Derivative Securities: Lecture 5
American Options and Black Scholes
PDE

Sources:
J. Hull
Avellaneda and Laurence
The Black Scholes PDE

- The hedging argument for assets with normal returns presented at the end of Lecture 4 gave rise to the Black Scholes PDE

\[
\frac{\partial C(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C(S,t)}{\partial S^2} + (r - q)S \frac{\partial C(S,t)}{\partial S} - rC(S,t) = 0
\]

\(r=\text{interest rate, } q=\text{dividend yield, } \sigma = \text{volatility. The volatility is the annualized standard deviation of returns (it is not a market price or, rate, but rather a model input).}\)

- We introduce a method for solving this PDE numerically on a grid.
Finite-difference scheme, or "trinomial tree"

\[ S_n^j = S_0 e^{j \Delta x}, \quad -M \leq j \leq +M \]

\[ C_n^j \leftrightarrow C(S_n^j, n \Delta t), \quad 0 \leq n \leq N \]
Change of variables

\[ S = S_0 e^x \]

\[ S \frac{\partial C}{\partial S} = S \frac{\partial C}{\partial x} \frac{\partial x}{\partial S} = S \frac{\partial C}{\partial x} \frac{1}{S} = \frac{\partial C}{\partial x} \]

\[ S^2 \frac{\partial^2 C}{\partial S^2} = S^2 \frac{\partial}{\partial S} \left( \frac{1}{S} \right) \frac{\partial C}{\partial S} = S^2 \frac{\partial}{\partial S} \left( \frac{1}{S} \frac{\partial C}{\partial x} \right) \]

\[ = S \frac{\partial}{\partial x} \left( \frac{1}{S} \frac{\partial C}{\partial x} \right) \]

\[ = \frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x} \]

**BS equation in log-price**

\[ \frac{\partial C}{\partial t} + \left( r - q - \frac{1}{2} \sigma^2 \right) \frac{\partial C}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 C}{\partial x^2} - rC = 0 \]
Taylor expansion & symmetric finite-difference approximations for derivatives

\[ f(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + \ldots \]

\[ f(-x) = f(0) - f'(0)x + \frac{1}{2} f''(0)x^2 + \ldots \]

\[ 
\begin{align*}
\therefore \\
 f(x) - f(-x) &= 2f'(0)x + o(x^2) \\
 f(x) + f(-x) &= 2f(0) + f''(0)x^2 + o(x^3)
\end{align*}
\]

\[ 
\begin{align*}
\therefore \\
 f'(0) &= \frac{f(x) - f(-x)}{2x} + o(x) \\
 f''(0) &= \frac{f(x) + f(-x) - 2f(0)}{x^2} + o(x)
\end{align*}
\]

Symmetric finite difference approximations for first and second derivatives
Discretization of the PDE

\[
\begin{align*}
\frac{\partial C(S,t)}{\partial t} & \leftrightarrow \frac{C_{n+1}^j - C_n^j}{\Delta t} \\
\frac{\partial C(S,t)}{\partial x} & \leftrightarrow \frac{C_{n+1}^{j+1} - C_n^{j-1}}{2\Delta x} \\
\frac{\partial^2 C(S,t)}{\partial x^2} & \leftrightarrow \frac{C_{n+1}^{j+1} + C_n^{j-1} - 2C_n^j}{(\Delta x)^2}
\end{align*}
\]

Here we do not use symmetric differences

Here use symmetric differences

\[
\begin{align*}
\frac{C_{n+1}^j - C_n^j}{\Delta t} + (r - q - \frac{\sigma^2}{2}) \frac{C_{n+1}^{j+1} - C_n^{j-1}}{2\Delta x} + \frac{\sigma^2}{2} \frac{C_{n+1}^{j+1} + C_n^{j-1} - 2C_n^j}{(\Delta x)^2} - rC_n^j &= 0
\end{align*}
\]
From PDE to recursive scheme

