Eulerian mean flow from an instability of convective plumes

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The dynamical origin of large-scale flows in systems driven by concentrated Archimedean forces is considered. A two-dimensional model of plumes, such as those observed in thermal convection at large Rayleigh and Prandtl numbers, is introduced. From this model, we deduce the onset of mean flow as an instability of a convective state consisting of parallel vertical flow supported by buoyancy forces. The form of the linear equation governing the instability is derived and two modes of instability are discussed, one of which leads to the onset of steady Eulerian mean flow in the system. We are thus able to link the origin of mean flow precisely to the profiles of the unperturbed plumes. The form of the nonlinear partial differential equation governing the Eulerian mean flow, including nonlinear effects, is derived in one special case. The extension to three dimensions is outlined. © *2000 American Institute of Physics.* [S1054-1500(00)01101-0]

While mean flows are observed in numerous flows driven by buoyancy forces, the origin of the stresses needed to set up such flows is not well understood, particularly when the system is well beyond the onset of convective instability. In this paper we treat the generation of mean flow by discrete, thin plumes. Such plumes are often observed in convection at high Rayleigh numbers. To simplify our analysis we focus on two-dimensional flow, and study the stability of a periodic array of parallel plumes. Two modes of instability are deduced, one of which is associated with an Eulerian mean flow. By introducing a boundary-value problem that isolates the latter mode of instability, we treat higher-order terms in an expansions and derive an equation for the evolution of the mean flow. The extension of the method to three dimensions is discussed.

I. INTRODUCTION

It is now well-established that a large-scale or mean shear flow is a natural accompaniment of many systems driven by buoyancy or Archimedean forces. Mean flows, in either the Eulerian or Lagrangian sense, can be important contributors to transport, and so their origin in flows driven by Archimedean forces is of practical importance. Here we understand an Archimedean force to be a density variation caused by a single scalar field such as temperature or salt concentration. In this paper we are concerned with the dynamical processes driving these mean flows. We shall study this problem in a very simple two-dimensional thermal model in which the basic unperturbed state is a parallel flow driven by a horizontal gradient of temperature. We further assume that the horizontal scale of this basic state is small compared to some vertical scale, presumably the distance between two horizontal walls containing the fluid. We shall not, however, explicitly consider the flow near these walls. Thus our model is focused on the ‘‘central region’’ of flow, and does not consider the interaction of the basic state with the walls. We are then able to crudely model perturbations of slender plumes, but not the processes by which the plumes are created, or the results of collisions of plumes with a boundary. We shall allow, however, for the breaking of top–bottom (TB) symmetry. An example of this symmetry breaking is thermal convection between constant temperature walls, but with a fluid whose viscosity depends upon temperature. In an isoviscous fluid the symmetry can be broken by the boundary conditions, e.g., by introducing salt water into fresh at one wall with a compensating fresh water flow at the opposite wall, or by varying the temperature horizontally on one wall. Since we shall be concerned only with the central region, we adopt the simplest isoviscous model but allow the basic state to break TB symmetry.

Krishnamurti and Howard[1] first observed mean flows in classical isoviscous thermal convection, as a component of a convective state consisting of tilted rolls. They also found that the mean Reynolds stresses driving the flow were balanced by viscous stresses of the mean flow in the steady state, and noted the distinction between Eulerian mean flow, which can involve no net mean transport of a given fluid parcel, and Lagrangian mean flow, where, by definition, particle transport occurs.

In order to understand the origin of a mean flow in their experiment, Howard and Krishnamurti[2] considered a truncated system of equations, which enlarged the Lorenz model of convection to allow a mean-flow mode. In a thorough study of the bifurcation structure of the resulting five-dimensional system, they confirmed the origin of the Eulerian mean flow as a secondary bifurcation from the basic conductive state, which accompanies the onset of tilting of steady rolls created by the basic Bénard instability. A tertiary bifurcation results in unsteady tilted rolls and the onset of Lagrangian mean flow. Tilted rolls can be viewed as precursors of the fully developed tilted plumes we study below.

In 2D with flow in the $x-z$ plane, z upward, tilted rolls

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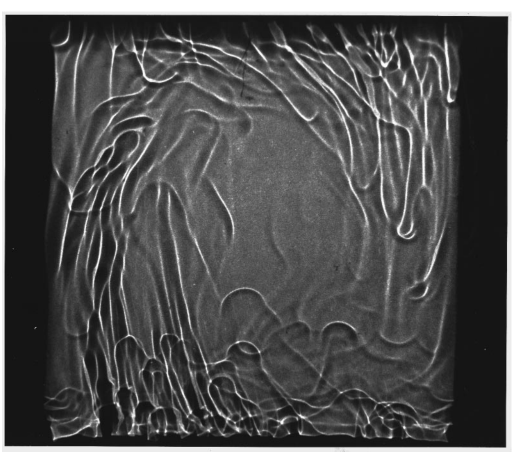
create a nonzero horizontal average of uw, leading to a force which is minus the z-derivative of the latter quantity. This force is in equilibrium with the viscous force of the mean flow, here the horizontal mean of u. Many subsequent papers have examined various aspects of mean flow structure, using low-order ODE models or full numerical simulation in convection and magnetoconvection see, e.g., Refs. 3–7,14,15. Related Archimedean instabilities, associated with larger scales of motion, occur in doubly diffusive systems as instabilities of salt fingers; see, e.g., Ref. 6. The linear stability of these fingers, with negligible diffusivity of the salt, has been discussed in Ref. 7. There a vertically uniform unperturbed state similar to that considered below was found to be unstable, and in certain cases the mode of instability involved horizontal mean flow.

For fluids with variable viscosity, Solomatov[8] divided the thermal convective states into three regimes. The first is essentially isoviscous, with upwelling plume and downwelling plumes of roughly the same properties. When the contrast is in the range 10^4-10^5 plumes of both types coexist, but are different in structure. For still higher contrasts a ''stagnant lid'' regime is reached, where essentially isoviscous convection by upwelling plumes occurs below a lid of very viscous fluid, where the transport is mainly by conduction. Recent computations have gone as high as 10^5 (Ref. 9) and 10^6 (Ref. 10), and observed strong mean flows at Rayleigh numbers up to 10^4.

The present paper was motivated by recent experiments in glycerol, a common working fluid with a significant dependence of viscosity upon temperature.[11,12] In these experiments the mean center (or interior) temperature was measured and a mean flow observed in an 18 cm cube at Rayleigh numbers 10^6-10^9 and Prandtl numbers 10^2-10^3, with a modest viscosity contrast ~200. A shadowgraph of the plume structure, clearly showing a mean flow (in fact, a cellular flow across the diagonal plane) is shown in plate I.



Note that there is a small but noticeable difference in the boundary-layer structure and the plume geometry of the top and bottom walls. The thickness of the thermal layers differed by a factor ~3 in these experiments. In Ref. 11, a model allowing calculation of the average center temperature, well outside the thin thermal boundary layers on each wall, was deduced from a boundary-layer analysis of the wall region.

These experiments suggested that an analysis of the onset of mean flow in fully-developed convection might be possible by taking slender plumes as the underlying steady flow, in the place of marginally convective Bénard cells. The relative ease of setting up the mean flow in the presence of top–bottom asymmetry suggested a parallel investigation of the influence of TB symmetry on the Reynolds stresses responsible for the mean flow. In such an approach marginal stability can still be investigated, but about a state quite different from that of marginal convection.[1] Even at high Rayleigh numbers, it is observed that convective flows may or may not drive mean flows, so a criterion based upon the properties of plumes is of some interest.

There seems to be little hope of finding useful exact solutions of the nonlinear equations, and the simplified model proposed below, where wall effects are neglected, becomes attractive. We shall formulate a 2D model and discuss the unperturbed plumes in Sec. II. In Sec. III a basic invariance of slender plumes associated with a horizontal shift of their positions is discussed, and shown to be unrelated to the generation of an Eulerian mean flow. In Sec. IV the mean-flow instability is identified as a degenerate bifurcation of the solutions of the linear stability problem, at a zero eigenvalue of geometric multiplicity two. In Sec. V the form of the linear mean-flow equation is derived, and the mechanism is shown by an example to lead to instability in a case with TB symmetry.

Some properties of the partial differential equation governing the mean flow are given in Sec. V, including the form of the nonlinear terms and the role of breaking the TB symmetry. Finally, in Sec. VI we outline an extension of the theory to three dimensions.

