-Lee-Naor).

The Goemans–Linial question

How does $\alpha(n)$ grow with n?

- $\alpha(n) \lesssim (\log n)^{\frac{1}{2}+o(1)}$ (Arora-Lee-Naor).

Does every finite negative-type metric space embed in L_1 by a bilipschitz map?

The Goemans–Linial question

How does $\alpha(n)$ grow with n?

- $\alpha(n) \lesssim (\log n)^{\frac{1}{2}+o(1)}$ (Arora-Lee-Naor).

Does every finite negative-type metric space embed in L_1 by a bilipschitz map?

The answer is no:

- $\alpha(n) \gtrsim (\log \log n)^c$ (Khot-Vishnoi)

The Goemans–Linial question

How does $\alpha(n)$ grow with n?

- $\alpha(n) \lesssim (\log n)^{\frac{1}{2}+o(1)}$ (Arora-Lee-Naor).

Does every finite negative-type metric space embed in L_1 by a bilipschitz map?
The answer is no:

- $\alpha(n) \gtrsim (\log \log n)^c$ (Khot-Vishnoi)
- $\alpha(n) \gtrsim (\log n)^{c'}$ (with $c' \approx 2^{-60}$) (Cheeger-Kleiner-Naor)

The Heisenberg group

Let $H^{2k+1} \subset M_{k+2}$ be the $(2k+1)$–dimensional nilpotent Lie group

$$H^{2k+1} = \left\{ \begin{pmatrix} 1 & x_1 & \dots & x_k & z \\ 0 & 1 & 0 & 0 & y_1 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & y_k \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x_i, y_i, z \in \mathbb{R} \right\}.$$

The Heisenberg group

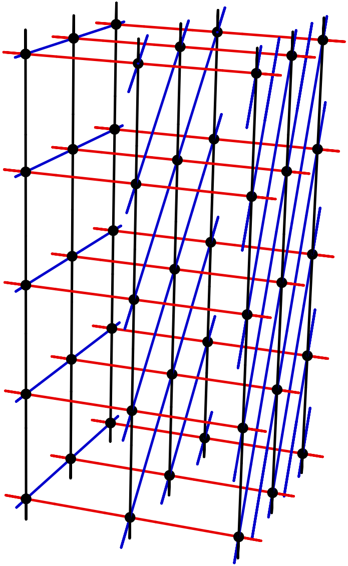
Let $H^{2k+1} \subset M_{k+2}$ be the $(2k+1)$–dimensional nilpotent Lie group

$$H^{2k+1} = \left\{ \begin{pmatrix} 1 & x_1 & \dots & x_k & z \\ 0 & 1 & 0 & 0 & y_1 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & y_k \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; x_i, y_i, z \in \mathbb{R} \right\}.$$

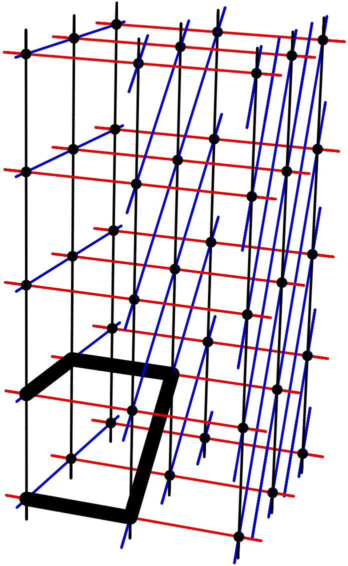
This contains a lattice

$$H^{\mathbb{Z}}_{2k+1} = \langle x_1, \dots, x_k, y_1, \dots, y_k, z \mid [x_i, y_i] = z, \text{ all other pairs commute} \rangle.$$

