Calculus III Part 1

Name: Solutions

1. (a) \( \mathbf{u} \times \mathbf{v} = \langle -\sqrt{2}, \sqrt{2}, 0 \rangle \)

(b) \( \mathbf{u} \cdot \mathbf{v} = 2 \)

(c) Let \( \theta \in [0, \pi] \) be the angle between the two vectors

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \cdot ||\mathbf{v}||} = \frac{2}{2\sqrt{2}} \implies \theta = 45^0
\]

(d)

\[
\frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2} \mathbf{v} = \mathbf{v}
\]

(e)

\[
|| (\mathbf{u} \times \mathbf{v}) \cdot (1, 0, 0) || = \sqrt{2}
\]

2. (a) \( \mathbf{j} \)

(b) \( -\mathbf{i} \)

(c) \( 0 \)

(d) \( \mathbf{i} \)

(e) \( -\mathbf{j} + \mathbf{k} = \langle 0, -1, 1 \rangle \)

3. Unit tangent vector, \( \mathbf{T} \), gives the direction of the velocity, and unit normal vector, \( \mathbf{N} \), gives the direction of the normal acceleration which is responsible for the change of the direction of the velocity.

Recall there are two parts of acceleration: tangential acceleration \( \mathbf{\tilde{a}}_T \) changes the magnitude of the velocity only, and it is parallel to the velocity; and normal acceleration \( \mathbf{\tilde{a}}_N \) changes the direction of the velocity only (that is the reason why \( \mathbf{T} \) has to be normalized).

Here is a picture that illustrates \( \mathbf{\tilde{a}}_T \) and \( \mathbf{\tilde{a}}_N \). Suppose the motion is circular, and we can look at velocities at two instances, \( \mathbf{v}(t) \) and \( \mathbf{v}(t + \Delta t) \), separated by time \( \Delta t \).
To find the difference of two velocities, we shift $\vec{v}(t)$ and $\vec{v}(t + \Delta t)$ so that the two tails are coincide, so the blue line $\overrightarrow{BC} = \vec{v}(t + \Delta t) - \vec{v}(t) \approx \vec{a}\Delta t$. Now mark a point $D$ on $\overline{AC}$ such that $\overline{AD} = \overline{AB} = |\vec{v}(t)|$. Now we find that $\Delta POQ \sim \Delta BAD$, because both are isosceles and $\angle POQ = \angle BAD$. That is because $\overline{OP} \perp \overline{AB}$ and $\overline{OQ} \perp \overline{AD}$.

Let $\Delta t \to 0$, hence $\angle BAD \to 0$, so $\angle DBA \to \pi/2$, i.e. $\overline{BD} \perp \overline{AB}$, so $\overline{BD} \parallel \overline{OP}$, i.e. $\overline{BD}$ is in the normal direction. Since $\overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{DC}$ and clearly $|\overrightarrow{DC}| = |\vec{v}(t + \Delta t)| - |\vec{v}(t)|$, it is natural to define $\vec{a}_T$ and $\vec{a}_N$ so that

$$\vec{a} = \vec{a}_T + \vec{a}_N$$

and

$$\overrightarrow{DC} = \vec{a}_T \Delta t \quad \overrightarrow{BD} = \vec{a}_N \Delta t$$

Since $\vec{a}_N$ is perpendicular to the motion, centripetal forces do no work, i.e. $\vec{a}_N$ doesn’t contribute to the change of the speed.

Furthermore if the particle moves in a circular motion with constant speed, i.e. $\overline{CD} = 0$, using $\Delta POQ \sim \Delta BAD$, we get

$$\frac{\overline{OP}}{\overline{PQ}} = \frac{\overline{AB}}{\overline{BD}} \implies \frac{v \Delta t}{r} = \frac{a \Delta t}{v} \implies a = \frac{v^2}{r}$$

For arbitrary “smooth” motion in 3D, we can always approximate the trajectory at every instance by circle with radius, $1/\text{curvature}$. And $T$ is tangent to the circle, $N$ radially points to the center of the circle, and $B$ gives the normal direction of the plane in which the circle lies. We take $B = T \times N$ but not $N \times T$ because under right hand rule $B$ also gives the direction of the rotation of the particle. Two for the price of one.