We shall deal in this paper only with the onset of Eulerian mean flow. This is analogous to studying the addition of a parallel flow $[U(z),0]$ to the two-dimensional cellular flow of the form $(u,w)=(\psi_z,-\psi_x),\psi=\sin(ax)\sin(by)$, where, since thin plumes are considered, we would take $b\ll a$. The mean flow causes the ''channels'' to develop between the cells, where transport of scalar and vector fields is enhanced; see, e.g., Ref. 13. In a real flow, plume creation is a random process and transport would be chaotic in both space and time. Our purpose in the present investigation is to indicate how mean flow is generated in the simplest periodic geometry, with the understanding that the same dynamical effects would apply for irregular patterns. The averaging process needed in that case is discussed briefly in Sec. VI.

II. FORMULATION OF THE MODEL

We shall deal here with thermal convection between horizontal planes $z=0,H$. We consider the customary equations of thermal convection of a Boussinesq fluid of constant viscosity. In a fluid of constant viscosity and isothermal walls, the Boussinesq approximation leads to equations which possess the top–bottom symmetry mentioned above. Indeed, the equations are invariant under a shift of origin of the temperature scale, so we may assume that the wall temperatures satisfy $T_2 = -T_1$. Then the equation and boundary conditions are invariant under the transformation $z,w,T \to -z,-w,-T$ where w is the vertical velocity component. In such a fluid, regardless of the Rayleigh and Prandtl numbers, the convective fields associated with the top and bottom walls are effectively indistinguishable. As mentioned in the Introduction, we shall here allow top–bottom symmetry to be broken, although this will be introduced through the choice of the unperturbed state of model flow. We shall do this without reference to whatever boundary conditions at the wall might cause the broken symmetry; see Sec. II B. [The presence of the walls will appear in one way only, as a condition of zero net vertical mass flux, see (15).]

The equations of motion are then

$$\mathbf{q}_t + \mathbf{q}\cdot\nabla\mathbf{q} + \frac{1}{\rho}\nabla p - \nu\nabla^2\mathbf{q} = g\alpha T\mathbf{i}_z, \tag{1}$$

$$\partial_t T + \mathbf{q}\cdot\nabla T - \kappa_T\nabla^2 T = 0, \tag{2}$$

$$\nabla\cdot\mathbf{q} = 0. \tag{3}$$

All symbols are conventional, α being the coefficient of thermal expansion of the fluid and κ_T the thermal diffusivity.

A. Dimensionless variables

We now convert (1)–(3) to suitable dimensionless form. We shall be dealing with temperature fields whose horizontal scale of variation, measured by a length L, is small compared to H, $L/H \ll 1$. At the same time we shall exploit a high Prandtl number limit to neglect the diffusion of temperature, and adjust the Rayleigh number (based on the horizontal scale) to allow the nonlinear inertial terms to be neglected to first order. With those goals a reference scale of velocity is suggested by the balance between the viscous force associated with a horizontal scale L and the buoyancy,

$$U_{\mathrm{ref}} = g\alpha T_{\mathrm{ref}} L^2 \nu. \tag{4}$$

Here T_{ref} is a reference temperature which will be fixed later. With $\mathbf{q} = (u,w)$, we define dimensionless variables,

$$\begin{aligned}(x^*,y^*) &= (x/L, z/H), \quad t^* = tL/U_{\mathrm{ref}}, \quad u^* = uH/(LU_{\mathrm{ref}}),\\ w^* &= w/U_{\mathrm{ref}}, \quad T^* = T/T_{\mathrm{ref}}, \quad p^* = pH/(\rho U_{\mathrm{ref}}\nu).\end{aligned} \tag{5}$$

Dropping stars, the dimensionless equations become

$$\delta(u_t + uu_x + wu_z) + p_x - u_{xx} - \lambda^2 u_{zz} = 0, \tag{6}$$

$$\delta(w_t + uw_x + ww_z) + \lambda^2 p_z - w_{xx} - \lambda^2 w_{zz} = T, \tag{7}$$

$$T_t + uT_x + wT_z = \delta^{-1}P_r^{-1}(T_{xx} + \lambda^2 T_{zz}), \tag{8}$$

$$u_x + w_z = 0. \tag{9}$$

Here δ is a Reynolds number based upon λU_{ref}, L, and ν, P_r is the Prandtl number based upon ν, and λ is a slenderness parameter:

$$\delta = g\alpha L^4 T_{\mathrm{ref}}/(H\nu) = U_{\mathrm{ref}} L\lambda/\nu, \tag{10}$$

$$\lambda = H/L, \quad P_r = \nu/\kappa_T. \tag{11}$$

The parameter δ may also be expressed, in terms of the Rayleigh number R_a, as

$$\delta = \lambda^4 R_a/P_r, \quad R_a = g\alpha H^3 T_{\mathrm{ref}}/(\nu\kappa_T), \tag{12}$$

where κ_T is the (constant) thermal diffusivity.

We shall assume that

$$\delta \ll 1, \quad \lambda \ll 1, \quad \delta P_r \gg 1. \tag{13}$$

The last condition is essential, for we shall be neglecting the diffusion of temperature throughout.

A third small parameter, the amplitude ϵ of the perturbation of the basic state, is introduced below. The relative ordering of the assumed small parameters is not immediately obvious and will be discussed below. We use ϵ as the basic ordering parameter.

B. The unperturbed plumes

We define the *reduced system* to be (6)–(9) with $\lambda = \delta^{-1}P_r^{-1} = 0$. This simpler system, which we be used to discuss perturbations of slender nonconductive temperature fields, is then

$$\begin{aligned}&\delta(u_t + uu_x + wu_z) + p_x - u_{xx} = 0,\\ &\delta(w_t + uw_x + ww_z) - w_{xx} = T,\\ &T_t + uT_x + wT_z = 0, \quad u_x + w_z = 0.\end{aligned} \tag{14}$$

We shall say $\Sigma = (u,w,p,T)$ is *admissible* provided that (a) Σ solves (14), (b) Σ is periodic with period 2 in x, and (c) w has zero net vertical mass flux,

$$\langle w\rangle \equiv \int_0^2 w\, dx = 0. \tag{15}$$

An *unperturbed plume* will be an admissible $\Sigma = \Sigma_0$ of the form $(u_0, w_0, p_0, T_0) = (0, w_0(x), Gz, T_0(x))$. Thus $T_0(x)$ may be chosen to be any periodic function with period 2 and

$$\begin{aligned}&G - d^2 w_0/dx^2 = T_0(x), \quad w_0(x+2) = w_0(x),\\ &\int_0^2 w_0\, dx = 0.\end{aligned} \tag{16}$$

The constant pressure gradient G is needed to balance any net buoyancy of the assumed temperature profile, and thus allow condition (15) to be satisfied. In fact, we see immediately upon integration over an interval of periodicity that

$$G = \frac{1}{2}\int_0^2 T_0\, dx. \tag{17}$$

We remark that, in particular cases discussed below, we will impose additional restrictions on the positions of the critical points where w_0 vanishes. This is because perturbations can change the plume streamline topology near such points.

A simple example of a T_0 is the ''top-hat'' temperature profile:

$$T_0(x) = \begin{cases} 1, & \text{if } 0 \leqslant |x| < \mu,\\ 0, & \text{if } \mu \leqslant |x| < 1.\end{cases} \tag{18}$$

Here $\mu, 0 < \mu < 1$, is the half-width of the upward plume of temperature T_{ref}, as a fraction of the period. Here $G = \mu$ and

$$-d^2w_0/dx^2=\begin{cases}(1-\mu), & \text{if } 0\leqslant|x|<\mu,\\ -\mu, & \text{if } \mu\leqslant|x|<1.\end{cases} \tag{19}$$

Solving with the conditions (16), the unique continuously differentiable solution is

$$w_0=\begin{cases}(\mu-1)x^2/2+C, & \text{if } 0\leqslant|x|<\mu,\\ \mu(x-1)^2/2+\mu(\mu-1)/2+C, & \text{if } \mu\leqslant|x|<1,\end{cases} \tag{20}$$

where

$$C=\mu(\mu-1)(\mu-2)/6. \tag{21}$$

We note that here $w_0(\mu)\equiv w_\mu=\mu(\mu-1)(2\mu-1)/3$. If $\mu<1/2$, w_0 vanishes at $1-\sqrt{(1-\mu^2)/3}>\mu$, while if $\mu>1/2$, w_0 vanishes at $\sqrt{\mu(2-\mu)/3}<\mu$.

