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Growth of anti-parallel vorticity in Euler flows

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Abstract

In incompressible Euler flows, vorticity is intensified by line stretching, a process that can come either from the action of shear, or from advection with curvature. Focusing on the latter process, we derive some estimates on the maximal growth of vorticity in axisymmetric flow without swirl, given that vorticity support volume or kinetic energy is fixed. This leads to consideration of locally 2D anti-parallel vortex structures in three dimensions. We exhibit a class of line motions which lead to infinite vorticity in a finite time, with only a finite total line stretching. If the line is replaced by a locally 2D Euler flow, we obtain a class of models of vorticity growth which are similar to the paired vortex structures studied by Pumir and Siggia. We speculate on the mechanisms which can suppress the nonlinear effects necessary for the finite-time singularity exhibited by the moving line problem.

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1. Introduction

The question of the global regularity of three-dimensional solutions of the incompressible Euler equations continues to be of considerable interest to both mathematicians and fluid dynamicists, see e.g. the papers of Constantin, Gibbon, and Hou in this volume. According to the seminal Beale–Majda–Kato criterion, singularity formation must be accompanied by infinite (integral of) maximum vorticity, which in turn requires that some vortex tubes stretch to zero cross-section. The question of global regularity thus depends upon how fast vorticity can grow through line stretching.

In the present note, we re-examine anti-parallel vortex structures as a mechanism for the self-stretching of vorticity. We will also be interested in the existence of Euler flows which maintain quasi-two-dimensionality even as vorticity grows. In a perfect fluid, vortex lines are material and therefore move with the velocity created by the self-same vorticity, as described by the Biot–Savart law. Let us consider a curve $\mathcal{C}(t)$, restricted for simplicity to a plane, moving in the plane with velocity $\mathbf{u}(\zeta_0, t)$. Here ζ_0 is a Lagrangian parameter of the line, here arc length is measured from a reference point at $t = 0$.

Resolving $\mathbf{u}$ into tangential and normal components relative to the curve, a point $\mathbf{x}(\zeta_0, t)$ of $\mathcal{C}$ moves according to

$$\left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\zeta_0} = u(\zeta_0, t)\mathbf{n} + w(\zeta_0, t)\mathbf{t}, \tag{1}$$

where $(\mathbf{n}, \mathbf{b}, \mathbf{t})$ is the orthonormal triad of normal, tangent, and binormal vectors to the curve. As is well known, the equations of motion of the curve can be expressed for given u, w as a pair of equations for the Jacobian $J = \frac{\partial \zeta}{\partial \zeta_0}(\zeta_0, t)$ and the curvature $\kappa(\zeta, t)$, where ζ is the current arc length:

$$\left.\frac{\partial J}{\partial t}\right|_{\zeta_0} = w_\zeta J - Ju\kappa, \tag{2}$$

$$\left.\frac{\partial \kappa}{\partial t}\right|_{\zeta_0} - w\kappa_\zeta - \kappa^2 u - u_{\zeta\zeta} = 0. \tag{3}$$

Note that it is derivatives in ζ, not ζ_0, which occur in (3). The two terms on the right of (2) we may call, in order, the *shear* stretching and the *expansive* stretching terms.

Since shear stretching involves the tangential component w, it can be caused by the global vorticity only if the nearby lines

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are suitably skew to the line to be stretched. Shear-induced stretching can play a significant role in the amplification of vorticity, as was recognized by Pelz, see e.g. [1]. Expansive stretching is available to locally parallel but curved vortex lines. It is the basis for much of the line stretching in numerical experiments utilizing paired anti-parallel vortex structures, see e.g. [2–4].

We here focus on the intensification of vorticity by expansive stretching. We shall first consider stretching of vorticity in the simplest of 3D Euler flows, namely axisymmetric flow without swirl. In that case we may formulate and solve a maximization problem under global constraints on volume and energy. Next, we set $w = 0$ in (2), (3) and close the system with an equation for u. The resulting equations of motion of a line by its normal are solved, and it is found that finite time singularities, involving only finite total stretching, may occur. If we regard the line as the locus of a locally 2D three-dimensional Euler flow, we make contact with the calculations in [3]. We discuss an attempt to formulate this problem in terms of generalized partial differential equations, and the limitations on growth of vorticity for more general quasi-2D flows. We shall omit most details and refer to [12] for supporting calculations.