A lattice in H^3

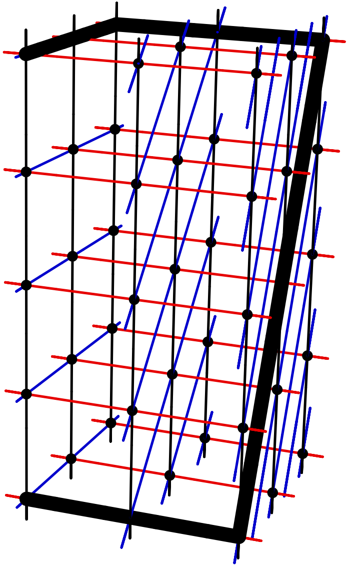


A lattice in H^3



$z = xyx^{-1}y^{-1}$

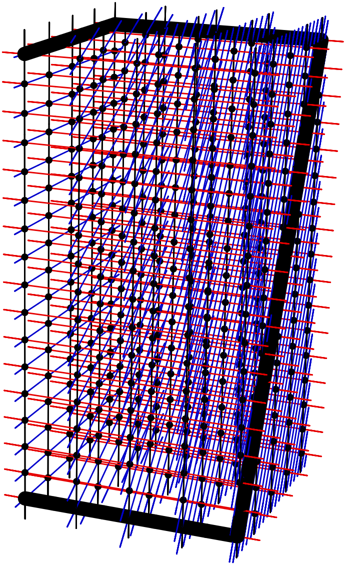
A lattice in H^3



$$z = xyx^{-1}y^{-1}$$

$$z^4 = x^2y^2x^{-2}y^{-2}$$

A lattice in H^3

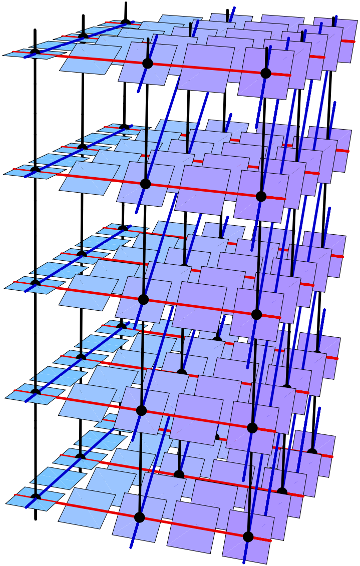


$$z = xyx^{-1}y^{-1}$$

$$z^4 = x^2y^2x^{-2}y^{-2}$$

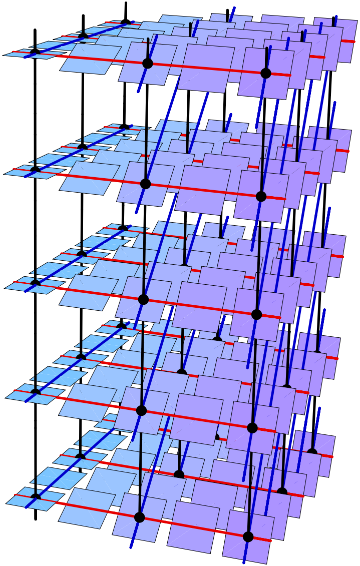
$$z^{n^2} = x^ny^nx^{-n}y^{-n}$$

From Cayley graph to sub-riemannian metric



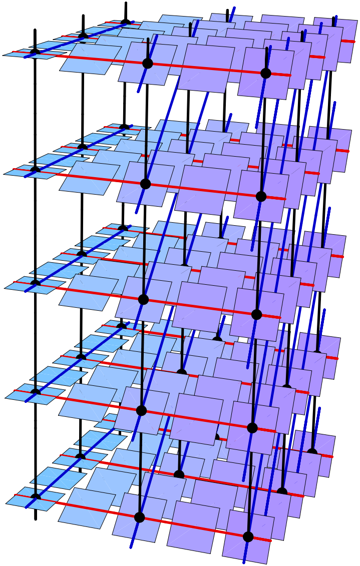
- $d(u, v) = \inf\{\ell(\gamma) \mid \gamma \text{ is a horizontal curve from } u \text{ to } v\}$

