Derivative Securities: Lecture 7
Further applications of Black-Scholes and Arbitrage Pricing Theory

Sources:
J. Hull
Avellaneda and Laurence
Black’s Formula

- Sometimes it’s easier to think in terms of forward prices. Recalling that in Black-Scholes:

\[
d_{1,2} = \frac{1}{\sigma \sqrt{T}} \ln \left( \frac{S}{K} \right) + \frac{r-q}{\sigma \sqrt{T}} \pm \frac{\sigma \sqrt{T}}{2}
\]

\[
= \frac{1}{\sigma \sqrt{T}} \ln \left( \frac{F}{K} \right) \pm \frac{\sigma \sqrt{T}}{2}, \quad F = Se^{(r-q)T}
\]

\[
BSCall(S,T,K,r,q,\sigma) = e^{-qT} \text{SN}(d_1) - e^{-rT} \text{KN}(d_2)
\]

\[
= e^{-rT} e^{(r-q)T} \text{SN}(d_1) - e^{-rT} \text{KN}(d_2)
\]

\[
= e^{-rT} \cdot BSCall(F,T,K,0,0,\sigma)
\]

- Similarly,

\[
BSPut(S,T,K,r,q,\sigma) = e^{-rT} \cdot BSPut(F,T,K,0,0,\sigma)
\]
Symmetry between puts and calls in Black’s formula

\[
\text{Call}(F, T, K, r, \sigma) = e^{-rT} \left[ F \cdot N\left( \frac{\ln(F / K)}{\sigma \sqrt{T}} + \frac{\sigma \sqrt{T}}{2} \right) - K \cdot N\left( \frac{\ln(F / K)}{\sigma \sqrt{T}} - \frac{\sigma \sqrt{T}}{2} \right) \right]
\]

\[
\text{Put}(F, T, K, r, \sigma) = e^{-rT} \left[ K \cdot N\left( \frac{\ln(K / F)}{\sigma \sqrt{T}} + \frac{\sigma \sqrt{T}}{2} \right) - F \cdot N\left( \frac{\ln(K / F)}{\sigma \sqrt{T}} - \frac{\sigma \sqrt{T}}{2} \right) \right]
\]

\[
\text{Call}(F, T, K, r, \sigma) = \text{Put}(K, T, F, r, \sigma)
\]

Call = ``exchanging cash for stock'', Put = ``exchanging stock for cash''
Options on Foreign Exchange

Example: Price a 90-day USD call/EUR put on 10,000,000 USD with strike 1.40. US rate=0.41%, EUR rate=1.50%. Current exchange rate: EUR= USD 1.3702, volatility: 16%.

\[ F = (1.3702) \times \frac{1 + \left( \frac{90}{360} \right) 0.0041}{1 + \left( \frac{90}{360} \right) 0.0150} = 1.36648 \]

\[ V(\text{per EUR notional}) = \frac{1}{1 + 0.25 \cdot 0.0041} \text{BSPut}(1.36648, 0.25, 1.40, 0, 0, 0.15) \]

\[ = \frac{1}{1.001025} 0.062903 = \$0.062839 \]

Option Value \[ = \$0.062839 \times \frac{10,000,000}{1.40} = \$448,847.10 = EUR 327,577.8 \]
Options on Futures

• Consider an option on a futures contract. Typically, the option gives the right, but not the obligation to enter/exit a futures contract at a specified price and is cash-settled.

• Example: call option on a Eurodollar futures contract with strike 97 in 1 month. Futures settle in 3 months.

  -- in or before 1 month: can exercise and get long a futures. The payoff will be max(f-97,0) times the notional amount, cash-settled.

• Similar for a put option, with cash settlement of max(97-f,0).
Derivation (I)

- Assuming that the option fair value depends on futures price and time-to-maturity,

\[ C = C(f,t), \quad t = \text{calendar time}, T = \text{maturity} \]

\[
\Delta C = \frac{\partial C}{\partial t} \Delta t + \frac{\partial C}{\partial f} \Delta f + \frac{1}{2} \frac{\partial^2 C}{\partial f^2} (\Delta f)^2 + o(\Delta t)
\]

- Exposure to futures: \( \frac{\partial C}{\partial f} \cdot \Delta f \)

- Assume that you `hedge’ by taking a position in \( -\frac{\partial C}{\partial f} \) contracts and in one option, the latter financed at rate \( r \).

