Calculus III – 2008 Spring

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Final Exam Solutions

① $x^2+y^2-z^2=1$, point $(1,1,-1)$

$z = -\sqrt{x^2+y^2-1} = f(x,y)$

$f_x = \dfrac{-x}{\sqrt{x^2+y^2-1}}$ $f_y = \dfrac{-y}{\sqrt{x^2+y^2-1}}$

$f_x(1,1) = -1$ $f_y(1,1) = -1$

Tangent plane: $z - z_0 = f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$$z+1 = -(x-1)-(y-1)$$

$$(x-1)+(y-1)+(z+1)=0$$

$$x+y+z=1$$

② $f(x,y) = 6xy^2 - 2x^3 - 3y^4$

$$\begin{cases} f_x = 6y^2 - 6x^2 = 0 \\ f_y = 12xy - 12y^3 = 0 \end{cases}$$

$$\begin{cases} 6(y^2-x^2) = 0 & \Rightarrow x^2 = y^2 \\ 12y(x-y^2) = 0 & \Rightarrow y=0 \text{ or } x=y^2 \text{ or both} \end{cases}$$

Critical points: $(0,0), (1,1), (1,-1)$

$D = f_{xx} f_{yy} - f_{xy}^2$

$f_{xx} = -12x$

$f_{yy} = 12x - 36y^2$

$f_{xy} = 12y$

Critical point	D	f_{xx}	Classification
$(0,0)$	0	0	inconclusive
$(1,1)$	144	-12	local max
$(1,-1)$	144	-12	local max

③ (i) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy$

$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\,dr\,d\theta$

$= \int_0^{2\pi} \left[-e^{-r^2/2}\right]_{r=0}^{r=\infty} d\theta$

$= \int_0^{2\pi} d\theta$

$= 2\pi$

(ii) $2\pi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2}\, e^{-y^2/2}\,dx\,dy$

$= \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \int_{-\infty}^{\infty} e^{-y^2/2}\,dy$

$= \left[\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right]^2$

$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$

④ $\vec{F} = \left\langle \frac{1}{y^2+1},\ -\frac{2xy}{(y^2+1)^2} + ze^{yz},\ ye^{yz} + 2z \right\rangle$

Suppose $\vec{F} = \nabla f(x,y,z)$. Then

$f_x = \frac{1}{y^2+1} \qquad f_y = \frac{-2xy}{(y^2+1)^2} + ze^{yz} \qquad f_z = ye^{yz} + 2z$

$f(x,y,z) = \frac{x}{y^2+1} + g(y,z)$

$f_y = \frac{-2xy}{(y^2+1)^2} + g_y \Rightarrow g_y = ze^{yz} \Rightarrow g(y,z) = e^{yz} + h(z)$

$f(x,y,z) = \frac{x}{y^2+1} + e^{yz} + h(z)$

$f_z = ye^{yz} + h'(z) \Rightarrow h'(z) = 2z \Rightarrow h(z) = z^2$

Therefore, $\vec{F} = \nabla f$ for $f(x,y,z) = \frac{x}{y^2+1} + e^{yz} + z^2$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(1,0,2\pi) - f(1,0,0)$$
$$= (1+1+4\pi^2) - (1+1+0)$$
$$= 4\pi^2$$

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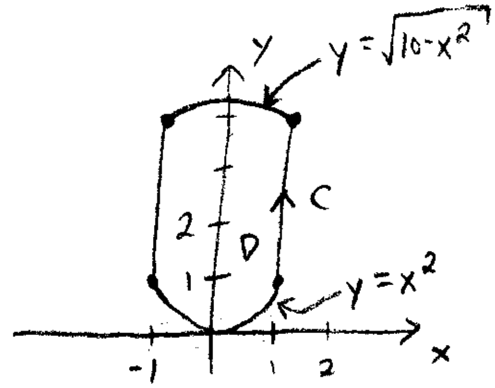
$$\oint_C [\ln(1+x^5) - \tfrac{1}{2}y^2]\,dx + xy\,dy$$



