Solutions to Final Examination

V63.0121
Calculus I

December 19, 2008
1. (10 Points) Compute the following derivatives. Please leave your answers unsimplified.

(i) \( \frac{d}{dx} (\ln (x \sin x + 1)) \)

Solution. We need the chain rule and the product rule:

\[
\frac{d}{dx} (\ln (x \sin x + 1)) = \frac{1}{x \sin x + 1} \cdot \frac{d}{dx} (x \sin x + 1) = \frac{x \cos(x) + \sin(x)}{x \sin(x) + 1}
\]

(ii) \( \frac{d}{dx} \left( \arctan \left( \frac{x + 1}{e^x + 1} \right) \right) \)

Solution. Again by the chain rule,

\[
\frac{d}{dx} \left( \arctan \left( \frac{x + 1}{e^x + 1} \right) \right) = \frac{1}{1 + \left( \frac{x + 1}{e^x + 1} \right)^2} \cdot \frac{d}{dx} \left( \frac{x + 1}{e^x + 1} \right) = \frac{1}{1 + \left( \frac{x + 1}{e^x + 1} \right)^2} \left\{ \frac{(1 + e^x) - e^x(x + 1)}{(1 + e^x)^2} \right\}
\]
2. (10 Points) Find the y-intercept of the line that is tangent to the ellipse \(4x^2 + 9y^2 = 900\) at the point \((12, 6)\). Put your answer in the box.

*Hint.* Implicit differentiation may help here.

*Solution.* Implicitly differentiating gives us

\[
8x + 18y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{4x}{9y}
\]

At the point \((12, 6)\), we have

\[
\frac{dy}{dx}\bigg|_{(12,6)} = -\frac{4 \cdot 12}{9 \cdot 6} = -\frac{8}{9}
\]

The equation of the line through \((6, 12)\) with slope \(-\frac{8}{9}\) is

\[
y - 6 = -\frac{8}{9} (x - 12) \implies y = -\frac{8}{9} x + \frac{8}{9} \cdot 12 + 6
\]

The y-intercept is

\[
\frac{8}{9} \cdot 12 + 6 = \frac{96}{9} + \frac{54}{9} = \frac{150}{9} = \frac{50}{3} = 16\frac{2}{3}
\]

\[\text{y-intercept: } \frac{16\frac{2}{3}}{}\]
3. (10 Points) Evaluate the following limits. Put your answers in the boxes. Show your work.

(i) (5 points) \( \lim_{{x \to 1}} \frac{x^4 - 1}{x^2 - 1} \).

Solution. We can factor the numerator:

\[
\lim_{{x \to 1}} \frac{x^4 - 1}{x^2 - 1} = \lim_{{x \to 1}} \frac{(x^2 - 1)(x^2 + 1)}{x^2 - 1} = \lim_{{x \to 1}} (x^2 + 1) = 1^2 + 1 = 2
\]

Alternatively, we could use L'Hôpital's Rule:

\[
\lim_{{x \to 1}} \frac{x^4 - 1}{x^2 - 1} \xrightarrow{H} \lim_{{x \to 1}} \frac{4x^3}{2x} = \lim_{{x \to 1}} 2x^2 = 2
\]

▲

(ii) (5 points) \( \lim_{{x \to 0}} (\cos x)^{1/x^2} \).

Solution. Let the limit be \( L \), if it exists. Then

\[
\ln L = \lim_{{x \to 0}} \ln(\cos x)^{1/x^2} = \lim_{{x \to 0}} \frac{\ln(\cos x)}{x^2}
\]

This limit is of the form \( \frac{0}{0} \) and so we can try L'Hôpital's Rule:

\[
\lim_{{x \to 0}} \frac{\ln(\cos x)}{x^2} \xrightarrow{H} \lim_{{x \to 0}} \frac{-\tan x}{2x} = \lim_{{x \to 0}} \frac{-\sec^2 x}{2} = -\frac{1}{2}
\]

So \( L = e^{-1/2} \).