The top-hat profile is very useful for explicit computations, as we shall see below. Note that (18) is a case of broken TB symmetry, except for the special case $\mu=1/2$. That case is special in another way, for then the discontinuity of temperature occurs at the zero of w_0.

A more general plume configuration is the ''double-hat'' profile,

$$T_0=\begin{cases}1, & \text{if } 0\leqslant|x|<\mu,\\ 0, & \text{if } \mu\leqslant|x|<1-\nu,\\ -\alpha, & \text{if } 1-\nu\leqslant|x|<1,\end{cases} \tag{22}$$

where $0<\mu<\nu<1$. Here α is a second parameter. It is convenient to denote the top-hat profile by $T_0^{(1)}(x;\mu)$ and then write (22) as

$$T_0^{(2)}=T_0^{(1)}(x;\mu)-\alpha T_0^{(1)}(x-1;\nu). \tag{23}$$

Thus

$$w_0^{(2)}=w_0^{(1)}(x;\mu)-\alpha w_0^{(1)}(x-1;\nu), \tag{24}$$

The explicit form is

$$w_0=\begin{cases}(\mu-\alpha\nu-1)x^2/2+A, & \text{if } 0\leqslant|x|<\mu,\\ (\mu-\alpha\nu)x^2/2+Bx+C, & \text{if } \mu\leqslant|x|<1-\nu,\\ (\alpha+\mu-\alpha\nu)(x-1)^2/2+D, & \text{if } 1-\nu\leqslant|x|<1,\end{cases} \tag{25}$$

where

$$\begin{aligned}&A=\mu^3/6-\alpha\nu^3/6-\mu/6+\alpha\nu/6+\mu(1-\mu)/2, B=-\mu,\\ &C=A+\mu^2/2,\quad D=A+\mu(\mu-1)/2+\alpha\nu(\nu-1)/2.\end{aligned} \tag{26}$$

The case of TB symmetry is $\alpha=1,\nu=\mu$:

$$w_0=\begin{cases}-x^2/2+\mu(1-\mu)/2, & \text{if } 0\leqslant|x|<\mu,\\ -\mu(x-1/2), & \text{if } \mu\leqslant|x|<1-\mu,\\ +(x-1)^2/2-\mu(1-\mu)/2, & \text{if } 1-\mu\leqslant|x|<1.\end{cases} \tag{27}$$

We point out that the double-hat profile provide a model of a convective field combining what can be described as distinct up- and down-plumes.

C. Remarks concerning instabilities of plumes

In the subsequent analysis we study instabilities of a slender plume. We shall show that two distinct instabilities are of interest. The first will be termed a ''shift'' instability, since it will result in a horizontal displacement of all fluid particles, equally at the same value of z. A new steady state obtained from a shift instability would have streamlines which are equivalent under parallel displacement in x, and cannot involve any Eulerian mean flow. However, we shall then identify a second instability which is associated with an Eulerian mean flow. In a new steady state obtained from the second instability, streamlines of upward moving and downward moving particles have opposite deflections. This can be understood by considering a simple combination of a plume $(0,w_0(x))$ and a small mean flow $\epsilon\sin 2\pi z$. The perturbed streamline passing through $(x_0,0)$ is given by

$$x=\epsilon[2\pi w_0 l(x_0)]^{-1}\sin 2\pi z, \tag{28}$$

provided that $w_0(x_0)\neq 0$. Note that, in the vicinity of any zero of a zero of $w_0(x)$, ''cat's-eye'' regions of closed streamlines are formed (see Fig. 2 below). Our object is to discuss both kinds of instability and see how they may be combined to understand the emergence of a mean flow from a perturbed plume.

III. SHIFTED SOLUTIONS

A. Shift invariance

A key ingredient in our analysis, for admissible solutions of (14), is a property of the plumes derived from the slenderness assumption. It is analogous to a well-known ''shift invariance'' of Prandtl's two-dimensional boundary-layer equations. These latter equations are

$$u_t+uu_x+vu_y+p_x-u_{yy}=0,\quad p=p(x,t),\quad u_x+v_y=0. \tag{29}$$

It is easily seen that, if $(u(x,y,t),v(x,y,t))$ solves (29) then so does

$$\begin{aligned}&u_s=u(x,y_s,t),\quad v_s=v(x,y_s,t)+\eta_t+u(x,y_s,t)\eta_x,\\ &p_s(x,t)=p,\end{aligned} \tag{30}$$

where $y_s(x,y,t)=y-\eta(x,t)$, η being an arbitrary function. The proof uses the chain rules,

$$\partial_t|_{x,y}=\partial_t|_{x,y_s}-\eta_t\partial_{y_s}|_{x,t},\quad \partial_x|_{t,y}=\partial_x|_{t,y_s}-\eta_x\partial_{y_s}|_{x,t}. \tag{31}$$

This can also be expressed by a transformation on the stream function, since $(u_s,v_s)=(\partial_y\psi_s,-\partial_x\psi_s)$ where $\psi_s=\psi(x,y-\eta,t)-\int\eta_t\,dx$.

We then have

$$u_{sx}+v_{sy}=-u_y(x,y-\eta,t)\eta_x+u_y(x,y-\eta,t)\eta_x+C=0, \tag{32}$$

$$\begin{aligned}&u_{st}+u_su_{sx}+v_su_{sy}+p_{sx}-u_{syy}\\ &\quad=-u_y(x,y_s,t)(\eta_t+u(x,y_s,t)\eta_x)\\ &\qquad+u_y(x,y_s,t)(\eta_t+u(x,y_s,t)\eta_x)+M|_{y=y_s}=0,\end{aligned} \tag{33}$$

where $C=u_x+v_y$ and $M=u_t+uu_x+vu_y+p_x-u_{yy}$, each evaluated at x,y_s,t, which completes the proof.

The function $\eta(x,t)$ determines a vertical shift of the entire flow field . The shifted boundary layer may be thought of as essentially parallel at any instant to a curve $y=\eta(x,t)$, away from its base position $y=0$. This shift invariance expresses the thinness of the boundary layer in physical variables, as well as the fact that the only viscous stress in the problem derives from the cross-stream variation of the velocity. This property figures prominently in the analysis of boundary-layer separation; see Ref. 14.

We shall state a corresponding result for (14). Since our plumes are vertical the shift will now be in the horizontal coordinate x.

Theorem: *Let $\Sigma=(u,w,p,T)$ be any admissible solution of the reduced equations (14). Then $\Sigma_s=(u_s,w_s,p_s,T_s)$ is also a solution, where*

$$u_s=u(x-\xi,z,t)+\xi_t+w(x-\xi,z,t)\xi_z,$$
$$w_s=w(x-\xi,z,t), \tag{34}$$

$$p_s=\partial_x u_s+\delta\int G_s\,dx, \quad T_s=T(x-\xi,z,t), \tag{35}$$

where ξ is a function of z,t and

$$G_s=\partial_t u_s+u_s\,\partial_x u_s+w_s\,\partial_z u_s, \tag{36}$$

provided that $\xi(z,t)$ satisfies

$$\xi_{tt}+(\chi(z,t)\xi_z)_z=0,$$
$$\chi(z,t)=\langle w^2\rangle\equiv\int_0^2 w^2(x,z,t)\,dx. \tag{37}$$

This result states an invariance property under the condition (37) on ξ, and therefore goes further than Prandtl boundary layer shift by including a constraint imposed by the x-momentum balance.

To prove the theorem we again may verify that the shifted solution Σ_s is a solution of the system, by substitution using formulas analogous to (31). The last three equations of (14), expressed in the new variables, may be shown to hold as in the shift of the Prandtl boundary layer. The new feature is the calculation of p_s, which enforces a condition that G_s have zero horizontal mean. We then have

$$\langle G_s\rangle=\langle u\rangle_t+\langle uw\rangle_z+\xi_{tt}+\langle w\xi_z\rangle_t+\langle\xi_t w\rangle_z+(\langle w^2\rangle\xi_z)_z=0. \tag{38}$$

Since Σ solves (14), the average of the reduced x-momentum equation yields $\langle u\rangle_t+\langle uw\rangle_z=0$. Since w satisfies (15), $\langle w\xi_z\rangle_t=\langle\xi_t w\rangle_z=0$. These expressions imply (37) and establish the theorem.

