Stability for chaotic sigma delta quantization

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Analog-to-digital (A/D) conversion

**Goal:** Represent audio signals by a sequence of bits, or by a sequence of $\pm 1$.

- Audio signals can be modeled as a sequence of their sample values $f_n = f(t_n) \in [-1, 1]$ at time $t_n$.

![Figure: the phrase "hi, how are you", at two time scales](image-url)
Quantization schemes

- **Pulse Code Modulation (PCM):** Replace each sample $f_n$ by $(q_j)_{j=1}^M$ the first $M$ bits in its binary expansion,
  \[ f_n \approx \sum_{j=1}^{M} q_j 2^{-j} \]

- **Sigma delta ($\Sigma\Delta$) quantization:** Sample audio at higher rate, then replace each sample $f_n$ by a single value $q_n \in \{-1, 1\}$ or $q_n \in \{-1, 0, 1\}$ such that $f_n$ is approximated by local averages of the $q_n$,
  \[ f_n \approx \sum_{j=n-M}^{j=n+M} c_{j-n} q_j \]
The standard second-order $\Sigma\Delta$ scheme can be reformulated as a dynamical system; set $u_0 = v_0 = 0$ and iterate for $n \geq 1$:

$$q_n = Q(u_{n-1} + \gamma v_{n-1}),$$
$$u_n = u_{n-1} + f_n - q_n,$$
$$v_n = v_{n-1} + u_n$$

- **One-bit quantization** $q_n \in \{-1, 1\}$, $Q(x) = \text{sign}(x)$
- **Tri-level quantization**: $q_n \in \{-1, 0, 1\}$,

$$Q(x) = \begin{cases} 
-1, & x < -0.5, \\
0, & -0.5 \leq x \leq 0.5, \\
1, & x > 0.5 
\end{cases}$$
Idle tones in $\Sigma\Delta$ quantization

- **Problem:** Periodicities in the output $q_n$ often occur in $\Sigma\Delta$ schemes, producing audible idle tones.
- One proposed solution: Modify the standard $\Sigma\Delta$ scheme by adding amplification to break up periodicities: fix $\lambda > 1$, and consider

\[
q_n = Q(u_{n-1} + \gamma v_{n-1}), \\
u_n = \lambda u_{n-1} + f_n - q_n \\
v_n = v_{n-1} + u_n,
\]

- This modification is called *chaotic* $\Sigma\Delta$ quantization in practice. So far a proof of stability was missing. By stability, we mean that the iterates $(u_n, v_n)$ do not blow up.
Output of quantization scheme

- Here is a sample of a sequence of points \((u_n, v_n)\) when the input sequence \(f_n\) is constant.

Figure: Output of standard scheme (left) versus chaotic scheme (right).
Proof of stability

- Ozgur Yilmaz proved the standard second-order $\Sigma \Delta$ scheme was to be stable within a specific convex region, as shown below.
- If $(u_n, v_n) \in S_\alpha$ and $|f_n| < \alpha$, then $(u_{n+1}, v_{n+1}) \in S_\alpha$.

Figure: $\Gamma_{B_1}$ and $\Gamma_{B_2}$ are the graphs of two quadratic functions, symmetric about the origin.
Proof of stability

- Restricting the input sequence \( (f_n) \) to \( |f_n| \leq \alpha < 1 \), then 
  \( \delta_n = |f_n - q_n| \) can take values from \( L = 1 - \alpha \) to \( H = 1 + \alpha \), 
  where \( |f_n| \leq \alpha < 1 \), we can rewrite the system:

\[
(u_n, v_n) = \begin{cases} 
(\lambda u_{n-1} - \delta_n, \lambda u_{n-1} + v_{n-1} - \delta_n); & \text{if } q_n = 1, \\
(\lambda u_{n-1} + \delta_n, \lambda u_{n-1} + v_{n-1} + \delta_n); & \text{if } q_n = -1 
\end{cases}
\]  

(1)

- To extend this, we suppose \( |f_n| \leq \alpha' < \alpha < 1 \), where 
  \( \alpha' = \alpha - \epsilon(\alpha) \), for \( \epsilon \) a small nonnegative value dependent on \( \alpha \).

- If \( (u_n, v_n) \in S_\alpha \), and \( |f_n| \leq \alpha' < \alpha \), then \( (u_{n+1}, v_{n+1}) \in S_\alpha \).
Bounds on expansion parameter $\lambda$

- Previous constraints on $C$ imply that our stability results will only hold for $\lambda \leq 1 + \frac{\epsilon L}{2H}$. However, in practice, values of lambda could hypothetically be much larger than this.

Figure: Lambda as a function of $\alpha$ for fixed $\epsilon$. 
Extensions of the chaotic $\Sigma\Delta$ scheme

- **Trilevel Quantizer**: A slight adjustment to the conditions on $C$, $\lambda$, and $\gamma$ allowed us to extend the proof of stability of the system to the case where $q_n$ can take values of 0, 1, or -1.

- **Finite Memory Quantizer**: We similarly extended the proof to the "leaky" scheme, described by

  $$(u_n, v_n) = (\beta \lambda u_{n-1} + f_n - q_n, \beta v_{n-1} + \beta \lambda u_{n-1} + f_n - q_n)$$

  where $\beta \leq 1$. 
Open problem

- There is still no rigorous proof that the second order $\Sigma\Delta$ scheme with expansion parameter $\lambda > 1$ is chaotic.