The 1-day PNL is

\[ PNL = -rC \Delta t + \Delta C - \frac{\partial C}{\partial f} \Delta f \]
• Notice that the futures position requires no financing (except for margin, which we do not take into account). Hence,

\[ PNL = -rC\Delta t + \frac{\partial C}{\partial t} \Delta t + \frac{\partial C}{\partial f} \Delta f + \frac{1}{2} \frac{\partial^2 C}{\partial f^2} (\Delta f)^2 - \frac{\partial C}{\partial f} \Delta f + o(\Delta t) \]

\[ = -rC\Delta t + \frac{\partial C}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 C}{\partial f^2} (\Delta f)^2 + o(\Delta t) \]

\[ = -rC\Delta t + \frac{\partial C}{\partial t} \Delta t + \frac{f^2 \sigma^2}{2} \frac{\partial^2 C}{\partial f^2} \Delta t + \frac{f^2}{2} \frac{\partial^2 C}{\partial f^2} \left[ \left( \frac{\Delta f}{f} \right)^2 - \sigma^2 \Delta t \right] + o(\Delta t) \]

\[ = \left( -rC + \frac{\partial C}{\partial t} + \frac{f^2 \sigma^2}{2} \frac{\partial^2 C}{\partial f^2} \right) \Delta t + \frac{f^2}{2} \frac{\partial^2 C}{\partial f^2} \left[ \left( \frac{\Delta f}{f} \right)^2 - \sigma^2 \Delta t \right] + o(\Delta t) \]
Derivation (III)

• If we assume that

\[ E\left[ \left( \frac{\Delta f}{f} \right)^2 \right] = \sigma^2 \Delta t + o(\Delta t) \]

the profit/loss from the hedged strategy would be on average

\[ PNL = \left( \frac{\partial C}{\partial t} + \frac{\sigma^2 f^2}{2} \frac{\partial^2 C}{\partial f^2} - rC \right) \Delta t + o(\Delta t) \]

• We conclude that the fair value of the option should satisfy the PDE

\[ \frac{\partial C(f,t)}{\partial t} + \frac{\sigma^2 f^2}{2} \frac{\partial^2 C(f,t)}{\partial f^2} - rC(f,t) = 0, \quad 0 < t < T \]
Black vs. Black Scholes PDE

• The PDE is a special case of the Black-Scholes PDE with $r=q$ and therefore no first-order term.

• For European-style options, this leads to the boundary-value problem

$$\frac{\partial C(f,t)}{\partial t} + \frac{\sigma^2 f^2}{2} \frac{\partial^2 C(f,t)}{\partial f^2} - rC(f,t) = 0, \quad 0 < t < T$$

$$C(f,T) = F(f), \quad [F(f) = (f - K)^+ \text{ (call)}, \quad F(f) = (K - f)^+ \text{ (put)}]$$

This has Black’s formula as an exact solution.

• For American options, the PDE is solved numerically, e.g. with the trinomial scheme.

• If the interest rate is positive, American-style options which are deep-in-the-money should be exercised before expiration.
Multi-period models & beyond Black-Scholes

• In a previous lecture, we presented option pricing as an expected value over a probability measure.

• Using a hedging argument, we also presented a PDE approach for pricing options and showed (numerically) that this lead to the same prices in the case of European options.

• To connect both methods and increase our understanding of pricing of derivatives, we need to consider multi-period models and a more general setting.
Assumptions

- N trading dates
- On each trading date, we know the interest rate for the next period, the dividend yield for the next period and the price level, i.e.

\[(S_t, q_t, r_t), \ t = t_1, t_2, \ldots, t_N\]

- \(S\) and \(q\) can represent vectors, i.e., multiple securities
Arbitrage opportunities

• An arbitrage opportunity consists of a trading strategy which has zero initial cost and which gives rise to non-negative cash-flows in all future states of the market with at least one state with positive cash flows.

• In a 1-period model, we saw that the absence of arbitrage opportunities implies the existence of a probability measure in the set of states at time $T$ such that

$$V_0 = \frac{1}{1 + r\Delta t} E^p \{V_T\}$$

all traded securities satisfy

In particular,

$$F_{0,T} = E^p \{S_T\}$$
The Hahn-Banach Theorem

• Let $X$ be a finite space representing the future states of a market, and let $\Pi(F)$ be a linear functional defined on functions of future states:

$$\Pi(\lambda F + \mu G) = \lambda \Pi(F) + \mu \Pi(G)$$

Here, the functional represents the price of a derivative with payoff $F(x)$ outcome $x$ is realized.