One way to memorize the formula for curvature $\kappa$ is to think the special case above: uniform circular motion.

We learned for uniform circular motion with constant speed $v$ and radius $r$, the magnitude of the acceleration is given by

$$a = \frac{v^2}{r} = \kappa v^2$$

and

$$\vec{a} = \frac{d\vec{v}}{dt} = v \frac{d\vec{T}}{dt}$$
so it is natural to define $\kappa$ as

$$\kappa = \left| \frac{d\vec{T}/dt}{v} \right| = \left| \frac{d\vec{T}}{ds} \right|$$

The presentation given above is of course not a proof, but a good trick to use on an exam. Special cases help memorizing formulas.

(a)

$$\vec{T}' = \left\langle \frac{4t}{(t^2 + 2)^2}, -\frac{4t}{(t^2 + 2)^2}, -\frac{2(t^2 - 2)}{(t^2 + 2)^2} \right\rangle$$

$$\vec{N} = \left\langle \frac{2t, -2t, -(t^2 - 2)}{\sqrt{8t^2 + (t^2 - 2)^2}} \right\rangle = \left\langle \frac{2t, -2t, -(t^2 - 2)}{t^2 + 2} \right\rangle$$

(b)

$$\kappa = \left| \frac{\left\langle \frac{4t}{(t^2 + 2)^2}, -\frac{4t}{(t^2 + 2)^2}, -\frac{2(t^2 - 2)}{(t^2 + 2)^2} \right\rangle}{\frac{1}{2}t^2 + 1} \right| = \frac{2}{\frac{1}{2}t^2 + 1} = \frac{4}{(t^2 + 2)^2}$$

(c)

$$\vec{B}(t = 0) = \langle 0, 1, 0 \rangle \times \langle 0, 0, 1 \rangle = \hat{i}$$

4. (a) No
   (b) Yes
   (c) Should read $\frac{\partial z}{\partial t}$ not $\frac{dz}{dt}$. ANS yes
   (d) Yes
   (e) Should read $f(x, y)$ is a non-constant function... ANS yes (cf problem 6 below)

5. (a) Clearly

$$f(0, 0) = f(0, x) = f(0, y) = 0$$

so all points on the $x$ and $y$ axes give the same value, so (a) goes with (4).

(b) Similarly $f(0, y) = 0$ for all $y$, and we already used (4), so (b) goes with (8).

(c) For fixed $f$, if $f > 0$, the level curve is

$$\frac{y^2}{(\sqrt{f})^2} - \frac{x^2}{(\sqrt{f})^2} = 1$$

If $f < 0$

$$\frac{x^2}{(\sqrt{-f})^2} - \frac{y^2}{(\sqrt{-f})^2} = 1$$

So (c) goes to (1)

(e) $f$ is invariant under $x \to x + a$, and $y \to y + a$, for any $a$, so the contour plot has to have this property, i.e. symmetric under shifting the plot by the vector $(1, 1)$, so (e) goes to (7).
(d) We can do the following transformation

\[
\begin{align*}
    x + y &= u \\
    x - y &= v
\end{align*}
\]

then

\[ f = \frac{x - y}{x^2 + y^2 + 1} = \frac{v}{u^2 + v^2 + 1} \]

which is almost (b).

If you know the transformation

\[
\begin{align*}
    x + y &= u \\
    x - y &= v
\end{align*}
\]

means to rotate \(x\) and \(y\) axes by \(45^0\), then you know the answer. Otherwise use the same trick

\[ f(0,0) = f(x, x) \]

so the line \(y = x\) must be one of the level curve, and we already used (1), (7), so it has to go to (3).