B. Shift instability

Applying Theorem 1 to an unperturbed plume Σ_0 yields the following result: To linear terms in ξ, (37) yields

$$\xi_{tt}+\langle w_0^2\rangle\xi_{zz}=0, \tag{39}$$

since w_0 is independent of z. Thus, with $\xi=e^{ikz+\sigma t}$ we extract an instability having growth rate $\sigma_s=|k|\sqrt{\langle w_0^2\rangle}$. The growth of σ with k indicates a poorly posed initial-value problem, which will have to be resolved outside the reduced equations. We note in passing one way which this can be done. Suppose that we take (6) in its entirety for the x-momentum equation of the ‘‘reduced system,’’ while still omitting all other terms in λ^2. This is of course not a fully consistent procedure but it slightly enlarges the reduced system. Then Theorem 1 again obtains with an additional term in G_s and an altered equation for ξ:

$$\xi_{tt}+\langle w_0^2\rangle\xi_{zz}-\gamma\xi_{zzt}=0, \quad \gamma=\lambda^2/\delta. \tag{40}$$

In this case the shift instability has the growth rate

$$\sigma_s(k)=\tfrac{1}{2}\left[-k^2\gamma+\sqrt{k^4\gamma^2+4\langle w_0^2\rangle}\right], \tag{41}$$

which grows monotonically from 0 to $\langle w_0^2\rangle/\gamma$ as $|k|$ increases from 0, thus making the initial-value problem well posed.

Since we shall be investigating the linear instability of an unperturbed plume, we note that, if $\Sigma\approx\Sigma_0+\Sigma'$ is the linear approximation in powers of ξ, we have

$$\Sigma'=\Bigg(\xi_t+w_0\xi_x,-w_{0x}\xi,w_{0x}\xi_z + \int(2w_0\xi zt+w_0^2-\overline{w_0^2})dx,-T_{0x}\xi\Bigg), \tag{42}$$

given that ξ satisfies (37).

Since our aim in this paper is an understanding of the origin of mean flow in this model, it is important to observe that the mean flow in the shift instability, ξ_t, is a transient which disappears if and when ξ becomes stationary.

IV. LINEAR ANALYSIS AND THE MEAN-FLOW INSTABILITY

In this section we shall discuss a bifurcation from the unperturbed plumes which is not a shift instability. We examine this by linearizing (6)–(9). Let the expansion in ϵ for fixed δ,λ be $\Sigma=(u,w,p,T)=\Sigma_0+\epsilon\Sigma'+\ldots$, $\epsilon\ll 1$. The linearized equations are then, dropping primes,

$$\delta(u_t+w_0u_z)+p_x-u_{xx}-\lambda^2u_{zz}=0,$$
$$\delta(w_t+w_0w_z+u\,dw_0/dx)+\lambda^2p_z-w_{xx}-\lambda^2w_{zz}=T, \tag{43}$$
$$T_t+w_0T_z+u\,dT_0/dx=0, \quad u_x+w_z=0.$$

If we set $\delta=\lambda=0$ in (43) we have the limiting system

$$p_x-u_{xx}=0, \quad -w_{xx}=T, \quad w_0T_z+u\,dT_0/dx=0,$$
$$u_x+w_z=0. \tag{44}$$

We shall now consider in some detail the solutions of (44).

Since, from (44), we have $-w_0w_{xxz}=w_0u_{xxx}=w_0T_z=-T_t-u\,dT_0/dx$ and $-d^2w_0/dx^2=T_0$, u,w satisfy

$$w_0u_{xxx}=w_{xxt}+uw_{0xxx}, \quad u_x+w_z=0. \tag{45}$$

Thus we see that the linearized shift solution $u=\xi_t(z,t)+\xi_z(z,t)w_0(x)$, $w=-\xi(z,t)w_{0x}$, $T=-\xi(z,t)T_{0x}$ is one solution of (45). We now seek other solutions, under the additional assumption that they do not depend upon t, so that (45) provides a single equation for u,

$$w_0u_{xxx}=uw_{0xxx}. \tag{46}$$

In other words, we now seek linear instability modes on a time scale long compared to the natural time scale of the shift instability, the latter given by the solution $u=-\xi_zw_0$.

The question we ask is, are there any other solutions of (46)? In order to eliminate additive solutions of the form $(u,w,p,T)=(0,f(x),0,-f_{xx})$ we shall in what follows specify the perturbation to have a mean in z which is zero.

A. Specific cases

The answer to the last question depends upon the choice of w_0, so we first examine some cases explicitly. Consider first the top-hat profile (18). Here (46) takes the form

$$w_0 u_{xxx}=u[\delta(x-\mu)-\delta(x+\mu)], \quad -1\leqslant x<+1. \tag{47}$$

It is easily seen by direct calculation that any continuously differentiable solution of (47) is a multiple of w_0 and thus a shift solution. Indeed we must have $u_{xxx}=0$, $x\neq\pm\mu$, together with the jump conditions

$$[u_{xx}](\mu)=u(\mu)/w_\mu, \quad [u_{xx}](-\mu)=-u(-\mu)/w_\mu, \tag{48}$$

$[\cdot]$ denoting the jump from left to right and $w_\mu=w_0(\mu)=\mu(\mu-1)(2\mu-1)/3$. Solving with

$$u=\begin{cases} A(x+1)^2+B(x+1)+C, & \text{if } -1\leqslant x<-\mu,\\ Dx^2+Ex+F, & \text{if } -\mu\leqslant x<\mu,\\ A(x-1)^2+B(x-1)+C, & \text{if } \mu\leqslant x<1,\end{cases} \tag{49}$$

where we have already imposed the periodicity conditions $u(-1)=u(+1)$, $u_x(-1)=u_x(+1)$, $u_{xx}(-1)=u_{xx}(+1)$, we obtain $B=E=0$,

$$C=(\mu^2-1)A/3, \quad D=(1-1/\mu)A, \qquad F=(\mu-1)(\mu-2)A/3. \tag{50}$$

We thus find that (49) is $2w_0/\mu$, so the only steady solution is a shift.

To see that this is not always true, we consider next the double-hat profile with $\alpha=1$, $\nu=\mu$. The equation for u is then

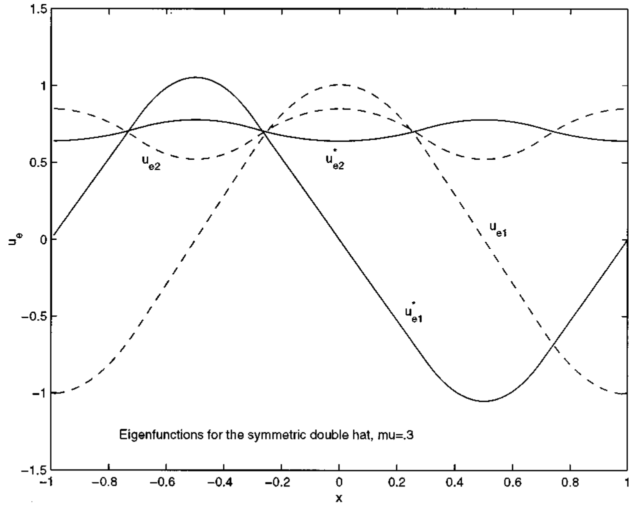


FIG. 1. The normalized eigenfunctions of the double hat for $\mu=0.3$.

$$u_{xxx}-uw_\mu^{-1}[\delta(x-\mu)-\delta(x+1-\mu)]=0. \tag{51}$$

Here $w_\mu=\mu(1-2\mu)/2$. Since w_0 is now odd in x about $x=1/2$, we look for a solution u which is *even* with respect to $x=1/2$. Assume

$$u=\begin{cases} Ax^2+B, & \text{if } 0\leqslant x<\mu,\\ C(x-1/2)^2+D, & \text{if } \mu\leqslant x<1-\mu,\\ A(x-1)^2+B, & \text{if } 1-\mu\leqslant x<1,\end{cases} \tag{52}$$

and extend this as an even function about 0, $u(-x)=u(x),)\leqslant x<1$. The conditions are that u,u_x be continuous at $x=\mu$, $1-\mu$, that $[u_{xx}](\mu)=u(\mu)/w_\mu$, $[u_{xx}](1-\mu)=-u(1-\mu)/w_\mu$, and finally that

$$\int_0^1 u\,dx=U, \tag{53}$$

a given number.