From Cayley graph to sub-riemannian metric



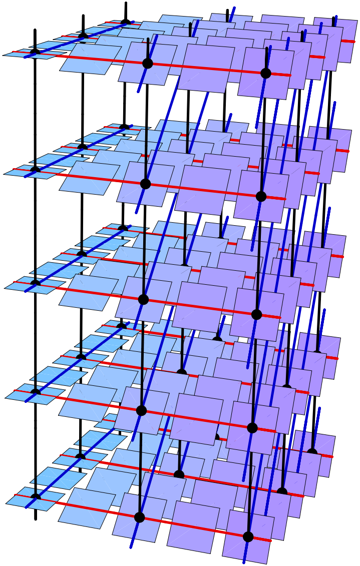
- $d(u, v) = \inf\{\ell(\gamma) \mid \gamma$ is a horizontal curve from u to $v\}$
- The map $s_t(x, y, z) = (tx, ty, t^2 z)$ scales the metric

From Cayley graph to sub-riemannian metric



- $d(u, v) = \inf\{\ell(\gamma) \mid \gamma$ is a horizontal curve from u to $v\}$
- The map $s_t(x, y, z) = (tx, ty, t^2 z)$ scales the metric
- The ball of radius ϵ is approximately an $\epsilon \times \epsilon \times \epsilon^2$ box.

From Cayley graph to sub-riemannian metric



- $d(u,v) = \inf\{\ell(\gamma) \mid \gamma$ is a horizontal curve from u to $v\}$
- The map $s_t(x,y,z) = (tx, ty, t^2 z)$ scales the metric
- The ball of radius ϵ is approximately an $\epsilon \times \epsilon \times \epsilon^2$ box.
- The z–axis has Hausdorff dimension 2

Embeddings of the Heisenberg group

Theorem (Pansu, Semmes)

There is no bilipschitz embedding from H^{2k+1} to $\mathbb{R}^N$.

Embeddings of the Heisenberg group

Theorem (Pansu, Semmes)

There is no bilipschitz embedding from H^{2k+1} *to* $\mathbb{R}^N$*.*

Theorem (Pansu)

Every Lipschitz map $f : H^{2k+1} \to \mathbb{R}^N$ *is Pansu differentiable almost everywhere.*

Embeddings of the Heisenberg group

Theorem (Pansu, Semmes)

There is no bilipschitz embedding from H^{2k+1} to $\mathbb{R}^N$.

Theorem (Pansu)

Every Lipschitz map $f : H^{2k+1} \to \mathbb{R}^N$ is Pansu differentiable almost everywhere.

That is, on sufficiently small scales, f is close to a homomorphism.

Embeddings of the Heisenberg group

Theorem (Pansu, Semmes)

There is no bilipschitz embedding from H^{2k+1} to $\mathbb{R}^N$.

Theorem (Pansu)

Every Lipschitz map $f : H^{2k+1} \to \mathbb{R}^N$ is Pansu differentiable almost everywhere.

That is, on sufficiently small scales, f is close to a homomorphism. But any homomorphism sends z to 0 – so any Lipschitz map to $\mathbb{R}^N$ collapses the z direction.

H^{2k+1} does not embed in L_1

Pansu's theorem does not work for L_1 because Lipschitz maps to L_1 may not be differentiable anywhere.

H^{2k+1} does not embed in L_1

Pansu's theorem does not work for L_1 because Lipschitz maps to L_1 may not be differentiable anywhere.

Example

The map $f : [0,1] \to L_1([0,1])$

$$f(t) = \mathbf{1}_{[0,t]},$$

is an isometric embedding that cannot be approximated by a linear map.

Regardless, Cheeger and Kleiner showed:

Theorem (Cheeger-Kleiner)

There is no bilipschitz embedding from the unit ball $B \subset H^{2k+1}$ to L_1.

Regardless, Cheeger and Kleiner showed:

Theorem (Cheeger-Kleiner)

There is no bilipschitz embedding from the unit ball $B \subset H^{2k+1}$ to L_1.

The proof involves a version of differentiation based on cut metrics.