2. Axisymmetric flow without swirl

This special class of Euler flows, probably the simplest allowing vortex stretching, is worth considering from the viewpoint established above. How fast can vorticity grow in this class of flows? Does vorticity necessarily become indefinitely large somewhere as $t \to \infty$? What bounds on growth can be given?

Any axisymmetric flow having no swirl has a vorticity field of the form $(0, 0, \omega_\theta(z, r, t))$ in cylindrical polar coordinates z, r, θ. We deal here only with flows in R^3 for which the initial vorticity is contained within a finite volume. Such a flow is known to exist globally in time, and a very direct proof of a bound exponential in time on the maximum vorticity is given in [5]. The proof utilizes the constancy of the volume support of the vorticity in an Euler flow, and also the material invariance of $r^{-1}\omega_\theta(\mathbf{x}, t)$.

The exponential bound is not however sharp; the bound on the growth of vorticity may be improved by making further use of the special geometry. The sustained growth of vorticity must involve continual expansion of a vortex ring. The expansive stretching of this ring must be due to nearby rings. Thus, if we want to find the fastest growth a ring can attain given the initial vorticity field, we can, at each instant in time, assemble the available vorticity in a kind of toroidal "cocoon" about the selected growing ring, termed below the *core ring*. We remark that, for simplicity in constructing the cocoon producing fastest growth of the core ring, we shall in fact allow rings larger than the core, as long as global constraints are met. The core ring itself should be thought of as a "test" vortex tube of small cross-section and circulation.

2.1. Construction of a t^2 bound

Let the initial vorticity have a finite initial support of volume V_0. Suppose that $-c_1 \le \omega_\theta(z, r, 0) \le c_2$ for some positive constants c_1, c_2, and let the region of the support where $\omega_\theta \ge 0$ have volume V_{0+}, that where $\omega_\theta < 0$ have volume $V_{0-} = V_0 - V_{0+}$. We suppose that $r^{-1}|\omega_\theta(\mathbf{x}, 0)| \le C$.

Consider a core ring of radius r at time t, lying on the plane $z = 0$. Taking the z axis as the axis of symmetry, we may assume the ring has radius r at time t, and lies on the plane $z = 0$. It is clear that to maximize the rate of growth at time t of the ring in question, we can take rings of negative vorticity $\omega_\theta = -Cr$ distributed over a volume $V/2$ in $z \ge 0$, and rings of positive vorticity $\omega_\theta = +Cr$ distributed over a volume $V/2$ in $z \le 0$. Indeed, we can have no stronger vorticity and any deviation from an optimal equal partition will be sub-optimal. Note that θ increases counterclockwise looking onto the x, y plane from $z > 0$, so by the right-hand-rule a negative ω_θ in $z > 0$ induces a positive u_r at the core ring.

Consider now the value of u_r induced at the core ring by a ring of radius ρ and cross-sectional area $2\pi\rho \mathrm{d}A$ carrying vorticity $-C\rho$ at height $z = \zeta > 0$. From the Biot–Savart law one finds

$$u_r(r, 0, t) \le \frac{C\rho^2|\zeta|}{4\pi}\left[\int_{-\pi}^{+\pi} H^{-3/2}\mathrm{d}\psi\right]\mathrm{d}A \tag{4}$$

where $H = (r-\rho)^2 + 2r\rho(1-\cos\psi) + \zeta^2$. Since $1 - \cos\psi \ge k^2\psi^2, ; |\psi| \le \pi,\ k = \sqrt{2}/\pi$, we may make this substitution and carry out the integral with the range extended from $[-\pi, +\pi]$ to $[-\infty, +\infty]$, to obtain

$$u_r(r, 0, t) \le \frac{C|\zeta|\rho^{3/2}}{4\sqrt{r}}((r-\rho)^2+\zeta^2)^{-1}\mathrm{d}A. \tag{5}$$