A unique solution of the last problem, having the form (52), can be found:

$$u=\frac{3U}{2(2\mu+1)}\begin{cases} -x^2/\mu+\mu+1, & \text{if } 0\leqslant x<\mu,\\ -2(x-1/2)^2/(2\mu-1)+\mu+1/2, & \text{if } \mu\leqslant x<1-\mu,\\ -(x-1)^2/\mu+\mu+1, & \text{if } 1-\mu\leqslant x<1.\end{cases} \tag{54}$$

Let us write the equation (51) as $Lu=0$, and adopt the L^2 inner product on the interval $(-1,1)$, with norm $\|\cdot\|$. Setting $u_{e1}=w_0/\|w_0\|$ and $u_{e2}=u/\|u\|$, where u is given by (54), the eigenspace of the zero eigenvalue of L for the symmetric double hat is thus spanned by u_{e1},u_{e2}. Both eigenfunctions are even in x.

The adjoint operator to L, L^* is defined by

$$L^*u=-u_{xxx}-uw_\mu^{-1}[\delta(x-\mu)-\delta(x+1-\mu)]=0. \tag{55}$$

Since the geometric multiplicity of the eigenvalue 0 is the same for L and L^*, for the symmetric double hat we can find two distinct eigenfunctions u^*_{e1},u^*_{e2} given by

$$u=u^*_{e1}=\begin{cases} (2\mu-1)x, & \text{if } 0\leqslant x<\mu,\\ (x-1/2)^2+\mu^2-1/4, & \text{if } \mu\leqslant x<1-\mu,\\ -(2\mu-1)(x-1), & \text{if } 1-\mu\leqslant x<1;\end{cases} \tag{56}$$

$$u=u^*_{e2}=\begin{cases} -x^2/\mu+\mu+1, & \text{if } 0\leqslant x<\mu,\\ -2(x-1/2)^2/(2\mu-1)+\mu+1/2, & \\ \quad \text{if } \mu\leqslant x<1-\mu, & \\ -(x-1)^2/\mu+\mu+1, & \text{if } 1-\mu\leqslant x<1.\end{cases} \tag{57}$$

Note that u_{e1}^* is odd in x and u_{e2}^* is even in x, and that only u_{e2}, u_{e2}^* have nonzero means. We show normalized (in the above norm) eigenfunctions in Fig. 1. In Fig. 2 we show the streamlines of the perturbed symmetric double-hat plume $Q_0+\epsilon Q_{e2}$ with $\epsilon=0.05$ and $U=-\sin \pi z$.

The geometric multiplicity of the eigenvalue 0 is in fact equal to two for a family of double-hat plumes. These were obtained numerically using MATLAB routines. If we fix $\mu=0.3$ and vary ν, a, the profiles in Table I have multiplicity two.

The symmetric double-hat is special in that u_{e2} has a first derivative which vanishes at $x=\pm 1/2$, and this is a condition not satisfied by u_{e1} in general. If we add the condition $\int_{-1/2}^{+1/2} u\, dx=0$, we obtain, as is easily checked, a problem for perturbations of the symmetric double hat which allows only the trivial solution $u=0$. Thus we can regard the eigensolution as ''forced'' by the assumed nonvanishing mean value. We then treat the linear problem in the reduced interval $|x|<1/2$ with perturbation $u_1=U+u_1$, where $\partial u_1/\partial x(\pm 1/2)=0$, $\int_0^1 u_1\, dx=0$; then u_1 is found by a forced linear problem. This opens the way to a much simpler analysis than would be realized by bifurcation theory at a degenerate eigenvalue. We shall term the reduced interval the ''cell'' of the computation, and the resulting perturbation a theory of the ''plume-in-cell.'' In effect we have isolated a problem where a single upwelling plume may be subjected to an arbitrary mean flow U imposed on the leading perturbation u_1, with zero mean flow imposed on all higher-order terms in the expansion of u. In this way the shift instability is expelled from the problem. Moreover, as long as the higher-order theory focuses on the reduced equations (14), the shift instability can be added at any time.

B. Linear theory of plume-in-cell: Subscript 2

We now turn to the expansion of the mean-flow mode we have obtained above with respect to $\delta, \lambda \sim \epsilon$, assuming the flow is steady. If we expand Σ' satisfying (43) as $\Sigma' = \Sigma_1 + \delta \Sigma_2 + O(\epsilon^2)$ we obtain

$$-d^2 w_2/dx^2 - T_2 = -w_0\, \partial w_1/\partial z - u_1\, dw_0/dx,$$
$$w_0\, \partial T_2/\partial z + u_2\, dT_0/dx = 0. \tag{58}$$

Thus

$$Lu_2 = w_0\, \partial^2 u_1/\partial x\, \partial z - (\partial u_1/\partial_z) dw_0/dx \equiv f_2. \tag{59}$$

Working now in the interval $|x|<1/2$, we have [cf. (54)]

$$u=u_1=\frac{3U}{2(2\mu+1)}\begin{cases} -x^2/\mu+\mu+1, & \text{if } 0\leqslant |x|<\mu, \\ -2(x-1/2)^2/(2\mu-1)+\mu+1/2, & \\ \quad \text{if } \mu \leqslant |x| < 1/2. & \end{cases} \tag{60}$$

With

$$w_0=\begin{cases} -x^2/2+\mu(1-\mu)/2, & \text{if } 0\leqslant |x|<\mu, \\ -\mu(x-1/2), & \text{if } \mu\leqslant |x|<1/2, \end{cases} \tag{61}$$

we see that

$$f_2=\begin{cases} 2\mu x & \text{if } 0\leqslant |x|<\mu, \\ \dfrac{2\mu}{2\mu-1}(x^2-x+\mu^2) & \text{if } \mu\leqslant |x|<1/2. \end{cases} \tag{62}$$

Solving (59) with (62) subject to the conditions $\partial u_2/\partial x(\pm 1/2)=0$, $\int_0^1 u_2\, dx=0$, we obtain u_2 and can compute the leading approximation to $\langle uw \rangle$:

$$\langle uw \rangle \sim \epsilon \delta \langle u_2 w_0 \rangle \equiv \epsilon\, \delta \kappa_1(\mu) DU. \tag{63}$$

We show the function $\kappa_1/\|w_0\|^2$ in Fig. 3. Note that values out to $\mu \approx 0.38$ are positive, which will enable an instability in the mean-flow equation. The normalization using w_0 in this figure is dictated by the rather small values of w_0 obtained from the equilibrium solution, i.e., the fact that U_{ref} is not a very good indication of the actual size of the unperturbed vertical velocity.

V. THE EQUATION FOR THE MEAN FLOW

We now turn to the derivation of the full expansion of the perturbed plume-in-cell, that is, for the symmetric double hat with zero shift, and to the calculation of the form of the nonlinear equation governing the evolution of the mean flow. We accordingly restrict attention to plumes with this symmetry and to the interval $|x|\leqslant 1/2$.

The equations are then

$$\delta(u_t+uu_x+wu_z)+p_x-u_{xx}-\lambda^2 u_{zz}=0, \tag{64}$$

$$\delta(w_t+uw_x+ww_z)+\lambda^2 p_z-w_{xx}-\lambda^2 w_{zz}=T, \tag{65}$$

$$T_t+uT_x+wT_z=\delta^{-1}P_r^{-1}(T_{xx}+\lambda^2 T_{zz}), \quad u_x+w_z=0. \tag{66}$$

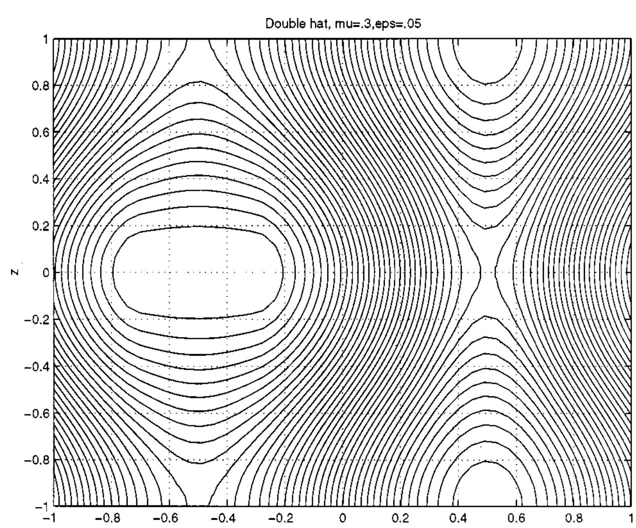


FIG. 2. Streamlines of $Q+0+\epsilon Q_{e2}$ where Q_0 is the symmetric double-hat profile $\mu=\nu=0.3$, $a=1$, $\epsilon=0.05$, and the mean flow is $U=-\sin \pi z$.