Cut metrics

Let X be a set. A *cut metric* on X is a semimetric of the form

$$d_S(i,j) = |\mathbf{1}_S(i) - \mathbf{1}_S(j)| \qquad \text{where } S \subset X.$$

Cut metrics

Let X be a set. A *cut metric* on X is a semimetric of the form

$$d_S(i,j) = |\mathbf{1}_S(i) - \mathbf{1}_S(j)| \qquad \text{where } S \subset X.$$

The metric induced by any map $f : X \to L_1$ is a linear combination of cut metrics:

Lemma

If $f : X \to L_1$, then there is a measure μ (the cut measure) on 2^X such that

$$d(f(x), f(y)) = \int d_S(x,y)\, d\mu(S).$$

Proof: H^{2k+1} does not embed in L_1

We can study maps $f : H^{2k+1} \to L_1$ by studying cuts in H^{2k+1}.

Proof: H^{2k+1} does not embed in L_1

We can study maps $f : H^{2k+1} \to L_1$ by studying cuts in H^{2k+1}. Open sets in H^{2k+1} have Hausdorff dimension $2k+2$ and any surface that separates two open sets has Hausdorff dimension at least $2k+1$, so we let area $= \mathcal{H}^{2k+1}$, vol $= \mathcal{H}^{2k+2}$.

Proof: H^{2k+1} does not embed in L_1

We can study maps $f : H^{2k+1} \to L_1$ by studying cuts in H^{2k+1}. Open sets in H^{2k+1} have Hausdorff dimension $2k+2$ and any surface that separates two open sets has Hausdorff dimension at least $2k+1$, so we let area $= \mathcal{H}^{2k+1}$, vol $= \mathcal{H}^{2k+2}$.

Lemma

If $B \subset H^{2k+1}$ is the unit ball and $f : B \to L_1$ is Lipschitz, then the cut measure μ is supported on sets S with $\operatorname{area}(\partial S) < \infty$ and

$$\int \operatorname{area}(\partial S)\, d\mu(S) \lesssim \operatorname{vol}(B) \operatorname{Lip}(f).$$

Proof: H^{2k+1} does not embed in L_1

Theorem (Franchi-Serapioni-Serra Cassano)

If area $\partial S < \infty$, *then near almost every* $x \in \partial S$, ∂S *is close to a plane containing the z–axis (the tangent plane at* x.*)*

Proof: H^{2k+1} does not embed in L_1

Theorem (Franchi-Serapioni-Serra Cassano)

If area $\partial S < \infty$, *then near almost every* $x \in \partial S$, ∂S *is close to a plane containing the z–axis (the tangent plane at* x.*)*

Cheeger and Kleiner show:

- For almost every $x \in B$, there is a neighborhood B' of x such that most of the cuts are close to vertical on B'.

Proof: H^{2k+1} does not embed in L_1

Theorem (Franchi-Serapioni-Serra Cassano)

If area $\partial S < \infty$, *then near almost every* $x \in \partial S$, ∂S *is close to a plane containing the z–axis (the tangent plane at* x.*)*

Cheeger and Kleiner show:

- For almost every $x \in B$, there is a neighborhood B' of x such that most of the cuts are close to vertical on B'.
- Therefore, $f|_{B'}$ is close to a map that is constant on vertical lines.

Proof: H^{2k+1} does not embed in L_1

Theorem (Franchi-Serapioni-Serra Cassano)

If area $\partial S < \infty$, *then near almost every* $x \in \partial S$, ∂S *is close to a plane containing the z–axis (the tangent plane at* x.*)*

Cheeger and Kleiner show:

- For almost every $x \in B$, there is a neighborhood B' of x such that most of the cuts are close to vertical on B'.
- Therefore, $f|_{B'}$ is close to a map that is constant on vertical lines.
- So f is not a bilipschitz map.

Quantitative nonembeddability

Cheeger, Kleiner, and Naor quantified this result:

Theorem (Cheeger-Kleiner-Naor)

Let $B \subset H^3$ be the ball of radius 1. There is a $\delta > 0$ such that for any $\epsilon > 0$ and any 1–Lipschitz map $f : B \to L_1$, there is a ball B' of radius at least ϵ such that $f|_{B'}$ is $\asymp |\log \epsilon|^{-\delta}$–close to a map that is constant on vertical lines.