▲
4. (15 Points) Let \( f(x) = x^4 + 4x^3 - 2 \). Explain your answers on each of these parts.

(i) (4 points) The derivative of \( f \) is \( f'(x) = 4x^3 + 12x^2 \). On which intervals is \( f \) increasing? decreasing?

\[ f'(x) = 4x^3 + 12x^2 = 4x^2(x + 3). \]
\[ This \ is \ zero \ when \ x = 0 \ or \ x = -3. \ On \ the \ interval \ (-\infty, -3), \ f'(x) < 0, \ so \ f \ is \ decreasing. \ On \ the \ intervals \ (-3, 0) \ and \ 0, \infty, \ f'(x) > 0. \ So \ f \ is \ decreasing \ on \ (-\infty, -3) \ and \ increasing \ on \ [-3, \infty). \]

(ii) (4 points) The second derivative of \( f \) is \( f''(x) = 12x^2 + 24x \). On which intervals is \( f \) concave up? concave down?

\[ f''(x) = 12x^2 + 24x = 12x(x + 2). \]
\[ This \ is \ zero \ when \ x = 0 \ or \ x = -2. \ On \ the \ interval \ (-\infty, -2), \ f''(x) > 0, \ so \ f \ is \ concave \ up. \ On \ the \ interval \ (-2, 0), \ f''(x) < 0, \ so \ f \ is \ concave \ down. \ On \ the \ interval \ (0, \infty), \ f''(x) > 0, \ so \ f \ is \ concave \ up. \ Thus \ f \ is \ concave \ up \ on \ (-\infty, -2) \ and \ [0, \infty), \ and \ concave \ down \ on \ [-2, 0]. \]
(iii) (4 points) Find the absolute maximum and minimum values of $f$ on $[-4, 1]$.

Solution. By the closed interval method, we need only check the values of $f$ at the endpoints of $[-4, 1]$ (that is, at $-4$ and $1$), and the critical points within $(-4, 1)$ (that is, at $-3$ and $0$).

We have

\[
\begin{align*}
    f(-4) &= (-4)^4 + 4(-4)^3 - 2 = -2 \\
    f(-3) &= (-3)^4 + 4(-3)^3 - 2 = -29 \\
    f(0) &= -2 \\
    f(1) &= 3 \\
\end{align*}
\]

So the maximum value is $f(1) = 3$, while the minimum value is $f(-3) = -29$. ▲

(iv) (3 points) Sketch the graph of $f$. Label all the critical points and inflection points.

![Graph of f with labeled critical points and inflection points]
5. (12 Points) A department store is fencing off part of the store for children to meet and be photographed with Santa Claus. They have decided to fence off a rectangular region of fixed area 800 ft\(^2\). Fire regulations require that there be three gaps in the fencing: 6 ft openings on the two facing sides and a 10 ft opening on the remaining wall (the fourth side of the rectangle will be against the building wall). Find the dimensions that will minimize the length of fencing used.

\[\text{Dimensions: } 40 \text{ ft} \times 20 \text{ ft}\]

\textbf{Solution.} Let } x \text{ the side with the 10 ft gap and } y \text{ the side with the 6 ft gap. We want to minimize }

\[f = (x - 10) + 2(y - 6) = x + 2y - 2\]

subject to the constraint that } xy = 800. Isolating } y = \frac{800}{x} \text{ gives us a function }

\[f(x) = x + 2 \cdot \frac{800}{x} - 2 = x + \frac{1600}{x} - 2\]

The domain of this function is } [10, \infty), \text{ because we need at least 10 ft to have a gap. To find the critical points, we have }

\[f'(x) = 1 - \frac{1600}{x^2}\]

So } f'(x) = 0 \text{ when } x = 40 \text{ (we discard the negative root since it's not in our domain). Now } f''(x) = \frac{3200}{x^3}, \text{ which is always positive on our domain. So the unique critical point is the global minimum of } f. \text{ Thus the dimensions of the rectangle that minimize fence are } 40 \text{ ft} \times 20 \text{ ft.} \]
6. (12 Points) A cannonball is shot into the air. Its velocity is given as a function \( f(t) \) m/s, where \( t \) measured in seconds since 1:00PM. We know that \( f(t) \) takes the following values:

<table>
<thead>
<tr>
<th>( t )</th>
<th>( f(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10.0</td>
</tr>
<tr>
<td>7.5</td>
<td>6.46</td>
</tr>
<tr>
<td>15</td>
<td>5.00</td>
</tr>
<tr>
<td>22.5</td>
<td>3.88</td>
</tr>
<tr>
<td>30</td>
<td>2.93</td>
</tr>
<tr>
<td>37.5</td>
<td>2.09</td>
</tr>
<tr>
<td>45</td>
<td>1.34</td>
</tr>
<tr>
<td>52.5</td>
<td>0.646</td>
</tr>
<tr>
<td>60</td>
<td>0</td>
</tr>
</tbody>
</table>

(i) (2 points) What does the integral \( I = \int_{0}^{60} f(t) \, dt \) represent?

**Solution.** The integral represents the distance traveled between 1:00PM and 1:01PM.

For the two parts below, let \( L_n \) be the Riemann sum for \( I \) using \( n \) subintervals and left endpoints, \( R_n \) be the Riemann sum for \( I \) using \( n \) subintervals and right endpoints, and \( M_n \) be the Riemann sum for \( I \) using \( n \) subintervals and midpoints.

(ii) (2 points) Write out the terms in \( M_4 \). You may leave your answer unsimplified.

**Solution.** We have
\[
M_4 = 6.46 \cdot 15 + 3.88 \cdot 15 + 2.09 \cdot 15 + 0.646 \cdot 15
\]

(iii) (2 points) Assume that \( f(t) \) is decreasing for all \( t \geq 0 \). Without computing, put these in order from least to greatest: \( L_2, R_4, R_2, I, L_4 \). Put your answers in the boxes. No justification is necessary.

**Hint.** A picture might help your thinking here.

\[
R_2 < R_4 < I < L_4 < L_2
\]

(iv) (6 points). It turns out \( f(t) = 10 - \frac{10}{\sqrt{60}} \sqrt{t} \). Compute \( I \) exactly.

**Solution.** We have
\[
I = \left[ 10t - \frac{10}{\sqrt{60}} \cdot \frac{2}{3} t^{3/2} \right]_0^{60} = 10 \cdot 60 - \frac{20 \cdot 10}{3 \sqrt{60}} \cdot 60 \cdot \sqrt{60} = 600 - 400 = 200
\]
7. (10 Points) Find the following indefinite integrals. Your answer should be the most general antiderivative.

(i) (5 points) \( \int \frac{12x}{\sqrt{2x^2 + 5}} \, dx \)

Solution. Let \( u = 2x^2 + 5 \), so \( du = 4x \, dx \). Then

\[
\int \frac{12x}{\sqrt{2x^2 + 5}} \, dx = \int \frac{3 \, du}{\sqrt{u}} = 6 \sqrt{u} + C = 6 \sqrt{2x^2 + 5} + C
\]

(ii) (5 points) \( \int 2x \sin(3x) \, dx \)

Solution. Let \( u = 2x \), so \( du = \sin(3x) \, dx \). Then \( du = 2 \, dx \) and \( v = \frac{1}{3} \cos(3x) \). So

\[
\int 2x \sin(3x) \, dx = -\frac{2}{3} x \cos(3x) + \frac{2}{3} \int \cos(3x) \, dx = -\frac{2}{3} x \cos(3x) + \frac{2}{9} \sin(3x) + C
\]
8. (10 Points) Let

\[ F(y) = \int_0^y e^{-s^2} \, ds \]

(i) (5 points) In each of these, select “>” if the quantity in column A is greater, “<” if the quantity in column B is greater, “=” if the two quantities are the same, and “?” if it is impossible to determine which is greater. No justification is necessary. No partial credit will be given. Please fill in the circle completely.