TABLE I. Values of ν and a where the null space of the eigenvalue 0 is two-dimensional.

a	0.5	0.7	1	1.3	1.5	2
ν	0.41764	0.36570	0.30000	0.24984	0.22355	0.17542

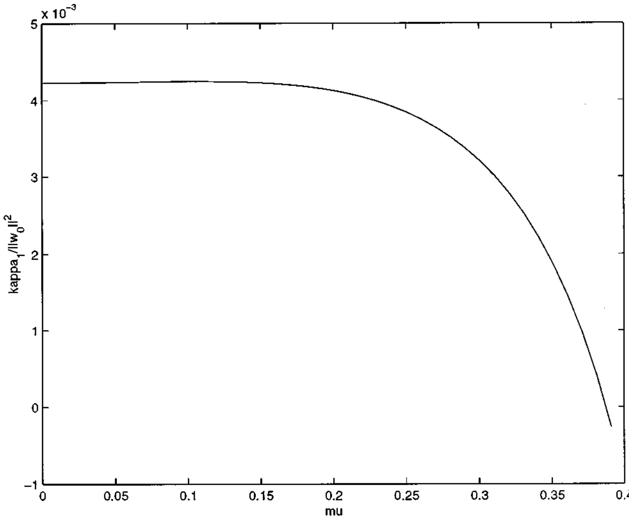


FIG. 3. The parameter $\kappa_1/\|w\|^2$ as a function of μ.

As a working hypothesis we shall take ϵ,δ,λ to be small and of the same relative order, measured in powers of ϵ. We consider the expansion through terms of formal order ϵ^4:

$$\begin{aligned}\Sigma=\Sigma_0&+\epsilon\Sigma_1+\epsilon\,\delta\Sigma_2+\epsilon^2\Sigma_3+\epsilon\lambda^2\Sigma_4\\&+\epsilon\,\delta^2\Sigma_5+\epsilon^2\,\delta\Sigma_6+\epsilon^3\Sigma_7 \qquad (67)\\&+\epsilon\,\delta\lambda^2\,\Sigma_8+\epsilon\,\delta^3\Sigma_9+\epsilon^2\lambda^2\Sigma_{10}\\&+\epsilon^2\,\delta^2\Sigma_{11}+\epsilon^3\,\delta\Sigma_{12}+\epsilon^4\Sigma_{13}+o(\epsilon^4). \qquad (68)\end{aligned}$$

The linear theory generates the six terms linear in ϵ, and $\Sigma_{1,2}$, have been discussed, the former introducing the variable $U(z,\tau)$, where we now include an as yet unspecified slow time variable τ.

For each term we may derive a governing equation $Lu_n=f_n$ where L is defined by (51), which is to be solved for u_n subject to the conditions $\int_{-1/2}^{1/2}u_n\,dx=0$, $\partial u_n/\partial x(\pm 1/2,z)=0$. We are interested here in the z-dependence of f_n as this will determine the form of the contribution of $\langle uw\rangle$ to the mean-field equation for $U(z,\tau)$. The various forcing terms and the functional contribution to the mean-field equation is given in Table II, where we now let $D=\partial_z$:

The "*" indicating that the term is not present with top–bottom symmetry can be determined by summing the numbers of ˆ and D symbols. The result must be odd if the terms survives with top–bottom symmetry. The coefficient of such a term is a pseudo-scalar with respect to this symmetry. The "†" identifies canceling symmetry with respect to $x=0$. For example, f_3 contains terms, such as $-u_1T_{1x}$, which are even with respect to $x=0$, so u_3 must be odd. In determining the functional dependence on U we have used the fact that, since $u_{nx}+Dw_n=0$, the z dependence of u_n must be of the form of D of a functional of U.

The mean-field equation for the symmetric double hat, through fourth-order terms, can thus be obtained from the following generally nonvanishing contributions to the Reynolds stress: $\langle w_0u_n\rangle=\kappa_n$, $n=2,8,9,12$; $\langle w_1u_n\rangle=\kappa_n^{(1)}$, $n=6$; $\langle w_2u_n\rangle=\kappa_n^{(2)}$, $n=3$. This leads to the following form for the mean-field equation:

TABLE II. Properties of contributing terms to the fifth-order mean-flow equation. Here $\hat{U}$ denotes the indefinite integral of U, and $D=\partial/\partial z$. Also $A_m=w_0u_{mxz}-u_{mz}w_{0x}$, $B_{mn}=D(u_mw_{nx}+w_mDw_n)$, $C_{mn}=w_0^{-1}(u_mT_{nx}+w_mDT_n)$. A "*" in the first column indicates that the contribution vanishes under integration over full period 2 by top–bottom symmetry. The "†" indicates that u_n is odd with respect to $x=0$, and so gives no contribution to $\langle w_0u_n\rangle$. p_n is obtained from the expansion of the x-momentum equation and is not given explicitly here.

Subscript n	Order	Functional dependence of u_n on U	f_n
1*	ϵ	U	0
2	$\epsilon\delta$	DU	A_1
3†	ϵ^2	$D(\hat{U}^2)$	$-C_{11}$
4*	$\epsilon\lambda^2$	D^2U	$-p_{1zz}-u_{1xzz}$
5*	$\epsilon\delta^2$	D^2U	A_2
6*†	$\epsilon^2\delta$	$D^2(U^2)$	$A_3-B_{11}-C_{12}-C_{21}$
7*	ϵ^3	$D\hat{U}^3$	$-C_{13}-C_{31}$
8	$\epsilon\delta\lambda^2$	D^3U	$-p_{2zz}-u_{2xzz}+A_4$
9	$\epsilon\delta^3$	D^3U	A_5
10†	$\epsilon^2\lambda^2$	$D^3(\hat{U}^2)$	$-p_{3zz}-u_{3xzz}-C_{14}-C_{41}$
11†	$\epsilon^2\delta^2$	$D^3(\hat{U}^2)$	$A_6-B_{12}-B_{21}-C_{22}-C_{15}-C_{51}$
12	$\epsilon^3\delta$	$D^2(\hat{U}^2)$	$A_7-B_{13}-B_{31}-C_{23}-C_{32}-C_{16}-C_{61}$
13†	ϵ^4	$D(\hat{U}^4)$	$-C_{33}-C_{17}-C_{71}$

$$\begin{aligned}\epsilon\,\delta U_t+N(U)+\delta\epsilon(\kappa_1\delta^2-\lambda^2)D^2U\\+\epsilon\delta^2(\kappa_8\lambda^2+\kappa_9\delta^2)D^4U=O(\epsilon^5),\end{aligned}\qquad (69)$$

where

$$\begin{aligned}N(U)=\epsilon^3\delta^2[\kappa_{12}D^3(\hat{U}^3)+\kappa_6^{(1)}D(\hat{U}D^2\hat{U}^2)\\+D(UD\hat{U}^2)].\end{aligned}\qquad (70)$$

Now to obtain terms of order ϵ^5 throughout we must make δ and λ of the same order and take $\kappa_1\delta^2-\lambda^2$ to be of order ϵ^2. Recall that we have established the positivity of κ_1 for sufficiently small μ; see Fig. 3. We also set $\epsilon^3t=\tau$ to define the slow time. Neglecting terms of order ϵ^6 or higher, the equation then assumes the form

$$U_\tau+D(\eta_1\hat{U}^2DU+\eta_2\hat{U}U^2+\chi_1DU+\chi_2D^3U)=0,\qquad (71)$$

where η_m, χ_m are $O(1)$ constants and $\chi_1=\kappa_1\epsilon^{-3}(\delta^2-\lambda^2)$.

We have no knowledge of the constants in (71) except for χ_1. If the instability is to be cut off at a large wavenumber to the order considered here, it is necessary that $\chi_2>0$. (Otherwise the instability can be cut off only by spatial derivatives of order six or higher.) If the equation is multiplied by U and integrated with respect to x from 0 to 2 we obtain, after several integrations by parts,

$$\begin{aligned}\frac{1}{2}d/d\tau\int_0^2u^2\,dx=\int_0^2\bigg[\chi_1(DU)^2-\chi_2(D^2U)^2\\+\eta_1\hat{U}^2(DU)^2-\frac{1}{3}\eta_2U^4\bigg]dx.\end{aligned}\qquad (72)$$

Thus if the instability saturates nonlinearly we should have $\eta_1<0$ and $\eta_2>0$.