We introduce local polar coordinates in the r, z plane, defined by $\rho - r = R\cos\Theta$, $\zeta = R\sin\Theta$. Then, since

$$\begin{aligned} u_r &\le \frac{C|\sin\Theta|(r+R\cos\Theta)^{3/2}\mathrm{d}R\mathrm{d}\Theta}{4\sqrt{r}} \\ &\le \frac{C}{4}|\sin\Theta|(r+R\cos\Theta)(1+R/r)^{1/2}\mathrm{d}R\mathrm{d}\Theta, \end{aligned} \tag{6}$$

we seek to maximize $U = \int_{\mathcal{A}} f(R, \Theta)\mathrm{d}R\mathrm{d}\Theta$, where

$$f = \frac{C}{4}|\sin\Theta|(r+R\cos\Theta)(1+R/r)^{1/2}, \tag{7}$$

subject to the volume constraint

$$V = \int_{\mathcal{A}} g(R, \Theta)\mathrm{d}R\mathrm{d}\Theta, \quad g = 2\pi(r+R\cos\Theta)R. \tag{8}$$

Here $\mathcal{A}$ is a set to be determined. It can be shown that $\mathcal{A}$ may be assumed to be mirror symmetric with respect to the plane $z = 0$, and star-like with respect to the core ring.

We may then formulate the optimization problem as the variational problem for the boundary $\mathcal{R}(\Theta)$, $0 \le \Theta \le \pi$, given by

$$\delta\int_0^\pi\int_0^{\mathcal{R}}(f(R, \Theta) - \lambda g(R, \Theta))\,\mathrm{d}R\Theta, \tag{9}$$

with scalar multiplier λ.

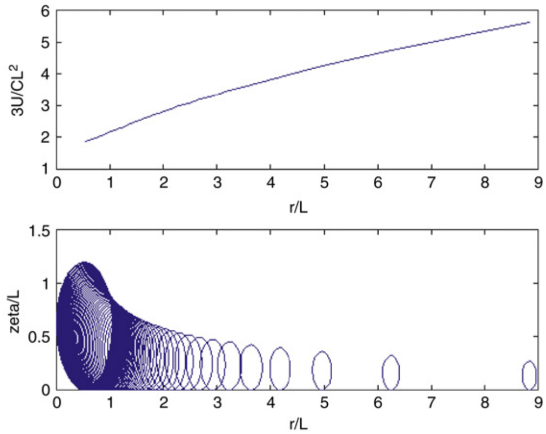

Fig. 1. Top: $\frac{3}{CL^2}\frac{\mathrm{d}r}{\mathrm{d}t}$ (as defined by (10)) versus r/L. Bottom: Cocoon shape for various position of the core ring. The cocoon is mirror symmetric with respect to the r/L line.

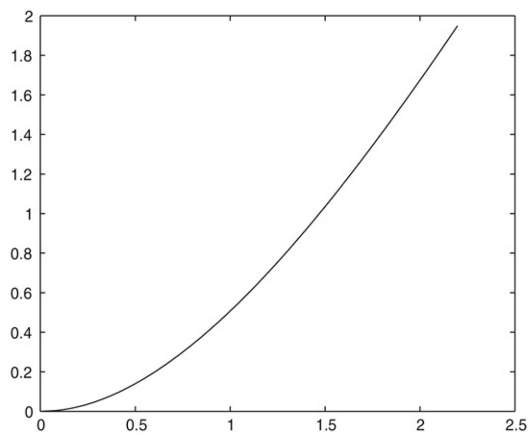

Fig. 2. $zA^{-1}\tau^{\gamma-1}$ versus $xA^{-1}\tau^{\gamma-1}$ for the case $\beta = 2, \gamma = \frac{1}{9}(1+\sqrt{19})$.

This variational problem may be easily solved. The extremal leads to the estimate on growth rate of $r* = r/L$, where $V = 2\pi L^3$, in the form

$$\frac{\mathrm{d}r*}{\mathrm{d}t} \le \sup U \le \frac{CLr*^2}{3}\mathcal{U}(r*). \tag{10}$$

We show this relation in Fig. 1, along with the cocoon boundaries at various values of r/L.