Quantitative nonembeddability

Cheeger, Kleiner, and Naor quantified this result:

Theorem (Cheeger-Kleiner-Naor)

Let $B \subset H^3$ be the ball of radius 1. There is a $\delta > 0$ such that for any $\epsilon > 0$ and any 1–Lipschitz map $f : B \to L_1$, there is a ball B' of radius at least ϵ such that $f|_{B'}$ is $\asymp |\log \epsilon|^{-\delta}$–close to a map that is constant on vertical lines.

Corollary

There is a $\delta > 0$ such that the Goemans-Linial integrality gap $\alpha(n)$ is bounded by

$$\alpha(n) \gtrsim (\log n)^{\delta}.$$

Quantitative nonembeddability

Cheeger, Kleiner, and Naor quantified this result:

Theorem (Cheeger-Kleiner-Naor)

Let $B \subset H^3$ be the ball of radius 1. There is a $\delta > 0$ such that for any $\epsilon > 0$ and any 1–Lipschitz map $f : B \to L_1$, there is a ball B' of radius at least ϵ such that $f|_{B'}$ is $\asymp |\log \epsilon|^{-\delta}$–close to a map that is constant on vertical lines.

Corollary

There is a $\delta > 0$ such that the Goemans-Linial integrality gap $\alpha(n)$ is bounded by

$$\alpha(n) \gtrsim (\log n)^{\delta}.$$

But δ is tiny – around 2^{-60}.

The main theorem

Theorem (Naor-Y.)

Let $k \geq 2$ and let $B \subset H^{2k+1}$ be the unit ball. Let $Z \in H^{2k+1}$ generate the z–axis. If $f : H^{2k+1} \to L_1$ is Lipschitz, then

$$\int_0^1 \left(\int_B \frac{\|f(x) - f(xZ^t)\|_1}{d(x, xZ^t)} \, dx \right)^2 \, \frac{dt}{t} \lesssim \mathrm{Lip}(f)^2.$$

The main theorem

Theorem (Naor-Y.)

Let $k \geq 2$ and let $B \subset H^{2k+1}$ be the unit ball. Let $Z \in H^{2k+1}$ generate the z–axis. If $f : H^{2k+1} \to L_1$ is Lipschitz, then

$$\int_0^1 \left(\int_B \frac{\|f(x) - f(xZ^t)\|_1}{d(x, xZ^t)} \, dx \right)^2 \frac{dt}{t} \lesssim \mathrm{Lip}(f)^2.$$

If f were bilipschitz, then this integral would be infinite, so

Corollary

B does not embed bilipschitzly in L_1.

The main theorem

Theorem (Naor-Y.)

Let $k \geq 2$ and let $B \subset H^{2k+1}$ be the unit ball. Let $Z \in H^{2k+1}$ generate the z–axis. If $f : H^{2k+1} \to L_1$ is Lipschitz, then

$$\int_0^1 \left(\int_B \frac{\|f(x) - f(xZ^t)\|_1}{d(x, xZ^t)} \, dx \right)^2 \frac{dt}{t} \lesssim \mathrm{Lip}(f)^2.$$

If f were bilipschitz, then this integral would be infinite, so

Corollary

B does not embed bilipschitzly in L_1.

And this gives sharp bounds on the scale of the distortion:

Corollary

When $k \geq 2$,

$$c_1(B_{\mathbb{Z}}^{2k+1}(n)) \asymp \sqrt{\log n}.$$

Reducing to surfaces

The sharp bound on Lipschitz embeddings follows from the following *horizontal–vertical isoperimetric inequality*:

Theorem (Naor-Y.)

Let $k \geq 2$ *and let* $S \subset H^{2k+1}$ *be a set with* $\text{area}\, \partial S < \infty$*. Let*

$$S \triangle T = (S \setminus T) \cup (T \setminus S)$$

Then

$$\int_0^\infty \left(\frac{\text{vol}(S \triangle SZ^t)}{d(0, Z^t)} \right)^2 \frac{dt}{t} \lesssim \text{area}(\partial S)^2.$$

Rectifiability and embeddings

- Cheeger-Kleiner-Naor: Surfaces in H^{2k+1} are rectifiable (vertical tangent planes almost everywhere)

Rectifiability and embeddings

- Cheeger-Kleiner-Naor: Surfaces in H^{2k+1} are rectifiable (vertical tangent planes almost everywhere), so maps to L_1 are differentiable (at sufficiently small scales, vertical lines collapse).