VI. PLUMES IN THREE DIMENSIONS

It is of interest to examine the corresponding problem in three dimensions, when the plumes are slender cylindrical objects. The problem is technically more complicated than the two-dimensional calculations, but the 2D model can nevertheless serve as a useful guide. The complications come both from the representation of unperturbed plumes and the form of the homogeneous problem for perturbations within a plume-in-cell model. We also need to decide how to derive the latter simplification in three dimensions.

Consider a convecting layer with horizontal coordinates x,y, ∇_H denoting the gradient in the horizontal. Let $T_0(x,y)$ be some unperturbed temperature field, arbitrary but having a horizontal average. For computations we might take T_0 as a linear combination of Fourier modes $T_{\mathbf{k}}=e^{i\mathbf{k}\cdot\mathbf{r}},\mathbf{r}=(x,y),\mathbf{k}=(k_x,k_y)$ over a set of $\mathbf{k}$ which includes $(0,0)$. Let $w_0(x,y)$ be the corresponding unperturbed plume structure, satisfying [cf. (16)] $G-\nabla_H^2 w_0=T_0$, and having zero horizontal average. The reduced system [cf. (14)] is then, writing the velocity as $(u,v,w)=(\mathbf{u},w)$,

$$e\delta(\mathbf{u}_t+\mathbf{u}\cdot\nabla_H\mathbf{u}+w\mathbf{u}_z)+\nabla_H p-\nabla_H^2\mathbf{u}=0,$$
$$\delta(w_t+\mathbf{u}\cdot\nabla_H w+ww_z)-\nabla_H^2 w=T, \tag{73}$$
$$T_t+\mathbf{u}\cdot\nabla_H T+wT_z=0,\quad \nabla_H\cdot\mathbf{u}+w_z=0.$$

To discuss the shift instability in three dimensions, we confine attention to $\delta=0$ and the linearized steady system,

$$\nabla_H p-\nabla_H^2\mathbf{u}=0,\quad -\nabla_H^2 w=T,\mathbf{u}\cdot\nabla_H T_0+w_0T_z=0,$$
$$\nabla_H\cdot\mathbf{u}+w_z=0. \tag{74}$$

A shift perturbation solving (74) will have the velocity

$$(u,v,w)=(\xi_z w_0+\psi_y,\eta_z w_0-\psi_x,-\xi w_{0x}-\eta w_{0y}), \tag{75}$$

where $\boldsymbol{\rho}=(\xi(z),\eta(z))$ is the horizontal shift function. The new streamfunction ψ is needed to ensure that horizontal viscous forces are balanced by a pressure. The z-component of the curl of the horizontal momentum equation yields

$$\nabla_H^4\psi+(\xi_z\partial_y-\eta_z\partial_x)\nabla_H^2 w_0=0. \tag{76}$$

We may assume that this equation can be solved, given T_0 as above, for a unique bounded ψ, and the pressure then has the form $p=\xi w_{0x}+\eta w_{0y}+\text{constant}$.

What is of particular interest is the choice of plume-in-cell model in three dimensions, allowing a direct analysis of a mean-flow instability. The main difficulty here is of course the fact that 2D periodic structures such as square or hexagonal cells are somewhat awkward to treat insofar as stability is concerned. It is clear that the obvious approximate model is a cylindrical cell, as in Fig. 4, the domain being $0<r=|\mathbf{r}|<1$ and, in the case of a top-hat temperature, $T_0=1$ in $0<r<\mu$ and zero otherwise. If the exact field is replaced by an array of up and down plumes of this type, an approximate averaging procedure must be devised to incorporate the results of the plume-in-cell calculation. This averaging can either can be based either on a statistical model, so the parameters of the cell are regarded as conditional expectations given that attention is restricted to a single plume, or else on averaging over many cells immersed in a common vertical mean pressure gradient. We next give an example of a plume-in-cell cylindrical geometry of the latter type and suggest that the boundary conditions are consistent in that computations over a range of μ indicate a unique solution of the perturbed fields in the presence of a prescribed horizontal mean $\mathbf{U}=\langle\mathbf{u}\rangle$.

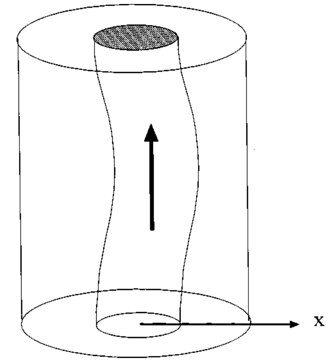


FIG. 4. A 3D ''up'' plume with a top-hat profile, perturbed by a mean flow.

Motivated by the symmetric double hat of the 2D theory, we imagine an array of equal numbers of equivalent ''up'' (temperature+1) and ''down'' (temperature−1) top-hat plumes in a zero vertical mean pressure gradient. In an ''up'' plume, for example, we assume w_0 vanishes at $r=1$. For the top-hat profile, we then have

$$w_0=\begin{cases}(\mu^2-r^2)/4-\mu^2\ln(\mu)/2, & \text{if } 0\leqslant r<\mu,\\ -\mu^2\ln(r)/2, & \text{if } \mu\leqslant r<1.\end{cases} \tag{77}$$

In the present case the ϵ-expansion with terms $\Sigma_n=(\mathbf{u}_n,w_n,p_n,T_n)$ produces a system of the form

$$\nabla_H p_n-\nabla_H^2\mathbf{u}_n=\mathbf{f}_n, \tag{78}$$

$$-\nabla_H^2 w_n-T_n=g_n, \tag{79}$$

$$\mathbf{u}_n\cdot\nabla_H T_0+w_0\,\partial_z T_n=h_n, \tag{80}$$

$$\nabla_H\cdot\mathbf{u}_n+\partial_z w_n=0. \tag{81}$$

Here the forcing terms $\mathbf{f}_n,g_n,h_n$ are known at each stage, with $\mathbf{u}_n=\nabla_H\phi_n+\nabla_H\times\mathbf{i}_z\psi_n$. Note that $\mathbf{u}_n, n\geqslant 2$ must in this case have zero average over the cell, but the leading term satisfies $uv=\mathbf{U}$ to introduce the mean horizontal velocity field. Also Σ is subject to other boundary conditions discussed below.

We now argue that the corresponding homogeneous system,

$$\nabla_H p-\nabla_H^2\mathbf{u}=0,\quad \nabla_H^2 w+T=0,\quad \mathbf{u}\cdot\nabla_H T_0+w_0\,DT=0,$$
$$\nabla_H\cdot\mathbf{u}+Dw=0, \tag{82}$$

has only the trivial solution for certain temperature profiles of physical interest. In this case we can obtain a unique expansion similar to the 2D case, with the added feature of a dependence upon the polar angle θ. We then have for the leading term of the horizontal velocity,

$$\mathbf{u}_1=\mathbf{U}(z,t)+\nabla_H\phi_1+\nabla_H\times\mathbf{i}_z\psi_1. \tag{83}$$

Also, in (78)–(81), $\mathbf{f}_1=g_1=0$ but

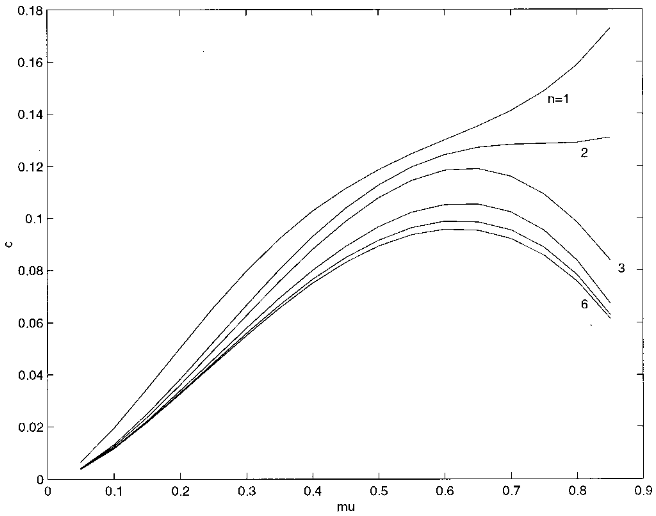


FIG. 5. Phase speed c vs μ for various wavenumbers n, 3D top-hat profile.

$$h_1 = -\mathbf{U}\cdot\nabla_H T_0 = \mathbf{U}\cdot\mathbf{i}_r \delta(r-\mu). \tag{84}$$

Thus we set $\phi_1, \psi_1 = f(\theta,z,t)\Phi_1(r), f_\theta(\theta,z,t)\Psi_1(r)$, where $f = \mathbf{U}\cdot\mathbf{i}_r$ arises naturally from (84). It is then seen that, for the horizontal average over the cell we have

$$\langle \nabla_H \phi_1 + \nabla_H \times \mathbf{i}_z \psi_1 \rangle = \tfrac{1}{2}(\Phi_1(1) - \Psi_1(1)), \tag{85}$$

and the boundary conditions must ensure that this vanishes.