Hint. It is **mathematically impossible** to compute \( F(y) \) exactly by antidifferentiation, so please do not try. That is not the point of this problem.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>Your answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) ( F(0) )</td>
<td>0</td>
<td>&gt; &lt; ● ?</td>
</tr>
<tr>
<td>2) ( F(1) )</td>
<td>( F(2) )</td>
<td>&gt; ● &lt; ?</td>
</tr>
<tr>
<td>3) ( F(-1) )</td>
<td>( F(-2) )</td>
<td>● &lt; &lt; ?</td>
</tr>
<tr>
<td>4) ( F'(0) )</td>
<td>0</td>
<td>● &lt; ● ?</td>
</tr>
<tr>
<td>5) ( F'(1) )</td>
<td>( F'(-1) )</td>
<td>&gt; &lt; ● ?</td>
</tr>
</tbody>
</table>

(ii) (5 points) Suppose \( y(t) = 9 \sin(\pi t) \) and let \( g(t) = F(y(t)) \). In other words,

\[ g(t) = \int_0^{9 \sin \pi t} e^{-s^2} \, ds \]

Find \( g' \left( \frac{1}{2} \right) \). Put your answer in the box.

**Solution.** We have \( g(t) = F(9 \sin \pi t) \), so

\[ g'(t) = F'(9 \sin \pi t) \cdot 9 \pi \cos(\pi t) = 9 \pi e^{-81 \sin^2 \pi t} \cos(\pi t). \]

Therefore

\[ g' \left( \frac{1}{2} \right) = 9 \pi e^{-81 \sin^2 (\pi / 2)} \cos \left( \frac{\pi}{2} \right) = 0 \]
9. (5 Points) **Evaluate the following.** No justification is necessary for this problem.

In the first three, express your answer as an integer or a fraction.

(i) \( \log_3(27) \)

\[ \text{Answer: } 3 \]

(ii) \( \log_4 \left( \frac{1}{2} \right) \)

\[ \text{Answer: } -\frac{1}{2} \]

(iii) \( \ln(1) \)

\[ \text{Answer: } 0 \]

In the next two, express your answer as an angle in radians.

*Note.* Remember that \( \arcsin \) is the inverse of \( \sin \), sometimes also written as \( \sin^{-1} \). But this is **not** the same as \( \frac{1}{\sin} \).

(iv) \( \arcsin \left( \frac{1}{2} \right) \)

\[ \text{Answer: } \frac{\pi}{6} \]

(v) \( \arctan(1) \)

\[ \text{Answer: } \frac{\pi}{4} \]
10. (6 Points) Determine whether the following statements are true (i.e. true in general) or false (i.e. not true in all cases). As long as there is one example where the statement does not hold, it is considered false. Please fill in the circle completely. No justification is necessary. No partial credit will be given.

(i) If \( f \) and \( g \) are continuous on \([a, b]\), then \( \int_a^b f(x)g(x) \, dx = \left( \int_a^b f(x) \, dx \right) \left( \int_a^b g(x) \, dx \right) \)

Solution. We use integration by parts to integrate a product.

(ii) If \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \).

Solution. Differentiability is a kind of super-continuity.

(iii) If \(-1 < x < 1\), then \( \arctan(x) = \frac{\arcsin(x)}{\arccos(x)} \)

Solution. Inverse functions do not work this way.

(iv) If \( \lim_{x \to 5} f(x) = 0 \) and \( \lim_{x \to 5} g(x) = 0 \), then \( \lim_{x \to 5} \left[ \frac{f(x)}{g(x)} \right] \) does not exist.

Solution. It is possible for the limit to still exist. For instance \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \).

(v) \( \lim_{x \to 1} \frac{x - 3}{x^2 + 2x - 4} = \frac{\lim_{x \to 1} (x - 3)}{\lim_{x \to 1} (x^2 + 2x - 4)} \)

Solution. Since the denominator approaches something other than 0, the limit of the quotient is the quotient of the limits.

(vi) If \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there is a point \( c \) in \((a, b)\) with \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

Solution. This is the statement of the Mean Value Theorem.