Rectifiability and embeddings

- Cheeger-Kleiner-Naor: Surfaces in H^{2k+1} are rectifiable (vertical tangent planes almost everywhere), so maps to L_1 are differentiable (at sufficiently small scales, vertical lines collapse).

Theorem (David-Semmes)

A set $E \subset \mathbb{R}^k$ is uniformly rectifiable if and only if E has a corona decomposition. (Roughly, for all but a few balls B, the intersection $B \cap E$ is close to the graph of a Lipschitz function with small Lipschitz constant.)

Rectifiability and embeddings

- Cheeger-Kleiner-Naor: Surfaces in H^{2k+1} are rectifiable (vertical tangent planes almost everywhere), so maps to L_1 are differentiable (at sufficiently small scales, vertical lines collapse).

Theorem (David-Semmes)

A set $E \subset \mathbb{R}^k$ is uniformly rectifiable if and only if E has a corona decomposition. (Roughly, for all but a few balls B, the intersection $B \cap E$ is close to the graph of a Lipschitz function with small Lipschitz constant.)

- Naor-Y.: Surfaces in H^{2k+1} are made of *uniformly rectifiable* pieces.

Decompositions in $\mathbb{R}^k$ and H^{2k+1}

Theorem (Y.)

If T is a mod-2 d–cycle in $\mathbb{R}^k$, $d < k$, it can be decomposed as a sum $T = \sum_i T_i$ such that $\operatorname{supp} T_i$ *is uniformly rectifiable and* $\sum_i \operatorname{mass} T_i \lesssim \operatorname{mass} T$.

Decompositions in $\mathbb{R}^k$ and H^{2k+1}

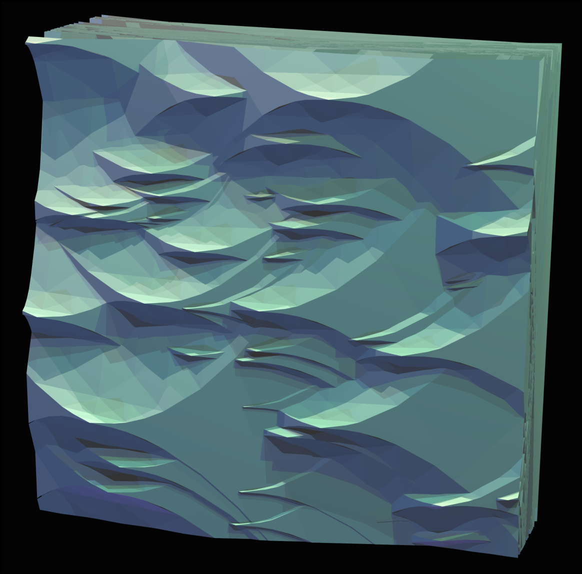
Theorem (Y.)

If T is a mod-2 d–cycle in $\mathbb{R}^k$, $d < k$, it can be decomposed as a sum $T = \sum_i T_i$ such that $\operatorname{supp} T_i$ is uniformly rectifiable and $\sum_i \operatorname{mass} T_i \lesssim \operatorname{mass} T$.

Theorem (Naor-Y.)

If $E \subset H^{2k+1}$, then E can be decomposed into sets E_i so that each ∂E_i has a corona decomposition that approximates ∂E_i by intrinsic Lipschitz graphs.

An intrinsic Lipschitz graph



The isoperimetric inequality for graphs

Theorem (Austin-Naor-Tessera, Naor-Y.)

If $k \geq 2$ and $S \subset B \subset H^{2k+1}$ is bounded by an intrinsic Lipschitz graph with bounded Lipschitz constant, then

$$\int_0^\infty \left(\frac{\mathrm{vol}(S \bigtriangleup SZ^t)}{d(0, Z^t)} \right)^2 \, \frac{dt}{t} \lesssim \mathrm{area}(\partial S)^2.$$

The isoperimetric inequality for graphs

Theorem (Austin-Naor-Tessera, Naor-Y.)

