Risk and Portfolio Management
Spring 2010

Option portfolios with several underlying assets
Option trades and portfolios: Many different styles

-- Carry trades using options (implied dividend vs. actual dividend, HTB)

-- Volatility surface trades (non-directional): trading different strikes on the same underlying asset

-- historical vol vs implied vol

-- Relative-value trades across names (non-directional)
  -- single-name option versus fair-value
  -- dispersion trading (index option versus components)

-- Directional volatility trades (long vol/short vol, etc)
Delta-neutral option position

-- Open position (long or short) and simultaneously trade the stock so as to be delta-neutral.
-- Adjust the Delta of the option as the stock/option prices move

\[
dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial \sigma} d\sigma + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} dS^2 + ...\]

\[
P & L \approx dC - \Delta dS + \Delta S r dt - \Delta S d\sigma dt - rC dt
\]

\[
= \left(\frac{\partial C}{\partial S} - \Delta\right) dS + \frac{\partial C}{\partial \sigma} d\sigma + \frac{S^2}{2} \frac{\partial^2 C}{\partial S^2} \left(\frac{dS^2}{S^2} - \sigma^2 dt\right)
\]

\[
- \left(\frac{\partial C}{\partial S} - \Delta\right) S (r - d) dt
\]

\[
+ \left(\frac{\partial C}{\partial t} + \frac{S^2 \sigma^2}{2} \frac{\partial^2 C}{\partial S^2} + (r - d) S \frac{\partial C}{\partial S} - rC\right) dt
\]

\[
\approx \frac{\partial C}{\partial \sigma} d\sigma + \frac{S^2}{2} \frac{\partial^2 C}{\partial S^2} \left(\frac{dS^2}{S^2} - \sigma^2 dt\right)
\]
Book-keeping: profit/loss from a delta-hedged option position

\[ P/L = \theta \cdot (n^2 - 1) + V \cdot d\sigma \]

or

\[ P/L = \frac{1}{2} \Gamma \cdot \left( \frac{(dI)^2}{I^2} - \sigma^2 dt \right) + V \cdot d\sigma \]
1-day P/L for Long Call/Short Stock

(Constant volatility=16%)

\[
P/L \approx \theta \cdot (n^2 - 1)
\]

\[\theta = \text{daily time - decay}, \quad n = \frac{\text{percent index change}}{\text{expected daily volatility}}\]
Assuming an implied volatility drop of 1%

Vol=15%

3.80 loss if stock does not move and volatility drops 1%
A closer look at the profit-loss due to a change in volatility

1% move in vol => 8% move in premium for a 6m ATM option
Measuring the Risk of a Portfolio
(assuming delta neutrality)

Portfolio of options on $N$ stocks

$n_{ij}$ contracts of option with underlying
stock $i$, expiration $T_j$, volatility $\sigma_{ij}$

$$\Delta \Pi = \sum_{ij} n_{ij} \left( C(S_i + \Delta S_i, T_j, K_{ij}, \sigma_{ij} + \Delta \sigma_{ij}) - C(S_i, T_j, K_{ij}, \sigma_{ij}) - \frac{\partial C_{ij}}{\partial S_i} \Delta S_i \right)$$

$$= \sum_{ij} n_{ij} \left( C(S_i (1 + R^{S_i}), T_j, K_{ij}, \sigma_{ij} (1 + R^{\sigma_{ij}})) - C(S_i, T_j, K_{ij}, \sigma_{ij}) - \frac{\partial C_{ij}}{\partial S_i} S_i R^{S_i} \right)$$

Need to define a joint distribution of stock returns and volatility returns to calculate statistics of PNL
Factor Models for Price/Vols

Consider only parallel vol shifts and use 30-day ATM volatilities

\[ R^{S_i} = \sum_{k=1}^{m} \beta_{ik} F_k + \varepsilon_i \]

\[ R^{\sigma_i} = \sum_{k=1}^{m} \gamma_{ik} F_k + \varsigma_i \]

Extract factors from PCA of augmented matrix

\[ C_{ij} = \langle R^{S_i} R^{S_j} \rangle, \quad D_{ij} = \langle R^{S_i} R^{\sigma_j} \rangle, \quad E_{ij} = \langle R^{\sigma_i} R^{\sigma_j} \rangle \]

\[ M = \begin{pmatrix} C & D \\ D' & E \end{pmatrix}, \quad M \in \mathbb{R}^{2N \times 2N} \]
Multivariate Analysis of Implied Vols

-- ATM constant maturity vols can be built using interpolation of variances

\[ \sigma_{30d}^2 = \frac{30 - T_1}{T_2 - T_1} \sigma_{T_1}^2 + \frac{T_2 - 30}{T_2 - T_1} \sigma_{T_2}^2 \]