The behavior for large $r*$ leads to an estimate on the vorticity: For axisymmetric flow with initial support volume V and initial vorticity satisfying $|\omega_\theta/r| \le C$, there is a constant C_1 depending only upon V, C such that

$$\sup|\omega_\theta| \le C\left(\frac{C}{8}\sqrt{V}t + C_1\right)^2. \tag{11}$$

Thus vorticity grows no faster than $O(t^2)$ for large time. We remark that the 2D "vortex couple", see [6], p. 535, and also [15], if formed into a toroidal structure, realizes kinematically a sub-optimal cocoon of constant volume.

2.2. Kinetic energy

In terms of basic scaling in r, the cocoon of constant volume is characterized by $J, a, \omega_\theta, U \sim r, 1/\sqrt{r}, r, \sqrt{r}$, a being a transverse dimension, and the kinetic energy is of order $ra^2(\omega_\theta^2 a^2) \sim r$. Thus the kinetic energy of the cocoon of constant volume grows with r. This suggests that a lower estimate of growth can be obtained by requiring that the kinetic energy of the cocoon be fixed.

If constant kinetic energy is imposed as the side constraint instead of constant volume, it can be seen that a, the lateral dimension of the resulting cocoon, must scale as $r^{-3/4}$. The optimizing cocoon for large r then can be shown to yield an $O(t^{4/3})$ growth estimate for ω_θ. The optimizing cocoon shrinks in volume, behaving as $1/\sqrt{r}$, and has a somewhat different shape from the cocoon of constant volume, but remains star-like.

What estimate can be obtained if volume and energy are simultaneously conserved? We have studied this question in a "thin-layer" version of the cocoon construction in the limit $r \to \infty$. Our results suggest that an optimizing cocoon under both volume and energy constraints consists of the cocoon under the energy constraint, with the same estimate on growth, but now having attached to it a filament or filaments (see Section 4) which contain the missing volume but have negligible energy. Thus we conjecture that a $t^{4/3}$ bound on growth is the best available from the cocoon construction. It is likely that the exponent $4/3$ can be reduced by other methods.

Since the Jacobian of the core vortex is proportional to r, and since the speed U of the cocoon is $\sim r^{1/2}$ under constant volume and $\sim r^{1/4}$ under constant energy, we see that the growth is ultimately associated with quasi-2D structures with $J \sim U^2$ and $J \sim U^4$ respectively. Of course these considerations are essentially kinematic and, even in the case of constant energy, need not have any implication for the actual dynamics. On the other hand it is of interest to understand what kind of growth can be realized in three dimensions under similar kinematic constraints by quasi-2D vortex structures. The remainder of this note will deal with this extension to three-dimensional structure.

3. Singular motion of a line by its normal

Motivated by the results just given, we augment the system (2), (3) (with $w = 0$) by

$$J = \alpha'(\zeta_0)(-u)^\beta. \tag{12}$$

Here $\beta \ge 2$, and we assume $u < 0$, i.e. the curve is moving opposite to the direction of $\mathbf{n}$. These assumptions are motivated by the kinematics of propagating, quasi 2D vortex structures, as will be discussed below. With (12) the equations may be reduced to the following equation for u:

$$u_{tt} + (\beta-2)\frac{u_t^2}{u} + \frac{u^2}{\beta\alpha'(\zeta_0)(-u)^\beta}\frac{\partial}{\partial\zeta_0}\frac{1}{\alpha'(\zeta_0)(-u)^\beta}\frac{\partial u}{\partial\zeta_0} = 0. \tag{13}$$

If $\mathcal{C}$ is initially a circle, it will remain a circle for all time. If its radius is $R(t)$ we see easily that $\mathrm{d}R/\mathrm{d}t = cR^{1/\beta}$ for some positive constant c, and so

$$R = \left(c(1-1/\beta)t + R(0)^{\frac{\beta-1}{\beta}}\right)^{\frac{\beta}{\beta-1}}. \tag{14}$$