If $k \geq 2$ and $S \subset B \subset H^{2k+1}$ is bounded by an intrinsic Lipschitz graph with bounded Lipschitz constant, then

$$\int_0^\infty \left(\frac{\mathrm{vol}(S \triangle SZ^t)}{d(0, Z^t)}\right)^2 \frac{dt}{t} \lesssim \mathrm{area}(\partial S)^2.$$

Theorem (Naor-Y.)

If $k \geq 2$ and $S \subset B \subset H^{2k+1}$ is a set such that ∂S has a corona decomposition, then

$$\int_0^\infty \left(\frac{\mathrm{vol}(S \triangle SZ^t)}{d(0, Z^t)}\right)^2 \frac{dt}{t} \lesssim \mathrm{area}(\partial S)^2.$$

This proves the main theorem for H^{2k+1} when $k \geq 2$.

What fails in H^3?

What fails in H^3?

- We can still decompose sets in H^3 into uniformly rectifiable pieces, but the isoperimetric inequality fails for intrinsic Lipschitz graphs in H^3.

What fails in H^3?

- We can still decompose sets in H^3 into uniformly rectifiable pieces, but the isoperimetric inequality fails for intrinsic Lipschitz graphs in H^3.
- In fact, graphs in H^3 satisfy a different inequality!

The three-dimensional case: a counterexample

Proposition

For any $\alpha > 1$, there is a half-space $S \subset B \subset H^3$ bounded by an intrinsic Lipschitz graph such that for any $p > 0$,

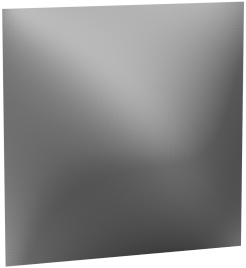
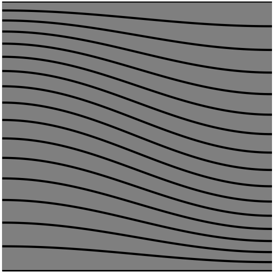
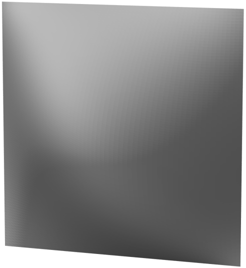
$$\int_0^\infty \left(\frac{\mathrm{vol}(S \triangle SZ^t)}{d(0, Z^t)} \right)^p \frac{dt}{t} \gtrsim \alpha^{4-p}.$$

The three-dimensional case: a counterexample

Proposition

For any $\alpha > 1$, there is a half-space $S \subset B \subset H^3$ bounded by an intrinsic Lipschitz graph such that for any $p > 0$,

$$\int_0^\infty \left(\frac{\operatorname{vol}(S \mathbin{\triangle} SZ^t)}{d(0, Z^t)} \right)^p \ \frac{dt}{t} \gtrsim \alpha^{4-p}.$$

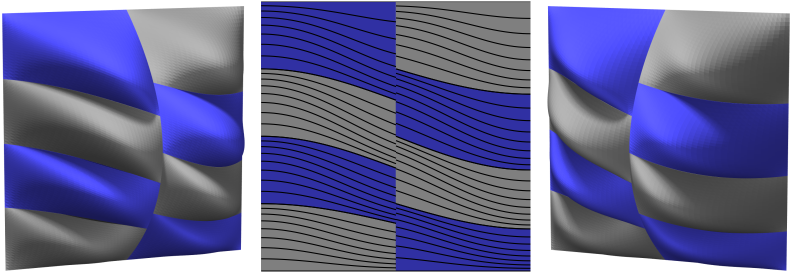




The three-dimensional case: a counterexample

Proposition

For any $\alpha > 1$, there is a half-space $S \subset B \subset H^3$ bounded by an intrinsic Lipschitz graph such that for any $p > 0$,

$$\int_0^\infty \left(\frac{\mathrm{vol}(S \triangle SZ^t)}{d(0, Z^t)}\right)^p \frac{dt}{t} \gtrsim \alpha^{4-p}.$$