• If $\Pi(\ )$ has the property that $F(x) > 0 \quad \forall x \in X \quad \Rightarrow \quad \Pi(F) > 0$, then

$$\Pi(F) = \sum_{x \in X} w(x)F(x) \quad \forall F$$

for some positive weights $\{w(x), \ x \in X\}$
Absence of arbitrage opportunities: the formalization of "fair value"

- In the one-period model, we found that the absence of arbitrage opportunities implied that there exists a probability measure on future states such that any derivative security with value $V_F$ at time 0 and value $F(S)$ at time $T$ should have a price $V_F$ satisfying

$$V_F(S_0) = \frac{1}{1 + rT} E^P \{ F(S_T) \}$$

- In a multi-period model, the same equation should hold conditionally on the current state of the market, i.e. if a security pays a cash-flow $F(S)$ at time $t_{n+1}$, its value at time $t_n$ should satisfy

$$V(S_{t_1}, S_{t_2}, \ldots, S_{t_n}; t_n) = \frac{1}{1 + r_n \Delta t} E^{P_{t_n,t_{n+1}}} \{ F(S_{t_{n+1}}) \}$$
The probability $p_{n,n+1}$

- The probability can be seen as a function of the observations up to time $t_n$ forecasting the price level at time $t_{n+1}$:

$$p_{n,n+1}(X) = p_{n,n+1}\{S_{t_{n+1}} = X \mid (S_{t_i}, d_{t_i}, r_{t_i}), i \leq n\}$$

- Main idea:
  The pricing probability from date $n$ to date $n+1$ must satisfy a constraint, associated with forward pricing.
Forwards and probability constraints

- If interest and dividend are known at date \( n \), then the forward price is known as well, and we must have

\[
0 = \frac{1}{1 + r_{n} \Delta t} E^{p_{n,n+1}} \{ S_{t_{n+1}} - F_{t_{n},t_{n+1}} \} \quad \therefore \quad E^{p_{n,n+1}} \{ S_{t_{n+1}} \} = F_{t_{n},t_{n+1}}
\]

\[
E^{p_{n,n+1}} \{ S_{t_{n+1}} \} = \frac{1 + r_{n} \Delta t}{1 + q_{n} \Delta t} S_{t_{n}}
\]

``currency-type model for the dividend”’” (for simplicity)
Probabilities for prices defined over paths

Given the 1-step probabilities $p_{n,n+1}$, this formula defines a probability over paths
Characterization of $P(\cdot)$

- Define the discounted total return process $T$ by

$$T_{t_n} \equiv \left( \prod_{j=0}^{n-1} \left( 1 + q_{t_j} \Delta t \right) \right) \cdot S_{t_n}, \quad T_{t_0} = S_{t_0}$$

- Then,

$$E^P \{ T_{t_{n+1}} \mid T_{t_n} \} = E^{P_{n,n+1}} \{ T_{t_{n+1}} \}$$

$$= \left( \prod_{j=0}^{n} \left( 1 + q_{t_j} \Delta t \right) \right) E^{P_{n,n+1}} \{ S_{t_{n+1}} \} = \left( \prod_{j=0}^{n} \left( 1 + q_{t_j} \Delta t \right) \right) \frac{1 + r_{t_n} \Delta t}{1 + q_{t_n} \Delta t} S_{t_n}$$

$$= \left( \prod_{j=0}^{n-1} \left( 1 + q_{t_j} \Delta t \right) \right) S_{t_n} = T_{t_n}$$

$$E^P \{ T_{t_{n+1}} \mid T_{t_n} \} = T_{t_n}$$
No-arbitrage theorem for multi-period asset pricing

- If there are no arbitrage opportunities, there exists a probability measure over paths of traded securities, \( P() \), such that the discounted total-return process of any traded security is a **martingale** under \( P \), i.e.,

\[
T_{t_n} \equiv \left( \prod_{j=0}^{n-1} \left( 1 + q_{t_j} \Delta t \right) \right) \cdot S_{t_n}, \quad T_{t_0} = S_{t_0}
\]

we have

\[
E^P \left\{ T_{t_{n+1}} \mid T_{t_n} \right\} = T_{t_n}
\]