When the curvature is *not* independent of ζ, more complicated behavior, including finite time singularities may occur. We consider here only solutions of (13) having the similarity form

$$u = -\tau^{-\gamma} A g(\sigma), \quad \sigma = \alpha(\zeta_0)\tau^{-\mu}. \tag{15}$$

Here A is an arbitrary constant, and

$$\tau = -t, \quad t < 0. \tag{16}$$

We take γ for the moment as an arbitrary positive number less than 1. The time of the hypothetical singularity is here stipulated to be $t = 0$. Substituting (15) into (13) we obtain a solution if

$$\mu = (\beta - 1)\gamma + 1. \tag{17}$$

The equation for g can then be integrated once. Applying the conditions $g(0) = 1$ (given the arbitrary constant A), and $g'(0) = 0$ (a symmetry condition), we obtain the following equation for g:

$$\mu\gamma\sigma g^{\beta-1} + \sigma^2\mu^2 g^{\beta-2} g' + \frac{1}{\beta A^{2\beta-2}}\frac{g'}{g^\beta} = 0. \tag{18}$$

A second integration gives

$$\mu\beta A^{2\beta-2}\sigma^2 g^{\frac{2\mu}{\gamma}} + g^{\frac{2}{\gamma}} = 1. \tag{19}$$

Let us regard $\mathcal{C}$ as oriented so that at $\sigma = 0$, $\mathbf{t}$ points in the direction of the positive x-axis. We define θ as the angle made by $\mathbf{t}$ with the z-axis, so that $\kappa = \frac{\partial\theta}{\partial\zeta}$. Then

$$\frac{\partial\theta}{\partial\sigma} = -A^{\beta-1}[g^{\beta-1}\gamma + \mu\sigma g^{\beta-2} g'] = 0, \tag{20}$$

and so, from (18)

$$\theta = -A^{1-\beta}\mu^{-1}\int g^{-\beta}\sigma^{-1}\mathrm{d}g. \tag{21}$$

Here, from (19),

$$\sigma = \frac{A^{1-\beta}}{\sqrt{\mu\beta}} g^{-\mu/\gamma}\sqrt{1-g^{2/\gamma}}. \tag{22}$$

So

$$\theta = \gamma\sqrt{\frac{\beta}{\mu}}\left[\frac{\pi}{2} - \sin^{-1}(g^{1/\gamma})\right]. \tag{23}$$

These formulas allow us to calculate the shape of the curve. At large arc length the curvature tends to zero and the asymptotes make an angle $\pi - 2\theta_\infty$ where

$$\theta_\infty = \frac{\gamma\pi}{2}\sqrt{\frac{\beta}{\mu}}. \tag{24}$$

Note that $\theta_\infty < \pi/2$ if $\gamma < 1$. Setting $\beta = 2$ and requiring that $\theta_\infty = \pi/3$ we find $\gamma = \frac{1}{9}(1+\sqrt{19}) = .5954$. As we shall see, it will be important for us that we take $\gamma > 1/2$. We show in Fig. 2 the shape of $\mathcal{C}$ for $\beta = 2, \gamma = \frac{1}{9}(1+\sqrt{19})$. When $\gamma = 1/2$, $\theta_\infty \approx 52°$. Since $\theta_\infty = \pi/2$ when $\gamma = 1$, we restrict this parameter to the interval $(1/2, 1)$.

The distribution of stretching along $\mathcal{C}$ can be calculated, and the total stretching experienced by the curve between some time $\tau = T > 0$ and $\tau = 0$ demonstrated to be finite if $0 < \gamma < 1$ and $\beta \geq 2$. If we specify $J(\zeta_0, T) = 1$ then $\alpha_0(\zeta_0)$ is determined, and the evolution of J may be calculated. One finds that the stretching is concentrated at the tip as $\tau \to 0$, with $J \to 1$ at points distant from the tip.