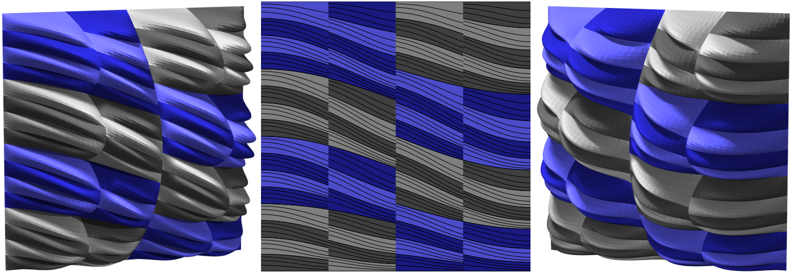


The three-dimensional case: a counterexample

Proposition

For any $\alpha > 1$, there is a half-space $S \subset B \subset H^3$ bounded by an intrinsic Lipschitz graph such that for any $p > 0$,

$$\int_0^\infty \left(\frac{\mathrm{vol}(S \triangle SZ^t)}{d(0, Z^t)}\right)^p \frac{dt}{t} \gtrsim \alpha^{4-p}.$$



The three-dimensional case: foliated corona decompositions

Proposition (Naor-Y.)

For any $S \subset H^3$,

$$\int_0^\infty \left(\frac{\mathrm{vol}(S \triangle SZ^t)}{d(0, Z^t)} \right)^4 \, \frac{dt}{t} \lesssim \mathrm{area}(\partial S)^4.$$

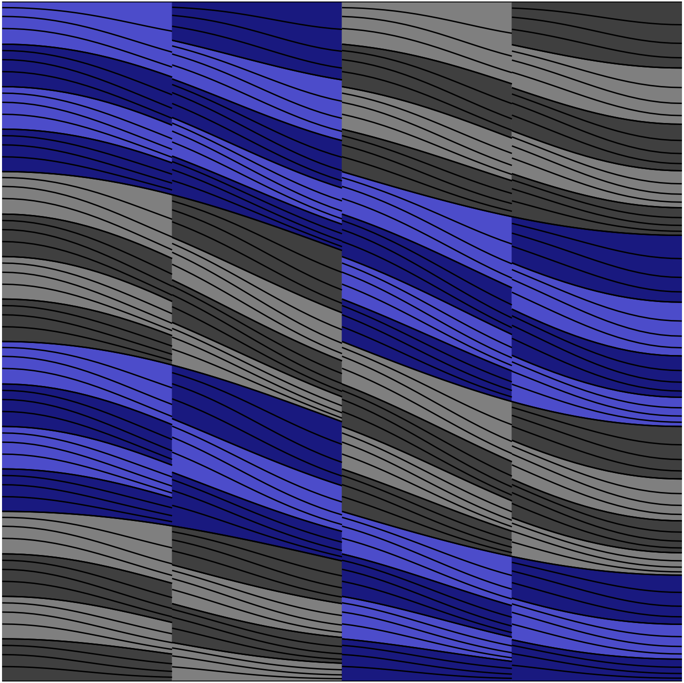
The three-dimensional case: foliated corona decompositions

Proposition (Naor-Y.)

For any $S \subset H^3$,

$$\int_0^\infty \left(\frac{\mathrm{vol}(S \triangle SZ^t)}{d(0, Z^t)} \right)^4 \frac{dt}{t} \lesssim \mathrm{area}(\partial S)^4.$$

The proof is based on *foliated corona decompositions*: decompositions of a graph into quadrilaterals of varying shapes and sizes on which the graph is nearly foliated by horizontal curves.



Question

- Uniform rectifiability in $\mathbb{R}^k$ has definitions in terms of singular integrals, β–coefficients, corona decompositions, the big-pieces-of-Lipschitz-graphs property, and many more. We've used corona decompositions to study one class of surfaces in the Heisenberg group – do the rest of the definitions also generalize?