The boundary conditions we suggest now will model the situation where, as in the 2D periodic cases studied above, each up cell is bordered by down cells, so that w changes sign across boundaries. We impose

$$w = \phi_n = \psi_n = \nabla_H^2 \psi_n = \nabla_H^2 \psi_n = 0, \quad r = 1. \tag{86}$$

For the top-hat profile, we have additional regularity conditions. Taking the z derivative of $-\nabla_H^2 w = T$, and using the continuity and temperature equations we obtain the homogeneous equation,

$$w_0 \nabla_H^4 \phi + \mathbf{u}\cdot\nabla_H T_0 = 0. \tag{87}$$

We then have that $\nabla_H^4 \phi = 0$, except at $r = \mu$, where the jump conditions

$$[\phi]_{r=\mu} = [\partial_r \phi]_{r=\mu} = [\partial_r^2 \phi]_{r=\mu} = 0, \tag{88}$$

and

$$[\partial_r^3 \phi]_{r=\mu} = K u_r(\mu), \quad K = w_0(\mu)^{-1} \tag{89}$$

prevail, $[Q(r)]_{r=\mu}$ here denoting $Q(\mu+) - Q(\mu-)$.

Now from $\nabla_H \times (\nabla_H p - \nabla_H^2 \mathbf{u}) = 0$ we obtain $\nabla_H^4 \psi = 0$, so with (86) we have only the trivial solution $\psi = 0$ in the homogeneous case.

For ϕ we must satisfy (87) and the six conditions from (86), (88), (89). From solutions of $\nabla_H^4 \phi = 0$ in separated form $\phi = f\Phi(r)$, $f_{\theta\theta} = -n^2 f$ we have, if $n \geq 1$,

$$\Phi = \begin{cases} C_1 r^n + C_2 r^{n+2}, & \text{if } 0 \leq r < \mu, \\ C_3 r^n + C_4 r^{n+2} + C_5 r^{-n} + C_6 r^{2-n} & \\ (\text{or } r\ln r \quad \text{if } n=1), & \text{if } \mu \leq r < 1. \end{cases} \tag{90}$$

The six conditions determine a matrix $A(K)$ whose determinant cannot vanish if the trivial solution is implied.

It is of interest to consider briefly the propagating wave solutions of the corresponding linear time-dependent problem,

$$T_t + w_0 T_z + \mathbf{u}\cdot\nabla_H T_0 = 0, \tag{91}$$

so that $w_0 - c$ replaces w_0 in (87), c being the phase velocity of the wave. In this case, since K appears only in the single condition (89), $\det(A((w_0(\mu)-c)^-1)) = 0$ is equivalent to

$$c = w_0(\mu)\frac{\det(A(K))}{\det(A(0))}. \tag{92}$$

Consequently, the condition $\det(A(K)) \neq 0$ that the quasi-steady perturbations be uniquely determined by the mean flow is the same as the condition that all traveling wave solutions have nonvanishing wave speed, so that such waves propagate to the boundaries of the domain. We show in Fig. 5 the wave speeds as a function of μ for several n, indicating that the expansion is consistent for these terms, at least for a substantial range of μ.

Unfortunately a general proof of uniqueness using energy methods does not seem possible. From (82), (86), (88), taking $\psi = 0$, we obtain

$$\begin{aligned} &\int \int_{r<1} [(\nabla_H^2 \phi)^2 + q\phi^2] dx\, dy = 0, \\ &q = \frac{1}{2}\nabla_H \cdot \left(\frac{\nabla_H \nabla_H^2 w_0}{w_0} \right). \end{aligned} \tag{93}$$

With conditions (86) we have a Poincaré inequality $\int\int(\nabla_H^2\phi)^2\, dx\, dy \geq C\int\int \phi^2\, dx\, dy, C > 0$, so that (93) implies $\phi = 0$ provided $q > -C$ for all $r < 1$. The last inequality does not generally hold for profiles of interest, as can be seen by considering a smoothed approximation to the top hat, for which q can have large positive and negative values in the vicinity of the plume boundary.

We recall that the plumes shown in plate I are three-dimensional. This picture reminds us that plumes originating at one wall might extend only a small distance into the bulk fluid before losing identity in the chaotic convective flow. Also it is clear that, to take one case, near the lower wall we expect to see mainly rising plumes. Various modifications of the plume-in-cell model can be devised to reflect these properties of the flow. In calculations not presented here, we have considered a model for a single ''up'' plume isolated in a chaotic sea of plumes of both types. Here, a cell model must account for the average ambient temperature in the vicinity of the plume. In this case we impose the condition $dw/dr(1) = 0$ of zero vertical stress at the outer wall (instead of relying, as we have done above, on the equal numbers of identical up- and down-plumes to cancel these stresses on average). Conditions on ψ, ϕ, corresponding to neighboring plumes being of the same type, were then imposed. For this problem we again obtained consistency of the problem in the sense of this section, for a range of μ of the top-hat profile.

VII. DISCUSSION

In general, with only a periodicity condition in x imposed in two dimensions, we have found that plumes of the type considered in this paper are always vulnerable to a shift instability. The latter produces a transient mean flow and a horizontal displacement of the fluid. In addition, steady bifurcation analysis indicates that the plume is susceptible to a second mode of instability, which mode carries a nonvanishing Eulerian mean flow in the steady case. This properties of this mean-flow instability depend very much on the detailed structure of the plume temperature profile.

In general, we can expect both modes to be present, depending upon the boundary conditions imposed on vertical walls. The evolution of plumes allowing both instabilities could be studied by perturbations of codimension two. In the discussion of Sec. V we chose instead to focus on the symmetric double-hat profile, so that a boundary-value problem could be formulated which excluded the shift instability. This allowed us to study the expansion in some detail and derive an equation for the mean flow. This procedure, if restricted to the reduced system $\lambda=0$, produces a solution of the reduced equations (14), and so we may apply Theorem I to obtain a shifted solution of the reduced system, with mean flow $\xi_t(z,t)+U(\tau,z)$. In this case (since $\lambda=0$) there is no linear cut-off at high wavenumber, and any saturation must come from nonlinearity.

Although we have focused on the symmetric double-hat plume, the plume-in-cell model might be applied to plumes which do not exhibit top–bottom symmetry. The boundary condition applied to u is only approximate, since it does not follow from symmetry. Up- and down-moving plumes could in this way be treated separately and then combined to obtain an equation for the mean flow. In this case all of the terms in Table II could appear. The lowest-order linear term is then of the form DU. It can be shown that this term can reduce the δ needed for the mean-flow instability. This suggests that the non-Boussinesq effects introduced in Refs. 11,12 might partially account for the ease with which mean flows were established in that experiment.

The ideas developed in the two-dimensional case carry over to three dimensions, where the plume geometry is richer, and there is an analogous cylindrical plume-in-cell model for analyzing the mean-flow instability.

We have not considered time-periodic eigenfunctions of the linear problem. In the experiments reported in Refs. 11,12 the mean flow is observed to be almost steady and the plume formation is an irregular process (in space and time). Time-dependence of the plume perturbations can create Lagrangian mean flow, with mean horizontal migrations of fluid particles. We have studied here only the Eulerian mean flow, and shown that for slender plumes how it can arise from an instability distinct from the shift instability.

When TB symmetry is broken, the leading $O(\epsilon\delta)$ contribution to the Reynolds stress appears to be similar in some respects to the anisotropic kinetic alpha (AKA) effect first studied by Frisch, She, and Sulem;[15] see also Ref. 16. The AKA instability was studied in the context of forced Navier–Stokes flows which lack *parity invariance*, meaning a lack of invariance to reflection in both velocity and coordinate vectors. Applied to Bénard convection, a reflection in temperature also occurs, and the TB invariance is then a particular case of parity invariance. The AKA mean flow is also driven by a force proportional to mean shear. On the other hand the AKA theory is a first-order theory, whereas we have obtained instability at second order in flows possessing TB symmetry.

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