If we regard $\mathcal{C}$ as the axis of a circular tube of incompressible fluid, stretching of $\mathcal{C}$ is accompanied by shrinking of the area of the cross section, and assuming this shrinkage is the same in all lateral directions, the radius of the tube will vary in proportion to $1/\sqrt{J}$. Thus the ratio of this radius to the radius of curvature of $\mathcal{C}$ varies as $\kappa/\sqrt{J}$. This is a quantity of order $\tau^{2\gamma-1}$. If $\gamma > 1/2$, The 3D tube has a non-self-similar development since the two radii grow as $\tau^{-\gamma}$ and $\tau^{\gamma-1}$; moreover "local quasi-two-dimensionality" is maintained as $\tau \to 0$. Note that nonexistence of 3D Euler singularities of self-similar form has been established by Chae, see [13].

Of course our interest here is that the "tube" is in fact a locally 2D Euler flow consisting of anti-parallel vortex structures moving according to (12). There are two main problems with such a scenario. First, the 2D propagation of a vortex structure of unchanging form according to (12) does not insure the same for a curved, quasi-2D variant with self-similar cross-sections, because of the failure of conservation of energy. A case in point is the vortex couple already mentioned and discussed in [3]. The result must be what we shall broadly classify as *core deformation*. Because of this deformation, the distribution of vorticity changes, (12) need not be sustained in the 3D problem, and no singularity can be inferred.

Second, the nonuniform stretching of vortex tubes leads to an axial pressure gradient, hence to axial flow within the tubes, and a disruption of area changes occuring during stretching. Some preliminary results, summarized below, suggest that this axial flow is unlikely to be a strong inhibitor of singularity formation, although it cannot be overlooked in a singularity construction involving quasi-2D vortex tubes.

4. Dynamics

The numerical simulations referred to above, as well as more recent ones (see [7–10]) indicate a flattening of the vorticity field and a kind of "tadpole" cross-section not unlike we have described for the cocoon under constraints of volume and energy. It is also interesting that the "vee"-shaped structure of our singular line is similar to some of the proposed singular flows [2]. Our estimates of growth have been essentially kinematic, and cannot address the ultimate dynamical growth. In [3] an attempt was made to calculate what we may refer to here as a "dynamical cocoon", meaning that the asymptotic dynamical evolution of a locally anti-parallel structure, 2D to first approximation, was sought. We have made a similar attempt for structures collapsing according to the moving line, under the working hypothesis that a system could be derived which would either indicate dynamically consistent singularities, or else provide an analytic example of depletion and extinction of the singularity.

Our approach utilized the scalings of the moving line, and contour averaging over closed streamlines of structures similar to the vortex couple [11]. The dominant flow is 2D, and it is assumed that the needed propagating dipole-like solutions exist. To first order, the transverse flow velocities are of order $\tau^{-\gamma}$. The velocity associated with expansive stretching and the shrinking of the cross-section of vortex tubes is smaller, of order $\tau^{1-\gamma}$ (recall $1/2 < \gamma < 1$). Evolution of the structure, including presumably core deformation on a time scale $s \sim -\ln \tau$, is obtained from compatibility conditions on the perturbed 2D system. The result is a system of generalized partial differential equations. A singular flow would be determined as a "fixed point" of the system, steady in the time scale s.

One case that can be calculated approximately is that of two thin anti-parallel vortex tubes. We find, using the model of [14], a system allowing tangential vorticity and velocity to be calculated simultaneously. On the other hand the collapse of the two vortices toward each other under mutual self-induction (see [6], p. 509) provides the core deformation and will presumably arrest the process. As yet we have no examples of a consistent fixed point solution of our system, and the non-existence or existence of the finite-time collapse remains open.

Finally, the Fourier spectrum of a collection of identical singularity forming vortex couples, averaged over orientation and lifetime, yields a $k^{-2/\gamma}$ spectrum for large wavenumber k, indicating a slope between -2 and -4. Such singular flows, should they exist, would have no effect on the $-5/3$ inertial spectrum of turbulence [16].

