Vector Analysis: V63.0224 Spring 2011
Syllabus

Week 1: Surfaces vs manifolds: the contrast between 19^{th} and 20^{th} views. Review of analysis topics: metric spaces, open/closed sets, continuity, topologies and topological spaces, metric vs topological properties. Product spaces, continuity of maps.

Week 2: Equivalence of topological spaces: homeomorphisms. Locally Euclidean spaces. Separation properties: Hausdorff spaces. Topological manifolds. Compactness in metric and topological spaces.

Week 3: Connected spaces, applications. RST equivalence relations, quotient spaces and quotient topologies. Examples: spheres, tori, projective spaces.

Week 4: Effective methods for computations with quotient spaces. Lifting and factoring maps through quotients. Gluing spaces together. Various models for projective spaces and n-dimensional tori. Invariance of domain theorem. Manifolds with boundaries.

Week 5: Coordinate independent framework for several variable Calculus. Norm estimates and "little oh" notation. Chain rule. Partial derivatives vs the "total derivative" Df. Criteria for existence of the total derivative Df.

Week 6: Smooth mappings: $\mathcal{C}^{(k)}$ and $\mathcal{C}^{\infty}$ maps. Equality of mixed partial derivatives. Directional derivatives, Mean Value theorem in $\mathbb{R}^n$, maxima and minima. Inverse mapping theorem, open mapping theorems and their significance.

Week 7: Implicit Function theorem and examples. The rank of a differentiable map; effective calaculations of rank. Other interpretations fo Implicit Function theorem. Smooth embedded manifolds in $\mathbb{R}^M$, examples, the classical matrix groups $\mathrm{GL}, \mathrm{SL}, \mathrm{O}(n), \mathrm{SO}(n)$ as smooth manifolds. Differentiable manifolds and the Implicit Function theorem. Charts and differentiable manifolds, embedded surfaces as manifolds.

Week 8: Smooth functions and smooth maps on manifolds and their combinatorial properties. The standard $\mathcal{C}^{\infty}$ structures on spheres, Euclidean spaces, tori, projective spaces. Detailed study of $\mathcal{C}^{\infty}$ structure on a quotient space: the projective plane $\mathbb{P}^2$. Diffeomorphisms. Partitions of the identity on $\mathcal{C}^{\infty}$ manifolds.

Week 9: $\mathcal{C}^{\infty}$ maps on manifolds, derivatives of smooth functions on manifolds and their transformation laws, chain rule on manifolds, directional derivatives and tangent vectors. Classical vs manifold interpretations of tangent vectors.

Week 10: The tangent space TM_p of a manifold; point derivations; Basis vectors $\partial/\partial x_i|_p$ in TM_p. Differential of a mapping $\phi : M \to N$, transformation law for tangent vectors. Vector fields on manifolds, differential operators on manifolds. Computational examples: vector fields on the sphere S^2.

Week 11: The dual V^* of a vector space, cotangent vectors on a manifold and the cotangent space TM_p^*, transformation laws under smooth mappings and change of coordinates. The differential df of a smooth function. Fields of cotangent vectors: smooth differential forms (1-forms). The rank-1 exterior derivative. Antiderivatives (primitives) of differential forms, necessary conditions for solving the equation $df = \omega$.

Week 12: Line integrals of differential forms and the antiderivative problem for 1-forms; path independence of line integrals, sufficient conditions for solving $df = \omega$. Examples. Multilinear algebra: multilinear forms and tensor products of vector spaces, tensor fields on manifolds, inner products and Riemannian structure on manifolds. Permutations and k-forms (rank-$k$antisymmetric tensors). Wedge product of forms, differential forms: smooth fields of k-forms on manifolds. The exterior derivative operation d on k forms and its algebraic properties.

Week 13: The antiderivative problem for differential forms: solving the equations $d\omega = \mu$ for k-forms. Existence of local solutions and the Poincare Lemma. Classical interpretations: the manifold interpretation of the classical vector operators div, grad, and curl. Integration of differential forms and Stokes theorem.