$P = \ln(1+x^5) - \frac{1}{2}y^2$

$Q = xy$

By Green's Theorem,

$$\oint_C [\ln(1+x^5) - \tfrac{1}{2}y^2]\,dx + xy\,dy$$
$$= \iint_D \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}\left(\ln(1+x^5) - \tfrac{1}{2}y^2\right)\right] dA$$
$$= \iint_D [y - (-y)]\,dA = \iint_D 2y\,dA$$
$$= \int_{-1}^{1}\int_{x^2}^{\sqrt{10-x^2}} 2y\,dy\,dx$$
$$= \int_{-1}^{1} y^2 \Big|_{y=x^2}^{y=\sqrt{10-x^2}} dx$$
$$= \int_{-1}^{1} (10 - x^2 - x^4)\,dx$$
$$= 10x - \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5 \Big|_{-1}^{1}$$
$$= (10 - \tfrac{1}{3} - \tfrac{1}{5}) - (-10 + \tfrac{1}{3} + \tfrac{1}{5}) = 2(10 - \tfrac{1}{3} - \tfrac{1}{5})$$
$$= 2\left(\frac{150-5-3}{15}\right) = 2\cdot\frac{142}{15} = \frac{284}{15}$$

(6) $z = f(x,y) = 2 - x^2 - y^2$

This surface intersects the xy-plane at

$$0 = 2 - x^2 - y^2$$

$$x^2 + y^2 = 2 \qquad \text{circle of radius } \sqrt{2}$$

Parametrize as $\quad x = x \quad y = y \quad z = 2 - x^2 - y^2$

$$\vec{r}_x = \langle 1, 0, -2x \rangle$$
$$\vec{r}_y = \langle 0, 1, -2y \rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \vec{i}(2x) - \vec{j}(-2y) + \vec{k}(1)$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{4x^2 + 4y^2 + 1}$$

Surface area $= \iint_D |\vec{r}_x \times \vec{r}_y| \, dx\, dy$ where D is inside of circle of radius $\sqrt{2}$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2+1} \; r\, dr\, d\theta$$
$$= \int_0^{2\pi} \frac{1}{8} \frac{2}{3} (4r^2+1)^{3/2} \Big|_{r=0}^{r=\sqrt{2}} d\theta$$
$$= \int_0^{2\pi} \frac{1}{12} \left[9^{3/2} - 1 \right] d\theta$$
$$= \int_0^{2\pi} \frac{1}{12} \left[3^3 - 1 \right] d\theta$$
$$= \int_0^{2\pi} \frac{1}{12} [26]\, d\theta$$
$$= 2\pi \cdot \frac{26}{12}$$
$$= \frac{13\pi}{3}$$

(7) By the Divergence Theorem,

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

where E is the solid region bounded by surface S (unit sphere)

$$\vec{F}(x,y,z) = \langle xy^2, yz^2, zx^2\rangle$$

$$\operatorname{div}\vec{F} = y^2 + z^2 + x^2$$

$$\iiint_E \operatorname{div}\vec{F}\, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^2\cdot\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{1}{5}\rho^5\sin\phi\Big|_{\rho=0}^{\rho=1} d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{1}{5}\sin\phi\, d\theta\, d\phi$$

$$= \int_0^{\pi} \frac{2\pi}{5}\sin\phi\, d\phi$$

$$= -\frac{2\pi}{5}\cos\phi\Big|_0^{\pi} = -\frac{2\pi}{5}(-1-1) = \frac{4\pi}{5}$$

(8)
(i) h
(ii) b
(iii) e
(iv) d
(v) f

(9)
(i) false
(ii) true
(iii) true
(iv) true
(v) true
(vi) false
(vii) true
(viii) false
(ix) true
(x) false