6. Recall \(\vec{u} = \langle \Delta x, \Delta y, \Delta z \rangle\)

\[
\Delta f = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \frac{\partial f}{\partial z} \Delta z = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \langle \Delta x, \Delta y, \Delta z \rangle
\]

So if one chooses \(\vec{u} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle\), then \(\Delta f\) is maximum, i.e. \(\vec{u} = \nabla f\) gives the direction that maximally increases \(f\). The direction perpendicular to \(\nabla f\) gives \(\Delta f = 0\), which makes up the tangent plane.

So the normal direction at point \((1, -1, 1)\) is

\[ (2x, 4y, 2z) \sim (1, -2, 1) \]

So the equation of the plane

\[ x - 2y + z = d \]

Since it passes \((1, -1, 1)\),

\[ x - 2y + z = 4 \]

7. (a)

\[
\begin{align*}
    4x^3 - 4y &= 0 \\
    4y^3 - 4x &= 0
\end{align*}
\]

\[ \implies x = y = \pm 1, 0 \]

ANS \((0, 0), (1, 1), (-1, -1)\)
(b) We are going to apply second derivative test. Recall second derivative test says suppose $f$ has continuous second derivatives and at the critical points if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \text{ and } f_{xx} > 0$$

then that critical point is a local minimum. Let’s use a crude argument to show why this test makes sense.

Suppose $(x_0, y_0)$ is a critical point. Let us compare $f(x_0, y_0)$ to its neighborhood, say $f(x_0 + \Delta x, y_0 + \Delta y)$

Let us use Taylor. First expand in $y$ then expand in $x$, and keep up to second order terms (because first order terms are zeros, for $(x_0, y_0)$ is a critical point. Because $f$ has continuous second derivatives, by Clairaut’s, expanding in $y$ then expanding in $x$ gives the same answer if we expand in $x$ then expand in $y$, i.e. following the upper left path is the same as following the lower right path.) We obtain

$$f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \Delta y^2$$

$$+ \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x_0, y_0) \Delta x^2$$

$$+ \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \Delta y^2$$

$$= f(x_0, y_0) + \frac{\partial^2 f}{\partial x^2}(x_0, y_0) \Delta x^2 + \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \Delta x \Delta y + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \Delta y^2$$

We want $f(x_0, y_0)$ to be truly a local minimum, then the sum after $f(x_0, y_0)$ had better to be positive for any direction $(\Delta x, \Delta y)$ we pick, i.e.

$$f_{xx}(\Delta x)^2 + 2f_{xy}\Delta x \Delta y + f_{yy}(\Delta y)^2 > 0$$

If we view above as a parabola in variable $\Delta x$, then we know the entire parabola lives above the $x$ axis iff the parabola is concave up and no real roots, so the requirements are

$$f_{xx} > 0 \& 4f_{xy}^2(\Delta y)^2 - 4f_{xx}f_{yy}(\Delta y)^2 < 0$$

That is what we want

$$f_{xx} > 0 \& f_{xx}f_{yy} - f_{xy}^2 > 0$$
And the requirements for local maximum are that the entire parabola lives below the $x$ axis, i.e. the parabola is concave down and no real roots.

Now we do the test on $(0,0)$, $(1,1)$, and $(-1,-1)$

$$f_{xx} = 12x^2, \quad f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 + 4 > 0$$

So $(1,1)$ and $(-1,-1)$ are minimum, and $(0,0)$ is inconclusive by the test, so we will have to use other methods. So we can stop here.

[If you have the luxury of time, you can work out the problem for extra credits:

Is $(0,0)$ a min, max or saddle point?

Hint: use the $45^0$ rotation transformation mentioned in problem 5(d) above with proper normalization (i.e. Jacobian = 1), so $f$ is reduced into a equation with 2nd degrees in $x$ and $y$, then go to polar coordinate to find a level curve passing through the origin, then rotate $45^0$ back to the normal $xy$ plane.

ANS: the level curve passing through the origin is given by

$$r^2 = \frac{4\sin2\theta}{2 - \sin^22\theta}$$

Hence $(0,0)$ is not min nor max, is a saddle point.]