-- WRDS has historical data on CM volatility surfaces parameterized by Deltas for standard maturities (Option Metrics)

-- Compute extreme values of standardized vol returns

-- Perform factor analysis (PCA) to explore the dimensionality of the cross-section

-- Dataset: 98 constituents of Nasdaq 100, from 9/4/2008 to 10/30/2009
Excerpt of the data used for the calculations

<table>
<thead>
<tr>
<th>DATES</th>
<th>AAPL</th>
<th>ADBE</th>
<th>ADSK</th>
<th>AKAM</th>
<th>ALTR</th>
<th>AMAT</th>
<th>AMGN</th>
<th>AMLN</th>
<th>AMZN</th>
<th>APOL</th>
</tr>
</thead>
<tbody>
<tr>
<td>20070904</td>
<td>45.2%</td>
<td>30.9%</td>
<td>32.7%</td>
<td>42.9%</td>
<td>30.4%</td>
<td>27.9%</td>
<td>29.4%</td>
<td>44.9%</td>
<td>37.9%</td>
<td>38.8%</td>
</tr>
<tr>
<td>20070905</td>
<td>48.0%</td>
<td>29.5%</td>
<td>32.3%</td>
<td>44.7%</td>
<td>31.0%</td>
<td>29.1%</td>
<td>31.3%</td>
<td>44.8%</td>
<td>41.1%</td>
<td>39.2%</td>
</tr>
<tr>
<td>20070906</td>
<td>45.7%</td>
<td>29.6%</td>
<td>31.9%</td>
<td>46.6%</td>
<td>30.9%</td>
<td>28.7%</td>
<td>31.6%</td>
<td>45.6%</td>
<td>39.6%</td>
<td>39.5%</td>
</tr>
<tr>
<td>20070907</td>
<td>46.2%</td>
<td>32.2%</td>
<td>33.8%</td>
<td>46.7%</td>
<td>32.0%</td>
<td>33.1%</td>
<td>32.9%</td>
<td>47.1%</td>
<td>40.4%</td>
<td>40.3%</td>
</tr>
<tr>
<td>20070910</td>
<td>45.6%</td>
<td>33.6%</td>
<td>34.3%</td>
<td>45.0%</td>
<td>32.7%</td>
<td>33.2%</td>
<td>33.5%</td>
<td>47.7%</td>
<td>41.8%</td>
<td>43.0%</td>
</tr>
<tr>
<td>20070911</td>
<td>45.9%</td>
<td>32.5%</td>
<td>33.3%</td>
<td>42.8%</td>
<td>31.3%</td>
<td>32.1%</td>
<td>27.8%</td>
<td>47.6%</td>
<td>41.0%</td>
<td>41.9%</td>
</tr>
<tr>
<td>20070912</td>
<td>44.5%</td>
<td>32.7%</td>
<td>34.0%</td>
<td>42.5%</td>
<td>31.9%</td>
<td>33.4%</td>
<td>26.7%</td>
<td>46.5%</td>
<td>41.3%</td>
<td>42.8%</td>
</tr>
<tr>
<td>20070913</td>
<td>43.1%</td>
<td>34.6%</td>
<td>33.6%</td>
<td>41.8%</td>
<td>31.3%</td>
<td>32.7%</td>
<td>25.1%</td>
<td>49.5%</td>
<td>42.3%</td>
<td>43.0%</td>
</tr>
<tr>
<td>20070914</td>
<td>42.1%</td>
<td>34.0%</td>
<td>32.6%</td>
<td>43.0%</td>
<td>31.4%</td>
<td>32.9%</td>
<td>27.6%</td>
<td>46.6%</td>
<td>42.2%</td>
<td>42.7%</td>
</tr>
<tr>
<td>20070917</td>
<td>44.2%</td>
<td>36.0%</td>
<td>33.9%</td>
<td>45.8%</td>
<td>34.2%</td>
<td>32.3%</td>
<td>27.9%</td>
<td>49.7%</td>
<td>43.9%</td>
<td>45.1%</td>
</tr>
<tr>
<td>20070918</td>
<td>40.1%</td>
<td>26.8%</td>
<td>30.3%</td>
<td>44.3%</td>
<td>29.1%</td>
<td>31.3%</td>
<td>25.7%</td>
<td>49.8%</td>
<td>42.2%</td>
<td>44.4%</td>
</tr>
<tr>
<td>20070919</td>
<td>39.8%</td>
<td>26.1%</td>
<td>31.9%</td>
<td>44.3%</td>
<td>29.7%</td>
<td>29.7%</td>
<td>28.2%</td>
<td>48.4%</td>
<td>41.0%</td>
<td>42.5%</td>
</tr>
<tr>
<td>20070920</td>
<td>38.5%</td>
<td>27.5%</td>
<td>31.3%</td>
<td>43.2%</td>
<td>29.6%</td>
<td>30.4%</td>
<td>27.5%</td>
<td>47.8%</td>
<td>42.5%</td>
<td>43.4%</td>
</tr>
</tbody>
</table>
Average Implied Volatility vs. QQV (Implied Vol of NDX-100)
QQ-plot: AAPL 30D vol shocks