The ultimate fate of the vorticity in axisymmetric flow without swirl could well be some configuration of thin anti-parallel vortex tubes and we examine now its possible structure. We shall assume that a thin bilayer sheet is attached to a "rim" at position $r \sim t^{4/3}$ representing the cocoon of constant energy. (In the estimates we omit all constants fixed by the initial conditions.) The cocoon originates from some finite radius $r_0 \gg V$, where V is the initial cocoon volume, thereafter sheds volume, forming the sheet. As $t \to \infty$ the cocoon contains negligible volume, is moving with velocity $\sim t^{1/3}$, and has a cross-section of dimension $\sim t^{-1}$. As the sheet is created, it will evolve slowly as a thin vortex layer, but we neglect this evolution. The time rate of change of cocoon volume is $\sim r^{-5/4}$ and this must balance $rHr^{1/4}$ where H is the half-thickness of the bilayer at position r. Thus $H \sim r^{-5/2}$. This shed vorticity carries away the volume at a negligible loss of energy as $r \to \infty$. The maximal sustained growth realizable in this symmetric flow remains an open question. Although such vortices may be rare in fully developed turbulence, general quasi-2D anti-parallel structures and expansive stretching can provide substantial vorticity growth in 3D Euler flows.

The key mechanism for suppression of singularity formation in the structures studied here is a core deformation which can alter the simple kinematic scaling given by (12). An interesting related question, which to our knowledge has not been studied, is the dynamical fate of dipole structures at large distance from the axis in the flow without swirl. Finally, it would be interesting to determine whether or not the flow *with* swirl, involving the additional circulation invariant and having no known bounds on vorticity growth, might be accessible by the methods of this paper.

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References

[1] Richard B. Pelz, Symmetry and the hydrodynamic blow-up problem, J. Fluid Mech. 444 (2001) 299–320.
[2] R.M. Kerr, Evidence for a singularity of the three-dimensional incompressible Euler equations, Phys. Fluids A 5 (1993) 1725–1746.
[3] Alain Pumir, Eric D. Siggia, Vortex dynamics and the existence of solutions to the Navier–Stokes equations, Phys. Fluids 30 (1987) 1606–1626.
[4] M.V. Melander, F. Hussain, Cross-linking ot two anti-parallel vortex tubes, Phys. Fluids A 1 (1989) 633–636.
[5] A.J. Majda, A.L. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, Cambridge, 2001.
[6] G.K. Batchelor, An Introduction to Fluid Dynamics, Cambridge, 1967.
[7] M. Shelley, DF.I. Meiron, S.A. Orszag, Dynamical aspects of vortex reconnection of perturbed ani-parallel vortex tubes, J. Fluid Mech. 246 (1993) 613–652.
[8] T.Y. Hou, R. Li, Dynamic depletion of vortex stretching and non-blowup of the 3-D incompressible Euler equations, J. Nonlinear Sci. 16 (2006) 639–664.
[9] T.Y. Hou, R. Li, Nonexistence of local self-similar blow-up for the 3-D incompressible Navier–Stokes equations, Discrete Contin. Dyn. Syst. Ser. A 18 (4) (2007) 637–642.
[10] T.Y. Hou, R. Li, Numerical study of nearly singular solutions of the 3-D incompressible euler equations, in: Proceedings of the 2006 Abel Symposium on Computational Mathematics (2006) (in press).
[11] Stephen Childress, Nearly two-dimensional solution of Euler's equations, Phys. Fluids 30 (1987) 944–953.
[12] Stephen Childress, Models of vorticity growth in Euler flows I, Axisymmetric flow without swirl and II, Almost 2-D dynamics, AML reports 05–07 and 06–07, courant institute of mathematical sciences (2007). Available at: http://www.math.nyu.edu/aml/amlreport.html.
[13] Dongho Chae, Nonexistence of self-similar singularities of the 3-D incompressible Euler equations, (2006). Preprint.
[14] T.S. Lundgren, W.T. Ashurst, Area-varying waves on curved vortex tubes with application to vortex breakdown, J. Fluid Mech. 200 (1989) 283–307.
[15] S.A. Chaplygin, On case of vortex motion in fluid, Trans. Sect. Imperial Moscow Soc. Friends Nat. Sci. 11 (N2) (1903) 11–14.
[16] D.D. Holm, Robert Kerr, Transient vortex events in the initial value problem for turbulence, Phys. Rev. Lett. 88 (2002) 244501-1–244501-4.