\[
\frac{C_{n+1}^j - C_n^j}{\Delta t} + (r - q - \frac{\sigma^2}{2}) \frac{C_{n+1}^j - C_n^j}{2\Delta x} + \frac{\sigma^2}{2} \frac{C_{n+1}^j + C_{n+1}^{j-1} - 2C_n^j}{(\Delta x)^2} - rC_n^j = 0
\]

\[
C_n^j = C_{n+1}^j + \left( \frac{\sigma^2 \Delta t}{2(\Delta x)^2} + \frac{(r - q - \sigma^2 / 2) \Delta t}{2\Delta x} \right) C_{n+1}^j + \left( 1 - \frac{\sigma^2 \Delta t}{(\Delta x)^2} \right) C_{n+1}^j + \left( \frac{\sigma^2 \Delta t}{2(\Delta x)^2} - \frac{(r - q - \sigma^2 / 2) \Delta t}{2\Delta x} \right) C_{n+1}^{j-1} - r\Delta t C_n^j
\]

\[
C_n^j = \frac{1}{1 + r\Delta t} \left( p_U C_{n+1}^{j+1} + p_M C_n^j + p_D C_{n+1}^{j-1} \right)
\]

\[
p_U = \frac{\sigma^2 \Delta t}{2(\Delta x)^2} + \frac{(r - q - \sigma^2 / 2) \Delta t}{2\Delta x}
\]

\[
p_M = 1 - \frac{\sigma^2 \Delta t}{(\Delta x)^2}
\]

\[
p_D = \frac{\sigma^2 \Delta t}{2(\Delta x)^2} - \frac{(r - q - \sigma^2 / 2) \Delta t}{2\Delta x}
\]
Interpreting the weights

• Notice that

\[ p_U + p_M + p_D = 1 \]

• Set

\[ \Delta x = \sigma_{\text{max}} \sqrt{\Delta t} \]
\[ \mu = r - q - \frac{\sigma^2}{2} \]
\[ p = \frac{\sigma^2 \Delta t}{2(\Delta x)^2} = \frac{\sigma^2}{2\sigma_{\text{max}}^2} \]

• The weights become

\[ p_U = p + \frac{\mu \sqrt{\Delta t}}{2\sigma_{\text{max}}} \]
\[ p_M = 1 - 2p \]
\[ p_D = p - \frac{\mu \sqrt{\Delta t}}{2\sigma_{\text{max}}} \]
Stability conditions & probabilities

\[ p < 1/2 \]
\text{and}
\[ \left| \frac{\mu}{\sqrt{\Delta t}} \right| \frac{1}{\sigma_{\text{max}}} < 1 \]
\[ \Rightarrow \quad p_U > 0, \ p_M > 0, \ p_D > 0 \]

- In this case, the discretization of the PDE corresponds to discounting over probabilities

\[ C_n^j = \frac{1}{1 + r\Delta t} \left( p_U C_{n+1}^{j+1} + p_M C_{n+1}^j + p_D C_{n+1}^{j-1} \right) \]

- This gives a simple and intuitive interpretation of the B-S PDE
European Options

• Value at expiration date

\[ C_N^j = \max\left(S_0 e^{j\sigma_{\max} \sqrt{\Delta t}} - K, 0\right), \quad -M \leq j \leq +M \quad \text{(call)} \]

\[ C_N^j = \max\left(K - S_0 e^{j\sigma_{\max} \sqrt{\Delta t}}, 0\right), \quad -M \leq j \leq +M \quad \text{(put)} \]

• Solve recursively

\[ C_n^j = \frac{1}{1 + r \Delta t} \left[p_U C_{n+1}^{j+1} + p_M C_n^j + p_D C_{n+1}^{j-1}\right], \quad -M < j < +M, \quad n = N - 1, N - 2, \ldots, 0 \]
These boundary conditions are called "radiation boundary conditions" or "zero-gamma" boundary conditions. They assume that there is no convexity at the boundary, so the values at the boundary will not affect the computation significantly.