X = student df = 4
Y = standardized
AAPL vol returns

y = 0.9674x - 0.0166
R^2 = 0.5776
QQ-plot: LLTC vol returns

$X=\text{student df}=4$

$Y=\text{standardized LLTC vol returns}$

$\gamma = 1.089 \times -0.0099$

$R^2 = 0.9893$
LLTC vs Student with df=1000
(just to see that tails are indeed fat!)

X=student df=1000
Y= standardized
LLTC vol returns
PCA Calculations

-- There are 98 stocks (implied volatilities)

-- We perform a dynamic PCA with window of 180 days

-- 365 successive calculations (spectrum, eigenvectors)
Spectrum on 5/22/2008
Eigenvalues on 12/1/2008
Evolution of 1st and 2nd eigenvalues from May 2008 to Oct 2009
Factor Model

\[
\frac{d\sigma_{ATM,i}}{\sigma_{ATM,i}} = k_i \left( \sum_{k=1}^{m} \gamma_{i,k} F_k + \sqrt{1 - \sum_{i=1}^{m} \gamma_{i,k}^2 G_k} \right)
\]

\[
\frac{d\sigma_i(x)}{\sigma_i(x)} = \frac{d\sigma_{ATM,i}}{\sigma_{ATM,i}} + \delta_i dx \quad x = \ln\left(\frac{K}{S}\right), \quad dx = -\frac{dS}{S}
\]

The motivation for the second equation is that we assume a parametric skew model

\[
\sigma(x) = \sigma_{ATM} \left(1 + \delta x + \gamma x^2 + \ldots\right)
\]
Alternative Approach using ETFs

\[ \frac{d\sigma_i}{\sigma_i} = \beta_i \frac{dS_i}{S_i} + \gamma_i \frac{d\sigma_{ETF(i)}}{\sigma_{ETF(i)}} + \zeta_i, \]

\[ ETF(i) = \text{ETF associated with stock } i \]

Model the ATM volatility returns as a function of the stock return and changes in the volatility of the sector.

Conjecture: there are fewer systematic factors that explain volatility returns than in the case of stock returns. (m<20)
Volatility skew of stocks and volatility skew of indexes

-- For equities, the implied volatility curve is decreasing in the strike price around ATM

-- The effect is more pronounced for indices and ETFs than for single names

-- Indexes are more skewed than single stocks, presumably due to "correlation risk"

-- Indexes implied vol curves have less convexity than single-stock implied volatility curves
AAPL 30D Vol 9/2/2008

BS Call Delta

Implied Vol

80 75 70 65 60 55 50 45 40 35 30 25 20

0.37 0.38 0.39 0.4 0.41 0.42 0.43 0.44 0.45
DIA 30D Vol 9/2/2008

Graph showing the relationship between BS Call Delta and Implied Vol.
AAPL 30D Skew vs. DIA 30D Skew
2/9/2008

AAPL Skew (Vol/VolATT)
DIA Skew (vol/volATM)

BS Call Delta

Implied Vol/ATM Vol

80 75 70 65 60 55 50 45 40 35 30 25 20
Modeling the Volatility Skew

\[ x = \ln(K / S) \]

\[ \sigma_{imp}(x, t) = \sigma_{imp}(0, t) \cdot (1 + \gamma x + \delta x^2 + ...) \]

**Proposition:** Under reasonable assumptions on model (stoch. vol),

\[
\text{If} \quad \frac{d\sigma_{atm}}{\sigma_{atm}} = \beta \frac{dS}{S} + \epsilon
\]

\[
\text{Then} \quad \gamma = \frac{\beta}{2}
\]

Can also check this directly on data.
Evolution of the slope of the 30-day implied volatility curve, 1996-2004

Avellaneda & Lee, 2005
Evolution of ratio [slope/leverage coefficient]
The ``roaring 90’s”!

\[ \frac{\text{slope}}{\text{leverage coefficient}} \]

Fair value line (SV) \( \gamma = \beta / 2 \)

Avellaneda & Lee, 2005