(More on this later...)
Function BSCall(ByVal S As Double, ByVal T As Double, ByVal K As Double, ByVal r As Double, ByVal q As Double, ByVal sigma As Double) As Double

' set mesh = 1 day
Dim dt As Double
dt = 1# / 252
' set number of time steps
Dim N As Integer
N = CInt(T / dt)
' set carry
Dim mu As Double
mu = r - q - 0.5 * sigma * sigma

' set sigma max for stability requirements
Dim smax As Double
smax = 2 * Abs(mu) * sqrt(dt)
If smax < sigma * sqrt(2) Then
    smax = sigma * sqrt(2)
End If
If smax = 0 Then
    BSCall = -9999
End If

This ensures that smax is large enough
'allocate arrays
Dim M As Integer
M = CInt(5 * sqrt(N))

Dim S() As Double
Dim C() As Double
Dim pC() As Double

ReDim C(1 To 2 * M + 1)
ReDim pC(1 To 2 * M + 1)
ReDim S(1 To 2 * M + 1)

'probabilities
Dim PU, PM, PD As Double
Dim p As Double
p = 0.5 * sigma * sigma / (smax * smax)

PU = p + 0.5 * mu * sqrt(dt) / smax
PM = 1 - 2 * p
PD = p - 0.5 * mu * sqrt(dt) / smax

This sets the vertical dimension

From the discretization of The PDE
'initialize call payoff
Dim D, E As Double
D = 1#/ (1 + r * dt)
E = Exp(smax * sqrt(dt))

S(1) = S * Exp(-M * smax * sqrt(dt))
For j = 2 To 2 * M + 1
S(j) = S(j - 1) * E
Next j

For j = 1 To 2 * M + 1
C(j) = Max(S(j) - K, 0)
Next j

'time loop
For K = 1 To N
'interior nodes
For j = 2 To 2 * M
pC(j) = PU * C(j + 1) + PM * C(j) + PD * C(j - 1)
pC(j) = pC(j) * D
Next j

'boundary nodes
pC(1) = 2 * pC(2) - pC(3)
pC(2 * M + 1) = 2 * pC(2 * M) - pC(2 * M - 1)

'copy array
For j = 1 To 2 * M + 1
C(j) = pC(j)
Next j

Next K

BSCall = C(M + 1)
End Function
Discussion

• This is called an explicit scheme, which means that we “roll back”, solving time n in terms of time n+1

• For this to work, we need smax large enough so that the “probabilities” are positive (stability)

• The requirement that M=5*sqrt(N) has to do with the fact that the grid must be large enough to avoid “feeling the boundary”

• The result at the end is the full vertical array at n=0, so we get more information than just the central node, if we wish.
American Options

- We must enforce the requirement that, at each node, the value of the option is greater than the payoff (intrinsic value

\[ C_n^j \geq \max \left( S_n^j - K, 0 \right) \quad \text{(call)} \]
\[ C_n^j \geq \max \left( K - S_n^j, 0 \right) \quad \text{(put)} \]

Let \( F(S) \) be the intrinsic value. Then,

\[ C_n^j = \max \left[ F(S_n^j), \frac{1}{1 + r\Delta t} \left( p_U C_{n+1}^{j+1} + p_M C_{n+1}^{j} + p_D C_{n+1}^{j-1} \right) \right] \]
Why is the numerical scheme correct?

- An American-style option is always greater than the IV

- Suppose that you know the value of the American option at time $t_{n+1} = (n+1)\Delta t$.

- A European option with payoff $F(S,t_{n+1}) = C^j_{n+1}$, $S = S_0e^{j\Delta t}$ expiring at time $t_{n+1}$ has a value at time $t_n = n\Delta t$ equal to

  $$V^j_n = \frac{1}{1 + r\Delta t} \left[ p_U C^j_{n+1} + p_M C^j_{n+1} + p_D C^{j-1}_{n+1} \right]$$

- An American option gives the right to exercise at time $t_n$ or to continue. If you continue, this is like holding the European-style derivative for one more time period. Therefore,

  $$C^j_n = \max \left[ IV\left(S^j_n\right), V^j_n \right] = \max \left[ IV\left(S^j_n\right), \frac{1}{1 + r\Delta t} \left[ p_U C^j_{n+1} + p_M C^j_{n+1} + p_D C^{j-1}_{n+1} \right] \right]$$
' time loop
For K = 1 To N
    'interior nodes
    For j = 2 To 2 * M
        pC(j) = PU * C(j + 1) + PM * C(j) + PD * C(j - 1)
        pC(j) = pC(j) * D
    Next j

    'boundary nodes
    pC(1) = 2 * pC(2) - pC(3)
    pC(2 * M + 1) = 2 * pC(2 * M) - pC(2 * M - 1)

    'copy array & compare with intrinsic value
    For j = 1 To 2 * M + 1
        C(j) = pC(j)
        If C(j) < Max(S(j) - K, 0) then
            C(j) = Max(S(j) - K, 0)
        End if
    Next j
Next K

BSCall = C(M + 1)
End Function

This guarantees that C is at least equal to the intrinsic value. Everything else is the same.
Pricing a 1-year call numerically

\[ T = 1 \text{ year} \]
\[ S = 100 \]
\[ K = 100 \]
\[ r = 0.10\% \]
\[ q = 11\% \]
\[ \Sigma = 16\% \]
\[ DT = 1/252 \]
\[ N = 252 \]
\[ M = 79 \]
\[ S_{\text{max}} = 22.6\% \]
\[ D_x = 1.42\% \]
\[ P_u = 0.2348, \ P_m = 0.5, \ P_d = 0.2652 \]
Numerical solution vs. Black-Scholes for European options

- Same parameters as previous example
- Compared BS with numerical scheme
- Adjust the time-step to produce acceptable error
- Use numerical code to price American options
Numerical solution as a surface
Early-exercise boundary

- Exercising a call option makes sense only if the stock is sufficiently high (and the dividend income is greater than the potential increase in the option price).

- Exercising a put option makes sense only if the stock is sufficiently low, so that the interest income from the cash received exceeds the expected gains from increase in option price.

- Therefore, the exercise regions for calls/puts will be of the form

\[
\begin{align*}
ER_{\text{call}} &= \{(S,t) : S > S^*(t)\} \\
ER_{\text{put}} &= \{(S,t) : S < S^{**}(t)\}
\end{align*}
\]
Exercise region for a put

Continuation region

Exercise region

S

K

T=T

t
Exercise region for a call

Exercise region

Continuation region

S

K

T=T
Applications of the Black-Scholes pricing model

• In the case of European-style options, we obtain a compact formula for the value of options:

\[ C_{eur} = BSCall(S, K, T, r, d, \sigma) \quad P_{eur} = BSPut(S, K, T, r, d, \sigma) \]

• In the case of American-style options, we have numerical scheme which depends on the same 6 parameters and gives the value with arbitrary precision.

• Of the 6 parameters, 5 of them are observable or derivable from the market (e.g. implied dividend)

• The volatility parameter is NOT observable or derivable from the market in an unequivocal way. It is an essential component of the model.
Implied Volatility

• The implied volatility of an option is the volatility that makes the Black-Scholes pricing formula true

\[ C = BSCall(S, T, K, r, d, \sigma_{imp}), \quad P = BSPut(S, T, K, r, d, \sigma_{imp}) \]

• Given \((S, K, T, r, q)\) and the price of an option, there is a unique implied vol associated with a given price. The reason is that

\[
\frac{\partial BSCall}{\partial \sigma} > 0, \quad \frac{\partial BSPut}{\partial \sigma} > 0
\]

• Usually computed from mid-prices (bid+offer)/2. We can also talk about a **bid implied vol** and an **offer implied vol**, associated with bid prices and offer prices.
Call Implied Volatility (SPY Dec 17 calls)

IDIV=2.50%
Put Implied Vols (SPY Dec 17 Calls)

IDIV=2.50%
Calls and Puts together

Red = calls
Blue = puts
Implied Volatility

• Implied volatility of OTM options are more stable than ITM

• IVOLS of calls and puts should be approximately equal due to the fact that we determined the dividend yields implicitly

• For SPY, the implied volatility is a decreasing function of the strike price. This is known as the volatility skew in the business.

• Volatilities are not constant across strikes, but they vary relatively smoothly.

• Option markets can be viewed as volatility markets, as we will soon see.